

ADDITIVE DECOMPOSITION OF SIGNED PRIMES

IMRE Z. RUZSA

ABSTRACT. Under the prime-tuple hypothesis, the set of signed primes is a sumset.

In memoriam Andreae Schinzel, excellentis mathematici et optimi viri.

1. INTRODUCTION

Ostmann’s “inverse Goldbach” problem (this term is probably Wirsing’s) asks whether the set P of (positive) primes is asymptotically a sumset, that is, whether there are sets A, B (having more than 1 elements each) such that $P \Delta (A+B)$ is finite, with Δ meaning symmetric difference.

Still unsolved, generally a negative answer is expected. See Elsholz[1, 2] for the best partial results.

The aim of this paper is to show that the situation changes if we admit negative primes.

Theorem 1. *Assuming the prime-tuple hypothesis there are infinite sets $A, B \subset \mathbb{Z}$ such that $A + B$ is exactly the collection of (positive and negative) primes satisfying $|p| > 3$; moreover, every prime p has exactly one representation as $p = a + b$, $a \in A$, $b \in B$.*

It is easily seen that $\pm 2, \pm 3$ must be omitted, and that if the summands have more than 1 elements, they both must be infinite.

Some details of the argument are easier written in terms of differences, so we reformulate the theorem as follows.

Theorem 2 (Variant with differences). *Assuming the prime-tuple hypothesis there are infinite sets A, B of positive integers such that $A - B$ is exactly the collection of (positive and negative) primes satisfying $|p| > 3$; moreover, every prime p has exactly one representation as $p = a - b$, $a \in A$, $b \in B$.*

2. THE PRIME-TUPLE HYPOTHESIS

The prime-tuple hypothesis, generally assumed to be true but hopeless, expresses that linear forms can simultaneously represent primes unless there is a congruence obstacle.

Conjecture (Prime-tuple hypothesis). Let a_i, b_i be integers, $a_i \neq 0$. There are infinitely many values of x such that all $a_i x + b_i$ are prime, unless there is a prime p such that for all x we have $p | a_i x + b_i$ for some i .

(Simplest case is the twin prime conjecture.)

We shall apply the following special case.

2020 *Mathematics Subject Classification.* 11P32, 11N99.

Supported by NKFI grants K-129335, K-119528, KKP-133819.

Corollary. *Given $d_i \in \mathbb{Z}$, $q > 1$ and t , the condition for the existence of infinitely many $x \equiv t \pmod{q}$ such that all $x + d_i$ are primes is that for all primes p :*

- *if $p|q$, then $p \nmid t + d_i$ for any i ;*
- *if $p \nmid q$, then the d_i do not contain a complete system of residues modulo p .*

This is the preceding applied to the forms $qx + t + d_i$.

3. THE PLAN

Assume we have finite sets A, B such that all elements of $A - B$ are primes. We want to represent a further prime r . How can we do it? Try

$$A' = A \cup \{x\}, B' = B \cup \{x - r\}.$$

This works if all elements of $B - x$ and $A - x + r$ are prime.

We find such an x if $B \cup (A + r)$ is not a complete system modulo any prime.

But: the inclusion of x and $x - r$ may spoil this property; we may build a trap to kill the plan after several steps.

Remedy: we will a priori restrict the possible residues mod p . We clearly cannot do this initially for all primes; we shall dynamically add more and more restrictions, that is, new elements compatible with given restrictions and new restrictions compatible with given elements.

4. PRIME-COMPATIBLE SETS OF RESIDUES

Definition. Let p be a prime and $U, V \subset \mathbb{Z}_p$ nonempty sets of residues. We say that U, V form a *prime-compatible pair*, if

$$(U \setminus V) - (V \setminus U) = \mathbb{Z}_p^*.$$

The residues of A and B modulo p must have this property.

Lemma. *Let p be a prime.*

(a) *Let $W \subset \mathbb{Z}_p$ be a nonempty set of residues. If*

$$(4.1) \quad |W| < \frac{p-1}{2} - \frac{\log p}{\log 4/3},$$

then there are prime-compatible sets modulo p satisfying $W \subset U$, $W \subset V$.

(b) *For $p \geq 7$ there are prime-compatible sets modulo p such that $1, 11 \in U \setminus V$, $6 \in V \setminus U$ and $|U \cap V| = 2$.*

Part (b) is motivated by the following consideration. By looking at the residues of A and B modulo 5 it is easy to see that the representations of 5 and -5 must have a common element; we set $1, 11 \in A$ and $6 \in B$.

Proof. (a): Write $|W| = n$. We assign the $p - n$ elements of $\mathbb{Z}_p \setminus W$ randomly to U or V , and estimate the probability that a $z \in \mathbb{Z}_p^*$ is represented as a difference of these new elements.

Consider the pairs $(z, 2z), (3z, 4z), \dots, ((p-2)z, (p-1)z)$. From these $(p-1)/2$ pairs at most n contain an element of W , at least $(p-1)/2 - n$ remain.

A pair represents z with probability $1/4$, so

$$\Pr(\text{none is good}) \leq (3/4)^{\frac{p-1}{2} - n}.$$

If this quantity is $< 1/p$, there is a choice that works for all z . This yields the condition

$$(3/4)^{\frac{p-1}{2}-n} < \frac{1}{p},$$

which can be rearranged as

$$\frac{p-1}{2} - n > \frac{\log p}{\log 4/3},$$

that is, (4.1).

(b): Let $U' = \{0, 1, 2, p-1\}$, $V' = \{0, 1, 3, 4, \dots, p-2\}$. This is a prime-compatible pair modulo p . This property is preserved by linear transformations. Define α, β by

$$3\alpha \equiv -10 \pmod{p}, \quad \beta = \alpha + 11.$$

The linear function $\alpha x + \beta$ maps 2 to 1, $p-1$ to 11 and $(p+1)/2$ to 6, so the sets $U = \alpha \cdot U' + \beta$, $V = \alpha \cdot V' + \beta$ are suitable. \square

The bound could be improved with little effort, but it is irrelevant for our application.

5. THE CONSTRUCTION

We prove the theorems.

Let r_1, r_2, \dots be the sequence of signed primes $\pm p$, $|p| > 3$, ordered by increasing absolute value ($p_1 = 5, p_2 = -5$, etc.) We shall construct sequences of sets A_n, B_n of nonnegative integers such that

$$A_1 \subset A_2 \subset \dots, \quad B_1 \subset B_2 \subset \dots,$$

$$A_n - B_n \supset \{r_1, r_2, \dots, r_n\},$$

$$|A_n| \leq n, \quad |B_n| \leq n,$$

$$|A_n| > |A_{n-1}|, \quad |B_n| > |B_{n-1}| \text{ infinitely often,}$$

and the numbers $a-b, a \in A_n, b \in B_n$ are all distinct primes. Clearly the sets $A = \bigcup A_n$, $B = \bigcup B_n$ will have the properties asserted in Theorem 2.

Let K be a constant such that

$$(5.1) \quad 2n < \frac{|r_n| - 1}{2} - \frac{\log |r_n|}{\log 4/3}$$

holds whenever $|r_n| \geq K/2$. Such a constant exists since $|r_n| \sim (n \log n)/2$.

We construct the sets recursively. In Step n we will have sets A_n, B_n and sets $U_p, V_p \subset \mathbb{Z}_p$ for all primes $p < \max(K, |r_{n+1}|)$ such that

$$A_n \pmod{p} \subset U_p, \quad B_n \pmod{p} \subset V_p,$$

and U_p, V_p form a prime-compatible pair.

The starting point is $A_2 = \{1, 11\}$, $B_2 = \{6\}$, $U_2 = \{1\}$, $V_2 = \{0\}$, $U_3 = \{1, 2\}$, $V_3 = \{0\}$, $U_5 = \{0, 1, 2\}$, $V_5 = \{1, 3, 4\}$, and for $7 \leq p < K$ the sets U_p, V_p are given in part (b) of the Lemma.

Assume we have A_{n-1}, B_{n-1} and $U_p, V_p \subset \mathbb{Z}_p$ for all primes $p < \max(K, |r_n|)$. We construct A_n, B_n and U_p, V_p for $\max(K, |r_n|) \leq p < \max(K, |r_{n+1}|)$. (This is either $p = |r_n|$, or there is no such prime.)

If $r_n \in A_{n-1} - B_{n-1}$, we put $A_n = A_{n-1}$, $B_n = B_{n-1}$. Otherwise we will set

$$A_n = A_{n-1} \cup \{x\}, \quad B_n = B_{n-1} \cup \{x - r_n\}$$

with some positive integer x . This x should have the following properties:

- (i) all elements of $x - B_{n-1}$ and $A_{n-1} - x + r_n$ are prime;
- (ii) $x \pmod{p} \in U_p$, $x - r_n \pmod{p} \in V_p$ for all primes $p < \max(K, |r_n|)$;
- (iii) no coincidence.

It is clear that condition (iii) excludes only finitely many values of x , hence if we find infinitely many that satisfy (i) and (ii), we are done.

First we fix $x \pmod{p}$ for $p < \max(K, |r_n|)$, $p \neq |r_n|$. ($p = |r_n|$ is a possibility for the first few primes.)

There are $u_p \in U_p \setminus V_p$, $v_p \in V_p \setminus U_p$, such that

$$u_p - v_p \equiv r_n \pmod{p}.$$

Impose

$$x \equiv u_p \pmod{p}, \quad x - r_n \equiv v_p \pmod{p}.$$

If $p = |r_n|$, we must have $u_p = v_p$. Now we use the 2 element of $U_p \cap V_p$, choosing different ones for p and $-p$.

According to the Corollary to the prime-tuple hypothesis, the conditions for the existence of values of x that satisfy condition (i) above is the following:

— if $p < \max(K, |r_n|)$, then $p \nmid u_p - b$ for $b \in B_{n-1}$ and $p \nmid u_p - (a + r_n) = v_p - a$ for $a \in A_{n-1}$;

— if $p \geq \max(K, |r_n|)$, then $B_{n-1} \cup (A_{n-1} + r_n)$ does not contain a complete system of residues modulo p .

To check the first, note that for $p \neq |r_n|$ we have $b \pmod{p} \in V_p$ and $u_p \notin V_p$; similarly $a \pmod{p} \in U_p$ and $v_p \notin U_p$. For $p = |r_n|$ this means that $A_{n-1} \cup B_{n-1}$ avoids the residue class of $U_p \cap V_p$ assigned to r_n . Indeed, in previous steps for r_j with $|r_j| \neq p$ we assigned elements of $U_p \setminus V_p$ and $V_p \setminus U_p$ to r_j , and for $|r_j| = p$ (which may be the case for $j = n - 1$) we used the other element of $U_p \cap V_p$.

To check the second, observe that $|B_{n-1} \cup (A_{n-1} + r_n)| < 2n$ and $\max(K, |r_n|) > 2n$ by (5.1).

Finally we construct U_p, V_p modulo $p = |r_n|$ (if necessary). The requirement is that they form a prime-compatible pair and contain the at most $2n$ residues of $A_n \cup B_n$. The existence of such sets follows from part (a) of the Lemma.

This ends the proof of the Theorems.

6. REMARKS

We don't need the full strength of the prime-tuple hypothesis; however, if the primes form a sumset, then *some* configurations must appear infinitely many times.

Problem 3. Can one prove, *without resorting to the prime-tuple hypothesis*, the existence of infinite sets $A, B \subset \mathbb{N}$ such that $A + B \subset P$?

Recent advances related to the twin prime conjecture imply that we can find infinite A and arbitrarily large finite B .

Granville[3] proved that under the prime-tuple hypothesis the set of primes contains very general sorts of sumsets.

Problem 4. Can one prove, *without resorting to the prime-tuple hypothesis*, the existence of infinite sets $A, B \subset \mathbb{N}$ such that no element of $A + B$ is divisible by any prime of form $4k + 1$?

For $4k - 1$ we have an easy example.

REFERENCES

1. Ch. Elsholtz, *The inverse goldbach problem*, Mathematika **48** (2001), 151–158.
2. ———, *Additive decomposability of multiplicatively defined sets*, Funct. Approx. Comment. Math. **35** (2006), 61–67.
3. A. Granville, *A note on sums of primes*, Canad. Math. Bull. **33** (1990), no. 4, 452–454.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, BUDAPEST, PF. 127, H-1364 HUNGARY

Email address: `ruzsa@renyi.hu`