



## Regular Articles

 $L_p$  Bernstein type inequalities for star like Lip  $\alpha$  domainsAndrás Kroó<sup>1</sup>

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## ABSTRACT

The goal of the present paper is to establish that *square root of the Euclidean distance to the boundary* is the universal measure suitable for obtaining  $L^p$  Bernstein type inequalities on *general star like Lip 1 domains*. This will be proved for *derivatives of any order*, every  $0 < p < \infty$  and generalized Jacobi type weights. A converse result will show that the “square root of the Euclidean distance to the boundary” in general is the best possible measure in the vicinity of any vertex of a convex polytope. In addition we will also consider *cuspidal* Lip $\alpha$ ,  $0 < \alpha < 1$  graph domains. It turns out that for such cuspidal domains the situation can change dramatically: instead of taking the square root we need to use the  $(\frac{1}{\alpha} - \frac{1}{2})$ -th power of the Euclidean distance to the boundary when  $0 < \alpha < 1$ , and this measure of the distance to the boundary is in general the best possible, as well.

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## 1. Introduction

The classical Bernstein and Markov inequalities for univariate algebraic polynomials of degree  $n$  give the following sharp upper bounds for their derivatives:

$$\|\sqrt{1-x^2}p'(x)\|_{C[-1,1]} \leq n\|p\|_{C[-1,1]}, \quad \|p'\|_{C[-1,1]} \leq n^2\|p\|_{C[-1,1]}.$$

Similar upper bounds for the derivatives of polynomials are known to hold in case of the  $L^p$  norm as well, see for instance [4]. Above inequalities and their numerous generalizations play crucial role in various branches of analysis and approximation theory, it would be difficult to overstate their importance to the field. Extensions of the Bernstein-Markov inequalities to the  $d$  dimensional unit ball  $B^d(0, 1) := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \leq 1\}$  were given by Baran [1] and Sarantopoulos [18] (Bernstein type estimate) and Kellogg [11] (Markov type estimate). Namely it was shown therein that

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$$\|\sqrt{1-|\mathbf{x}|^2}Dp(\mathbf{x})\|_{C(B^d(0,1))} \leq n\|p\|_{C(B^d(0,1))}, \quad \|Dp\|_{C(B^d)} \leq n^2\|p\|_{C(B^d(0,1))}, \quad (1)$$

where  $|\mathbf{x}| = (\sum_{1 \leq j \leq d} x_j^2)^{1/2}$  and  $Dp := (\sum_{1 \leq j \leq d} (\frac{\partial p}{\partial x_j})^2)^{1/2}$  is the  $l_2$  norm of the gradient of  $p \in P_n^d$ . Here as usual  $P_n^d$  stands for the space of real polynomials of  $d$  variables and degree at most  $n$  and  $B(\mathbf{a}, r)$  denotes the ball in  $\mathbb{R}^d$  centered at  $\mathbf{a}$  and radius  $r$ .

In a recent paper [12] the following analogue of the  $L^2$  Bernstein inequality on the unit ball was given:

$$\|\sqrt{1-|\mathbf{x}|^2}Dp\|_{L^2(B^d(0,1))} \leq M_n(d)\|p\|_{L^2(B^d(0,1))},$$

where  $M_n(d) = \sqrt{n(n+d)}$  if  $n$  is even, and  $M_n(d) = \sqrt{n(n+d)-d+1}$  if  $n$  is odd. It should be emphasized that all of the estimates listed above are sharp with equalities being attained for certain polynomials.

As can be seen above Bernstein type inequalities provide an estimate of order  $n$  (versus  $n^2$  in case of Markov inequality) for derivatives of polynomials of degree  $n$ . This significant reduction of order follows from inserting into the norms of derivatives the weights  $\sqrt{1-x^2}$  and  $\sqrt{1-|\mathbf{x}|^2}$  measuring the Euclidean distance to the boundary for domains  $[-1, 1]$  and  $B^d(0, 1)$ , respectively. It should be mentioned that Bernstein type upper bounds of order  $n$  are crucial for establishing Marcinkiewicz-Zygmund type estimates with discrete nodes of asymptotically optimal size. The present paper is devoted to the study of the multivariate  $L^p$  Bernstein type inequality for star like Lip 1 domains. This question was first considered in [8] where a certain measure of distance to the boundary of a convex set was used in order to derive a Bernstein type inequality. In particular, for the model case of bivariate triangle  $\Delta^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 \leq 1\}$  it is shown in [8] (see also that [3], [9])

$$\left\| \sqrt{x_1(1-x_1-x_2)} \frac{\partial q}{\partial x_1} \right\|_{L^p(\Delta^2)} \leq cn\|q\|_{L^p(\Delta^2)}, \quad q \in P_n^d, \quad 1 \leq p < \infty. \quad (2)$$

The factor  $\sqrt{x_1(1-x_1-x_2)}$  in the above inequality may seem to be correct since in the univariate case it reduces to the usual term  $\sqrt{x(1-x)}$  appearing in the Bernstein inequality on  $[0, 1]$ . However, this factor in the neighborhood of vertices of  $\Delta^2$  is substantially smaller than the Euclidean distance to the boundary of the triangle given by  $\min\{x_1, x_2, 1-x_1-x_2\}$ . This discrepancy was pointed out in a recent paper by Y. Xu [20] where it is shown that

$$\left\| \frac{\sqrt{x_1(1-x_1-x_2)}}{\sqrt{1-x_2}} \frac{\partial q}{\partial x_1} \right\|_{L_w^p(\Delta^2)} \leq cn\|q\|_{L_w^p(\Delta^2)}, \quad q \in P_n^d, \quad 1 \leq p < \infty, \quad (3)$$

where  $w$  is a so called doubling weight. Throughout this paper we denote by  $\|\cdot\|_{L_w^p(K)}$  the usual  $L^p$  norm with weight  $w$  on the domain  $K \subset \mathbb{R}^d$  (Lebesgue measure). When  $w = 1$  we simply delete  $w$  from the notation of the norm. One can easily observe that the term  $\frac{\sqrt{x_1(1-x_1-x_2)}}{\sqrt{1-x_2}}$  in (3) in the neighborhood of the vertex  $(0, 1)$  of the triangle  $\Delta^2$  provides a sharper measure of the distance to the boundary than  $\sqrt{x_1(1-x_1-x_2)}$  in (2). In fact since  $\frac{\sqrt{x_1(1-x_1-x_2)}}{\sqrt{1-x_2}} \sim \min\{x_1, x_2, 1-x_1-x_2\}$  around vertex  $(0, 1)$  this essentially corresponds to using Euclidean distance to the boundary. In addition, estimates similar to (3) are given in [20] for conic domains in  $\mathbb{R}^d$  and derivatives of order 1 and 2. Y. Xu [20] also poses a question whether (3) can be extended to all convex polytopes and derivatives of higher order.

One of the main goals of the present paper is to establish that *square root of the Euclidean distance to the boundary* is the universal measure suitable for obtaining  $L^p$  Bernstein type inequalities on *general star like Lip 1 domains*. This will be proved in Theorem 1 below for *derivatives of any order, every  $0 < p < \infty$*  and generalized Jacobi type weights. A converse result (see Theorem 3) will show that the “square root of the Euclidean distance to the boundary” in general is the best possible measure in the vicinity of any vertex of a convex polytope. In addition we will also consider *cuspidal*  $\text{Lip}\alpha, 0 < \alpha < 1$  graph domains. It turns

out that for such cuspidal domains the situation can change dramatically: instead of taking the square root we need to use the  $(\frac{1}{\alpha} - \frac{1}{2})$ -th power of the Euclidean distance to the boundary when  $0 < \alpha < 1$ , and this is sharp, as well.

## 2. $L^p$ Bernstein type inequality for star like Lip 1 domains

Now we turn our attention to the multivariate case and the space  $P_n^d$  of real algebraic polynomials of  $d$  variables and degree at most  $n$ . Throughout this paper we consider *star like domains*  $K \subset \mathbb{R}^d$  given by

$$K := \{t\mathbf{u} : \mathbf{u} \in S^{d-1}, 0 \leq t \leq \rho(\mathbf{u})\}, \quad (4)$$

where  $\rho(\mathbf{u}) : S^{d-1} \rightarrow \mathbb{R}_+$  is a positive continuous function. Here as usual  $S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$  denotes the unit sphere in  $\mathbb{R}^d$ . In addition, without restricting the generality we will always assume that  $K$  contains the unit ball, i.e.  $\rho(\mathbf{u}) \geq 1, \forall \mathbf{u} \in S^{d-1}$ . Furthermore,  $K$  is called a  $\text{Lip}_M 1$  domain if  $\rho \in \text{Lip}_M 1$ , that is

$$|\rho(\mathbf{u}_1) - \rho(\mathbf{u}_2)| \leq M|\mathbf{u}_1 - \mathbf{u}_2|, \quad \mathbf{u}_1, \mathbf{u}_2 \in S^{d-1}.$$

As usual  $\partial K$  will stand for the boundary of  $K$ , and let  $R \geq 1$  be the radius of the smallest ball centered at the origin containing  $K$ , that is  $1 \leq \rho(\mathbf{u}) \leq R, \mathbf{u} \in S^{d-1}$ . Given any  $\mathbf{x} \in K$  we denote by  $\tau_K(\mathbf{x}) := \inf_{\mathbf{y} \in \partial K} |\mathbf{x} - \mathbf{y}|$  the Euclidean distance from  $\mathbf{x}$  to the boundary of the domain. In addition,  $\|f\|_{L_w^p} := (\int_K |f|^p w)^{1/p}$  is the weighted  $L^p$  norm on  $K \subset \mathbb{R}^d$  with a nonnegative weight  $w$ . When  $w = 1$  the notation  $\|f\|_{L^p}$  is used below.

The Bernstein type inequalities for higher derivatives will involve gradients of polynomials  $p$  of order  $k$ . As usual let  $D_{\mathbf{u}}$  denote the derivative in the direction  $\mathbf{u} \in S^{d-1}$ . Then the  $k$ -th order gradient of  $f$  can be defined as

$$\partial^k f(\mathbf{x}) := \max_{\mathbf{u}_j \in S^{d-1}, 1 \leq j \leq k} |D_{\mathbf{u}_1} \dots D_{\mathbf{u}_k} f(\mathbf{x})|, \quad k \in \mathbb{N}.$$

Our results will be derived in a general setting of  $L^p$  norms endowed with multivariate Jacobi type weights. In order to introduce Jacobi type weights we will need to recall the notion of generalized trigonometric and algebraic polynomials of degree  $m$  [4], p. 392 which are defined as functions of the form, respectively

$$f(t) = c \prod_{1 \leq j \leq k} |\sin((t - t_j)/2)|^{r_j}, \quad r_j > 0, c \in \mathbb{R}, \quad m := \frac{1}{2} \sum_{1 \leq j \leq k} r_j,$$

$$g(x) = c \prod_{1 \leq j \leq k} |x - x_j|^{r_j}, \quad r_j > 0, c \in \mathbb{R}, \quad m := \sum_{1 \leq j \leq k} r_j.$$

**Definition.** Let  $K \subset \mathbb{R}^d, d \geq 1$  be any compact set. Given a nonnegative weight  $w$  on  $K$  and  $m > 0$  we will say that  $w$  is Jacobi type weight on  $K$  of degree  $m$  if there exist positive constants  $\alpha_w, \beta_w$  depending only on the weight so that for every line  $l(t) = \mathbf{u}t + \mathbf{a} \subset \mathbb{R}^d, t \in \mathbb{R}$  there exists a generalized algebraic polynomial  $g(t)$  of degree at most  $m$  such that

$$\alpha_w g(t) \leq w(l(t)) \leq \beta_w g(t), \quad \forall l(t) \in K. \quad (5)$$

Above definition essentially implicates that Jacobi type weights may vanish by polynomial rate.

**Theorem 1.** Let  $d, k \in \mathbb{N}, 0 < p < \infty, M, R \geq 1$ . Consider any  $\text{Lip}_M 1$  star like domain  $K \subset \mathbb{R}^d$  given by (4) with  $1 \leq \rho(\mathbf{u}) \leq R, \mathbf{u} \in S^{d-1}$ . Then for any Jacobi type weight  $w$  on  $K$  and every  $q \in P_n^d$  we have

$$\|\tau_K(\mathbf{x})^{\frac{k}{2}} \partial^k q(\mathbf{x})\|_{L_w^p} \leq c(MR)^{\frac{k(k+1)}{2} + \frac{d-1}{p}} n^k \|q\|_{L_w^p}, \quad (6)$$

where  $c > 0$  is a positive constant depending only on  $d, k, p$  and the weight  $w$ .

The proof of the above theorem will be based on several lemmas.

For any  $r > 0$ ,  $\mathbf{a} \in \mathbb{R}^d$  and  $\mathbf{u} \in S^{d-1}$  the cylinder  $L_r(\mathbf{a}, \mathbf{u})$  of radius  $r > 0$ , center  $\mathbf{a}$  and axis  $\mathbf{u}$  is given by

$$L_r(\mathbf{a}, \mathbf{u}) := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{a}|^2 < r^2 + \langle \mathbf{x} - \mathbf{a}, \mathbf{u} \rangle^2\}.$$

An addition,  $l_{\mathbf{x}}(\mathbf{u})$  will denote the line in  $\mathbb{R}^d$  in direction  $\mathbf{u} \in S^{d-1}$  through point  $\mathbf{x} \in \mathbb{R}^d$ . When the cylinder is centered at the origin a simpler notation  $L_r(\mathbf{u})$  will be used below.

Let us recall that  $K \subset \mathbb{R}^d$  is called a graph domain of width  $\delta > 0$  relative to  $L_r(\mathbf{u})$  if for any  $|\mathbf{z}| \leq r$  the line  $l_{\mathbf{z}}(\mathbf{u})$  intersects  $K$  along a single line segment  $[a(\mathbf{z}), b(\mathbf{z})] := l_{\mathbf{z}}(\mathbf{u}) \cap K$  of length  $\geq \delta$ .

The next auxiliary geometric lemma shows that Lip 1 domains are locally graph domains and the distance to the boundary in the Lip 1 domains can be measured along lines in corresponding cylinders.

**Lemma 1.** *Let  $K$  be a  $\text{Lip}_M 1$ ,  $M \geq 1$  star like domain given by (4) with  $1 \leq \rho(\mathbf{u}) \leq R$ ,  $\mathbf{u} \in S^{d-1}$ . Then for every  $\mathbf{u} \in S^{d-1}$  set  $K$  is a graph domain of width  $\sqrt{3}$  relative to the cylinder  $L_a(\mathbf{u})$  with  $a := \frac{1}{\pi M}$ . Moreover, for any  $\mathbf{z} \in L_{a/2}(\mathbf{u})$  we have*

$$\frac{1}{14R^2M} |\mathbf{z} - a(\mathbf{z})| |\mathbf{z} - b(\mathbf{z})| \leq \tau_K(\mathbf{z}) \leq \frac{2}{\sqrt{3}} |\mathbf{z} - a(\mathbf{z})| |\mathbf{z} - b(\mathbf{z})|. \quad (7)$$

**Proof.** We need to show that for every  $\mathbf{u} \in S^{d-1}$  and for any  $|\mathbf{z}| \leq \frac{1}{\pi M}$  the line  $l_{\mathbf{z}}(\mathbf{u})$  intersects  $K$  along a single line segment of length  $\geq \sqrt{3}$ . Hence it suffices to consider the 2-dimensional plane through  $l_{\mathbf{z}}(\mathbf{u})$  and the origin, that is without the loss of generality we may assume that  $d = 2$ ,  $\mathbf{u} = (0, 1)$  and hence

$$K = \{re^{i\phi} : 0 \leq r \leq \rho(e^{i\phi}), 0 \leq \phi \leq 2\pi\}, \quad \partial K = \{\rho(e^{i\phi})e^{i\phi} : 0 \leq \phi \leq 2\pi\}.$$

Furthermore, for any  $\frac{\pi}{2} - \frac{1}{2M} < \phi_2 < \phi_1 < \frac{\pi}{2} + \frac{1}{2M}$  we have using that  $\rho \in \text{Lip}_M 1$ ,  $\rho \geq 1$

$$\begin{aligned} \rho(e^{i\phi_2}) \cos \phi_2 - \rho(e^{i\phi_1}) \cos \phi_1 &= \rho(e^{i\phi_1})(\cos \phi_2 - \cos \phi_1) + \cos \phi_2(\rho(e^{i\phi_2}) - \rho(e^{i\phi_1})) \geq \\ (\phi_1 - \phi_2) \cos \frac{1}{2M} - M \sin \frac{1}{2M} |e^{i\phi_1} - e^{i\phi_2}| &\geq \frac{\sqrt{3}}{2}(\phi_1 - \phi_2) - \frac{1}{2}(\phi_1 - \phi_2) > \frac{\phi_1 - \phi_2}{3}. \end{aligned} \quad (8)$$

This means that  $x(\phi) := \rho(e^{i\phi}) \cos \phi$  is a strictly decreasing function of  $\phi \in (\frac{\pi}{2} - \frac{1}{2M}, \frac{\pi}{2} + \frac{1}{2M})$ . Similarly it can be shown that  $x(\phi)$  is a strictly increasing function of  $\phi \in (\frac{3\pi}{2} - \frac{1}{2M}, \frac{3\pi}{2} + \frac{1}{2M})$ . Thus for any  $|x| \leq \frac{1}{\pi M} \leq \sin \frac{1}{2M}$  the line  $\{(x, t), t \in \mathbb{R}\}$  intersects the domain along a single line segment of length at least  $2 \cos \frac{1}{2M} \geq 2 \cos \frac{1}{2} > \sqrt{3}$  which is the first statement of the lemma.

Now let us verify the second assertion of the lemma. Using (8) it easily follows that for any  $\frac{\pi}{2} - \frac{1}{2M} < \phi_2 < \phi_1 < \frac{\pi}{2} + \frac{1}{2M}$  we have setting  $y_j := \rho(e^{i\phi_j}) \sin \phi_j$ ,  $x_j := \rho(e^{i\phi_j}) \cos \phi_j$ ,  $j = 1, 2$

$$|y_2 - y_1| \leq \rho(e^{i\phi_1}) |\sin \phi_2 - \sin \phi_1| + |\rho(e^{i\phi_2}) - \rho(e^{i\phi_1})| \leq (R + M)(\phi_1 - \phi_2) \leq 3(R + M)|x_2 - x_1|,$$

and a similar estimate holds in  $\phi \in (\frac{3\pi}{2} - \frac{1}{2M}, \frac{3\pi}{2} + \frac{1}{2M})$ , as well. This means that when  $|x| \leq \frac{1}{\pi M} \leq \sin \frac{1}{2M}$  the boundary of  $K$  is given by  $\text{Lip}_{M_1} 1$  functions with  $M_1 := 3(R + M)$ . That is

$$K \cap L_a(\mathbf{u}) = \{(x, y) : |x| \leq a, g(x) \leq y \leq f(x)\}, \quad f, g \in \text{Lip}_{M_1} 1.$$

For any  $\mathbf{z} = (x_0, y_0) \in K \cap L_{a/2}(\mathbf{u})$ ,  $|x_0| \leq \frac{1}{2\pi M}$  set  $[a(\mathbf{z}), b(\mathbf{z})] := l_{\mathbf{z}}(\mathbf{u}) \cap K$ ,  $a(\mathbf{z}) = (x_0, A)$ ,  $b(\mathbf{z}) = (x_0, B)$ ,  $B > A$ . Consider the rhombus  $\Omega := \{(x, y) \in \mathbb{R}^2 : Q|x - x_0| + |y - \frac{A+B}{2}| \leq \frac{B-A}{2}\}$  with  $Q := 2\pi RM > M_1$ . Since  $\frac{B-A}{2} \leq R$  it easily follows that  $\Omega \subset L_a(\mathbf{u})$ . In addition, the  $\text{Lip}_{M_1} 1$  property of  $\partial K$  implies that  $\Omega \subset K \cap L_a(\mathbf{u})$ . This yields that  $\tau_{\Omega}(\mathbf{z}) \leq \tau_K(\mathbf{z})$  for  $\mathbf{z} \in \Omega \subset K$  and hence

$$\tau_K(\mathbf{z}) \geq \tau_{\Omega}(\mathbf{z}) \geq \frac{1}{\sqrt{1+Q^2}} \min\{|\mathbf{z} - a(\mathbf{z})|, |\mathbf{z} - b(\mathbf{z})|\} \geq \frac{1}{2R\sqrt{1+Q^2}} |\mathbf{z} - a(\mathbf{z})| |\mathbf{z} - b(\mathbf{z})|$$

which implies the lower bound in (7).

Since  $a(\mathbf{z}), b(\mathbf{z}) \in \partial K$  the needed upper bound easily follows by  $|b(\mathbf{z}) - a(\mathbf{z})| \geq \sqrt{3}$  verified above

$$\tau_K(\mathbf{z}) \leq \min\{|\mathbf{z} - a(\mathbf{z})|, |\mathbf{z} - b(\mathbf{z})|\} \leq \frac{2}{\sqrt{3}} |\mathbf{z} - a(\mathbf{z})| |\mathbf{z} - b(\mathbf{z})|. \quad \square$$

The next assertion will allow us to rotate cylinders while estimating directional derivatives leading to bounds for gradients of polynomials. As usual,  $B^d(\mathbf{a}, R)$  stands for the  $d$  dimensional Euclidean ball centered at  $\mathbf{a}$  and radius  $R$ .

**Lemma 2.** For any  $0 < a_1 < a \leq 1, R \geq 1$  and arbitrary  $\mathbf{u}, \mathbf{w} \in S^{d-1}$  such that  $|\mathbf{u} - \mathbf{w}| \leq \frac{a-a_1}{2R}$  we have

$$L_{a_1}(\mathbf{w}) \cap B^d(0, R) \subset L_a(\mathbf{u}). \quad (9)$$

**Proof.** Given any  $\mathbf{z} \in L_{a_1}(\mathbf{w}) \cap B^d(0, R)$  it suffices to verify the statement in the 2-dimensional plane through  $\mathbf{z}$  and the line  $l_{\mathbf{z}}(\mathbf{u})$ . Hence without the loss of generality we may assume that  $d = 2, \mathbf{u} = (1, 0)$  and  $\mathbf{w} = e^{i\phi}, 0 < \phi < \pi$ . Then setting  $\epsilon := \frac{a-a_1}{2R} < 1$  it easily follows that  $\sin \frac{\phi}{2} \leq \frac{\epsilon}{2}$  and hence  $0 < \phi < \frac{\pi\epsilon}{2}$ . Evidently,

$$L_{a_1}(\mathbf{w}) \cap B^2(0, R) \subset \{(x, y) \in \mathbb{R}^2 : |x| \leq R, |y - x \tan \phi| \leq a_1 \sec \phi\}.$$

This easily yields that for any  $(x, y) \in L_{a_1}(\mathbf{w}) \cap B^2(0, R)$  we have

$$|y| \leq R \tan \phi + a_1 \sec \phi \leq \epsilon R \sec \phi + a_1 \sec \phi \leq \frac{\epsilon R + a_1}{\sqrt{1-\epsilon^2}} \leq \frac{\epsilon R + a_1}{1-\epsilon} \leq a.$$

Note that since  $\mathbf{u} = (1, 0)$ , i.e.,  $L_a(\mathbf{u}) = \{(x, y) \in \mathbb{R}^2 : |y| \leq a\}$  it follows that  $(x, y) \in L_a(\mathbf{u})$ .  $\square$

The gradient  $\partial^k f(\mathbf{x})$  provides the maximal  $k$ -th order directional derivatives when the directions vary over all of the unit sphere  $S^{d-1}$ . Next we need to compare this to the maximal value of  $k$ -th order directional derivatives when the maximum is taken over a given spherical cap  $S_{\epsilon}^{d-1}(\mathbf{w}) := \{\mathbf{u} \in S^{d-1} : |\mathbf{u} - \mathbf{w}| \leq \epsilon\}$  of radius  $\epsilon$  centered at any  $\mathbf{w} \in S^{d-1}$ . Thus for a given  $\mathbf{w} \in S^{d-1}, \epsilon > 0$  and  $k \in \mathbb{N}$  set

$$\partial_{\epsilon, \mathbf{w}}^k f(\mathbf{x}) := \max_{\mathbf{u}_j \in S_{\epsilon}^{d-1}(\mathbf{w}), 1 \leq j \leq k} |D_{\mathbf{u}_1} \dots D_{\mathbf{u}_k} f(\mathbf{x})|.$$

**Lemma 3.** Let  $\mathbf{w} \in S^{d-1}, k \in \mathbb{N}, 0 < \epsilon \leq 1$ . Then for any  $f \in C^{k+1}(\mathbb{R}^d)$  and every  $\mathbf{x} \in \mathbb{R}^d$  we have

$$\partial^k f(\mathbf{x}) \leq \left( \frac{4\sqrt{d}}{\epsilon} \right)^k \partial_{\epsilon, \mathbf{w}}^k f(\mathbf{x}). \quad (10)$$

**Proof.** Since the statement of the lemma is rotation invariant it suffices to prove it for the case when  $\mathbf{w} = \mathbf{e}_1 := (1, 0, \dots, 0)$ . We will verify the lemma by induction on  $k$ .

So set first  $k = 1$ . By the elementary inequality

$$\max\{|x_1|, |x_1 + \epsilon x_j|, 2 \leq j \leq d\} \geq \frac{\epsilon}{2} \max_{1 \leq j \leq d} |x_j|, \quad x_j \in \mathbb{R}, \quad \epsilon > 0$$

we easily obtain

$$\begin{aligned} \partial^1 f(\mathbf{x}) &\leq \left( \sum_{1 \leq j \leq d} f_{x_j}^2 \right)^{\frac{1}{2}} \leq \sqrt{d} \max_{1 \leq j \leq d} |f_{x_j}| \leq \frac{2\sqrt{d}}{\epsilon} \max\{|f_{x_1}|, |f_{x_1} + \epsilon f_{x_j}|, 2 \leq j \leq d\} \\ &= \frac{2\sqrt{d}}{\epsilon} \max\{|D_{\mathbf{e}_1} f|, \sqrt{1 + \epsilon^2} |D_{\mathbf{u}_j} f|, 2 \leq j \leq d\} \leq \frac{2\sqrt{2d}}{\epsilon} \max\{|D_{\mathbf{e}_1} f|, |D_{\mathbf{u}_j} f|, 2 \leq j \leq d\}, \end{aligned}$$

where  $\mathbf{u}_1 := \mathbf{e}_1$ ,  $\mathbf{u}_j := \frac{\mathbf{e}_1 + \epsilon \mathbf{e}_j}{\sqrt{1 + \epsilon^2}} \in S^{d-1}$ ,  $2 \leq j \leq d$ . Note that  $|\mathbf{e}_1 - \mathbf{u}_j| \leq \sqrt{2}\epsilon$ ,  $2 \leq j \leq d$ . This means that  $\mathbf{u}_j \in S_{\sqrt{2}\epsilon}^{d-1}(\mathbf{e}_1)$ ,  $1 \leq j \leq d$ . Using this observation together with the previous estimate yields

$$\partial^1 f(\mathbf{x}) \leq \frac{4\sqrt{d}}{\epsilon} \partial_{\epsilon, \mathbf{e}_1}^1 f(\mathbf{x}), \quad (11)$$

which implies the needed estimate for  $k = 1$ .

Assume now that (10) holds for  $k - 1$ . For any fixed  $\mathbf{y} \in \mathbb{R}^d$  there exist specific  $\mathbf{u}_j \in S^{d-1}$ ,  $1 \leq j \leq k$  so that

$$\partial^k f(\mathbf{y}) = D_{\mathbf{u}_1} \dots D_{\mathbf{u}_k} f(\mathbf{x}).$$

Now set

$$g(\mathbf{x}) := D_{\mathbf{u}_2} \dots D_{\mathbf{u}_k} f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad D_{\mathbf{u}_1} g(\mathbf{y}) = \partial^k f(\mathbf{y}).$$

Applying (11) for the function  $g$  at the point  $\mathbf{y}$  yields that for some  $\mathbf{w}_1 \in S_{\epsilon}^{d-1}(\mathbf{e}_1)$  we have

$$\partial^k f(\mathbf{y}) = D_{\mathbf{u}_1} g(\mathbf{y}) \leq \partial^1 g(\mathbf{y}) \leq \frac{4\sqrt{d}}{\epsilon} \partial_{\epsilon, \mathbf{e}_1}^1 g(\mathbf{y}) = \frac{4\sqrt{d}}{\epsilon} D_{\mathbf{w}_1} g(\mathbf{y}). \quad (12)$$

Furthermore, interchanging the consecutive order of directional differentiation yields

$$D_{\mathbf{w}_1} g(\mathbf{y}) = D_{\mathbf{w}_1} D_{\mathbf{u}_2} \dots D_{\mathbf{u}_k} f(\mathbf{y}) = D_{\mathbf{u}_2} \dots D_{\mathbf{u}_k} D_{\mathbf{w}_1} f(\mathbf{y}) = D_{\mathbf{u}_2} \dots D_{\mathbf{u}_k} h(\mathbf{y}), \quad (13)$$

where  $h(\mathbf{x}) := D_{\mathbf{w}_1} f(\mathbf{x})$ ,  $\mathbf{x} \in \mathbb{R}^d$ . By the induction hypotheses we can apply (10) with order  $k - 1$  to the function  $h$ . Clearly this implies that

$$D_{\mathbf{u}_2} \dots D_{\mathbf{u}_k} h(\mathbf{y}) \leq \partial^{k-1} h(\mathbf{y}) \leq \left( \frac{4\sqrt{d}}{\epsilon} \right)^{k-1} \partial_{\epsilon, \mathbf{e}_1}^{k-1} h(\mathbf{y}).$$

Thus using the last estimate together with (12) and (13) we arrive at

$$\partial^k f(\mathbf{y}) \leq \left( \frac{4\sqrt{d}}{\epsilon} \right)^k \partial_{\epsilon, \mathbf{e}_1}^{k-1} h(\mathbf{y}) \leq \left( \frac{4\sqrt{d}}{\epsilon} \right)^k \partial_{\epsilon, \mathbf{e}_1}^k f(\mathbf{y})$$

which is the needed statement for the gradients of order  $k$ .  $\square$

Now we are in position to prove Theorem 1 which is one of the main new results of this paper.

**Proof of Theorem 1.** Set  $a := \frac{1}{4^k \pi M}$ ,  $\epsilon := \frac{a}{4R}$ . Throughout the proof  $c$  stands for possibly different positive constants depending only on  $d, k, p$  and the weight  $w$ , but independent of  $n$  and the domain. In particular, this constant might be absolute constants, as well. Now choose  $\mathbf{u} \in S^{d-1}$ . Then for arbitrary  $\mathbf{u}_j \in S_\epsilon^{d-1}(\mathbf{u})$ ,  $1 \leq j \leq k$  we will estimate the weighted  $L^p$  norm of directional derivatives  $D_{\mathbf{u}_1} \dots D_{\mathbf{u}_k} q(\mathbf{x})$  of any polynomial  $q \in P_n^d$ . This will be accomplished by successive application of the Bernstein inequality in the cylinders  $L_{4^{j-1}a}(\mathbf{u}_j)$ ,  $1 \leq j \leq k$ .

In [4], Theorem A.4.15, p. 407 the authors establish an  $L_w^p$  Bernstein type inequality for trigonometric polynomials with  $w$  being a general Jacobi type weight (5).

The standard substitution  $x = \cos t$  yields a corresponding statement for univariate algebraic polynomials: for any  $q \in P_n^1$  and any Jacobi type weight  $w$  of degree  $m$  on  $[a, b]$  we have

$$\|\sqrt{(x-a)(b-x)}q'\|_{L_w^p[a,b]} \leq c \left( \frac{\beta_w}{\alpha_w} \right)^{1/p} (1+1/p)(m+1)(n+m)\|q\|_{L_w^p[a,b]}, \quad p > 0, \quad (14)$$

where  $c > 0$  is an some absolute constant.

Furthermore by Lemma 1 the set  $K$  is a graph domain of width  $\sqrt{3}$  relative to the cylinder  $L_{2a}(\mathbf{u}_1)$  and for any  $\mathbf{z} \in L_a(\mathbf{u}_1)$  relations (7) hold. Then we can use the univariate Bernstein inequality applied to the polynomial  $q_1 := D_{\mathbf{u}_2} \dots D_{\mathbf{u}_k} q(\mathbf{x})$  restricted to the line segment  $[\mathbf{a}, \mathbf{b}] := l_{\mathbf{y}}(\mathbf{u}_1) \cap K$  with any fixed  $\mathbf{y} \in L_a(\mathbf{u}_1)$  and the Jacobi type weight  $\tau_K(\mathbf{x})^{\frac{p}{2}(k-1)}w$  which by (7) satisfies (5) with some parameters  $\alpha_w = \frac{1}{c}(R^2M)^{-\frac{p}{2}(k-1)}$ ,  $\beta_w = c$  and a polynomial  $g(t)$  of degree  $m+2$ . Thus we obtain applying (14) on the line segment  $[\mathbf{a}, \mathbf{b}]$

$$\begin{aligned} \int_{[\mathbf{a}, \mathbf{b}]} w \tau_K(\mathbf{x})^{\frac{kp}{2}} |D_{\mathbf{u}_1} q_1|^p &\leq \left( \frac{2}{\sqrt{3}} \right)^{\frac{p}{2}} \int_{[\mathbf{a}, \mathbf{b}]} w \tau_K(\mathbf{x})^{\frac{p}{2}(k-1)} (|a(\mathbf{x}) - \mathbf{x}| |\mathbf{x} - b(\mathbf{x})|)^{\frac{p}{2}} |D_{\mathbf{u}_1} q_1|^p \\ &\leq c(R^2M)^{\frac{p}{2}(k-1)} n^p \int_{[\mathbf{a}, \mathbf{b}]} w \tau_K(\mathbf{x})^{\frac{p}{2}(k-1)} |q_1|^p. \end{aligned}$$

Evidently, the last estimate holds for any line  $l_{\mathbf{y}}(\mathbf{u}_1) \subset L_a(\mathbf{u}_1)$  so by the Fubini theorem we obtain

$$\int_{L_a(\mathbf{u}_1) \cap K} \tau_K(\mathbf{x})^{\frac{kp}{2}} w |D_{\mathbf{u}_1} q_1|^p \leq c(R^2M)^{\frac{p}{2}(k-1)} n^p \int_{L_a(\mathbf{u}_1) \cap K} \tau_K(\mathbf{x})^{\frac{p}{2}(k-1)} w |q_1|^p.$$

Recalling that  $\mathbf{u}_j \in S_\epsilon^{d-1}(\mathbf{u}_1)$ ,  $1 \leq j \leq k$  it follows that  $|\mathbf{u}_1 - \mathbf{u}_2| \leq 2\epsilon = \frac{a}{2R}$ . Thus by Lemma 2  $L_a(\mathbf{u}_1) \cap B^d(0, R) \subset L_{4a}(\mathbf{u}_2)$ , hence

$$\int_{L_a(\mathbf{u}_1) \cap K} \tau_K(\mathbf{x})^{\frac{kp}{2}} w |D_{\mathbf{u}_1} q_1|^p \leq c(R^2M)^{\frac{p}{2}(k-1)} n^p \int_{L_{4a}(\mathbf{u}_2) \cap K} \tau_K(\mathbf{x})^{\frac{p}{2}(k-1)} w |q_1|^p.$$

Now this process can be iterated and thus setting  $q_j := D_{\mathbf{u}_{j+1}} \dots D_{\mathbf{u}_k} q(\mathbf{x})$ ,  $1 \leq j \leq k-1$ ,  $q_k := q$ ,  $\mathbf{u}_{k+1} := \mathbf{u}$  we have

$$\int_{L_{4^{j-1}a}(\mathbf{u}_j) \cap K} \tau_K(\mathbf{x})^{\frac{(k-j+1)p}{2}} w |D_{\mathbf{u}_j} q_j|^p \leq c(R^2M)^{\frac{p}{2}(k-j)} n^p \int_{L_{4^j a}(\mathbf{u}_{j+1}) \cap K} \tau_K(\mathbf{x})^{\frac{p}{2}(k-j)} w |q_j|^p, \quad 1 \leq j \leq k.$$

Therefore setting  $N := (RM)^{pk(k-1)/2}$  this chain of estimates yields

$$\begin{aligned} \int_{L_a(\mathbf{u}_1) \cap K} \tau_K(\mathbf{x})^{\frac{kp}{2}} w |D_{\mathbf{u}_1} \dots D_{\mathbf{u}_k} q(\mathbf{x})|^p &= \int_{L_a(\mathbf{u}_1) \cap K} \tau_K(\mathbf{x})^{\frac{kp}{2}} w |D_{\mathbf{u}_1} q_1|^p \leq \\ cNn^{pk} \int_{L_{4^k a}(\mathbf{u}) \cap K} w |q|^p &= cNn^{pk} \int_{L_{\pi/M}(\mathbf{u}) \cap K} w |q|^p \leq cNn^{pk} \int_K w |q|^p. \end{aligned}$$

Using again Lemma 2 with  $a_1 := \frac{a}{4}$  and relation  $|\mathbf{u} - \mathbf{u}_1| \leq \epsilon = \frac{a}{4R} < \frac{a-a_1}{2R} = \frac{3a}{8R}$  we obtain  $L_{\frac{a}{4}}(\mathbf{u}) \subset L_a(\mathbf{u}_1)$ . Thus applying in addition Lemma 3 it follows that for every  $\mathbf{u} \in S^{d-1}$

$$\int_{L_{\frac{a}{4}}(\mathbf{u}) \cap K} \tau_K(\mathbf{x})^{\frac{kp}{2}} w |\partial^k q|^p \leq \left( \frac{4\sqrt{d}}{\epsilon} \right)^{pk} \int_{L_{\frac{a}{4}}(\mathbf{u}) \cap K} \tau_K(\mathbf{x})^{\frac{kp}{2}} w |\partial_{\epsilon, \mathbf{u}}^k q|^p \leq c(MR)^{\frac{pk(k+1)}{2}} n^{pk} \int_K w |q|^p. \quad (15)$$

The last estimate provides the needed upper bound in the section  $L_{a/4}(\mathbf{u}) \cap K$ . Now it remains to address the question how many of these sections cover all of  $K$ . In order to estimate this we shall use a result by Böröczky and Wintsche [5], Corollary 1.2. According to this result  $RS^{d-1}$  can be covered by  $c \left( \frac{R}{\delta} \right)^{d-1} d^{\frac{3}{2}} \ln d$  sphere caps of radius  $\delta$ , where  $c$  is an absolute constant. Since  $K \subset B^d(0, R)$  this gives an estimate for the number of sections  $L_{\frac{a}{4}}(\mathbf{u}) \cap K$  needed to cover  $K$  with  $\delta := \frac{a}{4}$ , i.e.,

$$\int_K \tau_K(\mathbf{x})^{\frac{kp}{2}} w |\partial^k q(\mathbf{x})|^p \leq c(MR)^{\frac{pk(k+1)}{2} + d-1} n^{pk} \int_K w |q|^p.$$

Taking now the  $p$ -th root in the last estimate yields the statement of the theorem.  $\square$

### 3. $L^p$ Bernstein type inequalities for convex and graph domains

In case when domain  $K$  is convex we can derive from Theorem 1 a *domain independent* Bernstein type inequality. This can be done by using the John's maximal ellipsoid theorem [10] which states that given any convex domain  $K$  there exists an ellipsoid  $E$  centered at point  $\mathbf{c}$  so that  $E \subset K \subset \mathbf{c} + d(E - \mathbf{c})$ . We may assume without restricting the generality that  $\mathbf{c} = 0$  and thus  $E = A(B^d(0, 1))$  is the image of the unit ball  $B^d(0, 1)$  under the nonsingular linear transformation  $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfying  $E \subset K \subset dE$ . Since  $B^d(0, 1) \subset K$  and  $A^{-1}(K) \subset B^d(0, d)$  it easily follows that  $|A^{-1}(\mathbf{u})| \leq d, \forall \mathbf{u} \in S^{d-1}$ . Thus a change of variable  $\mathbf{y} = A^{-1}\mathbf{x}, \mathbf{x} \in \mathbb{R}^d$  can increase the size of  $\partial^k q, q \in P_n^d$  just by a factor of  $d^k$ . Similarly, the distance to the boundary  $\tau_K(\mathbf{x})$  can increase just by a factor of  $d$ , as well. This means that without the loss of generality we can assume that the convex body  $K$  satisfies  $B^d(0, 1) \subset K \subset B^d(0, d)$ , and thus  $1 \leq \rho(\mathbf{u}) \leq d, \mathbf{u} \in S^{d-1}$  in the representation (4). This means that Theorem 1 can be applied with  $R = d$  in the convex case.

Now we will show that the Lipschitz property holds with  $M = d^2$  in case when  $B^d(0, 1) \subset K \subset B^d(0, d)$ . Recall the notion of Minkowski functional of the star like domain  $K$  defined as

$$\phi_K(\mathbf{x}) := \inf \left\{ \alpha > 0 : \frac{\mathbf{x}}{\alpha} \in K \right\}, \quad \mathbf{x} \in \mathbb{R}^d.$$

When  $K$  is given by representation (4) it is easy to see that

$$\phi_K(\mathbf{u}) = \frac{1}{\rho(\mathbf{u})}, \quad \mathbf{u} \in S^{d-1}.$$



Furthermore, for convex domains it is known, see [17] that their Minkowski functionals are convex functionals satisfying  $\phi_K(\mathbf{x} + \mathbf{y}) \leq \phi_K(\mathbf{x}) + \phi_K(\mathbf{y})$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ . Recalling that in addition  $1 \leq \rho(\mathbf{u}) \leq d$ ,  $\mathbf{u} \in S^{d-1}$  it follows that

$$|\phi_K(\mathbf{u}_1) - \phi_K(\mathbf{u}_2)| \leq \phi_K(\mathbf{u}_1 - \mathbf{u}_2) = \frac{|\mathbf{u}_1 - \mathbf{u}_2|}{\rho((\mathbf{u}_1 - \mathbf{u}_2)/|\mathbf{u}_1 - \mathbf{u}_2|)} \leq |\mathbf{u}_1 - \mathbf{u}_2|,$$

and hence

$$|\rho(\mathbf{u}_1) - \rho(\mathbf{u}_2)| = |\phi_K(\mathbf{u}_1) - \phi_K(\mathbf{u}_2)|\rho(\mathbf{u}_1)\rho(\mathbf{u}_2) \leq d^2|\mathbf{u}_1 - \mathbf{u}_2|, \quad \mathbf{u}_1, \mathbf{u}_2 \in S^{d-1}.$$

This means that convex domain  $K$  is  $\text{Lip}_M 1$  with  $M = d^2$ .

Thus above observations combined with Theorem 1 result in the next domain independent Bernstein type inequality for convex sets.

**Corollary 1.** *Let  $d, k \in \mathbb{N}$ ,  $0 < p < \infty$ , and consider an arbitrary Jacobi type weight  $w$  defined on a convex body  $K \subset \mathbb{R}^d$  containing the unit ball. Then for every  $q \in P_n^d$  we have*

$$\|\tau_K(\mathbf{x})^{\frac{k}{2}} \partial^k q(\mathbf{x})\|_{L_w^p} \leq c_{w,k,d,p} n^k \|q(\mathbf{x})\|_{L_w^p},$$

where  $c_{w,k,d,p}$  is independent of  $n$ , polynomial  $q$ , and the domain  $K$ .

In case when  $K$  is a so called *piecewise graph domain* one can improve the factor  $(MR)^{\frac{k(k+1)}{2} + \frac{d-1}{p}}$  in the estimate (6) of Theorem 1. This improvement will be important for deriving Bernstein type inequalities for  $\text{Lip}_\alpha$  domains in the next section.

**Definition.**  $K$  is called a **graph domain** with respect to the cylinder  $L_r(\mathbf{a}, \mathbf{u})$  if for every  $\mathbf{x} \in L_r(\mathbf{a}, \mathbf{u})$  we have that  $l_{\mathbf{x}}(\mathbf{u}) \cap K = [d_1(\mathbf{x}), d_2(\mathbf{x})]$  is a finite segment with  $d_i(\mathbf{x}), i = 1, 2$  being continuous for  $\mathbf{x} \in L_r(\mathbf{a}, \mathbf{u})$  and

$$\delta_r(\mathbf{a}, \mathbf{u}) := \inf_{\mathbf{x} \in L_r(\mathbf{a}, \mathbf{u})} |d_1(\mathbf{x}) - d_2(\mathbf{x})| > 0. \quad (16)$$

Moreover,  $K \subset \mathbb{R}^d$  is a **piecewise graph domain** if it can be covered by finite number of cylinders so that  $K$  is a graph domain with respect to each of them.

The piecewise graph domain  $K \subset \mathbb{R}^d$  is called  $\text{Lip}_M 1$  if functions  $d_1(\mathbf{x}), d_2(\mathbf{x})$  in the above definition are  $\text{Lip}_M 1$ . For  $\text{Lip}_M 1$  piecewise graph domain we can verify now a stronger version of Theorem 1.

**Corollary 2.** *Let  $d, k \in \mathbb{N}$ ,  $0 < p < \infty$ ,  $M \geq 1$ . Then for any  $\text{Lip}_M 1$  piecewise graph domain  $K \subset \mathbb{R}^d$  and for every Jacobi type weight  $w$  on  $K$  we have*

$$\|\tau_K(\mathbf{x})^{\frac{k}{2}} \partial^k q(\mathbf{x})\|_{L_w^p} \leq cM^{\frac{k(k+1)}{2}} n^k \|q\|_{L_w^p}, \quad q \in P_n^d \quad (17)$$

where  $c > 0$  is a positive constant depending only on  $K, d, k, p$  and the weight  $w$ .

**Proof.** The proof of the above corollary is quite similar to that of Theorem 1 and therefore we only briefly outline the proof without including all the details. Since  $K \subset \mathbb{R}^d$  is a piecewise graph domain it can be covered by  $m$  open cylinders  $L_{a_k}(\mathbf{u}_k, \mathbf{c}_k), 1 \leq k \leq m$  of radius  $a_k > 0$ , center  $\mathbf{c}_k$  and axis  $\mathbf{u}_k$ , so that  $K$  is a graph domain with respect to each  $L_{a_k}(\mathbf{u}_k, \mathbf{c}_k)$ . Clearly we can choose  $b_k < a_k$  sufficiently close to  $a_k$  so that  $K$  is covered by  $L_{b_k}(\mathbf{u}_k, \mathbf{c}_k), 1 \leq k \leq m$ , as well. Now it suffices to establish the needed estimates for

the derivatives in each of the cylinders  $L_{b_k}(\mathbf{u}_k, \mathbf{c}_k)$ ,  $1 \leq k \leq m$ . Using that  $K$  is  $\text{Lip}_M 1$  similarly to Lemma 1 it can be shown that perturbing vectors  $\mathbf{u}_k$  by  $\frac{c}{M}$  will preserve the graph domain property i.e., invoking also Lemmas 2 and 3 we can establish estimate (15) for cylindrical sections  $L_{b_k}(\mathbf{u}_k, \mathbf{c}_k) \cap K$ . Moreover, since  $K$  can be covered by finite number of such cylinders there is no need to apply the covering result of Böröczky and Wintsche [5] used at the end of the proof of Theorem 1 and thus (15) essentially provides the final estimate.

The significance of Corollary 2 consists in the fact that for  $k = 1$  it yields a factor  $M$  on the right hand side of (17) which will allow us to pass from  $\text{Lip } 1$  domains to  $\text{Lip } \alpha$ ,  $0 < \alpha < 1$  domains in the next section.

#### 4. $L^p$ Bernstein type inequalities for $\text{Lip } \alpha$ , $0 < \alpha < 1$ graph domains

Now we will turn our attention to establishing order  $n$  Bernstein type inequalities for  $L^p$  norm on  $\text{Lip } \alpha$ ,  $0 < \alpha < 1$  **cuspidal** graph domains. Optimal order  $n$  Bernstein type inequalities are known to play a very important role in establishing Marcinkiewicz-Zygmund type discrete point sets of asymptotically optimal cardinality, see e.g. the survey paper [15] for details.

In case of the uniform norm on  $\text{Lip } \alpha$ ,  $0 < \alpha \leq 1$  star like cuspidal domains it was proved in [13] that

$$\|\tau_K(\mathbf{x})^{\frac{1}{\alpha}-\frac{1}{2}} \partial^1 q\|_{C(K)} \leq cn \|q\|_{C(K)}, \quad q \in P_n^d. \quad (18)$$

Thus inserting the weight  $\tau_K(\mathbf{x})^{\frac{1}{\alpha}-\frac{1}{2}}$  into the norm of derivatives yields the optimal  $n$  order upper bound for the derivatives of polynomials.

It seems to be quite likely that estimate (18) could be extended for the  $L^p$  norm with higher derivatives, as well. As mentioned above this would have important implications for Marcinkiewicz-Zygmund type inequalities. Thus we would like to propose the next

**Conjecture.** *Show that for every  $\text{Lip } \alpha$ ,  $0 < \alpha < 1$  star like cuspidal domain  $K \subset \mathbb{R}^d$  and any Jacobi type weight  $w$  on  $K$  we have*

$$\|\tau_K(\mathbf{x})^{(\frac{1}{\alpha}-\frac{1}{2})k} \partial^k q\|_{L_w^p} \leq cn^k \|q\|_{L_w^p}, \quad q \in P_n^d. \quad (19)$$

Note that when  $\alpha = 1$  estimate (19) is the same as the Bernstein type result of Theorem 1.

Applying Corollary 2 given in the previous section we can verify above conjecture for  $k = 1$  and  $\text{Lip } \alpha$ ,  $0 < \alpha < 1$  piecewise graph domains  $K \subset \mathbb{R}^d$ . This will be accomplished by approximating  $\text{Lip } \alpha$  domains by  $\text{Lip } 1$  domains using the Steklov transform and then invoking Corollary 2 for the  $\text{Lip } 1$  domains. This method is similar to the one used in [7] where a **tangential**  $L^p$  Bernstein type inequality was verified for  $\text{Lip } \alpha$ ,  $1 < \alpha < 2$  graph domains. (For  $\alpha = 2$  this was accomplished earlier, see [6].) We will need below an  $L^p$  Remez type inequality for so called doubling weights proved in [16] in case when  $1 \leq p < \infty$ . Therefore in contrast to Theorem 1 verified for every  $0 < p < \infty$  our next result is stated for  $1 \leq p < \infty$ .

**Theorem 2.** *Let  $1 \leq p < \infty$ . Then given any  $\text{Lip } \alpha$ ,  $0 < \alpha < 1$  piecewise graph domain  $K \subset \mathbb{R}^d$  and any Jacobi type weight  $w$  on  $K$  we have*

$$\|\tau_K(\mathbf{x})^{\frac{1}{\alpha}-\frac{1}{2}} \partial^1 q\|_{L_w^p} \leq cn \|q\|_{L_w^p}, \quad q \in P_n^d. \quad (20)$$

**Proof.** First let us note that when  $k = 1$  Corollary 2 yields the upper bound

$$\|\tau_K(\mathbf{x})^{\frac{1}{2}} \partial^1 q(\mathbf{x})\|_{L_w^p} \leq cMn \|q\|_{L_w^p}, \quad q \in P_n^d \quad (21)$$

for every  $\text{Lip}_M 1$  piecewise graph domain  $K \subset \mathbb{R}^d$ , where  $c > 0$  is a positive constant depending only on  $K$ ,  $d$ ,  $p$  and the weight  $w$ . The piecewise graph domain  $K \subset \mathbb{R}^d$  can be covered by  $m$  cylinders  $L_{b_k}(\mathbf{u}_k, \mathbf{c}_k)$ ,  $1 \leq$

$k \leq m$  of radius  $b_k > 0$  and axis  $\mathbf{u}_k$ , so that for some  $b_k < a_k$  the set  $K$  is a graph domain with respect to each  $L_{a_k}(\mathbf{u}_k, \mathbf{c}_k)$ ,  $1 \leq k \leq m$ . Now it suffices to establish the needed estimate in each of the cylinders  $L_{b_k}(\mathbf{u}_k, \mathbf{c}_k)$ ,  $1 \leq k \leq m$ . Without the loss of generality we may assume that  $\mathbf{u}_k = \mathbf{e}_d, \mathbf{c}_k = 0$  and work with the cylinder  $L_{b_k}(\mathbf{e}_d)$ , where  $\mathbf{e}_d = (0, \dots, 0, 1)$ . Since  $K \subset \mathbb{R}^d$  is a  $\text{Lip}\alpha$ ,  $0 < \alpha < 1$  piecewise graph domain it follows that

$$K \cap L_{a_k}(\mathbf{e}_d) = \{(\mathbf{x}, y) \in \mathbb{R}^d : \mathbf{x} \in B^{d-1}, r_2(\mathbf{x}) \leq y \leq r_1(\mathbf{x})\}$$

where  $B^{d-1}$  is a proper  $d - 1$  dimensional ball and  $r_j(\mathbf{x})$ ,  $j = 1, 2$  are  $\text{Lip}\alpha$ ,  $0 < \alpha < 1$  functions on  $B^{d-1}$  satisfying  $|r_1(\mathbf{x}) - r_2(\mathbf{x})| \geq c_0 > 0$ ,  $\mathbf{x} \in B^{d-1}$ . Clearly we may assume by rescaling the domain that  $c_0 > 8$ . Then it suffices to verify the theorem for the “upper part” of the cylindrical section  $K \cap L_{a_k}(\mathbf{e}_d)$  given by

$$D := \{(\mathbf{x}, y) \in \mathbb{R}^d : \mathbf{x} \in B^{d-1}, r_1(\mathbf{x}) - 4 \leq y \leq r_1(\mathbf{x})\}$$

and  $\tau_D(\mathbf{x})$  being replaced by  $\rho(\mathbf{x}) := r_1(\mathbf{x}) - y$ . In addition, since Lemmas 2 and 3 allow small perturbations of directional derivatives it is enough to prove the result on  $D$  for  $\frac{\partial}{\partial y}$ . It should be also noted the critical part of the domain  $D$  is given by

$$D_0 := \{(\mathbf{x}, y) \in \mathbb{R}^d : \mathbf{x} \in B^{d-1}, r_1(\mathbf{x}) - 1 \leq y \leq r_1(\mathbf{x})\}$$

because for the “inside part”  $D \setminus D_0$  partial derivatives  $\frac{\partial q}{\partial y}$  clearly satisfy an order  $n$  upper bound. Therefore setting

$$T := \int_{D_0} w \rho(\mathbf{x})^{p\gamma} \left| \frac{\partial q}{\partial y} \right|^p = \int_{B^{d-1}} \int_{r_1(\mathbf{x})-4}^{r_1(\mathbf{x})} w \rho(\mathbf{x})^{p\gamma} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x}$$

our goal is to verify that

$$T \leq cn^p \int_{B^{d-1}} \int_{r_1(\mathbf{x})-4}^{r_1(\mathbf{x})} w |q|^p dy d\mathbf{x} = cn^p \|q\|_{L_w^p(D)}^p.$$

This in view of the above observations will conclude the proof of the theorem.

Now we will consider the classical Steklov transform of  $r_1(\mathbf{x})$  defined for any  $\delta > 0$  by the relation

$$r_1^\delta(\mathbf{x}) := (2\delta)^{-d} \int_{\delta I^d} r_1(\mathbf{x} + \mathbf{z}) d\mathbf{z}, \quad I^d := (-1, 1)^d, \quad \mathbf{x} \in B^{d-1}.$$

Evidently, the  $\text{Lip}\alpha$  property of function  $r_1(\mathbf{x})$  implies that for any sufficiently small  $0 < \delta < c_K$  we have

$$\max_{\mathbf{x} \in B^{d-1}} |r_1^\delta(\mathbf{x}) - r_1(\mathbf{x})| \leq c_0 \delta^\alpha. \quad (22)$$

Furthermore, using again the  $\text{Lip}\alpha$  property it follows that partial derivatives of  $r_1^\delta(\mathbf{x})$  satisfy

$$\max_{\mathbf{x} \in B^{d-1}} \left| \frac{\partial}{\partial x_m} r_1^\delta(\mathbf{x}) \right| \leq c \delta^{\alpha-1}, \quad 1 \leq m \leq d.$$

Of course this immediately yields that function  $r_1^\delta(\mathbf{x})$  is  $\text{Lip}_M 1$  with  $M := c\delta^{\alpha-1}$ . Hence estimate (21) is applicable to

$$D^\delta := \{(\mathbf{x}, y) \in \mathbb{R}^d : \mathbf{x} \in B^{d-1}, r_1^\delta(\mathbf{x}) - 1 \leq y \leq r_1^\delta(\mathbf{x})\}$$

with  $M := c\delta^{\alpha-1}$ . Now setting  $a_j := \frac{2^j}{n^2}$ ,  $\gamma := \frac{1}{\alpha} - \frac{1}{2}$  and using in addition the univariate Remez inequality [16], (7.15), p. 66 with the Jacobi type weight  $w\rho(\mathbf{x})^{p\gamma}$

$$\begin{aligned} T &= \int_{B^{d-1}} \int_{r_1(\mathbf{x})-1}^{r_1(\mathbf{x})} w\rho(\mathbf{x})^{p\gamma} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x} \leq c \int_{B^{d-1}} \int_{r_1(\mathbf{x})-1}^{r_1(\mathbf{x})-\frac{2}{n^2}} w\rho(\mathbf{x})^{p\gamma} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x} \\ &\leq c \sum_{1 \leq j < 2 \log_2 n} \int_{B^{d-1}} \int_{r_1(\mathbf{x})-2a_j}^{r_1(\mathbf{x})-a_j} w\rho(\mathbf{x})^{p\gamma} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x} \leq c \sum_{1 \leq j < 2 \log_2 n} a_j^{p(\gamma-\frac{1}{2})} \int_{B^{d-1}} \int_{r_1(\mathbf{x})-2a_j}^{r_1(\mathbf{x})-a_j} w\rho(\mathbf{x})^{\frac{p}{2}} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x} \\ &\leq c \sum_{1 \leq j < 2 \log_2 n} a_j^{p(\gamma-\frac{1}{2})+\frac{1}{2}} \int_{B^{d-1}} \int_{r_1(\mathbf{x})-2}^{r_1(\mathbf{x})-a_j} w_1 \rho(\mathbf{x})^{\frac{p}{2}} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x}, \end{aligned}$$

where  $w_1 := w\rho(\mathbf{x})^{-\frac{1}{2}}$  is a Jacobi type weight. Now we will replace  $r_1(\mathbf{x})$  in the last integral above by its Styeklov transform  $r_1^\delta(\mathbf{x})$  with  $\delta := c_0^{-\frac{1}{\alpha}} a_j^{\frac{1}{\alpha}-2}$ . Then in view of (22) it follows that

$$|r_1^\delta(\mathbf{x}) - r_1(\mathbf{x})| \leq c_0 \delta^\alpha = a_{j-2} = \frac{a_j}{4}, \quad \mathbf{x} \in B^{d-1}.$$

Hence by the previous estimate

$$T \leq c \sum_{1 \leq j < 2 \log_2 n} a_j^{p(\gamma-\frac{1}{2})+\frac{1}{2}} \int_{B^{d-1}} \int_{r_1^\delta(\mathbf{x})-3}^{r_1^\delta(\mathbf{x})-a_{j-1}} w_1 \rho(\mathbf{x})^{\frac{p}{2}} \left| \frac{\partial q}{\partial y} \right|^p dy d\mathbf{x}.$$

Now let us recall that function  $r_1^\delta(\mathbf{x})$  is  $\text{Lip}_M 1$  with

$$M := c\delta^{\alpha-1} = c_1 a_j^{1-\frac{1}{\alpha}} = c_1 a_j^{\frac{1}{2}-\gamma}.$$

Thus we can apply Corollary 2 for the last integral with this Lipschitz constant yielding

$$\begin{aligned} T &\leq c \sum_{1 \leq j < 2 \log_2 n} a_j^{p(\gamma-\frac{1}{2})+\frac{1}{2}} M^p n^p \int_{B^{d-1}} \int_{r_1^\delta(\mathbf{x})-3}^{r_1^\delta(\mathbf{x})-a_{j-1}} w_1 |q|^p dy d\mathbf{x} = \\ &= cn^p \sum_{1 \leq j < 2 \log_2 n} a_j^{\frac{1}{2}} \int_{B^{d-1}} \int_{r_1^\delta(\mathbf{x})-3}^{r_1^\delta(\mathbf{x})-a_{j-1}} w_1 |q|^p dy d\mathbf{x} \leq cn^p \sum_{1 \leq j < 2 \log_2 n} a_j^{\frac{1}{2}} \int_{B^{d-1}} \int_{r_1(\mathbf{x})-4}^{r_1(\mathbf{x})-a_{j-2}} w_1 |q|^p dy d\mathbf{x}. \end{aligned}$$

Since  $y \leq r_1(\mathbf{x}) - a_{j-2}$  in the last integral above it follows that  $\frac{a_j}{4} = a_{j-2} \leq r_1(\mathbf{x}) - y = \rho(\mathbf{x})$ , i.e.,  $a_j \leq 4\rho(\mathbf{x})$ . Thus  $j \leq \log_2 4n^2\rho(\mathbf{x})$  in the above summation. Recalling also that  $w_1 := w\rho(\mathbf{x})^{-\frac{1}{2}}$  we obtain

$$T \leq cn^p \int_{B^{d-1}} \int_{r_1(\mathbf{x})-4}^{r_1(\mathbf{x})} w|q|^p \sum_{1 \leq j \leq \log_2 4n^2 \rho(\mathbf{x})} \left( \frac{a_j}{\rho(\mathbf{x})} \right)^{\frac{1}{2}} dy d\mathbf{x}.$$

Now it remains to observe that

$$\sum_{1 \leq j \leq \log_2 4n^2 \rho(\mathbf{x})} \left( \frac{a_j}{\rho(\mathbf{x})} \right)^{\frac{1}{2}} = \frac{1}{n\sqrt{\rho(\mathbf{x})}} \sum_{1 \leq j \leq \log_2 4n^2 \rho(\mathbf{x})} 2^{\frac{j}{2}} \leq \frac{1}{n\sqrt{\rho(\mathbf{x})}} \frac{2\sqrt{2}n\sqrt{\rho(\mathbf{x})}}{\sqrt{2}-1} < 7.$$

Thus we arrive at the required estimate

$$T \leq cn^p \int_{B^{d-1}} \int_{r_1(\mathbf{x})-4}^{r_1(\mathbf{x})} w|q|^p = cn^p \|q\|_{L_w^p(D)}^p. \quad \square$$

**Remark.** Theorem 2 verified above provides a Bernstein type estimate of order  $n$  for polynomials in  $P_n^d$  when the  $L^p$  norm of the gradients of polynomials on the  $\text{Lip}\alpha$ ,  $0 < \alpha < 1$  piecewise graph domains is considered with the weight  $\tau_K(\mathbf{x})^{\frac{1}{\alpha}-\frac{1}{2}}$ . This weight ensures the proper behavior of the derivatives in the vicinity of the boundary of the domain. Of course when  $\alpha = 1$  we get the usual  $\sqrt{\tau_K(\mathbf{x})}$  factor of Theorem 1. This raises the natural question if the term  $\tau_K(\mathbf{x})^{\frac{1}{\alpha}-\frac{1}{2}}$  in Theorem 2 can be replaced by a sharper factor  $\tau_K(\mathbf{x})^\eta$  with some  $\eta < \frac{1}{\alpha} - \frac{1}{2}$ ? However, if this was the case then a Remez type estimate (which allows neglecting the  $\frac{1}{n^2}$  neighborhood of the boundary in norm estimates) would imply a Markov type upper bound

$$\|\partial^1 q\|_{L_w^p(K)} \leq cn^{2\eta+1} \|q\|_{L_w^p(K)}, \quad q \in P_n^d,$$

where  $2\eta + 1 < \frac{2}{\alpha}$ . On the other hand as shown in [14] (see also [2]) the factor  $n^{\frac{2}{\alpha}}$  provides in general the best possible order in  $L^p$  Markov type estimates for  $\text{Lip}\alpha$ ,  $0 < \alpha < 1$  piecewise graph domains. Thus  $\tau_K(\mathbf{x})^{\frac{1}{\alpha}-\frac{1}{2}}$  must be in general the best possible measure of the distance to the boundary in Theorem 2, as well.

## 5. A converse result for convex polytopes

Theorem 1 and Corollary 1 provide complete analogues of Bernstein type inequality for  $\text{Lip}_M 1$  star like domains and convex domains  $K$  when the Euclidean distance  $\tau_K(\mathbf{x})$  is used as the proper measure of the distance to the boundary. This raises the following natural question: is it possible to replace  $\tau_K(\mathbf{x})$  in  $\|\tau_K(\mathbf{x})^{\frac{k}{2}} \partial^k q(\mathbf{x})\|_{L^p(w)}$  by an essentially larger function  $\phi(\mathbf{x})$ ? Namely we consider the model case of a convex polytope  $K \subset \mathbb{R}^d$  and verify that in case if  $\frac{\phi(\mathbf{x})}{\tau_K(\mathbf{x})}$  is unbounded at at least one of the vertices of  $K$  then a Bernstein type inequality similar to Theorem 1 can not hold with  $\phi(\mathbf{x})$  instead of  $\tau_K(\mathbf{x})$ .

**Theorem 3.** Let  $K \subset \mathbb{R}^d$ ,  $d \geq 2$  be a convex polytope,  $0 < p < \infty$ . Consider a positive function  $\phi \in C(K)$  such that  $\lim_{\mathbf{x} \rightarrow \mathbf{z}, \mathbf{x} \in \text{Int } K} \frac{\phi(\mathbf{x})}{\tau_K(\mathbf{x})} = \infty$  at one of the vertices  $\mathbf{z} \in \partial K$ . Then there exist  $q_n \in P_n^d$  so that

$$\lim_{n \rightarrow \infty} \frac{\|\phi^{\frac{d-1}{2}} \partial^{d-1} q_n\|_{L^p(K)}}{n^{d-1} \|q_n\|_{L^p(K)}} = \infty. \quad (23)$$

**Proof.** Given a convex polytope  $K \subset \mathbb{R}^d$  with vertex  $\mathbf{z} \in \partial K$  we may assume that its interior contains the origin,  $\mathbf{z} = \mathbf{e}_d = (0, \dots, 0, 1)$  and  $\forall \mathbf{x} = (x_1, \dots, x_d) \in K$ ,  $\mathbf{x} \neq \mathbf{z}$  we have  $x_d < 1$ . (This can be insured by a proper

affine transformation.) Then for polytope  $K$  there exist inscribed and superscribed cones with vertex  $\mathbf{z}$  that is there exists  $a > 1$  such that  $K_{1/a} \subset K \subset K_a$  where  $K_c, c > 1$  is the cone defined by

$$K_c := \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sqrt{x_1^2 + \dots + x_{d-1}^2} \leq c(1 - x_d), 0 \leq x_d \leq 1\}.$$

Now we will apply certain properties of Jacobi polynomials  $P_n^{(\alpha, \beta)}(x)$ ,  $\alpha, \beta > -1$  verified in [19], (7.34.1), p. 173. It is essentially shown there that

$$\int_0^1 (1-x)^\mu \left| P_n^{(\alpha, \beta)}(x) \right|^p dx \sim n^{-2\mu-2+\alpha p}, \quad 2\mu < \alpha p - \frac{3}{2}. \quad (24)$$

Moreover, by [19], (7.34.4) the lower bound in the above asymptotic relation holds for integration over the interval  $[1 - n^{-2}, 1]$ , i.e.,

$$\int_{[1-n^{-2}, 1]} (1-x)^\mu \left| P_n^{(\alpha, \beta)}(x) \right|^p dx > cn^{-2\mu-2+\alpha p}, \quad 2\mu < \alpha p - \frac{3}{2}. \quad (25)$$

(In fact, in [19] these asymptotic relations are verified for  $p = 1$  but they follow analogously for any  $p > 0$ , see also [2] and [14].)

Consider the polynomials  $q_n(x_1, \dots, x_d) := x_1 \cdot \dots \cdot x_{d-1} P_n^{(\alpha, \beta)}(x_d) \in P_{n+d-1}^d$ ,  $n \in \mathbb{N}$ . Evidently

$$|q_n(\mathbf{x})| \leq a^{d-1} (1 - x_d)^{d-1} |P_n^{(\alpha, \beta)}(x_d)|, \quad \mathbf{x} \in K_a.$$

Thus inclusion  $K \subset K_a$  clearly implies that

$$\begin{aligned} \|q_n\|_{L^p(K)}^p &\leq a^{p(d-1)} \int_{K_a} (1-x_d)^{p(d-1)} |P_n^{(\alpha, \beta)}(x_d)|^p dx = \int_0^1 \int_{B^{d-1}(0, a(1-x_d))} (1-x_d)^{p(d-1)} |P_n^{(\alpha, \beta)}(x_d)|^p dx_1 \dots dx_d \\ &= c_{a,d} \int_0^1 (1-x_d)^{(p+1)(d-1)} |P_n^{(\alpha, \beta)}(x_d)|^p dx_d. \end{aligned}$$

Thus applying the upper bound in (24) with  $\mu := (p+1)(d-1)$  and any  $\beta > -1, \alpha > 2d(1 + \frac{1}{p})$  yields the estimate

$$\|q_n\|_{L^p(K)}^p \leq c_{a,d} n^{\alpha p - 2 - 2(p+1)(d-1)}. \quad (26)$$

Now we need to establish a proper lower bound for  $\|\phi(\mathbf{x})^{\frac{d-1}{2}} \partial^{d-1} q_n(\mathbf{x})\|_{L^p(K)}$ . For this end set

$$D_n := K_{\frac{1}{2a}} \cap \{\mathbf{x} \in \mathbb{R}^d : 1 - n^{-2} \leq x_d < 1\}, \quad n \in \mathbb{N}, \quad \gamma(D) := \inf_{\mathbf{x} \in D} \frac{\phi(\mathbf{x})}{\tau_K(\mathbf{x})}, \quad D \subset \text{Int}K.$$

Note that with some  $c_a > 1$  depending only on  $a$

$$D_n \subset B^d\left(\mathbf{e}_d, \frac{c_a}{n^2}\right) = B^d\left(\mathbf{z}, \frac{c_a}{n^2}\right),$$

and thus condition  $\lim_{\mathbf{x} \rightarrow \mathbf{z}, \mathbf{x} \in \text{Int}K} \frac{\phi(\mathbf{x})}{\tau_K(\mathbf{x})} = \infty$  yields that  $\gamma(D_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Using that  $|\partial^{d-1}q_n(\mathbf{x})| \geq |\frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_{d-1}} q_n| = |P_n^{(\alpha, \beta)}(x_d)|$  and recalling inclusions  $D_n \subset K_{1/a} \subset K$  we have

$$\|\phi(\mathbf{x})^{\frac{d-1}{2}} \partial^{d-1} q_n(\mathbf{x})\|_{L^p(K)}^p \geq \int_{D_n} \phi(\mathbf{x})^{\frac{p(d-1)}{2}} |P_n^{(\alpha, \beta)}(x_d)|^p \geq \gamma(D_n)^{\frac{p(d-1)}{2}} \int_{D_n} \tau_K(\mathbf{x})^{\frac{p(d-1)}{2}} |P_n^{(\alpha, \beta)}(x_d)|^p$$

Furthermore a straightforward calculation implies that

$$\tau_K(\mathbf{x}) \geq \tau_{K_{1/a}}(\mathbf{x}) \geq c_a(1 - x_d), \quad \mathbf{x} \in D_n, \quad n \geq 2.$$

This together with the previous estimate yields

$$\begin{aligned} \|\phi(\mathbf{x})^{\frac{d-1}{2}} \partial^{d-1} q_n(\mathbf{x})\|_{L^p(K)}^p &\geq c_a \gamma(D_n)^{\frac{p(d-1)}{2}} \int_{D_n} (1 - x_d)^{\frac{p(d-1)}{2}} |P_n^{(\alpha, \beta)}(x_d)|^p \\ &= c_a \gamma(D_n)^{\frac{p(d-1)}{2}} \int_{[1-n^{-2}, 1]} \int_{B^{d-1}(0, \frac{1-x_d}{2a})} (1 - x_d)^{\frac{p(d-1)}{2}} |P_n^{(\alpha, \beta)}(x_d)|^p dx_1 \dots dx_d \\ &= c_a \gamma(D_n)^{\frac{p(d-1)}{2}} \int_{[1-n^{-2}, 1]} (1 - x_d)^{\frac{(p+2)(d-1)}{2}} |P_n^{(\alpha, \beta)}(x_d)|^p dx_d. \end{aligned}$$

Now applying relation (25) with  $\mu := \frac{(p+2)(d-1)}{2}$  and  $\alpha > 2d(1 + \frac{1}{p}), \beta > -1$  we arrive at

$$\|\phi(\mathbf{x})^{\frac{d-1}{2}} \partial^{d-1} q_n(\mathbf{x})\|_{L^p(K)}^p \geq c_a \gamma(D_n)^{\frac{p(d-1)}{2}} n^{-(p+2)(d-1)-2+\alpha p}.$$

Combining this last lower bound with the estimate (26) and taking  $p$ -th root we get

$$\frac{\|\phi(\mathbf{x})^{\frac{d-1}{2}} \partial^{d-1} q_n(\mathbf{x})\|_{L^p(K)}}{\|q_n\|_{L^p(K)}} \geq c_{a,p} \gamma(D_n)^{\frac{d-1}{2}} n^{d-1}.$$

It remains now to recall that as shown above  $\gamma(D_n) \rightarrow \infty$  as  $n \rightarrow \infty$  which together with the last lower bound yields the statement of the theorem.  $\square$

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