

RANDOM WALKS ON THE TWO-DIMENSIONAL K -COMB LATTICE

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ABSTRACT

We study the path behavior of the symmetric walk on some special comb-type subsets of \mathbb{Z}^2 which are obtained from \mathbb{Z}^2 by generalizing the comb having finitely many horizontal lines instead of one.

KEYWORDS

Random walk, 2-dimensional comb, strong approximation, 2-dimensional Wiener process, laws of the iterated logarithm, iterated Brownian motion

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 60F17, 60G50, 60J65; Secondary 60F15, 60J106

1. INTRODUCTION

The anisotropic random walk has a huge literature. Some important early work on this topic is due to Heyde [14, 15]. In our papers [7, 9] we gave an account of some of the relevant literature. In this paper we consider an anisotropic walk, which is defined as a nearest neighbor random walk on the square lattice \mathbb{Z}^2 of the plane with possibly unequal symmetric horizontal and vertical step probabilities, so that these probabilities depend only on the values of the vertical coordinates.

A very important special case is the simple random walk on the 2-dimensional comb lattice which is obtained from \mathbb{Z}^2 by removing all horizontal lines off the x -axis. More formally, consider the random walk $\{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \dots\}$ on \mathbb{C}^2 , which is the subset of \mathbb{Z}^2 , containing all vertical lines and the x -axis with the transition probabilities for $(k, j) \in \mathbb{Z}^2$, $N = 0, 1, 2, \dots$

$$\begin{aligned} P(C(N+1) = (k, j \pm 1) | C(N) = (k, j)) &= \frac{1}{2} \quad \text{if } j \neq 0 \\ P(C(N+1) = (k, \pm 1) | C(N) = (k, 0)) &= P(C(N+1) = (k \pm 1, 0) | C(N) = (k, 0)) = \frac{1}{4} \end{aligned} \quad (1.1)$$

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For a recent review of some related literature concerning this simple random walk we refer to Bertacchi [1] and Csáki et al. [6]. In the latter paper we established a simultaneous strong approximation for the two coordinates of the random walk $C(N) = (C_1(N), C_2(N))$ that reads as follows.

THEOREM A ([6]). On an appropriate probability space for the simple random walk

$$\{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \dots\}$$

on the two-dimensional comb lattice \mathbb{C}^2 , one can construct two independent standard Wiener processes $\{W_1(t); t \geq 0\}$, $\{W_2(t); t \geq 0\}$ so that, as $N \rightarrow \infty$, we have with any $\varepsilon > 0$

$$N^{-1/4}|C_1(N) - W_1(\eta_2(0, N))| + N^{-1/2}|C_2(N) - W_2(N)| = O(N^{-1/8+\varepsilon}) \quad \text{a.s.},$$

where $\eta_2(0, \cdot)$ is the local time process at zero of $W_2(\cdot)$.

In this paper we want to generalize this theorem, by permitting to have finitely many horizontal lines, instead of one, and also relax the requirements about the probabilities along these lines as follows. At first we select a finite set of permitted horizontal lines. We call this set B containing the horizontal lines at $y = m_j$, $j = 1, 2, \dots, K$. Denote by $\mathbb{C}_K^2(B)$ the subset of \mathbb{Z}^2 containing all vertical lines and only those horizontal lines which belong to our fix set B . Then we consider the random walk with the transition probabilities for $(k, j) \in \mathbb{C}_K^2(B)$, $N = 0, 1, 2, \dots$

$$P(C(N+1) = (k, \ell \pm 1) | C(N) = (k, \ell)) = \frac{1}{2} \quad \text{if } \ell \notin B$$

and for $m_j \in B$, $j = 1, 2, \dots, K$

$$\begin{aligned} P(C(N+1) = (k, m_j \pm 1) | C(N) = (k, m_j)) &= p_j \\ P(C(N+1) = (k \pm 1, m_j) | C(N) = (k, m_j)) &= \frac{1}{2} - p_j \end{aligned} \quad (1.2)$$

where $0 < p_j < \frac{1}{2}$.

Unless otherwise stated, we assume also that $C(0) = (0, 0)$. We will call the above random walk as K -comb walk on $\mathbb{C}_K^2(B)$. Introduce the notations $\alpha_j = 2p_j$, $j = 1, 2, \dots, K$ and

$$A_K = \sum_{j=1}^K \frac{1 - \alpha_j}{\alpha_j}. \quad (1.3)$$

In our paper [10] we considered more general (much bigger) B sets but only under the condition of $p_j = 1/4$ for all j -s. The main result of this paper is the following generalization of Theorem A:

THEOREM 1.1. On an appropriate probability space for the random walk $\{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \dots\}$ on the two-dimensional K -comb lattice $\mathbb{C}_K^2(B)$, one can construct two independent standard Wiener processes $\{W_1(t); t \geq 0\}$, $\{W_2(t); t \geq 0\}$ so that, as $N \rightarrow \infty$, we have with any $\varepsilon > 0$

$$N^{-1/4}|C_1(N) - W_1(A_K \eta_2(0, N))| + N^{-1/2}|C_2(N) - W_2(N)| = O(N^{-1/8+\varepsilon}) \quad \text{a.s.},$$

where $\eta_2(0, \cdot)$ is the local time process at zero of $W_2(\cdot)$.

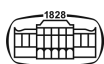
REMARK 1.1. Observe that the above result does not depend on the positions of the K horizontal lines, only on A_K defined in (1.3).

REMARK 1.2. In case $K = 1$ and $p_1 = 1/4$ Theorem 1.1 coincides with Theorem A.

The structure of this paper from now on is as follows. In Section 2 we give preliminary facts and results. In Section 3, first we redefine the walk on $\mathbb{C}_K^2(B)$ in terms of two independent simple symmetric walks, and prove Theorem 1.1. In Section 4 some consequences will be discussed.

2. PRELIMINARIES

In this section we list some well-known results, and some new ones which will be used in the rest of the paper. In case of the known ones we won't give the most general form of the results, just as much as we intend to use, while the exact reference will also be provided for the interested reader.



Let $\{X_i\}_{i \geq 1}$ be a sequence of independent i.i.d. random variables, with $P(X_i = \pm 1) = 1/2$. Then the simple symmetric random walk on the line is defined as $S(n) = \sum_{i=1}^n X_i$, and its local time is $\xi(j, n) = \#\{k : 0 < k \leq n, S(k) = j\}$, $n = 1, 2, \dots$ for any integer j .

For $\xi(n) = \sup_x \xi(x, n)$ we have Kesten's LIL for local time.

LEMMA A (Kesten [17]). For the maximal local time we have

$$\limsup_{n \rightarrow \infty} \frac{\xi(n)}{(2n \log \log n)^{1/2}} = 1 \quad \text{a.s.}$$

Let $(W(t), t \geq 0)$ be a standard Wiener process (called also standard Brownian motion). Its local time $(\eta(x, t), x \in \mathbb{R}, t \geq 0)$ (called Wiener local time or Brownian local time) is defined as

$$\eta(x, t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t I\{W(s) \in (x - \varepsilon, x + \varepsilon)\} ds,$$

where $I\{\cdot\}$ denotes the indicator function.

Concerning the increments of the Brownian motion, Brownian local time and their random walk counterparts we quote the following results from Csörgő and Révész [13], (see in [20] page 69), Csáki and Földes [8] and Csáki et al. [4].

LEMMA B. Let $0 < a_T \leq T$ be a non-decreasing function of T . Then, as $T \rightarrow \infty$, we have

$$\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |W(t + s) - W(t)| = O(a_T(\log(T/a_T) + \log \log T))^{1/2} \quad \text{a.s.}$$

$$\sup_{0 \leq t \leq T - a_T} |\eta(0, t + a_T) - \eta(0, t)| = O(a_T(\log(T/a_T) + \log \log T))^{1/2} \quad \text{a.s.}$$

$$\sup_{0 \leq t \leq T - a_T} |\xi(0, t + a_T) - \xi(0, t)| = O(a_T(\log(T/a_T) + \log \log T))^{1/2} \quad \text{a.s.,}$$

where in the last line t and T should be integers.

We quote the following simultaneous strong approximation result from Révész [19].

LEMMA C ([19]). On an appropriate probability space for a simple symmetric random walk $\{S(n); n = 0, 1, 2, \dots\}$ with local time $\{\xi(x, n); x = 0, \pm 1, \pm 2, \dots; n = 0, 1, 2, \dots\}$ one can construct a standard Wiener process $\{W(t); t \geq 0\}$ with local time process $\{\eta(x, t); x \in \mathbb{R}; t \geq 0\}$ such that, as $n \rightarrow \infty$, we have for any $\varepsilon > 0$

$$|S(n) - W(n)| = O(n^{1/4+\varepsilon}) \quad \text{a.s.}$$

and

$$\sup_{x \in \mathbb{Z}} |\xi(x, n) - \eta(x, n)| = O(n^{1/4+\varepsilon}) \quad \text{a.s.,}$$

simultaneously.

The following result about the uniformity of the local time is in Heyde [14], see also in Csáki and Révész [12].

LEMMA D ([12, 14]). For the simple symmetric walk for any $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \sup_x \frac{|\xi(x + 1, n) - \xi(x, n)|}{n^{1/4+\varepsilon}} = 0 \quad \text{a.s.}$$

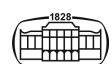
REMARK 2.1. In fact [12] deals with more general random walks, but we only need it for a simple symmetric random walk.

The following result is the so called exponential Kolmogorov inequality. It is a direct consequence of Doob's maximal inequality. Its proof can be found e.g. on page 139 of Williams [22].

LEMMA E. Let X_j , $j = 1, 2, \dots$ be i.i.d. random variables with $E(e^{\theta|X_j|}) < \infty$ for some $\theta > 0$ and $E(X_j) = 0$. Then for any $\lambda > 0$

$$P\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^j X_i\right| > \lambda\right) \leq e^{-\lambda\theta} \left((Ee^{\theta|X_j|})^n + (Ee^{-\theta|X_j|})^n\right).$$

The following result is a generalization of an inequality of Tóth [21].



LEMMA 2.1. Let $G_i, i = 1, 2, \dots$ be i.i.d. random variables with the common geometric distribution $P(G_i = k) = \alpha(1 - \alpha)^k, k = 0, 1, 2, \dots$ for some $0 < \alpha < 1$. Then for n big enough

$$P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j \left(G_i - \frac{1-\alpha}{\alpha} \right) \right| > \lambda\right) \leq 2 \exp\left(-\frac{\lambda^2 \alpha^2}{4(1-\alpha)n}\right)$$

for $0 < \lambda < na$ with some $a > 0$.

Proof. The common moment generating function of G_i -s for $i = 1, 2, \dots$ is

$$E(e^{\theta G_i}) = \frac{\alpha}{1 - e^{\theta(1-\alpha)}}$$

and $E(G_i) = \frac{1-\alpha}{\alpha}$. Then

$$E\left(e^{\theta(G_i - \frac{1-\alpha}{\alpha})}\right) = \frac{\alpha}{(1 - e^{\theta(1-\alpha)})e^{\theta(\frac{1-\alpha}{\alpha})}}.$$

Then for $\theta > 0$, small enough we have that

$$1 - e^{\theta(1-\alpha)} = 1 - (1-\alpha)(1 + \theta + \frac{\theta^2}{2} + O(\theta^3)), \quad (2.1)$$

and

$$e^{\theta(\frac{1-\alpha}{\alpha})} = 1 + \frac{1-\alpha}{\alpha}\theta + \left(\frac{1-\alpha}{\alpha}\right)^2 \frac{\theta^2}{2} + O(\theta^3), \quad (2.2)$$

where θ above might depend on α . Now with elementary calculation we get, that

$$(1 - e^{\theta(1-\alpha)})e^{\theta(\frac{1-\alpha}{\alpha})} = \alpha - \frac{1-\alpha}{2\alpha}\theta^2 + O(\theta^3) = \alpha \left(1 - \frac{1-\alpha}{2\alpha^2}\theta^2\right) + O(\theta^3). \quad (2.3)$$

and

$$E\left(e^{\theta(G_i - \frac{1-\alpha}{\alpha})}\right) = 1 + \theta^2 \frac{1-\alpha}{2\alpha^2} + O(\theta^3) = e^{\frac{(1-\alpha)\theta^2}{2\alpha^2}} + O(\theta^3) \leq e^{\frac{(1-\alpha)\theta^2}{\alpha^2}}, \quad (2.4)$$

again with θ small enough such that (2.1)–(2.4) should hold.

Considering now the moment generating function $E(e^{-\theta(G_i - \frac{1-\alpha}{\alpha})})$ and almost identical calculation results that

$$E\left(e^{-\theta(G_i - \frac{1-\alpha}{\alpha})}\right) \leq e^{\frac{(1-\alpha)\theta^2}{\alpha^2}}, \quad (2.5)$$

as well, again with some $\theta > 0$ small enough that (2.1)–(2.5) should hold.

Applying now the exponential Kolmogorov Inequality (Lemma E) we get that

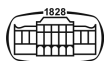
$$\begin{aligned} &P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j \left(G_i - \frac{1-\alpha}{\alpha} \right) \right| > \lambda\right) \\ &< e^{-\lambda\theta} \left(\left(Ee^{\theta(G_i - \frac{1-\alpha}{\alpha})} \right)^n + \left(Ee^{-\theta(G_i - \frac{1-\alpha}{\alpha})} \right)^n \right) \\ &\leq 2e^{-\lambda\theta} e^{n \frac{(1-\alpha)\theta^2}{\alpha^2}} \leq 2e^{-\frac{\lambda^2 \alpha^2}{4n(1-\alpha)}} \end{aligned} \quad (2.6)$$

where we selected $\theta = \frac{\lambda \alpha^2}{2n(1-\alpha)}$, which obviously can be done for n big enough. \square

3. PROOF OF THE THEOREM 1.1

First we are to redefine our random walk $\{C(N); N = 0, 1, 2, \dots\}$. It will be seen that the process described right below is equivalent to that given in the Introduction. For notational convenience we will say that the set B contains the levels $\{m_j, j = 1, 2, \dots, K\}$. To begin with, on a suitable probability space consider two independent simple symmetric (one-dimensional) random walks $S_1(\cdot)$, and $S_2(\cdot)$. We may assume that on the same probability space we have K sequences of i.i.d. geometric random variables $\{G_i(m_j), i = 1, 2, \dots, j = 1, 2, \dots, K\}$ which are independent from each other and $S_1(\cdot)$, and $S_2(\cdot)$, with

$$P(G_i(m_j) = k) = \alpha_j(1 - \alpha_j)^k, \quad k = 0, 1, 2, \dots \quad \text{with} \quad 0 < \alpha_j < 1, \quad j = 1, 2, \dots, K.$$



Recall, that the parameter of the geometric distribution above is $\alpha_j = 2p_j$ which is the probability that the walk makes a vertical step when it happens to be on the horizontal level m_j . For simplicity we will say that the sequence of geometric random variables $\{G_i(m_j), i = 1, 2, \dots\}$ belongs to the horizontal level m_j . We now construct our walk $C(N)$ as follows. We will take all the horizontal steps consecutively from $S_1(\cdot)$ and all the vertical steps consecutively from $S_2(\cdot)$. Consider our walk starting from the origin. First the walk moves vertically until it arrives at a level which belongs to B . (It is possible that no vertical step is needed, as the x -axis might belong to B). If this level is level $m_j \in B$, then it takes $G_1(m_j)$ horizontal steps from $S_1(\cdot)$. (Note that $G_1(m_j) = 0$ is possible with probability α_j). Then we again take vertical steps from $S_2(\cdot)$, as needed to get to a level belonging to B , then again some horizontal steps from $S_1(\cdot)$ as follows. If this level in B is the same as previously then we take $G_2(m_j)$ horizontal steps. However if it is another level, let's say $m_\ell \in B$ then it takes $G_1(m_\ell)$ horizontal steps from $S_1(\cdot)$, then the walk moves vertically taking steps from $S_2(\cdot)$ until it hits again a level in B , when again it moves horizontally taking steps from $S_1(\cdot)$. In general, whenever the walk arrives at a level in B then it takes some horizontal steps, the number of which is given by the next in line (first unused) geometric random variables belonging to that level.

Introduce the following notations:

$$H_N = \#\{k : 1 \leq k \leq N, C_1(k) \neq C_1(k-1)\} \quad (3.1)$$

$$V_N = \#\{k : 1 \leq k \leq N, C_2(k) \neq C_2(k-1)\} \quad (3.2)$$

thus H_N, V_N be the number of horizontal and vertical steps, respectively, from the first N steps of the just described process. Consequently, $H_N + V_N = N$, and

$$\begin{aligned} \{C(N); N = 0, 1, 2, \dots\} &= \{(C_1(N), C_2(N)); N = 0, 1, 2, \dots\} \\ &\stackrel{d}{=} \{(S_1(H_N), S_2(V_N)); N = 0, 1, 2, \dots\}, \end{aligned} \quad (3.3)$$

where $\stackrel{d}{=}$ stands for equality in distribution. Let $\xi_2(\cdot, \cdot)$ denote the local time of $S_2(\cdot)$, and

$$D_2(V_N) = \sum_{j=1}^K \xi_2(m_j, V_N) \frac{1 - \alpha_j}{\alpha_j}. \quad (3.4)$$

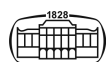
$D_2(V_N)$ is the expected occupation time of the levels belonging to B by $C(\cdot)$ in the first N steps.

LEMMA 3.1. For any $\varepsilon > 0$, as $N \rightarrow \infty$,

$$\max_{1 \leq i \leq N} |H_i - D_2(V_i)| = O(N^{1/4+\varepsilon}) \quad \text{a.s.}$$

Proof. Recall that H_N is the number of horizontal steps in our construction, and horizontal steps only occur on levels belonging to B . When the vertical walk arrives to such a level, m_j , it takes some horizontal steps, the number of which follows geometric distribution with expected value $\frac{1-\alpha_j}{\alpha_j}$. In V_N steps the vertical walk $S_2(\cdot)$ spends $\xi_2(m_j, V_N)$ steps on the level m_j , thus the number of horizontal steps on this level is the sum of $\xi_2(m_j, V_N)$ geometric random variables with common expected value $\frac{1-\alpha_j}{\alpha_j}$. The total number of horizontal steps is $\sum_{j=1}^K \sum_{i=1}^{\xi_2(m_j, V_N)} G_i(m_j)$. However this statement is slightly incorrect, as if the N -th step is a horizontal one, the corresponding last geometric random variable might remain truncated. Denote by H_N^+ the number of horizontal steps which includes all the steps of this last geometric random variable. Then

$$H_N^+ - D_2(V_N) = \sum_{j=1}^K \sum_{i=1}^{\xi_2(m_j, V_N)} \left(G_i(m_j) - \frac{1 - \alpha_j}{\alpha_j} \right),$$



where $G_i(m_j)$, $i = 1, 2, \dots$ are the i.i.d. geometric random variables, belonging to level m_j , as in Lemma 2.1. According to this lemma and Lemma A we have

$$\begin{aligned} \mathbf{P} \left(\max_{1 \leq \ell \leq N} |H_\ell^+ - D_2(V_\ell)| > \lambda \right) &= \mathbf{P} \left(\max_{1 \leq \ell \leq N} \left| \sum_{j=1}^K \sum_{i=1}^{\xi_2(m_j, V_\ell)} \left(G_i(m_j) - \frac{1 - \alpha_j}{\alpha_j} \right) \right| > \lambda \right) \\ &< \sum_{j=1}^K \mathbf{P} \left(\max_{1 \leq \ell \leq N} \left| \sum_{i=1}^{\xi_2(m_j, V_\ell)} \left(G_i(m_j) - \frac{1 - \alpha_j}{\alpha_j} \right) \right| > \frac{\lambda}{K} \right) \\ &\leq \sum_{j=1}^K 2 \exp \left(-\frac{\lambda^2 \alpha_j^2}{4K^2(1 - \alpha_j)N^{1/2+\epsilon}} \right) \leq 2K \exp \left(-\frac{\lambda^2 (\alpha^*)^2}{4K^2(1 - \alpha^*)N^{1/2+\epsilon}} \right), \end{aligned} \quad (3.5)$$

for any $\epsilon > 0$, with $\alpha^* = \min_{1 \leq j \leq K} \alpha_j$ as the function $\frac{\alpha^2}{1-\alpha}$ is increasing for $0 < \alpha < 1$, where we used the fact that

$$\max_{1 \leq \ell \leq N} \xi_2(m_j, V_\ell) \leq N^{1/2+\epsilon} \quad \text{a.s.}$$

if N is big enough.

Selecting $\lambda = N^{1/4+\epsilon}$ we get by the Borel–Cantelli lemma that for N large enough

$$\max_{1 \leq \ell \leq N} |H_\ell^+ - D_2(V_\ell)| = O(N^{1/4+\epsilon}) \quad \text{a.s.}$$

Now to estimate the difference of H_N and H_N^+ , we have to observe, that their difference is not more than one single geometric random variable, which happens to be the last one used up to N . Thus with α^* defined above we have

$$\mathbf{P}(|H_N^+ - H_N| \geq N^\epsilon) \leq \mathbf{P}(\max_{1 \leq j \leq K} \max_{1 \leq i \leq N} G_i(m_j) \geq N^\epsilon) \leq N \max_{1 \leq j \leq K} \mathbf{P}(G_1(m_j) \geq N^\epsilon) \leq N(1 - \alpha^*)^{N^\epsilon}$$

and hence by the Borel–Cantelli lemma

$$H_N^+ - H_N \leq N^\epsilon \quad \text{a.s.}$$

for all large N , proving our lemma. \square

Now observe that based on Lemma D

$$D_2(V_N) = \sum_{j=1}^K \xi_2(m_j, V_N) \frac{1 - \alpha_j}{\alpha_j} = \xi_2(0, V_N) \sum_{j=1}^K \frac{1 - \alpha_j}{\alpha_j} + O(N^{1/4+\epsilon}) \quad \text{a.s.}$$

Recall the notation $A_K = \sum_{j=1}^K \frac{1 - \alpha_j}{\alpha_j}$ given in (1.3). Using this, we have

$$D_2(V_N) = A_K \xi_2(0, V_N) + O(N^{1/4+\epsilon}) \quad \text{a.s.}$$

implying by Lemma 3.1 that

$$H_N = A_K \xi_2(0, V_N) + O(N^{1/4+\epsilon}) \quad \text{a.s.} \quad (3.6)$$

as well. Moreover by (3.6) and Lemma A we have that $H_N = O(N^{1/2+\epsilon})$ a.s., thus

$$|V_N - N| = O(N^{1/2+\epsilon}) \quad \text{a.s.},$$

implying by the last statement of Lemma B that

$$H_N = A_K \xi_2(0, N) + O(N^{1/4+\epsilon}) \quad \text{a.s.}$$

Then we have from (3.3), and Lemmas B and C that

$$\begin{aligned} C_1(N) &= S_1(H_N) = S_1(A_K \xi_2(0, N) + O(N^{1/4+\epsilon})) \\ &= W_1(A_K \xi_2(0, N) + O(N^{1/4+\epsilon})) + O(N^{1/8+\epsilon}) \\ &= W_1(A_K \eta_2(0, N) + O(N^{1/4+\epsilon})) + O(N^{1/8+\epsilon}) \\ &= W_1(A_K \eta_2(0, N)) + O(N^{1/8+\epsilon}) \quad \text{a.s.} \end{aligned} \quad (3.7)$$



and

$$C_2(N) = S_2(V_N) = S_2(N + O(N^{1/2+\varepsilon})) = W_2(N) + O(N^{1/4+\varepsilon}) \quad \text{a.s.}$$

proving Theorem 1.1. \square

4. CONSEQUENCES

Define the continuous version of our random walk process on \mathbb{C}_K^2 by linear interpolation, as follows:

$$\{C(xN) = (C_1(xN), C_2(xN)) : 0 \leq x \leq 1\}.$$

We have almost surely, as $N \rightarrow \infty$,

$$\sup_{0 \leq x \leq 1} \left\| \left(\frac{C_1(xN) - W_1(A_K \eta_2(0, xN))}{N^{1/4}(\log \log N)^{3/4}}, \frac{C_2(xN) - W_2(xN)}{(N \log \log N)^{1/2}} \right) \right\| \rightarrow 0.$$

We have the following laws of the iterated logarithm (for the first statement see Theorem 2.2 in Csáki et al. [5]).

$$\limsup_{n \rightarrow \infty} \frac{C_1(N)}{\sqrt{A_K} N^{1/4} (\log \log N)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.} \quad \text{and} \quad \limsup_{N \rightarrow \infty} \frac{C_2(N)}{(2N \log \log N)^{1/2}} = 1 \quad \text{a.s.}$$

As to the liminf behavior of the max functionals of the two components, we have the same results as for the two dimensional comb lattice [6]. These results are based on the corresponding ones for Wiener process and the iterated process $W_1(\eta_2(0, t))$ and the work of Chung [3], Hirsch [16], Bertoin [2], and Nane [18].

Based on [18], we get the following: Let $\rho(n), n = 1, 2, \dots$, be a non-increasing sequence of positive numbers such that $n^{1/4} \rho(n)$ is non-decreasing. Then we have almost surely that

$$\liminf_{N \rightarrow \infty} \frac{\max_{0 \leq k \leq N} C_1(k)}{N^{1/4} \rho(N)} = 0 \quad \text{or} \quad \infty$$

and

$$\liminf_{N \rightarrow \infty} \frac{\max_{0 \leq k \leq N} C_2(k)}{N^{1/2} \rho(N)} = 0 \quad \text{or} \quad \infty,$$

according as to whether the series $\sum_1^\infty \rho(n)/n$ diverges or converges.

As to the absolute values of the coordinates we get for the max functional of $|C_2(\cdot)|$ that

$$\liminf_{N \rightarrow \infty} \left(\frac{8 \log \log N}{\pi^2 N} \right)^{1/2} \max_{0 \leq k \leq N} |C_2(k)| = 1 \quad \text{a.s.} \quad (4.1)$$

On the other hand, for the max functional of $|C_1(\cdot)|$ we obtain from [6] the following result.

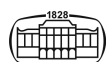
Let $\rho(n), n = 1, 2, \dots$, be a non-increasing sequence of positive numbers such that $n^{1/4} \rho(n)$ is non-decreasing. Then we have almost surely that

$$\liminf_{N \rightarrow \infty} \frac{\max_{0 \leq k \leq N} |C_1(k)|}{N^{1/4} \rho(N)} = 0 \quad \text{or} \quad \infty,$$

as to whether the series $\sum_{n=1}^\infty \rho^2(n)/n$ diverges or converges.

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