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Weakly saturated random graphs

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Abstract

As introduced by Bollobás, a graph G is weakly H -saturated if the complete graph K_n is obtained by iteratively completing copies of H minus an edge. For all graphs H , we obtain an asymptotic lower bound for the critical threshold p_c , at which point the Erdős–Rényi graph $\mathcal{G}_{n,p}$ is likely to be weakly H -saturated. We also prove an upper bound for p_c , for all H which are, in a sense, strictly balanced. In particular, we improve the upper bound by Balogh, Bollobás, and Morris for $H = K_r$, and we conjecture that this is sharp up to constants.

KEYWORDS

bootstrap percolation, random graph, weak saturation

1 | INTRODUCTION

The concept of *weak saturation* in graphs was introduced by Bollobás [12]. Given graphs G and H , the graph $\langle G \rangle_H$ is obtained by iteratively completing copies of H minus an edge, starting with G . Formally, set $G_0 = G$, and for $t \geq 1$, construct G_t by adding every edge not in G_{t-1} which if added to G_{t-1} creates a new copy of H . We let $\langle G \rangle_H = \bigcup_t G_t$ denote the result of this procedure. If $\langle G \rangle_H$ is the complete graph on the vertex set of G , that is, if all missing edges are eventually added, we say that G is *weakly H -saturated*, or that it *H -percolates*.

This process can be viewed as a type of cellular automaton [20, 26], of which bootstrap percolation (see, e.g., [1, 5, 13, 15, 17, 22, 25]) is a well-studied example. Balogh, Bollobás and Morris [6] introduced a random process called *graph bootstrap percolation*, taking G above to be the Erdős–Rényi [16] graph $\mathcal{G}_{n,p}$. The critical point p_c , at which $\mathcal{G}_{n,p}$ is likely to H -percolate, is defined formally as

$$p_c(n, H) = \inf\{p > 0 : \mathbb{P}(\langle \mathcal{G}_{n,p} \rangle_H = K_n) \geq 1/2\}.$$

The purpose of this work is to obtain general bounds for p_c . Our first main result (see Theorem 4) establishes a nontrivial lower bound, which holds for *all* H . This follows by a general extremal result (see Proposition 9) that lower bounds the number of edges in so-called witness graphs, which add a given edge. This extends the bound in [6], proved in the case that $H = K_r$, to all H . As an application (see Theorem 2), we locate p_c up to poly-logarithmic factors for all balanced (see Definition 1) graphs H , partially answering Prob. 1 in [6]. Our second main result (see Theorem 6) proves a sharper upper bound for all strictly balanced H . In particular, this improves the bound in [6] for K_r , when $r \geq 5$.

The primary focus in [6] is the case that $H = K_r$ is a complete graph (although some other graphs are also analyzed, see [6, Sec. 5]). Note that all graphs K_2 -percolate (any missing edge is added at time $t = 1$) so trivially $p_c(n, K_2) = 0$. A graph K_3 -percolates if and only if it is connected, so it follows $\mathbb{P}(\langle \mathcal{G}_{n,p} \rangle_{K_3} = K_n) \rightarrow \exp(-e^{-c})$ if $p = (\log n + c)/n$ by the fundamental work [16]. The next threshold of interest $p_c(n, K_4)$ is estimated in [6] up to constant factors, and the recent works [3, 4, 19] show that $p_c(n, K_4) \sim 1/\sqrt{3n \log n}$.

For $r \geq 5$, $p_c(n, K_r)$ is estimated in [6] up to poly-logarithmic factors. The upper bound for $p_c(n, K_r)$ proved in [6] holds for a more general class of graphs, which we now recall.

1.1 | Balanced graphs

For a graph H , we let v_H and e_H denote its number of its vertices and edges, respectively, and we put

$$\lambda = (e_H - 2)/(v_H - 2).$$

Note that H -percolation is trivial if $v_H \leq 3$ or if the minimum degree $\delta_H = 1$. In the latter case, p_c essentially coincides with the threshold for a copy of H minus an edge in $\mathcal{G}_{n,p}$ (see [6, Prop. 26]). That is, $p_c = \Theta(n^{-1/\lambda'})$, where

$$\lambda' = \min_{e \in E[H]} \max_{F \subset H \setminus e} e_F / v_F,$$

where $E[H]$ is the edge set of H , and F is a subgraph of H not containing the edge e . We therefore assume throughout this work that $\delta_H \geq 2$ and $v_H \geq 4$. In particular, this implies $\lambda \geq 1$. In fact, these assumptions hold for every graph satisfying the next definition.

Definition 1. We say that a graph H is *balanced* if $v_H \geq 4$, and $(e_F - 1)/(v_F - 2) \leq \lambda$ for all subgraphs $F \subset H$ with $3 \leq v_F < v_H$.

This is related to the notion of a *2-balanced* graph G , such that $(e_F - 1)/(v_F - 2)$ is maximized (over $F \subset G$ with $v_F \geq 3$) when $F = G$. This concept plays a role in, for example, [7, 14, 23, 24], where the maximal number of edges in an H -free subgraph (Turán's problem) of $\mathcal{G}_{n,p}$ is studied. Indeed, a graph H is balanced as above if and only if $H \setminus e$ is 2-balanced, for all edges $e \in E[H]$. It also follows that H is connected. See Appendix A.1 for a proof of these basic facts.

In [6], it is shown that $p_c(n, K_r) = n^{-1/(\lambda + o(1))}$, as $n \rightarrow \infty$. The upper bound holds for balanced graphs H (see [6, Prop. 3]). The lower bound, on the other hand, relies on the so-called *witness set algorithm*, which assigns to each $e \in E[\langle G \rangle_H]$ a *witness graph* $W_e \subset G$ such that $e \in E[\langle W_e \rangle_H]$. This algorithm yields an Aizenman–Lebowitz [1] type property (see Lemma 8), a standard tool from the theory of bootstrap percolation. A lower bound for p_c is obtained by the first moment method in [6] using this, together with the fact (Lemma 9 in [6]) that if $H = K_r$ then a witness graph on k vertices has at least $\lambda(k - 2) + 1$ edges. The proof, however, is somewhat abstract and lengthy. The authors state that “the proof is delicate, and does not seem to extend easily to other graphs.”

In this work, we present a short and simple proof (see Proposition 9 below) that works directly with the H -percolation dynamics, and naturally for *all* graphs H . Using this, we obtain the following result, which answers Prob. 1 in [6] in the case that H is balanced.

Theorem 2. *If H is balanced (see Definition 1) then $p_c(n, H) = n^{-1/\lambda+o(1)}$.*

We note that Bayraktar and Chakraborty [9] studied the case that $H = K_{r,s}$ is a complete bipartite graph. They find p_c up to poly-logarithmic factors in the range of r, s where $K_{r,s}$ is balanced, partially answering Prob. 5 in [6]. These results follow by Theorem 2, as a special case.

We also note that Theorem 2 is used in the recent work [8] to locate p_c when $H = \mathcal{G}_{k,1/2}$ is a random graph, answering Prob. 6 in [6].

1.2 | General lower bound

Theorem 2 follows by the upper bound in [6] and a general lower bound for $p_c(n, H)$, that holds for all H (satisfying our baseline assumptions $v_H \geq 4$ and $\delta_H \geq 2$), which we now describe.

Definition 3. For a graph H with $v_H \geq 4$, we put

$$\lambda_* = \min \frac{e_H - e_F - 1}{v_H - v_F},$$

minimizing over all subgraphs $F \subset H$ with $2 \leq v_F < v_H$.

Note that $\lambda_* > 0$ if and only if $\delta_H \geq 2$, and we continue to restrict our attention to graphs H with this property.

It is easy to see that $\lambda_* \leq \lambda$, with equality if and only if H is balanced (see Lemma 10). In Section 2.3 below, we show that a witness graph W_e for some $e \in E[\langle G \rangle_H]$ with $k \geq v_H$ vertices has at least $\lambda_*(k - v_H) + e_H - 1$ edges. Note that K_r is balanced, and so this reduces to $\lambda(k - 2) + 1$ in that case, recovering Lemma 9 in [6]. A general lower bound follows.

Theorem 4. *For any graph H with $v_H \geq 4$ and $\delta_H \geq 2$,*

$$p_c(n, H) \geq \Omega(n^{-1/\lambda_*}(\log n)^{1/\lambda_*-1}).$$

Note that, in the case that $H = K_4$, this lower bound includes the correct poly-logarithmic factor (recall that $p_c(n, K_4) \sim 1/\sqrt{3n \log n}$, as discussed above), since K_4 is balanced and so $\lambda_* = \lambda = 2$. In [6], the “double-dumbbell” $H = DD_r$ (two copies of K_r , $r \geq 4$, joined by a pair of disjoint edges) is given as an example of an (unbalanced) graph for which $p_c = n^{-1/\gamma+o(1)}$, with $\gamma \in (\lambda', \lambda)$ (recall λ' defined above Definition 1). We note that, in this instance, $\gamma = \left\lceil \frac{r}{2} \right\rceil + 1 / r = \lambda_*$.

On the other hand, Bidgoli, Mohammadian and Tayfeh-Rezaie [11] have studied the cases $H = K_{2,t}$. For these unbalanced graphs, even finding the correct power γ in $p_c = n^{-1/\gamma+o(1)}$ remains open, except in the specific case $H = K_{2,4}$ where $p_c = \Theta(n^{-10/13})$. However, note that $\lambda_* = 1$ for this graph.

Towards a full solution to Prob. 1 in [6], it would be interesting to determine the class of graphs for which $p_c = n^{-1/\lambda_*+o(1)}$. We note that Theorem 2 shows that this class includes all balanced graphs.

1.3 | Strictly balanced graphs

Finally, we turn our attention to the specific class of balanced graphs, which includes the cases $H = K_r$, $r \geq 5$, of natural interest.

Definition 5. We call H *strictly balanced* if the inequality $(e_F - 1)/(v_F - 2) < \lambda$ is strict in Definition 1.

Note that $\lambda > 1$ for strictly balanced graphs. Indeed, recall that we assume that $\delta_H \geq 2$, and consider any $F \subset H$ with three vertices and at least two edges.

For this class of graphs, we prove a sharper upper bound.

Theorem 6. *If H is strictly balanced then $p_c(n, H) \leq O(n^{-1/\lambda})$.*

Note that K_4 is balanced, but not strictly. For all $r \geq 5$, however, the graphs K_r are strictly balanced. It is somewhat tempting to suspect that $p_c(n, H) = \Theta(n^{-1/\lambda})$ for all strictly balanced graphs, but we would not go so far as to make that conjecture.

1.4 | Outline

The general lower bound in Theorem 4 is proved in Section 2, using Proposition 9 below, which lower bounds the number of edges in a witness graph. This result generalizes a result about cliques K_r , proved by a different strategy in [6, Lemma 9], to all graphs H .

Finally, in Section 3, we prove the upper bound in Theorem 6 for strictly balanced graphs. This is the most technical part of the article, where two rounds of the second moment method are required. First we bound the probability that a given edge is added by a specific type of witness graph, called an “ H -ladder,” with an appropriately chosen height. Then we show that these events are roughly independent enough to ensure that a large proportion of all edges are added in this way, and then full percolation follows by sprinkling. We note that H -ladder graphs were introduced in [6, Sec. 2]. A finer analysis of the ways in which pairs of such graphs can overlap is key to the sharper upper bound in Theorem 6.

1.5 | Notation

For a graph $G = (V, E)$, we denote its vertex set by $V[G] = V$ and its edge set by $E[G] = E$, with sizes $v_G = |V[G]|$ and $e_G = |E[G]|$. We write $F \subset G$ for a (not necessarily induced) subgraph F of G . We denote by $G \setminus e$ the graph obtained from G by keeping its vertex set, and deleting the edge e from its edge set. We similarly use $G \setminus \{e_1, \dots, e_k\}$ when the edges e_1, \dots, e_k are removed. For graphs G_1, G_2, \dots, G_k , we denote by $\bigcup_{i=1}^k G_i$ the graph with vertex set $\bigcup_{i=1}^k V[G_i]$ and edge set $\bigcup_{i=1}^k E[G_i]$, whereas $\bigcap_{i=1}^k G_i$ is the graph with vertex set $\bigcap_{i=1}^k V[G_i]$ and edge set $\bigcap_{i=1}^k E[G_i]$. For an edge $e = \{x, y\}$, for ease of notation, we often simply write e to denote the graph G with $V[G] = \{x, y\}$ and $E[G] = \{e\}$.

Throughout the paper we use the notation $f = O(g)$ or $f \leq O(g)$ for functions f and g if $f \leq cg$ for some universal constant c not depending on any of the arguments of f and g . Similarly, we use $f = \Omega(g)$ or $f \geq \Omega(g)$ for the opposite inequality, and $f = \Theta(g)$ when both of these statements hold.

2 | A GENERAL LOWER BOUND

In this section, we obtain a lower bound for p_c that holds for all graphs H (with $\delta_H \geq 2$ and $v_H \geq 4$). We first recall, in Sections 2.1 and 2.2, the results from [6] which we use. Then, in Section 2.3, we prove Theorem 4.

2.1 | Witness set algorithm

In [6, Sec. 3.1], the *witness set algorithm* (WSA) is introduced, which assigns a *witness graph* $W_e \subset G$ to each $e \in E[\langle G \rangle_H]$ such that $e \in E[\langle W_e \rangle_H]$. These graphs are defined in time with the percolation

dynamics in the following way. Let E_t denote the set of edges $E[G_t] \setminus E[G_{t-1}]$ added at time t . For any edge $e \in E[G]$, W_e consists of the single edge e with its two endpoints as vertices. Then, at step $t \geq 1$, the WSA defines simultaneously for each $e \in E_t$ the witness graph $W_e = \bigcup_{f \in E[H_e \setminus e]} W_f$, where H_e is a copy (chosen arbitrarily if not unique) of H completed by the addition of e at time t . Since $H_e \setminus e \subset G_{t-1}$, this procedure is well-defined.

Definition 7. For a witness graph W_e , we define $v_{W_e} - 2$, the number of vertices in W_e besides the endpoints of e , to be its *size*.

We note that the “size” of a graph sometimes refers to its number of edges. We will not follow this convention.

A key property of this construction is the following Aizenman–Lebowitz [1] type property (cf. Lemma 13 in [6]), as is easily observed.

Lemma 8. Suppose that W_e for some $e \in E[\langle G \rangle_H]$ is of size at least k for some $k \geq v_H - 2$. Then, for some $k' \in [k, e_H k]$, there is an $f \in E[\langle G \rangle_H]$ so that W_f is of size k' .

Proof. Let M_t be the maximal size of a witness graph W_f , for $f \in E[G_t]$. Note that $M_0 = 0$ and $M_1 = v_H - 2$ (assuming $E_1 \neq \emptyset$). Then for any $e \in E_{t+1}$, $t \geq 1$, W_e is of size at most $v_H - 2 + (e_H - 1)M_t \leq e_H M_t$, since $M_t \geq M_1 = v_H - 2$. Therefore $M_{t+1} \leq e_H M_t$. Hence, if $M_t \geq k$ for some t , then $M_s \in [k, e_H k]$ for some $s \leq t$. ■

2.2 | Red edge algorithm

In [6] a lower bound for p_c , in the special case of $H = K_r$, is obtained using Lemma 8 together with a lower bound for the number of edges in a witness graph. Specifically, it is shown that if W_e is of size k , then it has at least $\lambda k + 1$ edges. A key tool in this regard is the *red edge algorithm (REA)*, which is based on WSA (see Section 2.1 above). Informally, for a given edge $e \in E[\langle G \rangle_H] \setminus E[G]$ (not in G but eventually added by the H -percolation dynamics), REA describes the construction of the witness graph W_e one step at a time by running WSA, but ignoring steps that do not contribute to the construction of W_e . All involved edges which are not in G are colored red.

REA is discussed in Sec. 3.1 of [6]. For completeness, we briefly describe the construction here. First, we “slow down” the H -percolation dynamics, so that in each step a single new edge e' is added. Recall (see Section 2.1) that $H_{e'}$ is the copy of H which e' completes (that is, $W_{e'} = \bigcup_{f \in E[H_{e'} \setminus e']} W_f$). Let $e_1, \dots, e_m = e$ be the edges for which $W_{e_j} \subset W_e$ for $j \in \{1, \dots, m\}$ that are added (in that order by the “slowed down” dynamics) until finally e is added. Color all e_1, \dots, e_m red. Then

$$W_e = (H_1 \cup \dots \cup H_m) \setminus \{e_1, \dots, e_m\}, \quad (2.1)$$

where $H_j = H_{e_j}$. Hence, REA has m steps. In the j th step, a copy H_j of H is added and one of its edges e_j is colored red. Note that $e_j \notin \bigcup_{i < j} E[H_i]$, however, some of the other edges in H_j may already be in $\bigcup_{i < j} E[H_i]$.

2.3 | Proof of the lower bound

In this section, we give a proof of Theorem 4 that is based on the Aizenman–Lebowitz type property of Lemma 8, and the following lower bound on the number of edges of witness graphs, which we will prove later (as a special case of Lemma 12 below). Recall the definition of λ_* given in Section 1.

Proposition 9. *If W_e is a witness graph for an edge $e \in E[\langle G \rangle_H]$ on $k \geq v_H$ vertices, then W_e has at least $\lambda_*(k - v_H) + e_H - 1$ edges.*

The next lemma shows, in particular, that

$$\lambda_*(k - v_H) + e_H - 1 \geq \lambda_*(k - 2) + 1,$$

with equality if H is balanced.

Lemma 10. *We have $\lambda_* \leq \lambda$, with equality if and only if H is balanced.*

Proof. The case when F consists of a single edge $e \in E[H]$, with its two endpoints as vertices, shows that $\lambda_* \leq \lambda$. To see the second claim, note that, for $F \subset H$ with $3 \leq v_F < v_H$,

$$\lambda = \frac{e_H - \lambda(v_F - 2) - 2}{v_H - v_F} \leq \frac{e_H - e_F - 1}{v_H - v_F},$$

if and only if $\lambda(v_F - 2) \geq e_F - 1$. ■

Since K_r is balanced, and so $\lambda_* = \lambda$ in this case, we obtain Lemma 9 in [6] as a special case of Proposition 9. We note here that, as per [21], this result, in the special case of K_r , can also be alternatively obtained using [18].

We note here that once Lemma 8 and Proposition 9 have been established, the proof of our general lower bound for p_c follows straightforwardly, along the same lines as the proof of Proposition 8 in Sec. 3.2 of [6].

Proof of Theorem 4. Fix $e \in E[K_n]$. Let $p = \alpha n^{-1/\lambda_*} (\log n)^{1/\lambda_* - 1}$. We show that, for $\alpha > 0$ sufficiently small, $e \notin E[\langle \mathcal{G}_{n,p} \rangle_H]$ with high probability.

If $e \in E[\langle \mathcal{G}_{n,p} \rangle_H]$, then by Lemma 8 either (1) $e \in E[\mathcal{G}_{n,p}]$, (2) W_e is of size $k \in [v_H - 2, \log n]$, or else, (3) some W_f is of size $k' \in (\log n, e_H \log n]$. By Proposition 9, and the remark after it, a witness graph of size k has at least $\lambda_* k + 1$ edges. There are at most $k[O(k^{\lambda_*})]^k$ graphs with exactly $\lambda_* k + 1$ edges on a given set of $k + 2$ vertices (using $\binom{m}{\ell} \leq (me/\ell)^\ell$). Therefore, taking a union bound,

$$\begin{aligned} \mathbb{P}(e \in E[\langle \mathcal{G}_{n,p} \rangle_H]) &\leq p + p \sum_{k=v_H-2}^{\log n} k[O(np^{\lambda_*} k^{\lambda_*-1})]^k + n^2 p \sum_{k'=\log n}^{e_H \log n} k'[O(np^{\lambda_*} (k')^{\lambda_*-1})]^{k'} \\ &\leq p[O(\log n)]^2 (1 + n^{2+O(\log(\alpha))}) \ll 1, \end{aligned}$$

for α sufficiently small. ■

We turn now to the proof of Proposition 9. To this end, it is useful to define an increasing sequence $H_1 \subset \dots \subset H_m$ of auxiliary graphs associated with REA. (We note that something similar appears in [6], however, using hyper-graphs. For our purposes, it suffices to use graphs.) Recall (see Section 2.2) that in the j th step of REA a copy H_j of H is added and one of its new (not in $\bigcup_{i < j} H_i$) edges e_j is colored red. Let H_0 be the empty graph. Then the graph H_j is obtained from H_{j-1} by adding a new vertex v_j , which we associate with H_j . For each edge $e' \in E[H_j] \cap E[\bigcup_{i < j} H_i]$ we add an edge from v_j to

v_k , where $k = \max\{i < j : e' \in H_i\}$. Finally, delete any redundant edges, to ensure that H_j is a simple graph. Note that v_k is associated with the most recently (before step j) added copy of H containing e' . Finally, we put $H_e = H_m$.

Definition 11. For each connected component C of H_j , we refer to $C = \bigcup_{v_i \in V[C]} H_i$ as the corresponding *component* in $\bigcup_{i=1}^j H_i$.

Components of $\bigcup_{i=1}^j H_i$ can share vertices but not edges. Indeed, when H_j is added in step j of REA, all components in $\bigcup_{i=1}^{j-1} H_i$ with at least one edge in H_j are merged with H_j to obtain a component of $\bigcup_{i=1}^j H_i$. Also note that $\bigcup_{i=1}^m H_i$ has only one component (that is, H_e is connected). To see this, simply recall that $W_e = \bigcup_{f \in E[H_e \setminus e]} W_f$, and so note that each W_f has an edge $f \in H_e$, by the construction of W_e . Therefore we obtain Proposition 9 by the next result (which also implies Lemma 10 in [6] as a special case, once the terminology there is unpacked).

Lemma 12. Let W_e be a witness graph for an edge $e \in E[\langle G \rangle_H] \setminus E[G]$ on $k \geq v_H$ vertices. Then, after any $j \geq 1$ number of steps of the corresponding instance of REA, any component C of $\bigcup_{i=1}^j H_i$ has at least $\lambda_*(v_C - v_H) + e_H - 1$ non-red edges.

Definition 13. We let $\mathcal{V}_* \subset \{2, \dots, v_H - 1\}$ be the set for which λ_* is attained by some subgraphs $F \subset H$ with $v_F \in \mathcal{V}_*$. We put

$$\xi = \min \frac{e_H - e_F - 1}{v_H - v_F} - \lambda_*,$$

minimizing over $F \subset H$ with $v_F \in \{2, \dots, v_H - 1\} \setminus \mathcal{V}_*$.

Roughly speaking, we shall see that the most efficient way to add a new copy H_j of H , in any given step j of REA, is to ensure that the number of vertices in $V(H_j) \cap V(C')$ is in \mathcal{V}_* , for each of the components C' in $\bigcup_{i=1}^{j-1} H_i$ that have an edge in common with H_j (except in the simple case $H_j \setminus e_j \subset C'$, when only the red edge e_j is added to C' to obtain C). Note that all such components are merged with H_j in the j th step of REA to form a new component C . The quantity ξ is related to the minimum “cost” of a nonoptimal merge in REA.

We will also use the following distinction of steps in REA.

Definition 14. We call the j th step in REA a *type-1 step* if $H_j \setminus e_j$ is contained in some component of $\bigcup_{i=1}^{j-1} H_i$ as a subgraph. Otherwise, we call it a *type-2 step*. In particular, we include steps in which a new component is formed as a trivial instance of a type-2 step.

Proof of Lemma 12. The proof is by induction on the number of steps j taken. The base case $j = 1$ is trivial, since $H_1 \setminus e_1$ has v_H vertices and $e_H - 1$ edges. Likewise, the same reasoning applies if in some step $j > 1$ a new component is created. Hence suppose that in step $j > 1$, the addition of H_j causes exactly $h \geq 1$ (edge-disjoint) components C_1, \dots, C_h (each with at least one edge in H_j) to merge with H_j into a single component C . By assumption, we assume that the C_i have k_i vertices and $\lambda_*(k_i - v_H) + e_H - 1 + \ell_i$ non-red edges, for some $\ell_i \geq 0$. Let k denote the number of vertices in C . Note that

$$k = v_H + \sum_i (k_i - \varepsilon_i - \eta_i),$$

where ε_i is the number of vertices in $V[C_i] \cap V[H_j]$, and η_i is the number of vertices in $(V[C_i] \setminus V[H_j]) \cap \bigcup_{i' < i} V[C_{i'}]$ (i.e., other vertices in C_i that are in a previously considered

$C_{i'}$). To complete the proof, we show that C has at least $\lambda_*(k - v_H) + e_H - 1$ nonred edges. We distinguish two scenarios based on whether the j th step is a type-1 or type-2 step.

Case 1. If $H_j \setminus e_j$ is contained in some component of $\bigcup_{i=1}^{j-1} H_i$ as a subgraph, then necessarily $h = 1$, since the components C_i are edge-disjoint. In this case, the result follows immediately, since then $k = k_1$ and a single red edge (and no nonred edges) is added to form C .

Case 2. On the other hand, suppose that no C_i contains $H_j \setminus e_j$ as a subgraph. It is more convenient to start with H_j and color e_j red, and then merge the C_i with it one at a time. In these dynamics, any edge in $C_i \cap H_j$ that is red in C_i remains red after merging. Initially, we have the $e_H - 1$ nonred edges in H_j . In the i th substep (when we merge C_i), $k_i - \varepsilon_i - \eta_i$ vertices are added. Note that, by the choice of λ_* and ξ , at least

$$(v_H - \varepsilon_i)(\lambda_* + \xi \mathbf{1}_{\varepsilon_i \notin \mathcal{V}_*}) + \mathbf{1}_{\varepsilon_i = v_H},$$

of the $e_H - 1$ edges in $H_j \setminus e_j$ are not in C_i . Therefore, since the components $\{C_{i'}\}_{i' \leq i}$ are edge-disjoint, the number of non-red edges increases by at least

$$\lambda_*(k_i - \varepsilon_i) + \ell_i + (v_H - \varepsilon_i)\xi \mathbf{1}_{\varepsilon_i \notin \mathcal{V}_*} + \mathbf{1}_{\varepsilon_i = v_H}, \quad (2.2)$$

in the i th substep. Altogether, summing over all i , we find that C has k vertices and at least

$$(e_H - 1) + \lambda_* \sum_i (k_i - \varepsilon_i) + \sum_i [\ell_i + (v_H - \varepsilon_i)\xi \mathbf{1}_{\varepsilon_i \notin \mathcal{V}_*} + \mathbf{1}_{\varepsilon_i = v_H}],$$

nonred edges. Finally, note that the above is equal to

$$\begin{aligned} & (e_H - 1) + \lambda_*(k - v_H) + \sum_i \eta_i + \sum_i [\ell_i + (v_H - \varepsilon_i)\xi \mathbf{1}_{\varepsilon_i \notin \mathcal{V}_*} + \mathbf{1}_{\varepsilon_i = v_H}] \\ &= [\lambda_*(k - v_H) + e_H - 1] + \sum_i [\ell_i + \lambda_* \eta_i + (v_H - \varepsilon_i)\xi \mathbf{1}_{\varepsilon_i \notin \mathcal{V}_*} + \mathbf{1}_{\varepsilon_i = v_H}] \\ &\geq \lambda_*(k - v_H) + e_H - 1, \end{aligned} \quad (2.3)$$

as required. \blacksquare

3 | UPPER BOUND FOR STRICTLY BALANCED H

Next, we prove the upper bound in Theorem 6. We show that, for a strictly balanced graph H , with high probability $\langle \mathcal{G}_{n,p} \rangle_H = K_n$, if $p = (\alpha/n)^{1/\lambda}$ and $\alpha > 0$ is sufficiently large.

Two applications of the second moment method are involved. First we show that with probability bounded away from 0 (and tending to 1 as $\alpha \rightarrow \infty$) any given edge $e \in E[K_n]$ is added in $\langle \mathcal{G}_{n,p} \rangle_H$ due to a simple type of witness graph, called an H -ladder. These graphs were considered in [6]. The main difference here is that we consider *induced* H -ladders, resulting in an easier analysis of correlations (overlapping ladders). Then we show that the events that two given edges are added in $\langle \mathcal{G}_{n,p} \rangle_H$ by induced H -ladders (of suitable heights) are roughly independent. Hence, a significant proportion (tending to 1 as $\alpha \rightarrow \infty$) of all $\binom{n}{2}$ edges in K_n are included in $\langle \mathcal{G}_{n,p} \rangle_H$. Full percolation is then easily deduced (by Turán's Theorem and sprinkling).

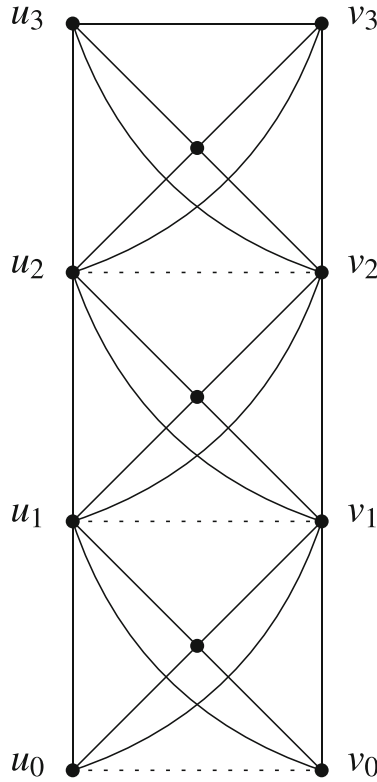


FIGURE 1 A K_5 -ladder of height $h = 3$ and size $k = (5 - 2)h = 9$ has $k + 2 = 11$ vertices and $\lambda k + 1 = \left[\binom{5}{2} - 2\right]h + 1 = 25$ edges.

3.1 | H -ladders

We consider (as in Sec. 2 of [6]) the following type of edge-minimal witness graph, where the associated graph (as in the discussion before Definition 11) is a path.

Definition 15. An H -ladder L of height h (see Figure 1) is a graph constructed using h copies S_i (called *steps*) of H minus two nonincident edges (called *rungs*) $\{u_{i-1}, v_{i-1}\}$ and $\{u_i, v_i\}$ such that, for each $1 < i \leq h$, we have that $V(S_i) \cap \bigcup_{j < i} V(S_j) = \{u_{i-1}, v_{i-1}\}$. We then obtain L as the union of the S_i and the *top rung* $\{u_h, v_h\}$. We call $(v_H - 2)h$ the *size*, h the *height* and $\{u_0, v_0\}$ the *base* of L . We often write $k = (v_H - 2)h$.

By induction on h , it is easy to see that L is a witness graph for its base edge. Note that L has $\lambda k + 1$ edges, the minimal possible number by Proposition 9. To see this note that each S_i has $e_H - 2 = \lambda(v_H - 2)$ edges. Since only the top rung $\{u_h, v_h\}$ is included in L , it has only $\lambda(v_H - 2)h + 1 = \lambda k + 1$ edges in total.

It is convenient, although slightly informal, to speak of vertices and edges of a ladder L that are “above” and “below” its various rungs, etc. In this sense, note that λ is the average number of edges “sent down the ladder” by vertices “above” the base.

Let us note here that we will, beginning in the next section, restrict to a specific class of H -ladders (see Definition 18 below) that is simpler and suffices for our purposes. However, before doing so, we first establish the following result that holds for H -ladders in general.

In [6] (see Lemma 6) it is shown that any subgraph $X \subset L$ containing $x + 2 < k + 2$ vertices of L , including those in its base, has at most λx edges. Equality is obtained if $X = \bigcup_{i \leq h'} S_i$ for some $1 \leq h' < h$. We prove the following estimate, which bounds the inefficiency of edge sharing in the other cases.

Since H is strictly balanced (see Definition 13),

$$\xi = \min \frac{e_H - e_F - 1}{v_H - v_F} - \lambda > 0,$$

minimizing over $F \subset H$ with $3 \leq v_F < v_H$. The case of F with $v_F = v_H - 1$ gives the bound $\xi \leq \delta_H - 1 - \lambda \leq \delta_H - 2$ (recall that $\delta_H \geq 2$, and so $\lambda \geq 1$).

Lemma 16. *Let L be an H -ladder of size $k = (v_H - 2)h$. Let X be a proper induced subgraph of L that contains x vertices above the base of L . Then X has at most $\lambda x - \xi \sigma$ edges, where σ is the number of steps $S_i \not\subset X$ of L such that X contains at least one vertex in $V[S_i] \setminus \{u_{i-1}, v_{i-1}\}$.*

Note that $\sigma = 0$ if and only if $x = 0$ or $X = \bigcup_{i \leq h'} S_i$ for some $1 \leq h' < h$. This result, in particular, implies Lemma 6 in [6] (without the condition $\lambda \geq 2$). Also note that we do not require that X contains the base vertices of L . This allows for an easier inductive proof, and will be useful for analyzing overlapping ladders with different bases (Lemma 21 below).

Proof. The proof is by induction on the height h of L . If $x = 0$ (and so also $\sigma = 0$) the statement is trivial. Thus we assume $x \geq 1$.

Base case. If $h = 1$ then $1 \leq x \leq v_H - 2$ and $\sigma = 1$ (since X is proper). There are $\lambda(v_H - 2) + 1$ edges in L .

Case 1a. If $x = v_H - 2$, then at least one vertex in the base of L is not in X . Hence there are at least $\delta_H - 1 \geq \xi + 1$ edges in $E[L] \setminus E[X]$, and so at most $\lambda x - \xi$ in $E[X]$.

Case 1b. If $1 \leq x < v_H - 2$, then there are at least $(\lambda + \xi)(v_H - 2 - x) + 1$ edges in $E[L] \setminus E[X]$, and so at most $\lambda x - \xi(v_H - 2 - x) \leq \lambda x - \xi$ in $E[X]$.

Inductive step. Suppose $h > 1$. Let $L' \subset L$ be the ladder of height $h - 1$ based at the first rung $\{u_1, v_1\}$ of L .

Case 2. If $S_1 \subset X$ or $V[X \cap S_1] \subset \{u_0, v_0\}$, then the result follows immediately by the inductive hypothesis applied to L' (since in either case $X \cap L' \subset L'$ is proper).

Case 3. Suppose that X contains $x_1 \geq 1$ vertices in $S_1 \setminus \{u_0, v_0\}$ and $S_1 \not\subset X$. Then, by the base case, X contains at most $\lambda x_1 - \xi - \mathbf{1}_{u_1, v_1 \in V[X]}$ edges in S_1 (since $h > 1$, the edge $\{u_1, v_1\} \notin E[L]$).

Case 3a. If $X \cap L' = L'$ (in which case $\sigma = 1$, and $u_1, v_1 \in V[X]$) the claim follows, since then there are $\lambda x_1 - \xi - 1$ edges in X below the first rung, $\lambda(k - x_1) + 1$ above, and so $\lambda x - \xi$ in total.

Case 3b. Otherwise, applying the inductive hypothesis to the remaining $x - x_1$ vertices of X in L' , it follows that there are at most $\lambda(x - x_1) - \xi(\sigma - 1)$ edges in $X \cap L'$. Hence L has at most $\lambda x - \xi \sigma$ edges. ■

3.2 | H -Ladders in $\mathcal{G}_{n,p}$

Having established Lemma 16, we turn to the upper bound for p_c . We first obtain a lower bound on the probability that a given edge $e \in E[K_n]$ is the base of an H -ladder of height h in $\mathcal{G}_{n,p}$. This gives a

lower bound on the probability that $e \in E[\langle \mathcal{G}_{n,p} \rangle_H]$. We then verify the approximate independence for different bases. This strategy thus involves two applications of the second moment method. As already discussed, we restrict to the case of induced H -ladders, since this simplifies the analysis of correlations. Furthermore, we also restrict our attention to a specific type of H -ladder, defined as follows.

Definition 17. Fix two nonincident edges e_t, e_b in H , and a copy T of $H \setminus \{e_t, e_b\}$ labeled in some arbitrary way such that

- the vertices in e_b are labelled by $\{1, 2\}$, and
- all other vertices (not in e_b) in H are labeled by $\{3, 4, \dots, v_H\}$.

We call T the *template*.

We fix T for the rest of this work. We use the template T to define a simple class of H -ladders, where, informally speaking, copies of T are stacked on top of each other.

Definition 18. A (labelled) H -ladder (see Definition 15) is *uniform* if, for each of its steps S_i , the function ϕ_i for which

- $\phi_i(u_{i-1}) = 1, \phi_i(v_{i-1}) = 2$, and
- $\phi_i(w_k) = k + 2$, for $w_k \in V[S_i] \setminus \{u_{i-1}, v_{i-1}\}$ of k th largest label,

is an isomorphism from S_i to the template T .

Note that, since T is fixed, there are exactly

$$\binom{k}{v_H - 2, \dots, v_H - 2} = \frac{k!}{(v_H - 2)!^h}$$

uniform H -ladders of size $k = (v_H - 2)h$ (and height h) on a given set of $k + 2$ vertices and with a given base e . Indeed, the conditions in Definition 18 imply that the only freedom in selecting such an H -ladder is in choosing which vertices are in each of the h sets $V[S_i] \setminus \{u_{i-1}, v_{i-1}\}$.

Definition 19. For $\varepsilon \in (0, 1)$, put

$$\alpha_\varepsilon = \exp \left[\left(\frac{v_H + 1}{\xi} + \frac{1 - \varepsilon}{4} \right) \frac{\log(v_H - 2)}{\varepsilon} \right], \quad (3.1)$$

and

$$\beta_\varepsilon = \frac{\varepsilon \xi}{\lambda(v_H + 1)(v_H - 2) \log(v_H - 2)}. \quad (3.2)$$

Note that $\alpha_\varepsilon \rightarrow \infty$ and $\beta_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$. We also note that $\alpha_\varepsilon, \beta_\varepsilon$ are chosen so that (3.4), (3.2), and (3.8) below simplify nicely (but are, of course, not the only choice of α, β for which the following lemmas hold).

First we show that a given edge in K_n is the base of an H -ladder with probability tending to 1 as $np^\lambda \rightarrow \infty$.

Lemma 20. Fix $\varepsilon \in (0, 1)$. Put $np^\lambda = \alpha_\varepsilon (v_H - 2)!^{1/(v_H - 2)}$. Then any given $e \in E[K_n]$ is the base of an induced uniform H -ladder of height $h = \beta_\varepsilon \log n$ in $\mathcal{G}_{n,p}$ with probability at least $\gamma_\varepsilon - o(1)$, where

$$\gamma_\varepsilon = 1 - \frac{1}{\alpha_\varepsilon^{v_H - 2} - 1}.$$

For ease of exposition, we write quantities such as $h = \beta_\epsilon \log n$ as is, instead of replacing them with their integer parts.

Proof. Let N_k denote the number of induced uniform H -ladders in $\mathcal{G}_{n,p}$ of size $k = (v_H - 2)h$ with a given base e . Noting that $k \ll n$ and $k^2 p \ll 1$ (and using the standard bound $\binom{n}{k} \geq (n - k)^k / k!$) we find that

$$\mathbb{E}N_k = \binom{n-2}{k} \frac{k!}{(v_H - 2)!^h} p^{\lambda k + 1} (1 - p)^{\binom{k+2}{2} - (\lambda k + 1)} \geq p \alpha_\epsilon^k (1 - o(1)). \quad (3.3)$$

Since, by (3.1) and (3.2),

$$\lambda \beta_\epsilon (v_H - 2) \log \alpha_\epsilon = 1 + \frac{(1 - \epsilon)\xi}{4(v_H + 1)} > 1, \quad (3.4)$$

it follows that $\mathbb{E}N_k \gg 1$.

Let L_1, L_2, \dots enumerate all uniform H -ladders in K_n of size k that are based at e . Let A_i be the event that L_i is an induced subgraph of $\mathcal{G}_{n,p}$. Following Sec. 4.3 of [2], and using symmetry,

$$\begin{aligned} \mathbb{P}(N_k = 0) &\leq \frac{\mathbb{E}(N_k^2)}{(\mathbb{E}N_k)^2} - 1 = \frac{1}{\mathbb{E}N_k} + \frac{1}{(\mathbb{E}N_k)^2} \sum_{i \neq j} \mathbb{P}(A_i \cap A_j) - 1 \\ &= \frac{1}{\mathbb{E}N_k} \left[1 + \sum_{i > 1} \mathbb{P}(A_i | A_1) \right] - 1. \end{aligned} \quad (3.5)$$

Since we are considering induced subgraphs of $\mathcal{G}_{n,p}$, for any $i \neq 1$, we have $\mathbb{P}(A_i | A_1) = 0$ unless $V[L_i] \neq V[L_1]$. Let S_i , $1 \leq i \leq h$, denote the steps of L_1 .

Case 1. First, we consider the case that L_i “breaks cleanly” from L_1 at one of its rungs, that is, $L_i \cap L_1 = e$ or $L_i \cap L_1 = \bigcup_{i \leq h'} S_i$, for some $1 \leq h' < h$.

If $L_i \cap L_1 = e$ then L_i is an H -ladder of height h that is edge-disjoint from L_1 . Hence $\mathbb{P}(A_i | A_1) \leq p^{\lambda k + 1}$. Similarly, if $L_i \cap L_1 = \bigcup_{i \leq h'} S_i$, for some $1 \leq h' < h$, then L_i and L_1 agree up to height h' . The part of L_i that is “above” the intersection $L_i \cap L_1$ is an H -ladder (based at the h' th rung $\{u_{h'}, v_{h'}\}$ of L_1) of height $h - h'$ that is edge-disjoint from L_1 . Hence, in this case, $\mathbb{P}(A_i | A_1) \leq p^{\lambda k' + 1}$, where $k' = (v_H - 2)(h - h')$. Summing over all such L_i , using (3.3),

$$\begin{aligned} \frac{1}{\mathbb{E}N_k} \sum \mathbb{P}(A_i | A_1) &\leq \frac{1}{\mathbb{E}N_k} \sum_{h'=0}^{h-1} \frac{n^{k'} p^{\lambda k' + 1}}{(v_H - 2)!^{h-h'}} \\ &\leq (1 + o(1)) \sum_{h'=0}^{h-1} \alpha_\epsilon^{-(v_H - 2)h'} \\ &\leq \frac{1 + o(1)}{1 - 1/\alpha_\epsilon^{v_H - 2}}. \end{aligned}$$

Case 2. Next, we show that all other cases are of lower order. If L_i does not “break cleanly” (as in Case 1) from L_1 then by Lemma 16, $\mathbb{P}(A_i | A_1) \leq p^{\lambda(k-x)+1+\xi\sigma}$, where x is the

number of vertices in $X = L_i \cap L_1$ above the base of L_1 , and $\sigma \geq 1$ is the number of $S_i \not\subset X$ such that $V[X] \cap (V[S_i] \setminus \{u_{i-1}, v_{i-1}\}) \neq \emptyset$.

For any such L_i , let $s \geq 0$ be the number of maximal subgraphs $\bigcup_{i=h_1}^{h_2} S_i \subset X$, $h_1 \leq h_2$. For convenience, we refer to these as the subladders that L_1 and L_i have in common. Let $y \geq 0$ denote the number of other vertices in X (not inside a common subladder). Note that $s + y \geq 1$ since $\sigma \geq 1$. We claim that

$$\sigma \geq \max\{1, (s - 1 + y)/v_H\} \geq (s + y)/(v_H + 1). \quad (3.6)$$

To see this, note that there are at least $2(s - 1)\mathbf{1}_{s \geq 1} + y$ vertices of X in steps $S_i \not\subset X$, since if $\bigcup_{i=h_1}^{h_2} S_i \subset X$ is maximal and $h_1 > 1$, then $S_{h_1-1} \not\subset X$ and $u_{h_1-1}, v_{h_1-1} \in V[X]$.

Next, we claim that, for given x, s, y , there are at most

$$\binom{h+1}{2s} s! \binom{h}{s} 2^s \binom{k}{y} \binom{n}{k-x} (k-x+y)! \leq [O(k^3)]^{s+y} n^{k-x},$$

ladders L_i to consider. To see this, observe that:

- (1) there are $\binom{h+1}{2s}$ ways to select the s common subladders (since this corresponds to selecting $2s$ rungs in L_1) and $s!$ ways to choose the order in which they can appear in L_i ,
- (2) there are at most $\binom{h}{s}$ possibilities for where the sub-ladders are located in L_i (choose a height for the top rung of each), and 2^s ways to decided whether the top rung (as it appears in L_1) of each sub-ladder is the top or bottom rung as it appears in L_i , and
- (3) the final three factors bound the choices for the $k - x + y$ other vertices in L_i and their locations in L_i .

Note that in (2) we need not also consider the possibility that the labels in a top or bottom rung of a common subladder are reversed (with respect to e) in L_i as compared with how they appear in L_1 . Indeed, doing so would either produce a nonuniform H -ladder, or else one that is equivalent to L_1 .

Hence, summing over all such L_i with given x, s, y , we find (using $\alpha_\epsilon > 1$, $x \geq 1$, $s + y \geq 1$, (3.3) and (3.6)) that

$$\begin{aligned} \frac{1}{\mathbb{E}N_k} \sum \mathbb{P}(A_i | A_1) &\leq \frac{1}{\mathbb{E}N_k} [O(k^3)]^{s+y} n^{k-x} p^{\lambda(k-x)+1+\xi\sigma} \\ &\leq O\left[(\log n)^3 \frac{(np^\lambda)^{k-x}}{\mathbb{E}N_k} p^{1+\xi/(v_H+1)}\right] \\ &\leq O[(\log n)^3 (v_H - 2)!^h p^{\xi/(v_H+1)}] \ll n^{-\vartheta_\epsilon}, \end{aligned}$$

where, by (3.2),

$$\vartheta_\epsilon = \frac{\xi}{\lambda(v_H + 1)} - \beta_\epsilon(v_H - 2) \log(v_H - 2) = \frac{(1 - \epsilon)\xi}{\lambda(v_H + 1)} > 0. \quad (3.7)$$

Since there are only $O(k^3)$ relevant x, s, y the same holds summing over all L_i (not included in Case 1).

Therefore, combining the two cases, we find that

$$\frac{1}{\mathbb{E}N_k} \sum_{i \geq 1} \mathbb{P}(A_i | A_1) \leq \frac{1 + o(1)}{1 - 1/\alpha_\epsilon^{v_H-2}},$$

and so, by (3.5),

$$\mathbb{P}(N_k > 0) \geq 1 - \frac{1 + o(1)}{\alpha_\epsilon^{v_H-2} - 1} = \gamma_\epsilon - o(1).$$

■

Next, by another application of the second moment method, we show that with high probability a significant proportion (tending to 1 as $\alpha \rightarrow \infty$) of edges in K_n are bases of H -ladders in $\mathcal{G}_{n,p}$, and so included in $\langle \mathcal{G}_{n,p} \rangle_H$.

Lemma 21. Fix $\epsilon \in (0, 1)$. Put $np^\lambda = \alpha_\epsilon(v_H - 2)!^{1/(v_H-2)}$. Then, with high probability, there are at least $(\gamma_\epsilon - \epsilon) \binom{n}{2}$ edges in K_n which are bases of induced uniform H -ladders of height $h = \beta_\epsilon \log n$ in $\mathcal{G}_{n,p}$.

Proof. Let e_1, e_2, \dots enumerate the edges of K_n and let E_i denote the event that e_i is the base of an induced uniform H -ladder of height h in $\mathcal{G}_{n,p}$. Similarly, let E'_i denote the event that e_i is the base of a (not necessarily induced) uniform H -ladder of height h in $\mathcal{G}_{n,p}$. We show that

$$\frac{1}{\left[\binom{n}{2} \mathbb{P}(E_1)\right]^2} \sum_{i \neq j} \mathbb{P}(E_i \cap E_j) \leq 1 + o(1),$$

from which, together with Lemma 20, the result follows (as then the number of such edges $\sum_i \mathbf{1}_{E_i}$, divided by its expectation $\binom{n}{2} \mathbb{P}(E_1)$, converges to 1 in probability, see again Sec. 4.3 of [2] and the technique used in (3.5)).

To this end, we bound the event $E_i \cap E_j$ by the union of events (1) $E'_i \circ E'_j$ that there are edge-disjoint (not necessarily induced) uniform H -ladders of heights h based at e_i and e_j , and, (2) E_{ij} that there is an induced uniform H -ladder of height h based at e_j that includes an edge of such a ladder based at e_i .

By the van den Berg–Kesten (BK) inequality [10] (the events E'_i are increasing) and symmetry,

$$\mathbb{P}(E'_i \circ E'_j) \leq \mathbb{P}(E'_1)^2 = (1 + o(1)) \mathbb{P}(E_1)^2,$$

where the last equality follows since $k^2 p \ll 1$. It thus suffices to show that

$$\sum_{i \neq j} \mathbb{P}(E_{ij}) \ll n^4,$$

since by Lemma 20, the probability $\mathbb{P}(E_1) \geq \gamma_\epsilon - o(1)$ (and so, in particular, bounded away from 0 as $n \rightarrow \infty$).

Let L_1 be a fixed uniform H -ladder of height h in K_n based at e_1 , and let A_1 denote the event that L_1 is an induced subgraph of $\mathcal{G}_{n,p}$. For $j > 1$, let B_j be the event that there is an induced uniform H -ladder in $\mathcal{G}_{n,p}$ of height h based at e_j that includes at least one edge in L_1 . As in the previous proof, let N_k denote the number of induced uniform H -ladders in

$\mathcal{G}_{n,p}$ of size $k = (v_H - 2)h$ with a given base e . Note that $\mathbb{E}(N_k) \leq p\alpha_\epsilon^k$. Hence, by symmetry, we have

$$\begin{aligned} \sum_{i \neq j} \mathbb{P}(E_{ij}) &\leq n^2 \sum_{j>1} \mathbb{P}(E_{1j}) \\ &\leq n^2 \mathbb{E}(N_k) \sum_{j>1} \mathbb{P}(B_j|A_1) \\ &\leq n^2 p\alpha_\epsilon^k \sum_{j>1} \mathbb{P}(B_j|A_1). \end{aligned}$$

Hence, it suffices to show that

$$p\alpha_\epsilon^k \sum_{j>1} \mathbb{P}(B_j|A_1) \ll n^2.$$

Finally, we estimate $\sum_{j>1} \mathbb{P}(B_j|A_1)$ by a union bound, considering the expected (conditioned on A_1) number of induced H -ladders L of height h based at some $e \neq e_1$ that include at least one edge of L_1 , and hence at least one vertex not in its base e_1 .

At this point, the argument is similar to the proof of Lemma 20, and so we only sketch the details. As before, we take two cases with respect to whether L and L_1 “intersect cleanly” (i.e., if $L \cap L_1 = \bigcup_{i \leq h'} S_i$ for some $1 \leq h' < h$, where S_i are the steps of L_1) or not. Note that if L and L_1 share at least one common edge, then we cannot have that $L \cap L_1 = e_1$, since if A_1 occurs then $e \notin E[\mathcal{G}_{n,p}]$.

Case 1. If L and L_1 “intersect cleanly” then there are $O(h)$ possibilities for where (i.e., the height at which) $L \cap L_1$ is located in L . Apart from this, by an argument similar to that in Case 1 in proof of Lemma 20, we see that the expected number of such L is at most $\binom{n}{2} O(h p \alpha_\epsilon^k / n^2) \leq O(k p \alpha_\epsilon^k)$. The compensating factor $1/n^2$ here is due to the fact that there are $(v_H - 2)h' + 2$ vertices, but only $\lambda(v_H - 2)h'$ edges, in $L \cap L_1$.

Case 2. Otherwise, if L and L_1 do not “intersect cleanly” then, arguing as in Case 2 in the proof of Lemma 20, the expected number of such L in this case is $\ll \binom{n}{2} n^{-\vartheta_\epsilon} p \alpha_\epsilon^k$, where $\vartheta_\epsilon > 0$ is as defined in (3.7).

Altogether,

$$\frac{p\alpha_\epsilon^k}{n^2} \sum_{j>1} \mathbb{P}(B_j|A_1) \leq O[(p\alpha_\epsilon^k)^2 (kn^{-2} + n^{-\vartheta_\epsilon})] = O[(p\alpha_\epsilon^k)^2 n^{-\vartheta_\epsilon}] \ll 1,$$

since, by (3.4) and (3.7),

$$\frac{2}{\lambda}(-1 + \lambda\beta_\epsilon(v_H - 2)\log \alpha_\epsilon) - \vartheta_\epsilon = -\frac{\vartheta_\epsilon}{2} < 0. \quad (3.8)$$

■

3.3 | The upper bound

With Lemma 21 at hand, we obtain our upper bound for p_c by an adaptation of the argument found at the end of Sec. 2 in [6].

Proof of Theorem 6. Let $p = (\alpha/n)^{1/\lambda}$. We show that for $\alpha > 0$ sufficiently large, $\langle \mathcal{G}_{n,p} \rangle_H = K_n$ with high probability.

Let $G = (V, E)$ be a graph. If only εn vertices $v \in V$ have degree $d_v \geq \delta(n-1)$, for some δ , then $|E| \leq [\varepsilon + (1-\varepsilon)\delta] \binom{n}{2}$. Hence, if $|E| > \gamma \binom{n}{2}$, there is a set $S \subset V$ of size satisfying $|S|/n \geq (\gamma - \delta)/(1 - \delta)$ so that all $v \in S$ have $d_v \geq \delta(n-1)$.

Therefore, by Lemma 21, for $\alpha > 0$ large (and so γ close to 1) with high probability there is a set S of size $\Omega(n)$ such that all neighborhoods N_v in $\langle \mathcal{G}_{n,p} \rangle_H$ of vertices $v \in S$ are larger than $(3/4)n$. As a result, all $u, v \in S$ have $|N_u \cap N_v| \geq n/2$. Also, for α large enough, all induced subgraphs of $\langle \mathcal{G}_{n,p} \rangle_H$ of size $n/4$ contain a copy of K_{v_H-2} by Turán's Theorem. Hence all edges between vertices in S are in $\langle \mathcal{G}_{n,p} \rangle_H$.

Once a percolating subgraph S of size $\Omega(n)$ has been established, the result follows easily by sprinkling, as in [6]. For completeness, we sketch the argument. Consider a random graph $\mathcal{G}_{n,p'}$ that is independent of $\mathcal{G}_{n,p}$ with $(\log n)/n \ll p' \ll p$. Since H is strictly balanced, such a p' exists, as $p = \Omega(n^{-1/\lambda})$ for some $\lambda > 1$ (see below Definition 5). Due to $(\log n)/n \ll p'$, with high probability, in the graph $\mathcal{G}_{n,p'}$, all vertices outside of S have at least $v_H - 2$ neighbors in S . Hence, $\langle \mathcal{G}_{n,p} \cup \mathcal{G}_{n,p'} \rangle_H = K_n$ with high probability. This implies the result, noting that $\mathcal{G}_{n,p} \cup \mathcal{G}_{n,p'}$ is a random graph with edge probability $1 - (1-p)(1-p') \sim p$. ■

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable, as no datasets were generated or analyzed.

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APPENDIX A: SUPPLEMENTARY FACTS

A.1 | Balanced graphs

We note here some basic facts about balanced graphs H . Recall Definition 1 and the definition of 2-balanced graphs, which appears below Definition 1. Also recall that we assume throughout this work that $\delta_H \geq 2$ and $v_H \geq 4$. Hence $e_H \geq v_H$.

Lemma 22. *For any graph H , we have that H is balanced if and only if $H \setminus e$ is 2-balanced for all edges $e \in E[H]$.*

Proof. Suppose that $H \setminus e$ is 2-balanced for all edges $e \in H$. Let F be a proper subgraph of H with $v_F \geq 3$. Let $e \in E[H] \setminus E[F]$. Since $H' = H \setminus e$ is 2-balanced, it follows that $(e_F - 1)/(v_F - 2) \leq (e_{H'} - 1)/(v_{H'} - 2) = \lambda$. Thus H is balanced.

On the other hand, if H is balanced, then for any proper subgraph F of some $H' = H \setminus e$ with $v_F \geq 3$, $(e_F - 1)/(v_F - 2) \leq \lambda = (e_{H'} - 1)/(v_{H'} - 2)$. Thus H' is 2-balanced. ■

Lemma 23. *For any graph H , if H is balanced then it is connected.*

Proof. Assuming that H is balanced, we show that there is at least one edge between any two non-empty sets V_1, V_2 that partition the vertex set of H . Let v_i and e_i be the number of

vertices and edges, respectively, in the subgraph of H induced by V_i , and e_{12} the number of edges in H between V_1 and V_2 , so that $e_H = e_1 + e_2 + e_{12}$. If either $v_i \leq 2$ or $e_1 + e_2 \leq 3$ then $e_{12} \geq 1$, since $\delta_H \geq 2$ and $e_H \geq v_H \geq 4$. Hence assume that both $v_i \geq 3$ and $e_1 + e_2 \geq 4$. Then both

$$\frac{e_i - 1}{v_i - 2} \leq \lambda = \frac{e_1 + e_2 + e_{12} - 2}{v_1 + v_2 - 2}.$$

Taking a weighted average, with weights $v_i - 2$, it follows that

$$\frac{e_1 + e_2 - 2}{v_1 + v_2 - 4} \leq \lambda,$$

and so

$$e_{12} \geq \left(\frac{v_1 + v_2 - 2}{v_1 + v_2 - 4} - 1 \right) (e_1 + e_2 - 2) > 0.$$

■