Geometric relative entropies and barycentric Rényi divergences

Milán Mosonyi, ^{1,2,3,*} Gergely Bunth, ^{1,2,†} and Péter Vrana^{1,4,‡}

¹MTA-BME Lendület Quantum Information Theory Research Group,
Budapest University of Technology and Economics,
Műegyetem rkp. 3., H-1111 Budapest, Hungary

²HUN-REN Alfréd Rényi Institute of Mathematics, Reáltanoda street 13-15, H-1053, Budapest

³Department of Analysis and Operations Research, Institute of Mathematics,
Budapest University of Technology and Economics,
Műegyetem rkp. 3., H-1111 Budapest, Hungary

⁴Department of Algebra and Geometry, Institute of Mathematics,
Budapest University of Technology and Economics,
Műegyetem rkp. 3., H-1111 Budapest, Hungary

Abstract

We give systematic ways of defining monotone quantum relative entropies and (multi-variate) quantum Rényi divergences starting from a set of monotone quantum relative entropies.

Interestingly, despite its central importance in information theory, only two additive and monotone quantum extensions of the classical relative entropy have been known so far, the Umegaki and the Belavkin-Staszewski relative entropies, which are the minimal and the maximal with these properties, respectively. Using the Kubo-Ando weighted geometric means, we give a general procedure to construct monotone and additive quantum relative entropies from a given one with the same properties; in particular, when starting from the Umegaki relative entropy, this gives a new one-parameter family of monotone (even under positive trace-preserving (PTP) maps) and additive quantum relative entropies interpolating between the Umegaki and the Belavkin-Staszewski ones on full-rank states.

In a different direction, we use a generalization of a classical variational formula to define multivariate quantum Rényi quantities corresponding to any finite set of quantum relative entropies $(D^{q_x})_{x\in\mathcal{X}}$ and real weights $(P(x))_{x\in\mathcal{X}}$ summing to 1, as

$$Q_P^{\mathrm{b},\mathbf{q}}((\varrho_x)_{x\in\mathcal{X}}) := \sup_{\tau\geq 0} \left\{ \mathrm{Tr}\,\tau - \sum_x P(x) D^{q_x}(\tau \| \varrho_x) \right\}.$$

We analyze in detail the properties of the resulting quantity inherited from the generating set of quantum relative entropies; in particular, we show that monotone quantum relative entropies define monotone Rényi quantities whenever P is a probability measure. With the proper normalization, the negative logarithm of the above quantity gives a quantum extension of the classical Rényi α -divergence in the 2-variable case ($\mathcal{X} = \{0,1\}$, $P(0) = \alpha$). We show that if both D^{q_0} and D^{q_1} are lower semi-continuous monotone and additive quantum relative entropies, and at least one of them is strictly larger than the Umegaki relative entropy then the resulting barycentric Rényi divergences are strictly between the log-Euclidean and the maximal Rényi divergences, and hence they are different from any previously studied quantum Rényi divergence.

CONTENTS

I. Introduction	2
II. Preliminaries	4
III. Quantum Rényi divergences	7
A. Classical and quantum divergences	7
B. Classical Rényi divergences	13
C. Quantum Rényi divergences	16
D. Weighted geometric means and induced divergences	22
IV. γ -weighted geometric relative entropies	26

^{*} milan.mosonyi@gmail.com

 $^{^{\}dagger}$ gbunthy@gmail.com

[‡] vranap@math.bme.hu

V.	Barycentric Rényi divergences	33
	A. Definitions	33
	B. Barycentric Rényi divergences are quantum Rényi divergences	37
	C. Homogeneity and scaling	41
	D. Monotonicity in α and limiting values	41
	E. Non-negativity and finiteness	44
	F. Monotonicity under CPTP maps and joint convexity	46
	G. Lower semi-continuity and regularity	48
	H. Finiteness and non-examples	49
VI.	Examples of barycentric Rényi divergences	51
	A. General relations	51
	B. Umegaki relative entropy and smaller/larger relative entropies	52
	C. Maximal relative entropy and a smaller relative entropy	53
	D. Maximal Rényi divergences vs. the barycentric maximal Rényi divergences	55
VII.	Conclusion and outlook	59
	Acknowledgments	60
Α.	Proof of Lemma III.13	60
В.	Proof of Lemma VI.17.	61
С.	Different approaches to multi-variate Rényi divergences	64
D.	Order of relative entropies and Rényi divergences	65
	References	66

I. INTRODUCTION

Dissimilarity measures of states of a system (classical or quantum) play a fundamental role in information theory, statistical physics, computer science, and various other disciplines. Probably the most relevant such measures for information theory are the *Rényi divergences*, defined for finitely supported probability distributions ϱ , σ on a set \mathcal{X} as

$$D_{\alpha}(\varrho \| \sigma) := \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} \varrho(x)^{\alpha} \sigma(x)^{1 - \alpha}$$
(I.1)

where $\alpha \in [0, +\infty) \setminus \{1\}$ is a parameter. (For simplicity, here we assume all probability distributions and quantum states to have full support; we give the formulas for the general case in Section III.) See, for instance, [18] for the role of the Rényi divergences and derived information measures (entropy, divergence radius, channel capacity) in classical state discrimination, as well as source- and channel coding. The limit $\alpha \to 1$ yields the Kullback-Leibler divergence, or relative entropy

$$\lim_{\alpha \to 1} D_{\alpha}(\varrho \| \sigma) = D(\varrho \| \sigma) := \sum_{x \in \mathcal{X}} [\varrho(x) \log \varrho(x) - \varrho(x) \log \sigma(x)]$$
 (I.2)

Interestingly, the relative entropy in itself also determines the whole one-parameter family of Rényi divergences, as for every $\alpha \in [0,1) \cup (1,+\infty)$,

$$D_{\alpha}(\varrho \| \sigma) = \frac{1}{1 - \alpha} \min_{\omega \in \mathcal{P}(\mathcal{X})} \left\{ \alpha D(\omega \| \varrho) + (1 - \alpha) D(\omega \| \sigma) \right\}, \tag{I.3}$$

where the optimization is over all finitely supported probability distributions ω on \mathcal{X} [19].

Due to the non-commutativity of quantum states, for any given $\alpha \in [0, +\infty)$, there are infinitely many quantum extensions of the classical Rényi α -divergence for pairs of quantum states, e.g., the measured, the maximal [56], or the Rényi (α, z) -divergences [7]. Of particular importance are the Petz-type [71] and the sandwiched [68, 80] Rényi divergences, which appear as exact quantifiers of the trade-off relations between the operationally relevant quantities in a number of information-theoretic problems, including

state discrimination, classical-quantum channel coding, entanglement manipulation, and more [6, 28, 29, 42, 48–52, 64, 65, 69]. While so far only these two families of quantum Rényi divergences have found such explicit operational interpretations, it is nevertheless useful, for a number of different reasons, to consider other quantum extensions as well. Indeed, apart from their study being interesting from the purely mathematical point of view of matrix analysis, some of these quantities serve as useful tools in proofs to arrive at the operationally relevant Rényi information quantities in various problems; see, e.g., the role played by the so-called log-Euclidean Rényi divergences $D_{\alpha,+\infty}$ in determining the strong converse exponent in various problems [48, 50–52, 64, 65], or the family of Rényi divergences $D_{\alpha}^{\#}$ introduced in [24], where it was used to determine the strong converse exponent of binary channel discrimination.

Of course, only quantum extensions with a number of good mathematical properties may be interesting for quantum information theory, the most important being monotonicity, i.e., that for any two states ϱ, σ , and completely positive trace-preserving (CPTP) map Φ , the data processing inequality (DPI) $D_{\alpha}(\Phi(\varrho)\|\Phi(\sigma)) \leq D_{\alpha}(\varrho\|\sigma)$ holds. Another desirable property is additivity $D_{\alpha}(\varrho_1 \otimes \varrho_2 \| \sigma_1 \otimes \sigma_2) = D_{\alpha}(\varrho_1 \| \sigma_1) + D_{\alpha}(\varrho_2 \| \sigma_2)$. However, quantum Rényi divergences without these properties might still be useful; indeed, $D_{\alpha,+\infty}$ is additive, but not monotone for $\alpha > 1$ (the range of α values for which it was used in [64, 65]), and $D_{\alpha}^{\#}$ is monotone, but not additive.

Quantum divergences with good mathematical properties also play an important role in the study of the problem of state convertibility, where the question is whether a set of states $(\varrho_i)_{i\in\mathcal{I}}$ can be mapped into another set of states $(g_i')_{i\in\mathcal{I}}$ with a joint quantum operation. This problem can be studied in a large variety of settings; single-shot or asymptotic, exact or approximate, with or without catalysts, allowing arbitrary quantum operations or only those respecting some symmetry or being free operations of some resource theory, any combination of these, and more. Necessary conditions for convertibility can be obtained using multi-variate functions on quantum states with suitable mathematical properties; for instance, for exact single-shot convertibility, $F((\varrho_i)_{i\in\mathcal{J}}) \geq F((\varrho_i')_{i\in\mathcal{J}})$ has to hold for any function Fwhose variables are indexed by a subset $\mathcal{J} \subseteq \mathcal{I}$ and which is monotone non-increasing under the joint application of an allowed quantum operation on its arguments; the same has to hold also for multi-copy or catalytic single-shot convertibility, if F is additionally additive on tensor products, and for asymptotic catalytic convertibility, if, moreover, F is lower semi-continuous in its variables. (We refer to [23] for the precise definitions of the various versions of state convertibility.) Many of the quantum Rényi divergences provide such functions on pairs of states (i.e., $|\mathcal{J}| = 2$), including the maximal Rényi α -divergences [56] with $\alpha \in [0, 2]$, and the Rényi (α, z) -divergences [7] for certain values of α and z [81]. In the converse direction, sufficient conditions in terms of Rényi divergences have been given for the convertibility of pairs of commuting states in [15, 41, 44, 66, 78], and these have been extended very recently in [23] to a complete characterization of asymptotic as well as approximate catalytic convertibility between finite sets of commuting states in terms of the monotonicity of the multi-variate Rényi quantities

$$Q_{\alpha}(\varrho_1, \dots, \varrho_r) := \operatorname{Tr}(\varrho_1^{\alpha_1} \cdot \dots \cdot \varrho_r^{\alpha_r}), \tag{I.4}$$

where $\alpha_1 + \ldots + \alpha_r = 1$ and either all of them are non-negative or exactly one of them is positive. No sufficient conditions, however, are known in the general noncommutative case.

Motivated by the above, in this paper we set out to give systematic ways to define quantum Rényi divergences with good mathematical properties (in particular, monotonicity), for two and for more variables. Other approaches to define multi-variate quantum Rényi divergences have been proposed very recently in [16, 27], which we briefly review in Section III D; here we only note that our approach is completely different from the previous ones. Indeed, while multi-variate quantum Rényi divergences in [16, 27] were defined by putting a non-commutative geometric mean (an iterated Kubo-Ando weighted geometric mean in [27]) of the arguments into the second argument of a Rényi (α , z)-divergence (sandwiched Rényi divergence in [16]), here we start with a collection of quantum relative entropies, and define corresponding multi-variate quantum Rényi divergences via a variational expression.

The structure of the paper is as follows. In Section II we give the necessary mathematical preliminaries. In Section III A we discuss in detail the definition and various properties of general multi-variate quantum divergences. Sections III B and III C contain brief reviews of the definitions and properties of the classical and the quantum Rényi divergences that we use in the paper. Section III D gives a high-level overview of the various ways we propose to define new quantum Rényi divergences from given ones, of which we work out in detail two approaches in this paper. We also illustrate other possible approaches in Figure 1 in Appendix C.

In Section IV we focus on quantum relative entropies. Interestingly, despite its central importance in information theory, only two additive and monotone quantum extensions of the classical relative entropy have been known so far, the Umegaki [79] and the Belavkin-Staszewski [9] relative entropies, which are the minimal and the maximal with these properties, respectively [56]. Here we give a general procedure to construct monotone and additive quantum relative entropies from a given one with the same properties;

in particular, when starting from the Umegaki relative entropy, this gives a new one-parameter family of monotone (even under positive trace-preserving (PTP) maps) and additive quantum relative entropies interpolating between the Umegaki and the Belavkin-Staszewski ones on full-rank states.

In Section V we use a generalization of the classical variational formula in (I.3) to define quantum extensions of the multi-variate Rényi quantities (I.4) corresponding to any set of quantum relative entropies $(D^{q_x})_{x \in \mathcal{X}}$ and real weights $(P(x))_{x \in \mathcal{X}}$ summing to 1, as

$$Q_P^{\mathbf{b},\mathbf{q}}((\varrho_x)_{x\in\mathcal{X}}) := \sup_{\tau\geq 0} \left\{ \operatorname{Tr} \tau - \sum_x P(x) D^{q_x}(\tau \| \varrho_x) \right\}. \tag{I.5}$$

We analyze in detail the properties of the resulting quantity inherited from the generating set of quantum relative entropies; in particular, we show that monotone quantum relative entropies define monotone Rényi quantities whenever P is a probability measure. We also show that the negative logarithm of $Q_P^{b,\mathbf{q}}((\varrho_x)_{x\in\mathcal{X}})$ is equal to the P-weighted (left) relative entropy radius of the $(\varrho_x)_{x\in\mathcal{X}}$, i.e.,

$$-\log Q_P^{\mathbf{b},\mathbf{q}}((\varrho_x)_{x\in\mathcal{X}}) = \inf_{\omega} \sum_x P(x) D^{q_x}(\omega \| \varrho_x), \tag{I.6}$$

where the infimum is taken over all states on the given Hilbert space, and therefore we call the quantities in (I.5) barycentric Rényi quantities. With the proper normalization, the quantities in (I.6) give quantum extensions of the classical Rényi divergences (I.1) in the 2-variable case ($\mathcal{X} = \{0,1\}$, $P(0) = \alpha$), which we denote by $D_{\alpha}^{\text{b,q}}$. In Section VI we study the relation of the resulting (2-variable) quantum Rényi divergences to the known ones. It has been shown in [64] that if $D^{q_0} = D^{q_1} = D^{\text{Um}}$ is the Umegaki relative entropy [79] then $D_{\alpha}^{\text{b,q}}$ is equal to the log-Euclidean Rényi divergence $D_{\alpha,+\infty}$. We show that if both D^{q_0} and D^{q_1} are lower semi-continuous monotone and additive quantum relative entropies, and at least one of them is strictly larger than D^{Um} then the resulting barycentric Rényi divergences are strictly between the log-Euclidean and the maximal Rényi divergences, and hence they are different from any previously studied quantum Rényi divergences. Figure 2 in Appendix D illustrates how the new quantum relative entropies and (2-variable) Rényi divergences considered in this paper are related to other such quantities considered in the literature before.

Appendices A and B contain proofs relegated from the main body of the paper.

II. PRELIMINARIES

For a finite-dimensional Hilbert space \mathcal{H} , let $\mathcal{B}(\mathcal{H})$ denote the set of all linear operators on \mathcal{H} , and let $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$, $\mathcal{B}(\mathcal{H})_{\geq 0}$, $\mathcal{B}(\mathcal{H})_{\geq 0}$, and $\mathcal{B}(\mathcal{H})_{> 0}$ denote the set of self-adjoint, positive semi-definite (PSD), non-zero positive semi-definite, and positive definite operators, respectively. For an interval $J \subseteq \mathbb{R}$, let $\mathcal{B}(\mathcal{H})_{\mathrm{sa},J} := \{A \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}} : \mathrm{spec}(A) \subseteq J\}$, i.e., the set of self-adjoint operators on \mathcal{H} with all their eigenvalues in J. Let $\mathcal{S}(\mathcal{H}) := \{\varrho \in \mathcal{B}(\mathcal{H})_{\geq 0}, \mathrm{Tr} \, \varrho = 1\}$ denote the set of density operators, or states. For an operator $X \in \mathcal{B}(\mathcal{H})$,

$$||X||_{\infty} := \max\{||X\psi|| : \psi \in \mathcal{H}, ||\psi|| = 1\}$$

denotes the operator norm of X (i.e., its largest singular value).

Similarly, for a finite set \mathcal{I} , we will use the notation $\mathcal{F}(\mathcal{I}) := \mathbb{C}^{\mathcal{I}}$ for the set of complex-valued functions on \mathcal{I} , and $\mathcal{F}(\mathcal{I})_{\geq 0}$, $\mathcal{F}(\mathcal{I})_{\geq 0}$, $\mathcal{F}(\mathcal{I})_{> 0}$ for the set of non-negative, non-negative and not constant zero, and strictly positive functions on \mathcal{I} . The set of probability density functions on \mathcal{I} will be denoted by $\mathcal{P}(\mathcal{I})$. When equipped with the maximum norm, $\mathcal{F}(\mathcal{I})$ becomes a commutative C^* -algebra, which we denote by $\ell^{\infty}(\mathcal{I})$. In the more general case when \mathcal{I} is an arbitrary non-empty set, we will also use the notations $\mathcal{P}_f(\mathcal{I})$ for the set of finitely supported probability measures, and $\mathcal{P}_f^{\pm}(\mathcal{I})$ for the set of finitely supported signed probability measures on \mathcal{I} , i.e.,

$$\mathcal{P}_f^{\pm}(\mathcal{I}) := \left\{ P \in \mathbb{R}^{\mathcal{I}} : |\operatorname{supp} P| < +\infty, \sum_{i \in \mathcal{I}} P(i) = 1 \right\}, \quad \operatorname{supp} P := \{ i \in \mathcal{I} : P(i) \neq 0 \}.$$

We also introduce the following subset of signed probability measures:

$$\mathcal{P}_{f,1}^{\pm}(\mathcal{I}) := \left\{ P \in \mathcal{P}_f^{\pm}(\mathcal{I}) : \exists x_+ \in \mathcal{X} \text{ s.t. } P(x_+) > 0 \text{ and } P(x) \le 0, x \in \mathcal{X} \setminus \{x_+\} \right\},$$

which plays an important role in the definition of multi-variate Rényi divergences; see Lemma III.18. For any non-empty set \mathcal{X} , let

$$\mathcal{B}(\mathcal{X},\mathcal{H}), \quad \mathcal{B}(\mathcal{X},\mathcal{H})_{>0}, \quad \mathcal{B}(\mathcal{X},\mathcal{H})_{>0}, \quad \mathcal{B}(\mathcal{X},\mathcal{H})_{>0}, \quad \mathcal{S}(\mathcal{X},\mathcal{H}),$$

denote the set of functions mapping from \mathcal{X} into $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})_{\geq 0}$, $\mathcal{B}(\mathcal{H})_{\geq 0}$, $\mathcal{B}(\mathcal{H})_{>0}$, and $\mathcal{S}(\mathcal{H})$, respectively. Elements of $\mathcal{S}(\mathcal{X},\mathcal{H})$ are called classical-quantum channels, or cq channels, and we will use the terminology generalized classical-quantum channels, or gcq channels, for the elements of $\mathcal{B}(\mathcal{X},\mathcal{H})_{\geq 0}$. We will normally use the notation $W = (W_x)_{x \in \mathcal{X}}$ to denote elements of $\mathcal{B}(\mathcal{X},\mathcal{H})_{\geq 0}$. We say that $W \in \mathcal{B}(\mathcal{X},\mathcal{H})$ is classical if there exists an orthonormal basis $(e_i)_{i \in \mathcal{I}}$ in \mathcal{H} such that $W_x = \sum_{i \in \mathcal{I}} \langle e_i, W_x e_i \rangle |e_i \rangle \langle e_i|$, $x \in \mathcal{X}$; we call any such orthonormal basis a W-basis. Equivalently, we may identify W with the collection of functions $((\widetilde{W}_x(i) := \langle e_i, W_x e_i \rangle)_{i \in \mathcal{I}})_{x \in \mathcal{X}} \in \mathcal{F}(\mathcal{X}, \mathcal{I})$, where we use the notations

$$\mathcal{F}(\mathcal{X},\mathcal{I}), \quad \mathcal{F}(\mathcal{X},\mathcal{I})_{\geq 0}, \quad \mathcal{F}(\mathcal{X},\mathcal{I})_{\geq 0}, \quad \mathcal{F}(\mathcal{X},\mathcal{I})_{> 0}, \quad \mathcal{P}(\mathcal{X},\mathcal{I}),$$

for the sets of functions mapping elements of \mathcal{X} into functions $f_x \in \mathcal{F}(\mathcal{I})$, $x \in \mathcal{X}$, on the finite set \mathcal{I} , such that the f_x are arbitrary/non-negative/non-negative and not constant zero/strictly positive/probability density functions on \mathcal{I} .

Operations on elements of $\mathcal{B}(\mathcal{X}, \mathcal{H})$ are always meant pointwise; e.g., for any $W, W^{(1)}, W^{(2)} \in \mathcal{B}(\mathcal{X}, \mathcal{H})$, $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and $\sigma \in \mathcal{B}(\mathcal{K})$,

$$VWV^* := (VW_xV^*)_{x \in \mathcal{X}},\tag{II.7}$$

$$W^{(1)} \hat{\otimes} W^{(2)} := \left(W_x^{(1)} \otimes W_x^{(2)} \right)_{x \in \mathcal{X}}, \tag{II.8}$$

$$W \mathbin{\hat{\otimes}} \sigma := W \otimes \sigma := (W_x \otimes \sigma)_{x \in \mathcal{X}}. \tag{II.9}$$

Note that here we only consider the (pointwise) tensor product of functions defined on the same set, and that this notion of tensor product is different from the one used to describe the parallel action of two cq channels, given by

$$W^{(1)} \otimes W^{(2)} := \left(W_{x_1}^{(1)} \otimes W_{x_2}^{(2)}\right)_{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2},$$

where $W^{(i)} \in \mathcal{B}(\mathcal{X}^{(i)}, \mathcal{H}^{(i)})$, i = 1, 2, and possibly $\mathcal{X}^{(1)} \neq \mathcal{X}^{(2)}$, $\mathcal{H}^{(1)} \neq \mathcal{H}^{(2)}$. The tensor product in (II.9) can be interpreted either in this setting, with $\mathcal{X}^{(1)} = \mathcal{X}$, $W^{(1)} = W$ and $\mathcal{X}^{(2)} = \{0\}$, $W_0^{(2)} = \sigma$, or as the pointwise tensor product between $W^{(1)} = W \in \mathcal{B}(\mathcal{X}, \mathcal{H})$ and the constant function $W^{(2)} \in \mathcal{B}(\mathcal{X}, \mathcal{H})$, $W_x^{(2)} = \sigma$, $x \in \mathcal{X}$.

The set of projections on \mathcal{H} is denoted by $\mathbb{P}(\mathcal{H}) := \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^*\}$. For $P, Q \in \mathbb{P}(\mathcal{H})$, the projection onto $(\operatorname{ran} P) \cap (\operatorname{ran} Q)$ is denoted by $P \wedge Q$. For a sequence of projections $P_1, \ldots, P_r \in \mathcal{B}(\mathcal{H})$ summing to I, the corresponding pinching operation is

$$\mathcal{B}(\mathcal{H}) \ni X \mapsto \sum_{i=1}^r P_i X P_i.$$

For a self-adjoint operator A, let $P_a^A:=\mathbf{1}_{\{a\}}(A)$ denote the spectral projection of A corresponding to the singleton $\{a\}\subset\mathbb{R}$. (Here and henceforth $\mathbf{1}_H$ stands for the characteristic (or indicator) function of a set H.) The projection onto the support of A is $\sum_{a\neq 0}P_a^A$; in particular, if A is positive semi-definite, it is equal to $\lim_{\alpha\searrow 0}A^\alpha=:A^0$. In general, we follow the convention that real powers of a positive semi-definite operator A are taken only on its support, i.e., for any $x\in\mathbb{R}$, $A^x:=\sum_{a>0}a^xP_a^A$. In particular, $A^{-1}:=\sum_{a>0}a^{-1}P_a^A$ stands for the generalized inverse of A, and $A^{-1}A=AA^{-1}=A^0$. For $A\in\mathcal{B}(\mathcal{H})_{\geq 0}$ and a projection P on \mathcal{H} we write $A\in\mathcal{B}(P\mathcal{H})_{>0}$ if $A^0\leq P$.

For two PSD operators ϱ , σ , we write $\varrho \perp \sigma$ if ran $\varrho \perp$ ran σ , which is equivalent to $\varrho \sigma = 0$, and further to $\langle \varrho, \sigma \rangle_{HS} := \text{Tr } \varrho \sigma = 0$, and to $\varrho^0 \sigma^0 = 0$. In particular, it implies $\varrho^0 \wedge \sigma^0 = 0$, but not the other way around.

For two finite-dimensional Hilbert spaces \mathcal{H}, \mathcal{K} , we will use the notations $\mathrm{PTP}(\mathcal{H}, \mathcal{K})$ and $\mathrm{CPTP}(\mathcal{H}, \mathcal{K})$ for the set of positive trace-preserving linear maps and the set of completely positive trace-preserving linear maps, respectively, from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$. We will also use the notation $\mathrm{P}^+(\mathcal{H}, \mathcal{K})$ for the set of (positive) linear maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ such that $\Phi(\varrho) \in \mathcal{B}(\mathcal{K})_{\geq 0}$ for all $\varrho \in \mathcal{B}(\mathcal{H})_{\geq 0}$. We will also consider (completely) positive maps of the form $\Phi: \mathcal{B}(\mathcal{H}) \to \ell^{\infty}(\mathcal{I})$ and $\Phi: \ell^{\infty}(\mathcal{I}) \to \mathcal{B}(\mathcal{H})$.

For a finite-dimensional Hilbert space \mathcal{H} and a natural number n, we denote by

$$POVM(\mathcal{H}, [n]) := \left\{ M = (M_i)_{i=0}^{n-1} \in \mathcal{B}(\mathcal{H})_{\geq 0}^{[n]} : \sum_{i=0}^{n-1} M_i = I \right\}$$

the set of n-outcome positive operator valued measures (POVMs) on \mathcal{H} , where

$$[n] := \{0, \dots, n-1\}.$$

Any $M \in \text{POVM}(\mathcal{H}, [n])$ determines a CPTP map $\mathcal{M}: \mathcal{B}(\mathcal{H}) \to \ell^{\infty}([n])$ by

$$\mathcal{M}(\cdot) := \sum_{i=0}^{n-1} (\operatorname{Tr} M_i(\cdot)) \mathbf{1}_{\{i\}}.$$

For a differentiable function f defined on an interval $J \subseteq \mathbb{R}$, let $f^{[1]}: J \times J \to \mathbb{R}$ be its first divided difference function, defined as

$$f^{[1]}(a,b) := \begin{cases} \frac{f(a) - f(b)}{a - b}, & a \neq b, \\ f'(a), & a = b, \end{cases} \quad a, b \in J.$$

If f is a continuously differentiable function on an open interval $J \subseteq \mathbb{R}$ then for any finite-dimensional Hilbert space \mathcal{H} , $A \mapsto f(A)$ is Fréchet differentiable on $\mathcal{B}(\mathcal{H})_{\mathrm{sa},J}$, and its Fréchet derivative (Df)[A] at a point $A \in \mathcal{B}(\mathcal{H})_{\mathrm{sa},J}$ is given by

$$(Df)[A](Y) = \sum_{i,j=1}^{r} f^{[1]}(a_i, a_j) P_i Y P_j, \qquad Y \in \mathcal{B}(\mathcal{H})_{sa},$$
 (II.10)

for any $P_1, \ldots, P_r \in \mathbb{P}(\mathcal{H})$ and $a_1, \ldots, a_r \in \mathbb{R}$ such that $\sum_{i=1}^r P_i = I$, $\sum_{i=1}^r a_i P_i = A$. See, e.g., [11, Theorem V.3.3] or [31, Theorem 2.3.1].

By log we denote the natural logarithm, and we use two different extensions of it to $[0, +\infty]$, defined as

$$\log x := \begin{cases} -\infty, & x = 0, \\ \log x, & x \in (0, +\infty), \\ +\infty, & x = +\infty, \end{cases} \qquad \widehat{\log} x := \begin{cases} 0, & x = 0, \\ \log x, & x \in (0, +\infty), \\ +\infty, & x = +\infty. \end{cases}$$

Throughout the paper we use the convention

$$0 \cdot (\pm \infty) := 0. \tag{II.11}$$

For a function $f:(0,+\infty)\to\mathbb{R}$, the corresponding operator perspective function [21, 22, 32] \mathcal{P}_f is defined on pairs of positive definite operators $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$ as

$$\mathcal{P}_f(\varrho,\sigma) := \sigma^{1/2} f\left(\sigma^{-1/2} \varrho \sigma^{-1/2}\right) \sigma^{1/2},\tag{II.12}$$

and it is extended to pairs of positive semi-definite operators ϱ , σ as $\mathcal{P}_f(\varrho,\sigma) := \lim_{\varepsilon \searrow 0} \mathcal{P}_f(\varrho + \varepsilon I, \sigma + \varepsilon I)$, whenever the limit exists. It is easy to see that for the transpose function $\tilde{f}(x) := xf(1/x), x > 0$, we have

$$\mathcal{P}_f(\varrho,\sigma) = \mathcal{P}_{\tilde{f}}(\sigma,\varrho),$$
 (II.13)

whenever both sides are well-defined. For any $\gamma \in (0,1)$, the choice $f_{\gamma} := \operatorname{id}_{[0,+\infty)}^{\gamma}$ gives the *Kubo-Ando* γ -weighted geometric mean, denoted by $\mathcal{P}_{f_{\gamma}}(\varrho,\sigma) =: \sigma \#_{\gamma} \varrho, \, \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$; see Section IV for more details.

The following is completely elementary; we state it explicitly because we use it multiple times in the paper.

Lemma II.1 For any $c \in \mathbb{R}$, the supremum of the function $[0, +\infty) \ni t \mapsto t - t \log t - tc =: f_c(t)$ is e^{-c} , attained uniquely at $t = e^{-c}$.

Proof We have $f'(t) = -\log t - c = 0 \iff t = e^{-c}$, f''(t) = -1/t < 0, $t \in (0, +\infty)$, from which the statement follows immediately.

The following minimax theorem is from [60, Corollary A.2].

Lemma II.2 Let X be a compact topological space, Y be an ordered set, and let $f: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function. Assume that

- (i) f(., y) is lower semi-continuous for every $y \in Y$ and
- (ii) f(x,.) is monotonic increasing for every $x \in X$, or f(x,.) is monotonic decreasing for every $x \in X$.

Then

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y), \tag{II.14}$$

and the infima in (II.14) can be replaced by minima.

The following might be known; however, we could not find a reference for it, so we provide a detailed proof. For simplicity, we include the Hausdorff property in the definition of a topological space, although the statement also holds without this.

Lemma II.3 Let X be a topological space, Y be an arbitrary set, and $f: X \times Y \to \mathbb{R} \cup \{\pm \infty\}$ be a function.

- (i) If f(.,y) is lower semi-continuous for every $y \in Y$ then $\sup_{y \in Y} f(.,y)$ is lower semi-continuous.
- (ii) If Y is a compact topological space, and f is lower semi-continuous on $X \times Y$ w.r.t. the product topology, then $\inf_{u \in Y} f(., y)$ is lower semi-continuous.
- (iii) If Y is a compact topological space, and f is continuous on $X \times Y$ w.r.t. the product topology, then $\inf_{y \in Y} f(.,y)$ and $\sup_{u \in Y} f(.,y)$ are both continuous.

Proof The assertion in (i) is trivial from the definition of lower semi-continuity.

To prove (ii), let $(x_i)_{i\in I}\subseteq X$ be a generalized sequence and $\overline{x}=\lim_{i\to\infty}x_i$. Since $f(x_i,.)$ is lower semi-continuous on Y, and Y is compact, there exists a $y_i\in Y$ such that $\inf_{y\in Y}f(x_i,y)=f(x_i,y_i)$. Let $(f(x_{\alpha(j)},y_{\alpha(j)}))_{j\in J}$ be a subnet such that

$$\liminf_{i \to \infty} f(x_i, y_i) = \lim_{i \to \infty} f(x_{\alpha(i)}, y_{\alpha(i)}).$$

Since Y is compact, there exists a subnet $(y_{\alpha(\beta(k))})_{k\in K}\subseteq Y$ converging to some $\overline{y}\in Y$. Then

$$\inf_{y \in Y} f(\overline{x}, y) \le f(\overline{x}, \overline{y}) \le \lim_{k \to \infty} f(x_{\alpha(\beta(k))}, y_{\alpha(\beta(k))}) = \lim_{j \to \infty} f(x_{\alpha(j)}, y_{\alpha(j)})$$
$$= \liminf_{i \to \infty} f(x_i, y_i) = \liminf_{i \to \infty} \inf_{y \in Y} f(x_i, y),$$

where the second inequality follows from the lower semi-continuity of f on $X \times Y$.

The assertion in (iii) follows immediately from (i) and (ii).

III. QUANTUM RÉNYI DIVERGENCES

A. Classical and quantum divergences

Let \mathcal{X} be an arbitrary non-empty set. By an \mathcal{X} -variable quantum divergence Δ we mean a function on collections of matrices

$$\mathcal{D}(\Delta) \subseteq \bigcup_{d \in \mathbb{N}} \mathcal{B}(\mathcal{X}, \mathbb{C}^d), \quad \Delta: \mathcal{D}(\Delta) \to \mathbb{R} \cup \{\pm \infty\},$$

that is invariant under isometries, i.e., if $V: \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ is an isometry then

$$V\left(\mathcal{D}(\Delta) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^{d_1})\right) V^* \subseteq \mathcal{D}(\Delta), \text{ and}$$

$$\Delta\left(VWV^*\right) = \Delta\left(W\right), \qquad W \in \mathcal{D}(\Delta) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^{d_1}).$$

Due to the isometric invariance, Δ may be extended to collections of operators on any finite-dimensional Hilbert space \mathcal{H} , by

$$\mathcal{D}_{\mathcal{H}}(\Delta) := \left\{ W \in \mathcal{B}(\mathcal{X}, \mathcal{H}) : \exists V : \mathcal{H} \to \mathbb{C}^d \text{ isometry: } VWV^* \in \mathcal{D}(\Delta) \right\},$$

$$\Delta(W) := \Delta(VWV^*), \qquad W \in \mathcal{D}_{\mathcal{H}}(\Delta),$$

where V in the second line is any isometry $V: \mathcal{H} \to \mathbb{C}^d$ such that $VWV^* \in \mathcal{D}(\Delta)$. The isometric invariance of Δ on $\mathcal{D}(\Delta)$ guarantees that this extension is well-defined, in the sense that the value of $\Delta(VWV^*)$ is independent of the choice of d and V. Clearly, this extension is again invariant under isometries, i.e., for any $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and $V: \mathcal{H} \to \mathcal{K}$ isometry, $VWV^* \in \mathcal{D}_{\mathcal{K}}(\Delta)$, and $\Delta(VWV^*) = \Delta(W)$. Note that this implies that Δ is invariant under extensions by pure states, i.e., for any $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and unit vector $\psi \in \mathcal{K}$, we have

$$W \otimes |\psi\rangle\langle\psi| \in \mathcal{D}_{\mathcal{H}\otimes\mathcal{K}}(\Delta)$$
 and $\Delta(W \otimes |\psi\rangle\langle\psi|) = \Delta(W)$. (III.15)

Remark III.1 A quantum divergence Δ is also automatically invariant under partial isometries in the sense that if $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and $V : \mathcal{H} \to \mathcal{K}$ is a partial isometry such that $(V^*V)W(V^*V) = W$ and $\tilde{V}^*W\tilde{V} \in \mathcal{D}_{\tilde{\mathcal{H}}}(\Delta)$, where $\tilde{V} : \tilde{\mathcal{H}} := (\ker V)^{\perp} \to \mathcal{H}$ is the natural embedding, then

$$\Delta \left(VWV^{*}\right) =\Delta \left(W\right) .$$

Indeed,

$$\Delta(VWV^*) = \Delta(V(\tilde{V}\tilde{V}^*)W(\tilde{V}\tilde{V}^*)V^*) = \Delta(\tilde{V}^*W\tilde{V}) = \Delta(\tilde{V}\tilde{V}^*W\tilde{V}\tilde{V}^*) = \Delta(W),$$

where the first and the last equalities follow from $(\tilde{V}\tilde{V}^*)W(\tilde{V}\tilde{V}^*)=W$, since $\tilde{V}\tilde{V}^*=V^*V$, the second equality from the invariance of Δ under the isometry $V\tilde{V}$, due to the assumption that $\tilde{V}^*W\tilde{V}\in\mathcal{D}_{\tilde{\mathcal{H}}}(\Delta)$, and the third equality from the invariance of Δ under the isometry \tilde{V} .

Analogously, an \mathcal{X} -variable classical divergence Δ is a function

$$\mathcal{D}(\Delta) \subseteq \bigcup_{d \in \mathbb{N}} \mathcal{F}(\mathcal{X}, [d]), \qquad \Delta : \mathcal{D}(\Delta) \to \mathbb{R} \cup \{\pm \infty\},$$

that is invariant under injective maps $f:[d]\to [d']$, in the sense that for any such map, and any $w=(w_x)_{x\in\mathcal{X}}\in\mathcal{D}(\Delta)$,

$$f_*w \in \mathcal{D}(\Delta), \qquad \Delta(f_*w) = \Delta(w),$$

where $f_*w := (f_*w_x)_{x \in \mathcal{X}}$ with $(f_*w_x)(i) := w_x(f^{-1}(i))$, $i \in \operatorname{ran} f$, and $(f_*w_x)(i) := 0$ otherwise. A classical divergence can be uniquely extended to collections of functions on any non-empty finite set \mathcal{I} as

$$\mathcal{D}_{\mathcal{I}}(\Delta) := \{ w \in \mathcal{F}(\mathcal{X}, \mathcal{I}) : \exists f : \mathcal{I} \to [d] \text{ injective: } f_* w \in \mathcal{D}(\Delta) \}$$

$$\Delta(w) := \Delta(f_* w), \qquad w \in \mathcal{D}_{\mathcal{I}}(\Delta),$$

where f in the second line is any injective map such that $f_*w \in \mathcal{D}(\Delta)$. The invariance property of Δ on $\mathcal{D}(\Delta)$ guarantees that this extension is well-defined, and also invariant under injective functions, i.e., for any $w \in \mathcal{D}_{\mathcal{I}}(\Delta)$ and injective function $f: \mathcal{I} \to \mathcal{J}$ into a finite set \mathcal{J} , $f_*w \in \mathcal{D}_{\mathcal{I}}(\Delta)$ and $\Delta(f_*w) = \Delta(w)$.

Remark III.2 Divergences in quantum information theory primarily serve as statistically motivated measures of how far away a collection of (classical or quantum) states are from each other; in this case the natural domain is all possible collections of density operators on an arbitrary finite-dimensional Hilbert space \mathcal{H} indexed by a fixed set \mathcal{X} . More generally, one may allow (non-zero) PSD operators as arguments; this is motivated partly by the fact that formulas e.g., for the Rényi divergences extend naturally to that setting, while from a more operational point of view, subnormalized states, for instance, might model the outputs of probabilistic protocols. For technical convenience, one may also define a divergence only on collections of PSD operators with equal supports, as is often done in matrix analysis; however, this is usually not well justified in quantum information theory, and is normally only done as an intermediate step in the definition of a divergence before extending it to more general collections of PSD operators by some procedure.

Many of the conditions on the domain of a divergence considered below (e.g., convexity in a fixed dimension, closedness under tensor products, etc.) are automatically satisfied with the above described domains.

Remark III.3 An \mathcal{X} -variable divergence Δ with $\mathcal{X} = [n]$ for some $n \in \mathbb{N}$, is called an n-variable divergence gence (classical or quantum). In particular, in the case $\mathcal{X} = [2] = \{0, 1\}$, we call Δ a binary divergence, and use the notations $\varrho := W_0$ and $\sigma := W_1$, and

$$\Delta(\varrho \| \sigma) := \Delta(W_0 \| W_1) := \Delta((W_0, W_1)).$$

Definition III.4 Let Δ be an \mathcal{X} -variable classical divergence. We say that a quantum divergence Δ^q defined on a domain $\mathcal{D}(\Delta^q) \supseteq \mathcal{D}$ is a quantum Δ -divergence on \mathcal{D} , if for any classical $W \in \mathcal{D}$, and any orthonormal basis $(e_i)_{i\in\mathcal{I}}$ jointly diagonalizing all W_x , $x\in\mathcal{X}$,

$$\left(\left(\left\langle e_i, W_x e_i \right\rangle \right)_{i \in \mathcal{I}} \right)_{x \in \mathcal{X}} \in \mathcal{D}_{\mathcal{I}}(\Delta) \quad and \quad \Delta^q \left(W \right) = \Delta \left(\left(\left\langle \left\langle e_i, W_x e_i \right\rangle \right\rangle_{i \in \mathcal{I}} \right)_{x \in \mathcal{X}} \right). \tag{III.16}$$

We say that Δ^q is a quantum extension of Δ , if for any $d \in \mathbb{N}$ and $w \in \mathcal{D}(\Delta) \cap \mathcal{F}(\mathcal{X}, [d])$,

$$\left(\sum_{i=0}^{d-1} w_x(i) |i\rangle\langle i|\right)_{x\in\mathcal{X}} \in \mathcal{D}(\Delta^q) \quad and \quad \Delta^q\left(\left(\sum_{i=0}^{d-1} w_x(i) |i\rangle\langle i|\right)_{x\in\mathcal{X}}\right) = \Delta(w), \tag{III.17}$$

where $(|i\rangle)_{i\in[d]}$ is the canonical orthonormal basis of $\mathbb{C}^{[d]}$.

Remark III.5 Clearly, the condition in Definition III.4 for Δ^q being a quantum extension of Δ is equivalent to (III.16) holding with $\mathcal{D}_{\mathcal{H}}(\Delta)$ in place of $\mathcal{D}(\Delta)$, for any non-empty finite set \mathcal{I} , any $w \in \mathcal{D}_{\mathcal{I}}(\Delta)$, and any orthonormal system $(|i\rangle)_{i\in\mathcal{I}}$ in an arbitrary finite-dimensional Hilbert space \mathcal{H} with dim $\mathcal{H} \geq |\mathcal{I}|$.

Any classical divergence Δ has a unique extension to collections of jointly diagonalizable operators. To see this, we will need the following simple observation:

Lemma III.6 Let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})$ be classical, and let $(e_i)_{i \in \mathcal{I}}$ and $(f_i)_{i \in \mathcal{I}}$ be orthonormal bases jointly diagonalizing all W_x . Then for any classical divergence Δ ,

$$((\langle e_i, W_x e_i \rangle)_{i \in \mathcal{I}})_{x \in \mathcal{X}} \in \mathcal{D}_{\mathcal{I}}(\Delta) \quad \Longleftrightarrow \quad ((\langle f_i, W_x f_i \rangle)_{i \in \mathcal{I}})_{x \in \mathcal{X}} \in \mathcal{D}_{\mathcal{I}}(\Delta),$$

and if $((\langle e_i, W_x e_i \rangle)_{i \in \mathcal{I}})_{x \in \mathcal{X}} \in \mathcal{D}_{\mathcal{I}}(\Delta)$ then

$$\Delta\left(\left(\left(\langle e_i, W_x e_i \rangle\right)_{i \in \mathcal{I}}\right)_{x \in \mathcal{X}}\right) = \Delta\left(\left(\left(\langle f_i, W_x f_i \rangle\right)_{i \in \mathcal{I}}\right)_{x \in \mathcal{X}}\right). \tag{III.18}$$

Proof For every $x \in \mathcal{X}$, let $(\lambda_{x,j})_{j \in \mathcal{J}_x}$ be the different eigenvalues of W_x . For every $j \in \times_{x \in \mathcal{X}} \mathcal{J}_x$, let

$$\mathcal{I}_{\underline{j}} := \{ i \in \mathcal{I} : \langle e_i, W_x e_i \rangle = \lambda_{x, j_x}, \ x \in \mathcal{X} \}, \quad \mathcal{I}_{\underline{j}}' := \{ i \in \mathcal{I} : \langle f_i, W_x f_i \rangle = \lambda_{x, j_x}, \ x \in \mathcal{X} \}.$$

Then

$$|\mathcal{I}_j| = \dim\left(\bigcap_{x \in \mathcal{X}} \ker(\lambda_{x,j_x} I - W_x)\right) = |\mathcal{I}'_j|,$$

and therefore there exists a bijection $h: \mathcal{I} \to \mathcal{I}$ such that for all $\underline{j} \in \times_{x \in \mathcal{X}} \mathcal{J}_x, i \in \mathcal{I}_{\underline{j}} \iff h(i) \in \mathcal{I}'_j$. The assertions then follow by the invariance of Δ under bijections.

Consider now an \mathcal{X} -variable classical divergence Δ , and define its commutative extension Δ^{cm} as

$$\mathcal{D}(\Delta^{\mathrm{cm}}) := \bigcup_{d \in \mathbb{N}} \left\{ \sum_{i=0}^{d-1} w_x(i) |e_i\rangle\langle e_i| : w \in \mathcal{D}(\Delta) \cap \mathcal{F}(\mathcal{X}, [d]), (e_i)_{i=0}^{d-1} \text{ ONB in } \mathbb{C}^d \right\},$$
(III.19)

$$\Delta^{\mathrm{cm}}\left(\sum_{i=0}^{d-1} w_x(i) |e_i\rangle\langle e_i|\right) := \Delta(w), \qquad w \in \mathcal{D}(\Delta) \cap \mathcal{F}(\mathcal{X}, [d]), \ d \in \mathbb{N}, \tag{III.20}$$

where $(e_i)_{i=0}^{d-1}$ in (III.20) is an arbitrary orthonormal basis in \mathbb{C}^d . It follows from Lemma III.6 that (III.20) is well-defined, and it is easy to see that Δ^{cm} is a quantum divergence that is a quantum extension of Δ in the sense of Definition III.4; in fact, it is the unique quantum extension with the smallest domain among all quantum extensions. Clearly, (III.19)–(III.20) can be equivalently rewritten as

$$\mathcal{D}(\Delta^{\mathrm{cm}}) := \bigcup_{d \in \mathbb{N}} \left\{ W \in \mathcal{B}(\mathcal{X}, \mathbb{C}^d) : W \text{ classical, } ((\langle e_i, W_x e_i \rangle)_{i \in [d]})_{x \in \mathcal{X}} \in D(\Delta) \right\},$$

$$\Delta^{\mathrm{cm}}(W) := \Delta \left(((\langle e_i, W_x e_i \rangle)_{i \in [d]})_{x \in \mathcal{X}} \right), \qquad W \in \mathcal{D}(\Delta^{\mathrm{cm}}) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^d), \ d \in \mathbb{N},$$
(III.22)

$$\Delta^{\mathrm{cm}}(W) := \Delta\left(\left(\left(\langle e_i, W_x e_i \rangle\right)_{i \in [d]}\right)_{x \in \mathcal{X}}\right), \qquad W \in \mathcal{D}(\Delta^{\mathrm{cm}}) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^d), \ d \in \mathbb{N}, \tag{III.22}$$

where $(e_i)_{i=0}^{d-1}$ in the above is any orthonormal basis jointly diagonalizing all W_x , $x \in \mathcal{X}$.

In the converse direction, if Δ is any quantum divergence such that every $W \in \mathcal{D}(\Delta)$ is classical then

$$\mathcal{D}(\Delta^{\mathrm{cl}}) := \bigcup_{d \in \mathbb{N}} \left\{ ((\langle e_i, W_x e_i \rangle)_{i \in [d]})_{x \in \mathcal{X}} : W \in \mathcal{D}(\Delta) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^d), (e_i)_{i=0}^{d-1} W\text{-basis} \right\},$$
(III.23)

$$\Delta^{\text{cl}}\left(((\langle e_i, W_x e_i \rangle)_{i \in [d]})_{x \in \mathcal{X}}\right) := \Delta(W), \qquad W \in \mathcal{D}(\Delta) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^d), \ d \in \mathbb{N}, \tag{III.24}$$

where $(e_i)_{i=0}^{d-1}$ in (III.24) is any W-basis, defines a classical divergence. It is easy to see that for any classical divergence Δ , $(\Delta^{\rm cm})^{\rm cl} = \Delta$, and for any quantum divergence Δ with commutative domain, $(\Delta^{\rm cl})^{\rm cm} = \Delta$. Thus, classical divergences can be uniquely identified with quantum divergences with commutative domain, whence for the rest we will also call the latter classical divergences. When we want to specify in which sense we use the terminology "classical divergence", we might write "classical divergence on functions" or "classical divergence on operators". Under the above identification, every classical divergence Δ is identified with $\Delta^{\rm cm}$, and hence for the rest we will also denote $\Delta^{\rm cm}$ by Δ .

Different quantum extensions of a classical divergence may be compared according to the following:

Definition III.7 For two \mathcal{X} -variable quantum divergences Δ_1 and Δ_2 , we write

$$\Delta_1 \le \Delta_2 \quad on \quad \mathcal{D}, \qquad if \qquad \Delta_1(W) \le \Delta_2(W), \quad W \in D,$$
 (III.25)

where we assume that $D \subseteq \mathcal{D}(\Delta_1) \cap \mathcal{D}(\Delta_2)$. When $D = \mathcal{D}(\Delta_1) \cap \mathcal{D}(\Delta_2)$, we omit it and write simply that $\Delta_1 \leq \Delta_2$.

For 2-variable divergences on pairs of non-zero PSD operators we also introduce the following strict ordering that will be useful when comparing quantum relative entropies and Rényi divergences:

Definition III.8 Let Δ_1, Δ_2 be quantum divergences with $\mathcal{D}(\Delta_1) = \mathcal{D}(\Delta_2) = \bigcup_{d \in \mathbb{N}} (\mathcal{B}(\mathbb{C}^d)_{\geq 0} \times \mathcal{B}(\mathbb{C}^d)_{\geq 0})$. We write

$$\Delta_1 < \Delta_2 \quad \text{if} \quad \varrho^0 \le \sigma^0, \ \varrho\sigma \ne \sigma\varrho \quad \Longrightarrow \quad \Delta_1(\varrho\|\sigma) < \Delta_2(\varrho\|\sigma).$$
 (III.26)

Remark III.9 For the rest, by expressions like "for all $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ ", or "for all $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ " we mean that the given property holds for any finite-dimensional Hilbert space \mathcal{H} , and we write it out explicitly when something is only supposed to be valid for a specific Hilbert space.

Before further discussing quantum extensions of classical divergences, we review a few important properties of general divergences. Let Δ be an \mathcal{X} -variable quantum divergence. We say that Δ is

- non-negative if $\Delta(W) \geq 0$ for all collections of density operators $W \in \mathcal{D}_{\mathcal{H}}(\Delta) \cap \mathcal{S}(\mathcal{X}, \mathcal{H})$, and it is strictly positive if it is non-negative and $\Delta(W) = 0 \iff W_x = W_y, \ x, y \in \mathcal{X}$, again for density operators;
- monotone under a given map $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ for some finite-dimensional Hilbert spaces \mathcal{H}, \mathcal{K} , if

$$W \in \mathcal{D}_{\mathcal{H}}(\Delta) \implies \Phi(W) \in \mathcal{D}_{\mathcal{K}}(\Delta), \quad \Delta(\Phi(W)) \leq \Delta(W),$$

where $\Phi(W) := (\Phi(W_x))_{x \in \mathcal{X}}$; in particular, it is monotone under CPTP maps/PTP maps/pinchings if monotonicity holds for any map in the given class for any two finite-dimensional Hilbert spaces \mathcal{H}, \mathcal{K} , and it is trace-monotone, if

$$W \in \mathcal{D}_{\mathcal{H}}(\Delta) \implies \operatorname{Tr} W := (\operatorname{Tr} W_x)_{x \in \mathcal{X}} \in \mathcal{D}(\Delta), \quad \Delta(\operatorname{Tr} W) \le \Delta(W);$$
 (III.27)

• jointly convex if for all $W^{(k)} \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $k \in [r]$, and probability distribution $(p_k)_{k \in [r]}$,

$$\sum_{k \in [r]} p_k W^{(k)} \in \mathcal{D}_{\mathcal{H}}(\Delta), \quad \Delta\left(\sum_{k \in [r]} p_k W^{(k)}\right) \le \sum_{k \in [r]} p_k \Delta\left(W^{(k)}\right), \quad (\text{III.28})$$

and it is jointly concave if $-\Delta$ is jointly convex;

• additive, if for all $W^{(k)} \in \mathcal{D}_{\mathcal{H}^{(k)}}(\Delta), k = 1, 2,$

$$W^{(1)} \mathbin{\hat{\otimes}} W^{(2)} \in \mathcal{D}_{\mathcal{H}^{(1)} \mathbin{\otimes} \mathcal{H}^{(2)}}(\Delta), \qquad \Delta \left(W^{(1)} \mathbin{\hat{\otimes}} W^{(2)}\right) = \Delta \left(W^{(1)}\right) + \Delta \left(W^{(2)}\right),$$

and subadditive (superadditive) if LHS \leq RHS (LHS \geq RHS) holds above;

• weakly additive, if for all $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$,

$$W^{\hat{\otimes}n} \in \mathcal{D}_{\mathcal{H}^{\otimes n}}(\Delta), \quad \Delta(W^{\hat{\otimes}n}) = n \Delta(W), \quad n \in \mathbb{N},$$

and weakly subadditive (superadditive) if LHS \leq RHS (LHS \geq RHS) holds above;

• block subadditive, if for any $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$, and any sequence of projections $P_0, \ldots, P_{r-1} \in \mathbb{P}(\mathcal{H})$ summing to I, if $P_iWP_i \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $i \in [r]$, then

$$\Delta\left(\sum_{i=0}^{r-1} P_i W P_i\right) \le \sum_{i=0}^{r-1} \Delta(P_i W P_i). \tag{III.29}$$

Conversely, if the inequality in the above always holds in the opposite direction then Δ is called block superadditive, and if it is always an equality, then Δ is block additive.

Remark III.10 We say that Δ is strictly trace-monotone, if equality in (III.27) implies the existence of a state ω and numbers λ_x , $x \in \mathcal{X}$ such that $W_x = \lambda_x \omega$, $x \in \mathcal{X}$.

Remark III.11 Note that if W is jointly convex or jointly concave, then for any $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and any density operator $\sigma \in \mathcal{S}(\mathcal{K})$, $W \otimes \sigma \in \mathcal{D}_{\mathcal{H} \otimes \mathcal{K}}(\Delta)$, according to (III.15) and (III.28).

Remark III.12 Note that $z \otimes X \mapsto zX$ gives a canonical identification between $\mathbb{C} \otimes \mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$. In particular, any additive quantum divergence Δ satisfies the scaling law

$$\Delta\left((t_x W_x)_{x \in \mathcal{X}}\right) = \Delta(t) + \Delta(W), \qquad W \in \mathcal{D}_{\mathcal{H}}(\Delta), \ t \in \mathcal{D}(\Delta) \cap \mathcal{F}(\mathcal{X}, [1]), \tag{III.30}$$

where $t = (t_x = t_x(0))_{x \in \mathcal{X}}$.

A quantum divergence Δ is called *(positive) homogeneous*, if for every $W \in \mathcal{D}(\Delta)$ and $t \in (0, +\infty)$, $tW \in \mathcal{D}(\Delta)$, and

$$\Delta\left((tW_x)_{x\in\mathcal{X}}\right) = t\,\Delta\left((W_x)_{x\in\mathcal{X}}\right).$$

It is well known that a 2-variable quantum divergence that is monotone non-decreasing under partial traces is jointly concave whenever it has the additional properties of block superadditivity and homogeneity. The extension to multi-variable divergences is straightforward; we give a detailed proof in Appendix A for completeness.

Lemma III.13 Assume that an \mathcal{X} -variable quantum divergence Δ is block superadditive, homogeneous, monotone non-decreasing under partial traces, and for every \mathcal{H} , $\mathcal{D}_{\mathcal{H}}(\Delta)$ is convex. Then Δ is jointly concave and jointly superadditive.

Vice versa, if Δ is jointly concave and it is stable under tensoring with the maximally mixed state, i.e., for any $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and any \mathcal{K} , $\Delta(W \otimes (I_{\mathcal{K}}/\dim \mathcal{K})) = \Delta(W)$, then for any $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and any CPTP map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ such that $\Phi(W) \in \mathcal{D}_{\mathcal{K}}(\Delta)$, we have $\Delta(\Phi(W)) \geq \Delta(W)$.

Any monotone classical divergence admits two canonical quantum extensions, the minimal and the maximal ones:

Example III.14 For a classical divergence Δ , let

$$\mathcal{D}(\Delta^{\text{meas}}) := \bigcup_{d \in \mathbb{N}} \left\{ W \in \mathcal{B}(\mathcal{X}, \mathbb{C}^d) : \mathcal{M}(W) = (\mathcal{M}(W_x))_{x \in \mathcal{X}} \in \mathcal{D}(\Delta), M \in \text{POVM}(\mathbb{C}^d, [n]), n \in \mathbb{N} \right\}$$
(III.31)

$$\Delta^{\text{meas}}(W) := \sup\{\Delta\left(\mathcal{M}(W)\right) : M \in \text{POVM}(\mathbb{C}^d, [n]), n \in \mathbb{N}\}, \qquad W \in \mathcal{D}(\Delta^{\text{meas}}). \tag{III.32}$$

It is easy to see that Δ^{meas} is a quantum divergence, and if Δ is monotone then Δ^{meas} is a quantum extension of Δ , called the measured, or minimal extension.

As introduced in [56] in the 2-variable case, a reverse test for $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})$ is a pair (w, Γ) with $w \in \mathcal{F}(\mathcal{X}, \mathcal{I})$ for some finite set \mathcal{I} , and $\Gamma : \ell^{\infty}(\mathcal{I}) \to \mathcal{B}(\mathcal{H})$ a (completely) positive trace-preserving map such that $\Gamma(w) = W$. For a classical divergence Δ , let

$$\mathcal{D}(\Delta^{\max}) := \cup_{d \in \mathbb{N}} \mathcal{B}(\mathcal{X}, \mathbb{C}^d), \tag{III.33}$$

$$\Delta^{\max}(W) := \inf \{ \Delta(w) : (w, \Gamma) \text{ is a reverse test for } W \text{ with } w \in \mathcal{D}(\Delta) \}, \qquad W \in \mathcal{D}(\Delta^{\max}).$$
 (III.34)

It is easy to see that Δ^{\max} is a quantum divergence, and if Δ is monotone then Δ^{\max} gives a quantum extension of Δ , called the maximal extension.

It is straightforward to verify from their definitions that both $\Delta^{\rm meas}$ and $\Delta^{\rm max}$ are monotone under PTP maps, and for any quantum extension Δ^q of Δ that is monotone under CPTP maps,

$$\Delta^{\text{meas}}(W) \le \Delta^q(W) \le \Delta^{\text{max}}(W), \qquad W \in \mathcal{D}(\Delta^{\text{meas}}) \cap \mathcal{D}(\Delta^q)$$

holds.

It is also clear that if Δ is additive then Δ^{\max} is subadditive, and if furthermore $\mathcal{D}(\Delta^{\text{meas}})$ is closed under tensor products then Δ^{meas} is superadditive, and the regularized measured and the regularized maximal Δ -divergences

$$\overline{\Delta}^{\text{meas}}(W) := \sup_{n \in \mathbb{N}} \frac{1}{n} \Delta^{\text{meas}}(W^{\hat{\otimes}n}) = \lim_{n \to +\infty} \frac{1}{n} \Delta^{\text{meas}}(W^{\hat{\otimes}n})$$
(III.35)

$$\overline{\Delta}^{\max}(W) := \inf_{n \in \mathbb{N}} \frac{1}{n} \Delta^{\max}(W^{\hat{\otimes}n}) = \lim_{n \to +\infty} \frac{1}{n} \Delta^{\max}(W^{\hat{\otimes}n}), \tag{III.36}$$

are quantum extensions of Δ that are weakly additive. (Here $\mathcal{D}(\overline{\Delta}^{\mathrm{meas}}) := \mathcal{D}(\Delta^{\mathrm{meas}})$.) Obviously, $\overline{\Delta}^{\mathrm{meas}}$ and $\overline{\Delta}^{\mathrm{max}}$ are monotone under CPTP maps, and for any quantum extension Δ^q of Δ that is monotone under CPTP maps, and any $W \in \mathcal{D}(\Delta^{\mathrm{meas}})$ such that $W^{\hat{\otimes}n} \in \mathcal{D}(\Delta^q)$, $n \in \mathbb{N}$, we have

$$\exists \, \overline{\Delta}^q(W) := \lim_{n \to +\infty} \frac{1}{n} \, \Delta^q(W^{\hat{\otimes}n}) \quad \Longrightarrow \quad \overline{\Delta}^{\text{meas}}(W) \leq \overline{\Delta}^q(W) \leq \overline{\Delta}^{\text{max}}(W).$$

In particular, if Δ^q is additive then

$$\overline{\Delta}^{\text{meas}}(W) \le \Delta^q(W) \le \overline{\Delta}^{\text{max}}(W)$$

for any W as above.

Remark III.15 If Δ is monotone non-decreasing under stochastic maps then its measured and maximal versions are naturally defined for $-\Delta$ instead, or equivalently, with inf and sup instead of sup and inf in (III.32) and (III.34), respectively.

We will furthermore consider properties that only concern one variable of a divergence. We formulate these only for the case when this is the second variable of a 2-variable divergence; the definitions in the general case can be obtained by straightforward modifications. In particular, we say that a 2-variable quantum divergence Δ is

• anti-monotone in the second argument (AM), if for all $\varrho, \sigma_1, \sigma_2 \in \mathcal{B}(\mathcal{H})$ such that $(\varrho, \sigma_1), (\varrho, \sigma_2) \in \mathcal{D}_{\mathcal{H}}(\Delta)$,

$$\sigma_1 \le \sigma_2 \implies \Delta(\rho \| \sigma_1) \ge \Delta(\rho \| \sigma_2);$$
 (III.37)

• weakly anti-monotone in the second argument, if for any $(\varrho, \sigma) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, we have $(\varrho, \sigma + \varepsilon I) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $\varepsilon \in (0, \kappa_{\varrho, \sigma})$, with some $\kappa_{\varrho, \sigma} > 0$, and

$$[0, \kappa_{\rho,\sigma}) \ni \varepsilon \mapsto \Delta(\varrho \| \sigma + \varepsilon I)$$
 is decreasing; (III.38)

• regular, if for any $(\varrho, \sigma) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, we have $(\varrho, \sigma + \varepsilon I) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $\varepsilon \in (0, \kappa_{\varrho, \sigma})$, with some $\kappa_{\varrho, \sigma} > 0$, and

$$\Delta(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} \Delta(\varrho\|\sigma + \varepsilon I); \tag{III.39}$$

• strongly regular, if for any $(\varrho, \sigma) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, and any sequence of operators $(\sigma_n)_{n \in \mathbb{N}}$ converging decreasingly to σ such that $(\varrho, \sigma_n) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $n \in \mathbb{N}$, we have

$$\Delta(\varrho \| \sigma) = \lim_{n \to +\infty} \Delta(\varrho \| \sigma_n); \tag{III.40}$$

Remark III.16 Note that

$$AM + regularity \implies strong \ regularity.$$

Indeed, assume that Δ is regular and anti-monotone in its second argument. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of positive semi-definite operators converging decreasingly to σ , such that $(\varrho, \sigma) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $(\varrho, \sigma_n) \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $n \in \mathbb{N}$. Then for every n such that $c_n \in (0, \kappa_{\varrho, \sigma})$,

$$\sigma \le \sigma_n = \sigma + \sigma_n - \sigma \le \sigma + \underbrace{\|\sigma_n - \sigma\|_{\infty}}_{=:c_n} I = \sigma + c_n I,$$

whence

$$\Delta(\varrho \| \sigma) \ge \Delta(\varrho \| \sigma_n) \ge \Delta(\varrho \| \sigma + c_n I), \qquad n \in \mathbb{N}.$$

By the regularity assumption, the RHS above tends to $\Delta(\varrho\|\sigma)$ as $n \to +\infty$, whence also $\lim_{n \to +\infty} \Delta(\varrho\|\sigma_n) = \Delta(\varrho\|\sigma)$. Thus, Δ is strongly regular.

Remark III.17 All the above properties may be defined also for classical divergences, by requiring the PSD operators in the definitions to be commuting.

B. Classical Rényi divergences

The classical divergences of particular importance to us are the 2-variable divergences called relative entropy, or Kullback-Leibler divergence, and the classical Rényi divergences. For a finite set \mathcal{I} , and $\varrho, \sigma \in \mathcal{F}(\mathcal{I})_{>0}$ the relative entropy of ϱ and σ is defined as

$$D(\varrho \| \sigma) := \begin{cases} \sum_{i \in \mathcal{I}} \left[\varrho(i) \widehat{\log} \varrho(i) - \varrho(i) \widehat{\log} \sigma(i) \right], & \text{supp } \varrho \subseteq \text{supp } \sigma, \\ +\infty, & \text{otherwise.} \end{cases}$$
(III.41)

The classical Rényi α -divergences [74] are defined for $\varrho, \sigma \in \mathcal{F}(\mathcal{I})_{\geq 0}$ and $\alpha \in (0,1) \cup (1,+\infty)$ as

$$D_{\alpha}(\varrho \| \sigma) := \frac{1}{\alpha - 1} \underbrace{\log Q_{\alpha}(\varrho \| \sigma)}_{=:\psi_{\alpha}(\varrho \| \sigma)} - \frac{1}{\alpha - 1} \log \sum_{i \in \mathcal{I}} \varrho(i), \tag{III.42}$$

$$Q_{\alpha}(\varrho \| \sigma) := \lim_{\varepsilon \searrow 0} \sum_{i \in \mathcal{I}} (\varrho(i) + \varepsilon)^{\alpha} (\sigma(i) + \varepsilon)^{1-\alpha}$$

$$= \begin{cases} \sum_{i \in \mathcal{I}} \varrho(i)^{\alpha} \sigma(i)^{1-\alpha} & \alpha \in (0,1) \text{ or supp } \varrho \subseteq \text{supp } \sigma, \\ +\infty, & \text{otherwise.} \end{cases}$$
(III.43)

For $\alpha \in \{0, 1, +\infty\}$, the Rényi α -divergence is defined by the corresponding limit, and it is easy to see that

$$D_0(\varrho \| \sigma) := \lim_{\alpha \searrow 0} D_{\alpha}(\varrho \| \sigma) = -\log \sum_{i \in \text{supp } \varrho} \sigma(i) + \log \sum_{i \in \mathcal{I}} \varrho(i), \tag{III.44}$$

$$D_1(\varrho \| \sigma) := \lim_{\alpha \to 1} D_{\alpha}(\varrho \| \sigma) = \frac{1}{\sum_{i \in \mathcal{I}} \varrho(i)} D(\varrho \| \sigma), \tag{III.45}$$

$$D_{+\infty}(\varrho \| \sigma) := \lim_{\alpha \to +\infty} D_{\alpha}(\varrho \| \sigma) = \log \inf \{ \lambda > 0 : \varrho \le \lambda \sigma \}.$$
 (III.46)

In particular, the Rényi 1-divergence is the same as the relative entropy up to normalization.

We extend the definitions of the Rényi divergences to the case when the second argument is zero as

$$D_{\alpha}(\varrho||0) := +\infty, \qquad \varrho \ge 0, \quad \alpha \in [0, +\infty), \tag{III.47}$$

and the definition of the relative entropy to the case when one or both arguments are zero as

$$D(0||\sigma) := 0 \quad \sigma \ge 0, \qquad D(\varrho||0) := +\infty \quad \varrho \ge 0. \tag{III.48}$$

For the study and applications of the (classical) Rényi divergences, the relevant quantity is actually Q_{α} (equivalently, ψ_{α}); the normalizations in (III.42) are somewhat arbitrary, and are mainly relevant only for the limits in (III.44)–(III.46). For instance, one could alternatively use the *symmetrically normalized* Rényi α -divergences defined for any $\varrho, \sigma \in \mathcal{F}(\mathcal{I})_{\geq 0}$ and $\alpha \in (0,1) \cup (1,+\infty)$ as

$$\tilde{D}_{\alpha}(\varrho \| \sigma) := \frac{1}{\alpha(1-\alpha)} \left[-\log Q_{\alpha}(\varrho \| \sigma) + \alpha \log \sum_{i \in \mathcal{I}} \varrho(i) + (1-\alpha) \log \sum_{i \in \mathcal{I}} \sigma(i) \right]$$

$$= -\frac{1}{\alpha(1-\alpha)} \log Q_{\alpha} \left(\frac{\varrho}{\sum_{i \in \mathcal{I}} \varrho(i)} \left\| \frac{\sigma}{\sum_{i \in \mathcal{I}} \sigma(i)} \right).$$

For $\alpha \in \{0,1\}$ these give

$$\tilde{D}_{1}(\varrho \| \sigma) := \lim_{\alpha \to 1} \tilde{D}_{\alpha}(\varrho \| \sigma) = D\left(\frac{\varrho}{\sum_{i \in \mathcal{I}} \varrho(i)} \left\| \frac{\sigma}{\sum_{i \in \mathcal{I}} \sigma(i)} \right) = D_{1}(\varrho \| \sigma) - \log \sum_{i \in \mathcal{I}} \varrho(i) + \log \sum_{i \in \mathcal{I}} \sigma(i),$$

$$\tilde{D}_{0}(\varrho \| \sigma) := \lim_{\alpha \searrow 0} \tilde{D}_{\alpha}(\varrho \| \sigma) = D\left(\frac{\sigma}{\sum_{i \in \mathcal{I}} \sigma(i)} \left\| \frac{\varrho}{\sum_{i \in \mathcal{I}} \varrho(i)} \right) = D_{1}(\sigma \| \varrho) + \log \sum_{i \in \mathcal{I}} \varrho(i) - \log \sum_{i \in \mathcal{I}} \sigma(i),$$

while

$$\tilde{D}_{+\infty}(\varrho \| \sigma) := \lim_{\alpha \to +\infty} \tilde{D}_{\alpha}(\varrho \| \sigma) = 0$$

is not very interesting.

As mentioned already in the Introduction, the Rényi α -divergences with $\alpha \in (0,1) \cup (1,+\infty)$ can be recovered from the relative entropy as

$$-\log Q_{\alpha}(\varrho \| \sigma) = \inf_{\omega} \left\{ \alpha D(\omega \| \varrho) + (1 - \alpha) D(\omega \| \sigma) \right\}, \tag{III.49}$$

where the infimum is taken over all $\omega \in \mathcal{P}(\mathcal{I})$ with supp $\omega \subseteq \text{supp } \varrho$, and it is uniquely attained at

$$\omega_{\alpha}(\varrho \| \sigma) := \sum_{i \in S} \frac{\varrho(i)^{\alpha} \sigma(i)^{1-\alpha}}{\sum_{j \in S} \varrho(j)^{\alpha} \sigma(j)^{1-\alpha}} \mathbf{1}_{\{i\}}, \tag{III.50}$$

where $S := \operatorname{supp} \varrho \cap \operatorname{supp} \sigma$, provided that $\operatorname{supp} \varrho \subseteq \operatorname{supp} \sigma$, or $\operatorname{supp} \varrho \cap \operatorname{supp} \sigma \neq \emptyset$ and $\alpha \in (0,1)$. The case $\alpha \in (0,1)$ was discussed in [19] in the more general setting where $\mathcal I$ is not finite, while the case $\alpha > 1$ was discussed in the finite-dimensional quantum case in [65]; see also Section V below.

It is natural to ask whether the concept of Rényi divergences can be generalized to more than two variables. Formulas (III.43) and (III.49) offer two different approaches to do that. In a very general setting, one may consider a set \mathcal{X} equipped with a σ -algebra \mathcal{A} . Then for any measurable $w \in \mathcal{F}(\mathcal{X}, \mathcal{I})$ and signed measure P on \mathcal{A} with $P(\mathcal{X}) = 1$, one may consider

$$\hat{Q}_{P}(w) := \lim_{\varepsilon \searrow 0} \sum_{i \in \mathcal{I}} \exp\left(\int_{\mathcal{X}} \log(w_{x}(i) + \varepsilon) \, dP(x)\right), \tag{III.51}$$

or

$$\hat{Q}_{P}(w) := \lim_{\varepsilon \searrow 0} \sum_{i \in \mathcal{I}} \exp\left(\int_{\mathcal{X}} \log((1 - \varepsilon)w_{x}(i) + \varepsilon) dP(x)\right),$$
(III.52)

where the latter is somewhat more natural when the w_x are probability density functions on \mathcal{I} . In the most general case, various issues regarding the existence of the integrals and the limits arise, which are important from a mathematical, but not particularly relevant from a conceptual point, and hence for the rest we will restrict our attention to the case where P is finitely supported. In that case the integrals always exist, and the $\varepsilon \searrow 0$ limit can be easily determined as

$$\hat{Q}_{P}(w) = \sum_{i \in \mathcal{I}} \left(\left(\prod_{x: w_{x}(i) > 0} w_{x}(i)^{P(x)} \right) \cdot \begin{cases} 0, & \text{if } \sum_{x: w_{x}(i) = 0} P(x) > 0, \\ 1, & \text{if } \sum_{x: w_{x}(i) = 0} P(x) = 0, \\ +\infty, & \text{if } \sum_{x: w_{x}(i) = 0} P(x) < 0, \end{cases} \right)$$
(III.53)

independently of whether (III.51) or (III.52) is used.

Alternatively, one may define

$$\tilde{Q}_{P}^{\mathrm{b,cl}}(w) := \sup_{\tau \in [0,+\infty)^{\mathcal{I}}} \left\{ \sum_{i \in \mathcal{I}} \tau(i) - \int_{\mathcal{X}} D(\tau \| w_x) \, dP(x) \right\},$$

which is well-defined at least when P is a probability measure, all the w_x are probability density functions, and $\mathcal{X} \ni x \mapsto D(\tau || w_x)$ is measurable. Again, we restrict to the case when P is finitely supported, but allow it to be a signed probability measure, in which case we use a slight modification of the above to define

$$Q_P^{\text{b,cl}}(w) := \sup_{\substack{\tau \in [0, +\infty)^{\mathcal{I}} \\ \text{supp } \tau \subseteq \bigcap_{v \in P(x) \geq 0} \text{ supp } w_x}} \left\{ \sum_{i \in \mathcal{I}} \tau(i) - \sum_{x \in \mathcal{X}} P(x) D(\tau \| w_x) \right\}. \tag{III.54}$$

We will show in Section V A that this is equivalent to (III.49) when $\mathcal{X} = \{0,1\}$, $P(0) = \alpha \in (0,1) \cup (1,+\infty)$. It is not too difficult to see that with the definition in (III.54), we have

$$Q_P^{\mathrm{b,cl}}(w) = +\infty \quad \Longleftrightarrow \quad \bigcap_{x:\, P(x)>0} \operatorname{supp} w_x \not\subseteq \bigcap_{x:\, P(x)<0} \operatorname{supp} w_x;$$

see Proposition V.39 for the more general quantum case. Thus, while (III.53) and (III.54) coincide when P is a probability measure, they may differ when P can take negative values. The following is straightforward to verify from (III.53) and (III.54):

Lemma III.18 Let $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$ and $w \in \mathcal{F}(\mathcal{X}, \mathcal{I})_{\geq 0}$, and assume that at least one of the following holds true:

- (i) supp $w_x = \text{supp } w_{x'}, \ x, x' \in \text{supp } P;$
- (ii) $P \in \mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X})$, i.e., either $P(x) \geq 0$, $x \in \mathcal{X}$, or there exists a unique x_+ with $P(x_+) > 0$ and $P(x) \leq 0$, $x \in \mathcal{X} \setminus \{x_+\}$.

Then $\hat{Q}_P(w) = Q_P^{b,cl}(w)$.

Definition III.19 For any $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$ and any $w \in \mathcal{F}(\mathcal{X}, \mathcal{I})_{\geq 0}$ such that $\hat{Q}_P(w) = Q_P^{\mathrm{b,cl}}(w)$, we call this common value the multi-variate Rényi Q_P of w, and denote it by $Q_P(w)$.

For any $P \in \left(\mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X})\right) \setminus \{\mathbf{1}_{\{x\}} : x \in \mathcal{X}\}$, we define the (symmetrically normalized) classical P-weighted Rényi-divergence of $w \in \mathcal{F}(\mathcal{X}, \mathcal{I})_{\geq 0}$ as

$$D_P(w) := \frac{1}{\prod_{x \in \mathcal{X}} (1 - P(x))} \left(-\log Q_P(w) + \sum_{x \in \mathcal{X}} P(x) \log \sum_{i \in \mathcal{I}} w_x(i) \right)$$
$$= \frac{1}{\prod_{x \in \mathcal{X}} (1 - P(x))} \left(-\log Q_P\left(\left(\frac{w_x}{\sum_{i \in \mathcal{I}} w_x(i)} \right)_{x \in \mathcal{X}} \right) \right).$$

In this case we also define

$$\widetilde{Q}_P(w) := s(P)Q_P(w),$$

where

$$s(P) := \begin{cases} -1, & P \in \mathcal{P}_f(\mathcal{X}), \\ 1, & P \in \mathcal{P}_{f,1}^{\pm}(\mathcal{X}). \end{cases}$$

Remark III.20 In the case when P is a probability measure, $Q_P(w)$ was introduced in classical decision theory, and called the Hellinger transform of P; see [75]. The case where P(x) > 0 for exactly one x was very recently considered in [66] in the context of (classical) Blackwell dominance of experiments, and in [23] in the case where all w are strictly positive, in the context of classical state convertibility.

Note that in the case when $\mathcal{X} = \{0,1\}$ and $\alpha := P(0) \in (0,1) \cup (1,+\infty)$, condition (ii) in Lemma III.18 is always satisfied, and we have

$$Q_P(w) = Q_{\alpha}(w_0 || w_1).$$

That is, the multi-variate Q_P give multi-variate extensions of the Q_α quantities.

Remark III.21 Note that when $\mathcal{X} = \{0,1\}$ and $\alpha := P(0) = 0$, neither $\hat{Q}_P(w)$ nor $Q_P^{\mathrm{b,cl}}(w)$ coincides with $Q_0(w_0||w_1)$ in general. The reason for this in the case of $\hat{Q}_P(w)$ is that the limits $\varepsilon \searrow 0$ and $\alpha \searrow 1$ are not interchangeable, while in the case of $Q_P^{\mathrm{b,cl}}(w)$, it is clear that it only depends on $(w_x)_{x \in \mathrm{supp}\, P}$, while Q_0 depends on w_0 (or at least its support) even though $0 \notin \mathrm{supp}\, P = \mathrm{supp}(0,1) = \{1\}$.

Recall that classical divergences can be identified with quantum divergences defined on commuting operators; in particular, monotonicity under (completely) positive trace-preserving maps makes sense for the former. For the purposes of applications, it is monotone divergences that are relevant, and monotonicity is closely related to joint convexity. The following is easy to verify; see, e.g., [66, Lemma 8].

Lemma III.22 \widetilde{Q}_P is jointly convex on $\mathcal{F}(\mathcal{X},\mathcal{I})_{>0}$ for any/some finite non-empty \mathcal{I} if and only if $P \in \mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X})$.

Corollary III.23 For any $P \in \mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X})$, \widetilde{Q}_P is jointly convex and monotone under positive trace-preserving maps, and D_P is monotone under positive trace-preserving maps whenever $P \in \left(\mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X})\right) \setminus \{\mathbf{1}_{\{x\}} : x \in \mathcal{X}\}.$

Proof It is straightforward to verify that \widetilde{Q}_P is homogeneous, block additive, and stable under tensoring with an arbitrary state, and hence the assertion follows from Lemma III.22 and Lemma III.13.

C. Quantum Rényi divergences

In this section we give a brief review of the (2-variable) quantum Rényi divergences most commonly used in the literature, which will also play an important role in the rest of the paper. We will discuss various ways to define multi-variate quantum Rényi divergences in Sections III D and V.

Definition III.24 For any $\alpha \in [0, +\infty]$, by a quantum Rényi α -divergence we mean a quantum divergence that is a quantum extension of the classical Rényi α -divergence. Similarly, by a quantum relative entropy we mean a quantum extension of the relative entropy.

Remark III.25 Note that for any $\alpha \in [0,1) \cup (1,+\infty)$, there is an obvious bijection between quantum extensions of Q_{α} and quantum extensions of D_{α} .

Remark III.26 Since 0 commutes with any other operator, any quantum Rényi α -divergence D^q_{α} must satisfy

$$D_{\alpha}^{q}(\rho||0) = +\infty, \qquad \rho \ge 0, \tag{III.55}$$

according to (III.47), and any quantum relative entropy D^q must satisfy

$$D^{q}(0||\sigma) = 0 \quad \sigma \ge 0, \qquad D^{q}(\varrho||0) = +\infty \quad \varrho \ge 0, \tag{III.56}$$

according to (III.48).

Since these values are fixed by definition, in the discussion of different quantum Rényi divergences and relative entropies below, it is sufficient to consider non-zero arguments most of the time.

Remark III.27 Note that there is a bijection between quantum extensions of the Rényi 1-divergence and quantum extensions of the relative entropy, given in one direction by $D^q(\varrho \| \sigma) := (\operatorname{Tr} \varrho) D_1^q(\varrho \| \sigma)$, and in the other direction by $D_1^q(\varrho \| \sigma) := D^q(\varrho \| \sigma) / \operatorname{Tr} \varrho$, for any non-zero ϱ .

The following examples of quantum Rényi α -divergences are well studied in the literature. We review them in some detail for later use.

Example III.28 For any $\alpha \in [0,1) \cup (1,+\infty)$ and $z \in (0,+\infty)$, the Rényi (α,z) -divergence of $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ is defined as [7]

$$D_{\alpha,z}(\varrho,\sigma) := \frac{1}{\alpha - 1} \log Q_{\alpha,z}(\varrho,\sigma) - \frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho,$$

$$Q_{\alpha,z}(\varrho,\sigma) := \begin{cases} \operatorname{Tr} \left(\varrho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \varrho^{\frac{\alpha}{2z}}\right)^{z}, & \alpha \in [0,1) \text{ or } \varrho^{0} \leq \sigma^{0}, \\ +\infty, & \text{otherwise.} \end{cases}$$
(III.57)

It is easy to see that it defines a quantum Rényi α -divergence in the sense of Definition III.24. $D_{\alpha,1}(\varrho\|\sigma)$ is called the Petz-type (or standard) Rényi α -divergence [71] of ϱ and σ , and $D_{\alpha,\alpha}(\varrho\|\sigma)$ their sandwiched Rényi α -divergence [68, 80]. The limit

$$D_{\alpha,+\infty}(\varrho\|\sigma) := \lim_{z \to +\infty} D_{\alpha,z}(\varrho\|\sigma)$$
 (III.58)

$$= \begin{cases} \frac{1}{\alpha - 1} \log \underbrace{\operatorname{Tr} P e^{\alpha P(\widehat{\log} \varrho)P + P(\widehat{\log} \sigma)P}}_{=:Q_{\alpha, +\infty}(\varrho \parallel \sigma))} - \frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho, & \alpha \in (0, 1) \text{ or } \varrho^{0} \leq \sigma^{0}, \\ +\infty, & \text{otherwise}, \end{cases}$$
(III.59)

where $P := \varrho^0 \wedge \sigma^0$, is also a quantum Rényi α -divergence, often referred to as the log-Euclidean Rényi α -divergence [7, 38, 63]. It is known [55, 61] that for any function $z : (1 - \delta, 1 + \delta) \to (0, +\infty]$ such that $\liminf_{\alpha \to 1} z(\alpha) > 0$, and for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$\lim_{\alpha \to 1} D_{\alpha, z(\alpha)}(\varrho \| \sigma) = \frac{1}{\operatorname{Tr} \varrho} D^{\operatorname{Um}}(\varrho \| \sigma) =: D_1^{\operatorname{Um}}(\varrho \| \sigma),$$

where the Umegaki relative entropy $D^{\mathrm{Um}}(\varrho \| \sigma)$ is defined as

$$D^{\mathrm{Um}}(\varrho \| \sigma) := \begin{cases} \mathrm{Tr}(\varrho \widehat{\log} \varrho - \varrho \widehat{\log} \sigma), & \varrho^0 \leq \sigma^0, \\ +\infty, & otherwise. \end{cases}$$
(III.60)

In particular, for any $z \in (0, +\infty]$, we define $D_{1,z}(\varrho \| \sigma) := D_1^{\mathrm{Um}}(\varrho \| \sigma)$.

For every $\alpha \in (0, +\infty)$ and $z \in (0, +\infty]$, the Rényi (α, z) -divergence is strictly positive [59, Corollary III.28]. The range of (α, z) -values for which $D_{\alpha, z}$ is monotone under CPTP maps was studied in a series of works [8, 25, 34, 71], and was finally characterized completely in [81]. It is clear from their definitions that for every $\alpha \in (0, +\infty)$ and $z \in (0, +\infty]$, the Rényi (α, z) -divergence is additive on tensor products.

Example III.29 For any $\alpha \in [0, +\infty]$ and $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, their measured Rényi α -divergence $D_{\alpha}^{\text{meas}}(\varrho \| \sigma)$ is defined as a special case of (III.35) with $\Delta = D_{\alpha}$, and their measured relative entropy $D_{\alpha}^{\text{meas}}(\varrho \| \sigma) = (\text{Tr } \varrho)D_{1}^{\text{meas}}(\varrho \| \sigma)$ is also a special case of (III.32) with $\Delta = D$. We have $D_{0}^{\text{meas}} = D_{0,1}$, $D_{1/2}^{\text{meas}} = D_{1/2,1/2}$ (see [70, Chapter 9]) and for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$D_{+\infty}^{\text{meas}}(\varrho\|\sigma) = D_{+\infty}^*(\varrho\|\sigma) := D_{+\infty,+\infty}(\varrho\|\sigma) := \lim_{\alpha \to +\infty} D_{\alpha,\alpha}(\varrho\|\sigma) = \log\inf\{\lambda \ge 0 : \varrho \le \lambda\sigma\}, \quad \text{(III.61)}$$

where the last quantity was introduced in [20] under the name max-relative entropy, and its equality to the limit above has been shown in [68, Theorem 5]. No explicit expression is known for D_{α}^{meas} for other α values.

Similarly, the regularized measured Rényi α -divergence $\overline{D}_{\alpha}^{\text{meas}}(\varrho \| \sigma)$ is obtained as a special case of (III.35). Surprisingly, it has a closed formula for every $\alpha \in (0, +\infty]$, given by

$$\overline{D}_{\alpha}^{\text{meas}}(\varrho \| \sigma) = \begin{cases}
D_{\alpha,\alpha}(\varrho \| \sigma), & \alpha \in [1/2, +\infty], \\
\frac{\alpha}{1-\alpha} D_{1-\alpha,1-\alpha}(\sigma \| \varrho) + \frac{1}{\alpha-1} \log \frac{\text{Tr } \varrho}{\text{Tr } \sigma} = D_{\alpha,1-\alpha}(\varrho \| \sigma), & \alpha \in (0, 1/2), \\
D_{\alpha,1-\alpha}(\varrho \| \sigma), & \alpha = 0;
\end{cases} (III.62)$$

see [37] for $\alpha = 1$, [63] for $\alpha \in (1, +\infty)$, and [30] for $\alpha = (1/2, 1)$; the last expression for $\alpha \in (0, 1/2)$ above was first observed by Péter Vrana in August 2022, to the best of our knowledge.

For every $\alpha \in (0,1)$, strict positivity of D_{α}^{meas} is immediate from the strict positivity of the classical Rényi α -divergence, which is a straightforward corollary of Hölder's inequality, and strict positivity of D_{α}^{meas} for $\alpha \in [1,+\infty]$ follows from this and the easily verifiable fact that $\alpha \mapsto D_{\alpha}^{\text{meas}}$ is monotone increasing. Strict positivity of $\overline{D}_{\alpha}^{\text{meas}}$ follows from $D_{\alpha}^{\text{meas}} \leq \overline{D}_{\alpha}^{\text{meas}}$.

For any $\alpha \in [0, +\infty]$, the measured Rényi α -divergence is superadditive on tensor products (see Example III.14), but not additive unless $\alpha \in \{0, 1/2, +\infty\}$; see, e.g., [32, Remark 4.27] and [62, Proposition III.13] for the latter. On the other hand, for every $\alpha \in [0, +\infty]$, the regularized measured Rényi α -divergence is not only weakly additive (see Example III.14) but even additive on tensor products, according to Example III.28 and (III.62).

Monotonicity of D_{α}^{meas} under PTP maps and of $\overline{D}_{\alpha}^{\text{meas}}$ under CPTP maps is obvious by definition for every $\alpha \in [0, +\infty]$ (see Example III.14). Moreover, $\overline{D}_{\alpha}^{\text{meas}}$, $\alpha \in [0, +\infty]$, are monotone even under PTP maps, according to [8, 40, 67] and (III.62). In particular, the Umegaki relative entropy D^{Um} is monotone under PTP maps.

Example III.30 For any $\alpha \in [0, +\infty]$ and $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, their maximal Rényi α -divergence $D_{\alpha}^{\max}(\varrho \| \sigma)$ is defined as a special case of (III.34) with $\Delta = D_{\alpha}$, and their maximal relative entropy $D^{\max}(\varrho || \sigma) =$

(Tr ϱ) $D_1^{\max}(\varrho \| \sigma)$ is also a special case of (III.34) with $\Delta = D$. Let $\varrho_{\sigma, ac} := \max\{0 \le C \le \varrho : C^0 \le \sigma^0\} = P\varrho P - P\varrho(P^\perp \varrho P^\perp)^{-1}\varrho P$ be the absolutely continuous part of ϱ w.r.t. σ [2], where $P := \sigma^0$, and let λ_i , $i \in [r]$, be the different eigenvalues of $\sigma^{-1/2}\varrho_{\sigma,ac}\sigma^{-1/2}$ with corresponding spectral projections P_i . Let $\mathcal{I} := [r] \cup \{r+1\}$, let $\tau_0 \in \mathcal{S}(\mathcal{H})$ be arbitrary, and

$$\tau_1 := \begin{cases} \frac{\varrho - \varrho_{\sigma, ac}}{\text{Tr}(\varrho - \varrho_{\sigma, ac})}, & \varrho^0 \not\leq \sigma^0, \\ \tau_0, & otherwise. \end{cases}$$

According to [56],

$$\hat{p}(i) := \begin{cases} \lambda_i \operatorname{Tr} \sigma P_i, & i \in [r], \\ \operatorname{Tr}(\varrho - \varrho_{\sigma, ac}), & i = r + 1, \end{cases} \quad \hat{q}(i) := \begin{cases} \operatorname{Tr} \sigma P_i, & i \in [r], \\ 0, & i = r + 1, \end{cases}$$
(III.63)

$$\hat{p}(i) := \begin{cases}
\lambda_{i} \operatorname{Tr} \sigma P_{i}, & i \in [r], \\
\operatorname{Tr}(\varrho - \varrho_{\sigma, ac}), & i = r + 1,
\end{cases}
\qquad \hat{q}(i) := \begin{cases}
\operatorname{Tr} \sigma P_{i}, & i \in [r], \\
0, & i = r + 1,
\end{cases}$$

$$\hat{\Gamma}(\mathbf{1}_{\{i\}}) := \begin{cases}
\frac{\sigma^{1/2} P_{i} \sigma^{1/2}}{\operatorname{Tr} \sigma P_{i}}, & i \in [r], \operatorname{Tr} \sigma P_{i} \neq 0, \\
\tau_{0}, & i \in [r], \operatorname{Tr} \sigma P_{i} = 0, \\
\tau_{1}, & i = r + 1,
\end{cases}$$
(III.63)

is a reverse test for (ϱ, σ) that is optimal for every $D_{\alpha}^{\max}(\varrho \| \sigma)$, $\alpha \in [0, 2] \cup \{+\infty\}$, and

$$Q_{\alpha}^{\max}(\varrho\|\sigma) = Q_{\alpha}(\hat{p}\|\hat{q}) = \operatorname{Tr}\mathcal{P}_{f_{\alpha}}(\varrho\|\sigma)$$
(III.65)

$$= \begin{cases} \operatorname{Tr} \sigma \left(\sigma^{-1/2} \varrho_{\sigma, ac} \sigma^{-1/2}\right)^{\alpha} = \operatorname{Tr} \sigma \#_{\alpha} \varrho, & \alpha \in [0, 1), \\ \operatorname{Tr} \sigma \left(\sigma^{-1/2} \varrho \sigma^{-1/2}\right)^{\alpha} = \operatorname{Tr} \sigma \#_{\alpha} \varrho, & \alpha \in (1, 2], \ \varrho^{0} \leq \sigma^{0}, \\ +\infty, & \alpha \in (1, 2], \ \varrho^{0} \nleq \sigma^{0}, \end{cases}$$
(III.66)

$$D^{\max}(\varrho \| \sigma) = (\operatorname{Tr} \varrho) D_1^{\max}(\varrho \| \sigma) = D(\hat{p} \| \hat{q}) = \operatorname{Tr} \mathcal{P}_{\eta}(\varrho, \sigma)$$
(III.67)

$$D^{\max}(\varrho \| \sigma) = (\operatorname{Tr} \varrho) D_1^{\max}(\varrho \| \sigma) = D(\hat{p} \| \hat{q}) = \operatorname{Tr} \mathcal{P}_{\eta}(\varrho, \sigma)$$

$$= \begin{cases} \operatorname{Tr} \sigma^{1/2} \varrho \sigma^{-1/2} \widehat{\log} \left(\sigma^{-1/2} \varrho \sigma^{-1/2} \right) = \operatorname{Tr} \varrho \widehat{\log} \left(\varrho^{1/2} \sigma^{-1} \varrho^{1/2} \right), & \varrho^0 \leq \sigma^0, \\ +\infty, & otherwise, \end{cases}$$
(III.67)

$$D_{+\infty}^{\max}(\varrho\|\sigma) = D_{+\infty}(\hat{p}\|\hat{q}) = D_{+\infty}^*(\varrho\|\sigma),\tag{III.69}$$

where $f_{\alpha} := \mathrm{id}_{[0,+\infty)}^{\alpha}$, $\eta(x) := x \log x$, $x \geq 0$. (For the expressions in terms of the perspective functions, see also [32, 35].) The expression in (III.68) is called the Belavkin-Staszewski relative entropy [9]. Note that the optimal reverse test above is independent of α . No explicit expression is known for D_{α}^{\max} when $\alpha \in (2, +\infty)$, in which case the above reverse test is known not to be optimal.

Strict positivity of D_{α}^{\max} for all $\alpha \in (0, +\infty]$ follows from that of D_{α}^{\max} and the inequality $D_{\alpha}^{\max} \leq D_{\alpha}^{\max}$, which is due to the monotonicity of the classical Rényi divergences under stochastic maps.

For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $\alpha \in (0, 2]$, we have

$$D_{\alpha}^{\max}(\varrho \| \sigma) = \lim_{\varepsilon \searrow 0} D_{\alpha}^{\max}(\varrho + \varepsilon I \| \sigma + \varepsilon I); \tag{III.70}$$

see, e.g., [32, 35, 56].

It is immediate from their definition that D_{α}^{\max} , $\alpha \in [0, +\infty]$, are subadditive on tensor products (see Example III.14). For $\alpha \in [0, 2] \cup \{+\infty\}$, D_{α}^{\max} is even additive, as one can easily verify from the representation $Q_{\alpha}^{\max}(\varrho \| \sigma) = \operatorname{Tr} \mathcal{P}_{f_{\alpha}}(\varrho \| \sigma)$, $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ in (III.65). However, additivity of D_{α}^{\max} is not known for $\alpha \in (2, +\infty)$. In particular, we have

$$\overline{D}_{\alpha}^{\max} \begin{cases} = D_{\alpha}^{\max}, & \alpha \in [0, 2] \cup \{+\infty\}, \\ \leq D_{\alpha}^{\max}, & \alpha \in (2, +\infty). \end{cases}$$

Remark III.31 Note that, with the notations of Example III.30,

$$\lim_{\alpha \searrow 0} Q_{\alpha}^{\max}(\varrho \| \sigma) = \operatorname{Tr} \sigma(\sigma^{-1/2} \varrho_{\sigma, ac} \sigma^{-1/2})^{0} = \operatorname{Tr} \sigma \sum_{i: \lambda > 0} P_{i},$$
(III.71)

while

$$Q_0^{\max}(\varrho \| \sigma) = Q_0(\hat{p} \| \hat{q}) = \sum_i \hat{p}(i)^0 \hat{q}(i) = \sum_{i: \lambda_i \operatorname{Tr} \sigma P_i > 0} \operatorname{Tr} \sigma P_i = \operatorname{Tr} \sigma \sum_{i: \lambda_i \operatorname{Tr} \sigma P_i > 0} P_i.$$
 (III.72)

Since

$$P_i \sigma^{-1/2} \rho_{\sigma \text{ ac}} \sigma^{-1/2} P_i = \lambda_i P_i$$

we see that $\lambda_i > 0 \implies P_i \not\perp \sigma^0 \iff \operatorname{Tr} \sigma P_i > 0$, and hence (III.71) and (III.72) are equal to each other, i.e.,

$$\lim_{\alpha \searrow 0} D_{\alpha}^{\max}(\varrho \| \sigma) = D_{0}^{\max}(\varrho \| \sigma), \qquad \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}.$$

Example III.32 For any quantum Rényi α -divergence D^q_{α} , its regularization on a pair $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ is defined as

$$\overline{D}_{\alpha}^{q}(\varrho\|\sigma) := \lim_{n \to +\infty} \frac{1}{n} D_{\alpha}^{q}(\varrho^{\otimes n} \|\sigma^{\otimes n}),$$

whenever the limit exists. If the limit exists for all $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, then \overline{D}_{α}^q is a quantum Rényi α -divergence that is weakly additive, and if D_{α}^q is monotone under CPTP maps then so is \overline{D}_{α}^q .

Remark III.33 Note that if D_1^q is an additive quantum 1-divergence then the corresponding quantum relative entropy D^q is not additive; instead, it satisfies

$$D^{q}(\varrho_{1} \otimes \varrho_{2} \| \sigma_{1} \otimes \sigma_{2}) = (\operatorname{Tr} \varrho_{2})D^{q}(\varrho_{1} \| \sigma_{1}) + (\operatorname{Tr} \varrho_{1})D^{q}(\varrho_{2} \| \sigma_{2})$$

for any $\varrho_k, \sigma_k \in \mathcal{B}(\mathcal{H}_k)_{\geq 0}$, k = 1, 2. Thus, the natural notion of regularization for a quantum relative entropy D^q on a pair $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ is

$$\overline{D}(\varrho\|\sigma):=(\operatorname{Tr}\varrho)\overline{D}_1^q(\varrho\|\sigma),$$

which is well-defined whenever $\overline{D}_1^q(\varrho||\sigma)$ is. Clearly, if $\overline{D}(\varrho||\sigma)$ is well-defined for all $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ then it gives a quantum relative entropy that is weakly additive, and if D^q is monotone under CPTP maps then so is \overline{D}^q .

Remark III.34 According to Remark III.14, for any given $\alpha \in [0, +\infty]$, and any quantum Rényi α -divergence D_{α}^{q} that is monotone under CPTP maps,

$$D_{\alpha}^{\text{meas}} \le D_{\alpha}^{q} \le D_{\alpha}^{\text{max}}.$$
 (III.73)

If the regularization of D^q_{α} is well-defined then we further have

$$\overline{D}_{\alpha}^{\text{meas}} \leq \overline{D}_{\alpha}^{q} \leq \overline{D}_{\alpha}^{\text{max}}; \tag{III.74}$$

in particular, this is the case if D^q_{α} is additive, when we also have $\overline{D}^q_{\alpha} = D^q_{\alpha}$. Likewise, for any quantum relative entropy D^q that is monotone under CPTP maps,

$$D^{\text{meas}} \le D^q \le D^{\text{max}},$$
 (III.75)

and if the regularization of D^q is well-defined then we further have

$$D^{\mathrm{Um}} = \overline{D}^{\mathrm{meas}} \leq \overline{D}^q \leq \overline{D}^{\mathrm{max}} = D^{\mathrm{max}}. \tag{III.76}$$

It is also known that

$$D^{\text{meas}} < D^{\text{Um}} < D^{\text{max}}$$
: (III.77)

see [32, Theorem 4.18] for the first inequality (also [10] for a slightly weaker statement), and [32, Theorem 4.3] for the second inequality.

Remark III.35 Note that $D_{+\infty}^{\text{meas}} = \overline{D}_{+\infty}^{\text{meas}} = \overline{D}_{+\infty}^{\text{max}} = D_{+\infty}^{\text{max}}$ is the unique quantum extension of $D_{+\infty}$ that is monotone under (completely) positive trace-preserving maps, as it was observed in [76], and this unique extension also happens to be additive. On the other hand, for any other $\alpha \in [0, +\infty)$, there are infinitely many different monotone and additive quantum Rényi α -divergences; see, e.g., Example III.28.

Remark III.36 According to Remark III.12, any additive quantum Rényi α -divergence D^q_{α} satisfies the scaling law

$$D_{\alpha}^{q}(t\varrho||s\sigma) = D_{\alpha}^{q}(\varrho||\sigma) + D_{\alpha}(t||s) = D_{\alpha}^{q}(\varrho||\sigma) + \log t - \log s.$$
 (III.78)

In particular, this holds for $D_{\alpha,z}$, $\alpha \in [0,+\infty)$, $z \in (0,+\infty]$, and D_{α}^{\max} , $\alpha \in [0,2] \cup \{+\infty\}$. It is easy to verify that D_{α}^{\max} also satisfies (III.78) for every $\alpha \in (2,+\infty)$, where additivity is not known, and D_{α}^{\max} also satisfies (III.78) for every $\alpha \in [0,+\infty]$, even though they are not additive unless $\alpha \in \{0,1/2,+\infty\}$.

Note that a quantum Rényi 1-divergence D_1^q satisfies the scaling law (III.78) if and only if the corresponding quantum relative entropy D^q satisfies the scaling law

$$D^{q}(t\varrho||s\sigma) = tD^{q}(\varrho||\sigma) + (\operatorname{Tr}\varrho)D(t||s), \tag{III.79}$$

which in turn equivalent to

$$D^{q}(t\varrho||\sigma) = (t\log t)\operatorname{Tr}\varrho + tD^{q}(\varrho||\sigma), \tag{III.80}$$

$$D^{q}(\varrho \| s\sigma) = D^{q}(\varrho \| \sigma) - (\log s) \operatorname{Tr} \varrho. \tag{III.81}$$

Remark III.37 By definition, a quantum Rényi α -divergence D^q_{α} is trace-monotone, if

$$D_{\alpha}^{q}(\varrho \| \sigma) \ge D_{\alpha}(\operatorname{Tr} \varrho \| \operatorname{Tr} \sigma) \quad (= \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma) \tag{III.82}$$

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, and it is strictly trace-monotone if equality holds in (III.82) if and only if $\varrho = \sigma$. Likewise, a quantum relative entropy D^q is trace-monotone, if

$$D^{q}(\varrho \| \sigma) \ge D(\operatorname{Tr} \varrho \| \operatorname{Tr} \sigma) \quad (= (\operatorname{Tr} \varrho) \log \operatorname{Tr} \varrho - (\operatorname{Tr} \varrho) \log \operatorname{Tr} \sigma). \tag{III.83}$$

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, and it is strictly trace-monotone if equality holds in (III.83) if and only if $\varrho = \sigma$. Obviously, any trace-monotone Rényi α -divergence or relative entropy is non-negative. Moreover, is is easy to see that if a quantum Rényi α -divergence D^q_{α} satisfies the scaling law (III.78) then it is non-negative (strictly positive) if and only if it is (strictly) trace-monotone, and similarly, if a quantum relative entropy D^q satisfies the scaling law (III.79) then it is non-negative (strictly positive) if and only if it is (strictly) trace-monotone.

Remark III.38 If a quantum relative entropy D^q satisfies the trace monotonicity (III.83) then for any $\tau, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$D^{q}(\tau \| \sigma) \ge -(\operatorname{Tr} \tau) \log \frac{\operatorname{Tr} \sigma}{\operatorname{Tr} \tau} \ge \operatorname{Tr} \tau \left(1 - \frac{\operatorname{Tr} \sigma}{\operatorname{Tr} \tau} \right) = \operatorname{Tr} \tau - \operatorname{Tr} \sigma, \tag{III.84}$$

and equality holds everywhere when $\tau = \sigma$. As an immediate consequence of this, for any $\sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$\operatorname{Tr} \sigma = \max_{\tau \in \mathcal{B}(\mathcal{H})_{>0}} \{ \operatorname{Tr} \tau - D^q(\tau \| \sigma) \} = \max_{\tau \in \mathcal{B}(\sigma^0 \mathcal{H})_{>0}} \{ \operatorname{Tr} \tau - D^q(\tau \| \sigma) \},$$
(III.85)

$$\log \operatorname{Tr} \sigma = \max_{\tau \in \mathcal{B}(\mathcal{H})_{\geq 0}} \left\{ \log \operatorname{Tr} \tau - \frac{1}{\operatorname{Tr} \tau} D^{q}(\tau \| \sigma) \right\} = \max_{\tau \in \mathcal{B}(\sigma^{0}\mathcal{H})_{\geq 0}} \left\{ \log \operatorname{Tr} \tau - \frac{1}{\operatorname{Tr} \tau} D^{q}(\tau \| \sigma) \right\}.$$
 (III.86)

Note that τ is a maximizer for (III.85) if and only if $\operatorname{Tr} \tau = \operatorname{Tr} \sigma$ and $D^q(\tau || \sigma) = 0$ (since the second inequality in (III.84) holds as an equality if and only if $\operatorname{Tr} \tau = \operatorname{Tr} \sigma$), and if D^q also satisfies the scaling property (III.79) then τ is a maximizer for (III.86) if and only if $D\left(\frac{\tau}{\operatorname{Tr} \tau} || \frac{\sigma}{\operatorname{Tr} \sigma}\right) = 0$. If D^q is strictly trace monotone then $\tau = \sigma$ is the unique maximizer for all the expressions in (III.85)-(III.86).

The variational formula (III.85) has already been noted in [77, Lemma 6] in the case $D^q = D^{\text{Um}}$.

Remark III.39 It is easy to see from their definitions that D^{meas} , D^{Um} , and D^{max} are all regular and anti-monotone in their second argument (due to the operator monotonicity of log and operator anti-monotonicity of the inverse [11]), i.e.,

$$D^{q}(\varrho \| \sigma + \varepsilon I) \nearrow D^{q}(\varrho \| \sigma) \quad as \quad \varepsilon \searrow 0.$$
 (III.87)

By Remark III.16, they are also strongly regular. It is clear from (III.60) and (III.68) that for any fixed $\varepsilon > 0$, $\mathcal{B}(\mathcal{H})_{\geq 0}^2 \ni (\varrho, \sigma) \mapsto D^q(\varrho \| \sigma + \varepsilon I)$ is continuous when q = Um or q = max. Hence, by (III.87), D^{Um} and D^{max} are both jointly lower semi-continuous in their arguments. In particular, the classical relative entropy is jointly lower semi-continuous, whence D^{meas} , as the supremum of lower semi-continuous functions, is also jointly lower semi-continuous.

Remark III.40 It is clear from (III.60) and (III.68) that D^{Um} and D^{max} are block additive. For D^{meas} , we have block sub-additivity. Indeed, let $\varrho_k, \sigma_k \in \mathcal{B}(\mathcal{H}_k)_{\geq 0}$, k = 1, 2. For any $(M_i)_{i=0}^{n-1} \in \mathrm{POVM}(\mathcal{H}_1, [n])$, $(N_i)_{i=0}^{n-1} \in \mathrm{POVM}(\mathcal{H}_2, [n])$,

$$D\left((\operatorname{Tr} \varrho_{1} M_{i})_{i=0}^{n-1} \| (\operatorname{Tr} \sigma_{1} M_{i})_{i=0}^{n-1} \right) + D\left((\operatorname{Tr} \varrho_{2} N_{i})_{i=0}^{n-1} \| (\operatorname{Tr} \sigma_{2} N_{i})_{i=0}^{n-1} \right)$$

$$\geq D\left((\operatorname{Tr} \varrho_{1} M_{i} + \operatorname{Tr} \varrho_{2} N_{i})_{i=0}^{n-1} \| (\operatorname{Tr} \sigma_{1} M_{i} + \operatorname{Tr} \sigma_{2} N_{i})_{i=0}^{n-1} \right)$$

$$= D\left((\operatorname{Tr} (\varrho_{1} \oplus \varrho_{2})(M_{i} \oplus N_{i}))_{i=0}^{n-1} \| (\operatorname{Tr} (\sigma_{1} \oplus \sigma_{2})(M_{i} \oplus N_{i}))_{i=0}^{n-1} \right),$$

where the inequality follows from the joint subadditivity of the relative entropy (a consequence of joint convexity and homogeneity). Taking the supremum over $(M_i)_{i=0}^{n-1} \in POVM(\mathcal{H}_1, [n]), (N_i)_{i=0}^{n-1} \in POVM(\mathcal{H}_2, [n]),$ and then over n, we get

$$D^{\text{meas}}(\varrho_{1}\|\sigma_{1}) + D^{\text{meas}}(\varrho_{2}\|\sigma_{2})$$

$$\geq \sup_{n\in\mathbb{N}} \sup_{M,N} D\left((\text{Tr}(\varrho_{1}\oplus\varrho_{2})(M_{i}\oplus N_{i}))_{i=0}^{n-1} \| (\text{Tr}(\sigma_{1}\oplus\sigma_{2})(M_{i}\oplus N_{i}))_{i=0}^{n-1} \right)$$

$$= \sup_{n\in\mathbb{N}} \sup_{T\in\text{POVM}(\mathcal{H}_{1}\oplus\mathcal{H}_{2},[n])} D\left(((\text{Tr}(\varrho_{1}\oplus\varrho_{2})T_{i})_{i=0}^{n-1} \| (\text{Tr}(\sigma_{1}\oplus\sigma_{2})T_{i})_{i=0}^{n-1} \right)$$

$$= D^{\text{meas}}(\varrho_{1}\oplus\varrho_{2}\|\sigma_{1}\oplus\sigma_{2}).$$

The first equality above follows from the simple fact that for any $T \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $\text{Tr}(\varrho_1 \oplus \varrho_2)T = \text{Tr}(\varrho_1 \oplus \varrho_2)(T_1 \oplus T_2)$, where $T_k = P_k T P_k$, with P_k the projection onto \mathcal{H}_k in $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Example III.41 The measured and the maximal extensions of the multi-variate Rényi divergences can be obtained as special cases of Example III.14. In detail, for any $P \in \mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X})$,

$$\begin{split} \widetilde{Q}_P^{\text{meas}}(W) &:= \sup \{ \widetilde{Q}_P(\mathcal{M}(W)) : \ M \in \text{POVM}(\mathcal{H}, [n]), \ n \in \mathbb{N} \}, \\ \widetilde{Q}_P^{\text{max}}(W) &:= \inf \{ \widetilde{Q}_P(w) : \ (w, \Gamma) \ \textit{reverse test for } w \}, \end{split} \qquad W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0},$$

give the measured and the maximal P-weighted Rényi \widetilde{Q}_P -divergences. Clearly, $\widetilde{Q}_P^{\text{meas}}$ is the smallest and $\widetilde{Q}_P^{\text{max}}$ is the largest monotone quantum extension of the classical Q_P .

The measured and the maximal P-weighted Rényi divergences can be expressed as

$$\begin{split} D_P^{\text{meas}}(W) &:= \sup \left\{ D_P(\mathcal{M}(W)) : \ M \in \text{POVM}(\mathcal{H}, [n]), \ n \in \mathbb{N} \right\} \\ &= \frac{1}{\prod_{x \in \mathcal{X}} (1 - P(x))} \left(-\log s(P) \widetilde{Q}_P^{\text{meas}} \left(\left(\frac{W_x}{\text{Tr} \, W_x} \right)_{x \in \mathcal{X}} \right) \right), \\ D_P^{\text{max}}(W) &:= \inf \left\{ D_P(w) : \ (w, \Gamma) \ \textit{reverse test for } w \right\} \\ &= \frac{1}{\prod_{x \in \mathcal{X}} (1 - P(x))} \left(-\log s(P) \widetilde{Q}_P^{\text{max}} \left(\left(\frac{W_x}{\text{Tr} \, W_x} \right)_{x \in \mathcal{X}} \right) \right) \end{split}$$

for any $P \in \mathcal{P}_f(\mathcal{X}) \cup \mathcal{P}_{f,1}^{\pm}(\mathcal{X}) \setminus \{\mathbf{1}_{\{x\}} : x \in \mathcal{X}\}\$ and $W \in \mathcal{B}(\mathcal{X},\mathcal{H})_{\geq 0}$. It is clear that D_P^{meas} is superadditive and D_P^{max} is subadditive under tensor products, and hence their regularized versions are given as

$$\overline{D}_{P}^{\text{meas}} := \sup_{n \in \mathbb{N}} \frac{1}{n} D_{P}^{\text{meas}}(W^{\hat{\otimes}n}) = \lim_{n \to +\infty} \frac{1}{n} D_{P}^{\text{meas}}(W^{\hat{\otimes}n}),$$

$$\overline{D}_{P}^{\text{max}} := \inf_{n \in \mathbb{N}} \frac{1}{n} D_{P}^{\text{max}}(W^{\hat{\otimes}n}) = \lim_{n \to +\infty} \frac{1}{n} D_{P}^{\text{max}}(W^{\hat{\otimes}n}).$$

Clearly, D_P^{meas} is the smallest and D_P^{max} is the largest quantum extension of the classical D_P that is monotone under (completely) positive trace-preserving maps, and if $\overline{D}_P^q(W) := \lim_n \frac{1}{n} D_P^q(W^{\hat{\otimes} n})$ exists for some monotone quantum extension of D_P and some $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ then

$$\overline{D}_P^{\text{meas}}(W) \le \overline{D}_P^q(W) \le \overline{D}_P^{\text{max}}(W).$$

D. Weighted geometric means and induced divergences

For a collection of functions $w \in \mathcal{F}(\mathcal{X}, \mathcal{I})_{\geq 0}$ on a finite set \mathcal{I} , and a finitely supported probability distribution $P \in \mathcal{P}_f(\mathcal{X})$, the P-weighted geometric mean of w is naturally defined as

$$G_P(w) := \left(\prod_{x \in \text{supp } P} w_x(i)^{P(x)}\right)_{i \in \mathcal{I}} = \left(\exp\left(\sum_{x \in \mathcal{X}} P(x) \log w_x(i)\right)\right)_{i \in \mathcal{I}},$$
(III.88)

where in the second expression we use the convention (II.11). This can be extended in the obvious way to the case where $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$ is a signed probability measure, provided that $w_x(i) > 0$ for all $i \in \mathcal{I}$ whenever P(x) < 0. Extending this notion to non-commutative variables has been the subject of intensive research in matrix analysis; without completeness, we refer to [1, 3, 12–14, 36, 43, 46, 47, 54, 58, 72, 73] and references therein. Normally, the definition of a matrix geometric mean includes a number of desirable properties (e.g., monotonicity in the arguments or joint concavity, at least when all weights are positive), whereas for our purposes the following minimalistic definition is more suitable:

Definition III.42 For a non-empty set \mathcal{X} and a finitely supported signed probability measure $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$, a non-commutative P-weighted geometric mean is a function

$$G_P^q: \mathcal{D}(G_P^q) := \bigcup_{d \in \mathbb{N}} \left\{ W \in \mathcal{B}(\mathcal{X}, \mathbb{C}^d)_{>0} : W_x^0 = W_y^0, x, y \in \operatorname{supp} P \right\} \to \bigcup_{d \in \mathbb{N}} \mathcal{B}(\mathbb{C}^d)_{>0},$$

such that

(i)
$$W \in \mathcal{D}(G_P^q) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^d)_{\geq 0} \implies G_P^q(W) \in \mathcal{B}(\mathbb{C}^d)_{\geq 0};$$

(ii)
$$W, \tilde{W} \in \mathcal{D}(G_P^q), W_x = \tilde{W}_x, x \in \operatorname{supp} P \implies G_P^q(W) = G_P^q(\tilde{W});$$

(iii) G_P^q is covariant under isometries, i.e., if $V: \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ is an isometry then

$$G_P^q(VWV^*) = VG_P^q(W)V^*, \qquad W \in \mathcal{D}(G_P^q) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^{d_1})_{\geq 0};$$

(iv) if $W \in \mathcal{D}(G_P^q)$ is such that $W_x = \sum_{i=1}^d w_x(i) |i\rangle\langle i|$, $x \in \mathcal{X}$, are diagonal in the same ONB $(|i\rangle)_{i=1}^d$ then

$$G_P^q(W) = G_P(w),$$

with the latter given in (III.88).

If a non-commutative P-weighted geometric mean G_P^q is regular in the sense that

$$G_P^q(W) = \lim_{\varepsilon \searrow 0} G_P^q \left((W_x + \varepsilon W_P^0)_{x \in \mathcal{X}} \right), \qquad W \in \mathcal{D}(G_P^q),$$

where

$$W_P^0 := \bigvee_{x \in \operatorname{supp} P} W_x^0,$$

then it is automatically extended to collections of PSD operators with possibly different supports as

$$G_P^q(W) := \lim_{\varepsilon \searrow 0} G_P^q \left((W_x + \varepsilon W_P^0)_{x \in \mathcal{X}} \right), \tag{III.89}$$

for any $W \in \mathcal{B}(\mathcal{X}, \mathbb{C}^d)$ such that the limit above exists; the collection of all such W is denoted by $\overline{\mathcal{D}}(G_{\mathcal{P}}^q)$.

Remark III.43 A non-commutative geometric mean G_P^q is also automatically covariant under partial isometries on $\mathcal{D}(G_P^q)$ in the sense that if $W \in \overline{\mathcal{D}}(G_P^q) \cap \mathcal{B}(\mathcal{X}, \mathbb{C}^{d_1})$ and $V : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ is a partial isometry such that $W_P^0 \leq V^*V$ then

$$G_P^q(VWV^*) = VG_P^q(W)V^*.$$

The proof in the case $W \in \mathcal{D}(G_P^q)$ goes by a straightforward modification of the argument in Remark III.1, and the extension to the case $W \in \overline{\mathcal{D}}(G_P^q)$ is obvious.

Remark III.44 Using the isometric invariance, a non-commutative P-weighted geometric mean can be uniquely extended to any $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ such that $W_x^0 = W_y^0$, $x, y \in \text{supp } P$ (we denote the collection of all such W by $\mathcal{D}_{\mathcal{H}}(G_P^q)$), and then further extended using (III.89), provided that G_P^q is regular; we denote this extension also by G_P^q , and the set of $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ for which it is well-defined by $\overline{\mathcal{D}}_{\mathcal{H}}(G_P^q)$.

Remark III.45 In the 2-variable case, i.e., when $\mathcal{X} = \{0,1\}$, we identify P with the number $\gamma := P(0)$, and use the notation $G^q_{\gamma}(W_0||W_1)$ instead of $G^q_{P}(W)$.

Example III.46 For any $\gamma \in \mathbb{R}$, $z \in (0, +\infty)$, and $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ with $\varrho^0 = \sigma^0$, let

$$\begin{split} G_{\gamma,z}(\varrho\|\sigma) &:= \left(\varrho^{\frac{\gamma}{2z}}\sigma^{\frac{1-\gamma}{z}}\varrho^{\frac{\gamma}{2z}}\right)^{z},\\ \widetilde{G}_{\gamma,z}(\varrho\|\sigma) &:= \left(\sigma^{\frac{1-\gamma}{2z}}\varrho^{\frac{\gamma}{z}}\sigma^{\frac{1-\gamma}{2z}}\right)^{z},\\ \widehat{G}_{\gamma,1}(\varrho\|\sigma) &:= \sigma\#_{\gamma}\varrho := \sigma^{1/2}\left(\sigma^{-1/2}\varrho\sigma^{-1/2}\right)^{\gamma}\sigma^{1/2},\\ \widehat{G}_{\gamma,z}(\varrho\|\sigma) &:= \left(\sigma^{\frac{1}{z}}\#_{\gamma}\varrho^{\frac{1}{z}}\right)^{z},\\ \widehat{G}_{\gamma,+\infty}(\varrho\|\sigma) &:= \lim_{z\to +\infty}\widehat{G}_{\gamma,z}(\varrho,\sigma) = \varrho^{0}e^{\gamma\widehat{\log}\varrho + (1-\gamma)\widehat{\log}\sigma}. \end{split}$$

For $\gamma \in (0,1)$, $\#_{\gamma}$ is the Kubo-Ando γ -weighted geometric mean [45] (see Section IV for a more detailed exposition), and the last equality is due to [38, Lemma 3.3]. It is easy to see that these all define 2-variable non-commutative γ -weighted geometric means. The last one of the above quantities can be immediately extended to more than two variables as

$$\widehat{G}_{P,+\infty}(W) := W_P^0 e^{\sum_x P(x) \widehat{\log} W_x}. \tag{III.90}$$

For any non-commutative P-weighted geometric mean G_P^q ,

$$Q_P^q(W) := Q_P^{G_P^q}(W) := \operatorname{Tr} G_P^q(W), \qquad W \in \overline{\mathcal{D}}_{\mathcal{H}}(G_P^q)$$
 (III.91)

defines an extension of the classical Q_P to $\overline{\mathcal{D}}_{\mathcal{H}}(G_P^q)$ for any finite-dimensional Hilbert space \mathcal{H} ; see the end of Section III B for the definition of the classical quantity. This may potentially be further extended by

$$Q_P^q(W) := Q_P^{G_P^q}(W) := \lim_{\varepsilon \searrow 0} \operatorname{Tr} G_P^q((W_x + \varepsilon W_P^0)_{x \in \mathcal{X}}), \tag{III.92}$$

provided that the limit exists; this is obviously the case when $W \in \overline{\mathcal{D}}_{\mathcal{H}}(G_P^q)$, but there might be other W for which the limit in (III.92) exists, even though the limit in (III.89) does not. Among others, the Rényi (α, z) -divergences, the log-Euclidean Rényi divergences and the maximal Rényi divergences can be given in this way for the following parameter ranges:

$$Q_{\alpha,z}(\varrho\|\sigma) = Q_{\alpha}^{G_{\alpha,z}}(\varrho\|\sigma) = Q_{\alpha}^{\tilde{G}_{\alpha,z}}(\varrho\|\sigma), \qquad \alpha \in (0,1) \cup (1,+\infty), \quad z \in (0,+\infty),$$
 (III.93)

$$Q_{\alpha,+\infty}(\varrho\|\sigma) = Q_{\alpha}^{\widehat{G}_{\alpha,+\infty}}(\varrho\|\sigma), \qquad \alpha \in (0,1) \cup (1,+\infty),$$
(III.94)

$$Q_{\alpha}^{\max}(\varrho\|\sigma) = Q_{\alpha}^{\widehat{G}_{\alpha,1}}(\varrho\|\sigma), \qquad \qquad \alpha \in (0,1) \cup (1,2], \tag{III.95}$$

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$; see Examples III.28 and III.30, and [64]. The equality in (III.95) follows from (III.66) and the fact that $\sigma \#_{\alpha} \varrho = \sigma \#_{\alpha} \varrho_{\sigma,ac}$, $\alpha \in (0,1)$.

Remark III.47 For every $\alpha \in (0,1)$, $[1,+\infty] \ni z \mapsto Q_{\widehat{G}_{\alpha,z}}(\varrho \| \sigma)$ interpolates monotone increasingly between $Q_{\alpha}^{\max}(\varrho \| \sigma)$ and $Q_{\alpha,+\infty}(\varrho \| \sigma)$, according to [4, Corollary 2.4], and hence

$$[1,+\infty]\ni z\mapsto D_{\widehat{G}_{\alpha,z}}(\varrho\|\sigma):=\frac{1}{\alpha-1}\log Q_{\widehat{G}_{\alpha,z}}(\varrho\|\sigma)-\frac{1}{\alpha-1}\log\operatorname{Tr}\varrho$$

interpolates monotone decreasingly between $D_{\alpha}^{\max}(\varrho \| \sigma)$ and $D_{\alpha,+\infty}(\varrho \| \sigma)$.

Next, we explore some ways to define (potentially) new quantum Rényi divergences and relative entropies from combining some given quantum Rényi divergences/relative entropies with some non-commutative weighted geometric means.

The simplest way to do so is to take any \mathcal{Y} -variable \tilde{P} -weighted quantum Rényi quantity $Q_{\tilde{P}}^q$, for every $y \in \mathcal{Y}$, a $P^{(y)} \in \mathcal{P}_f^{\pm}(\mathcal{X})$ and a non-commutative $P^{(y)}$ -weighted geometric mean $G_{P^{(y)}}^{q_y}$, and define

$$Q_P^{q,\mathbf{q}}(W) := Q_{\tilde{P}}^q\left(\left(G_{P^{(y)}}^{q_y}(W)\right)_{y \in \mathcal{Y}}\right), \qquad W \in \bigcap_{y \in \operatorname{supp} \tilde{P}} \overline{\mathcal{D}}(G_{P^{(y)}}^{q_y}),$$

where

$$P(x) := \sum_{y \in \text{supp } \tilde{P}} \tilde{P}(y) P^{(y)}(x), \qquad x \in \mathcal{X}.$$

This can be further extended as in (III.92), or as

$$Q_P^{q,\mathbf{q}}(W) := \lim_{\varepsilon \searrow 0} Q_{\tilde{P}}^q \left(\left(G_{P^{(y)}}^{q_y} (W + \varepsilon W_{P^{(y)}}^0) \right)_{y \in \mathcal{Y}} \right), \qquad W \in \bigcup_{d \in \mathbb{N}} \mathcal{B}(\mathcal{X}, \mathbb{C}^d)_{\geq 0},$$

provided that the limit exists.

Such quantities have been defined before in [27] for $\mathcal{X} = [n+1]$ as

$$Q_P^{\gamma,z}(W) := Q_{P(0),z}\left(W_0 \left\| \left(W_1^{\frac{P(1)}{z\kappa_1}} \#_{\gamma_1} \left(W_2^{\frac{P(2)}{z\kappa_2}} \#_{\gamma_2} \dots \left(W_{n-1}^{\frac{P(n-1)}{z\kappa_{n-1}}} \#_{\gamma_{n-1}} W_n^{\frac{P(n)}{z\kappa_n}}\right)\right)\right)^{\frac{z}{1-P(0)}}\right),$$

where $P \in \mathcal{P}([n+1])$ is a probability distribution, $\gamma_i \in (0,1)$, $i \in \{1,\ldots,n-1\}$ are arbitrarily parameters, $\gamma_n := 0$, and $\kappa_i := \gamma_1 \cdot \ldots \cdot \gamma_{i-1} \cdot (1-\gamma_i)$, $i = 1,\ldots,n$. Here, $\mathcal{Y} = [2]$, $P^{(0)} = \mathbf{1}_{\{0\}}$, $P^{(1)}(0) = 0$, $P^{(1)}(k) = P(k)/(1-P(0))$, $k = 1,\ldots,n$. It was shown in [27] that these quantities are monotone under CPTP maps.

Similar quantities were considered in [16] in the study of relative submajorization of (subnormalized) state families; namely

$$D_{\alpha,\alpha}(W_0||G_P^q(W_1,\ldots,W_{n+1})),$$

where $D_{\alpha,\alpha}$ is the sandwiched Rényi α -divergence, and G_P^q is any non-commutative geometric mean satisfying a number of desirable properties. (In fact, [16] dealt with a more general setting where P need not be finitely supported, and hence infinitely many W_x may enter the expression in the second argument of $D_{\alpha,\alpha}$.) A new family of (2-variable) quantum Rényi α -divergences was also proposed in [16] as

$$D_{\alpha}^{*,\gamma}(\varrho\|\sigma) := \frac{1}{1-\gamma} D_{\frac{\alpha-\gamma}{1-\gamma},\frac{\alpha-\gamma}{1-\gamma}}\left(\varrho\|\sigma\#_{\gamma}\varrho\right),\,$$

where $\alpha > 1$ and $\gamma \in (0,1)$. This concept can be immediately generalized as

$$D_{\alpha}^{q_1, G_{\gamma}^{q_0}}(\varrho \| \sigma) := \frac{1}{1 - \gamma} D_{\frac{\alpha - \gamma}{1 - \gamma}}^{q_1} \left(\varrho \| G_{\gamma}^{q_0}(\varrho \| \sigma) \right) \tag{III.96}$$

for any $\alpha \in (0, +\infty)$ and $\gamma \in (0, \min\{1, \alpha\})$, where $D^{q_1}_{\frac{\alpha-\gamma}{1-\gamma}}$ is a quantum Rényi $\frac{\alpha-\gamma}{1-\gamma}$ -divergence, and $G^{q_0}_{\gamma}$ is an arbitrary non-commutative γ -weighted geometric mean. The special case $\alpha=1$ gives that any quantum relative entropy D^{q_1} and any non-commutative γ -weighted geometric mean $G^{q_0}_{\gamma}$ with some $\gamma \in (0,1)$ defines a quantum relative entropy via

$$D^{q_1, G_{\gamma}^{q_0}}(\varrho \| \sigma) := \frac{1}{1 - \gamma} D^{q_1}(\varrho \| G_{\gamma}^{q_0}(\varrho \| \sigma)). \tag{III.97}$$

In Section IV we will study in detail the relative entropies of the form (III.97) when $G_{\gamma}^{q_0} = \#_{\gamma}$ is the Kubo-Ando γ -weighted geometric mean.

Remark III.48 If $D^{q_1}_{\frac{\alpha-\gamma}{1-\gamma}}$ satisfies the scaling law (III.78) then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $Q_{G^{q_0}_{\gamma}(\varrho||\sigma)} = \operatorname{Tr} G^{q_0}_{\gamma}(\varrho||\sigma) \neq 0$, (III.96) can be rewritten as

$$\begin{split} D_{\alpha}^{q_{1},G_{\gamma}^{q_{0}}}(\varrho\|\sigma) &= \frac{1}{1-\gamma} \left[D_{\frac{\alpha-\gamma}{1-\gamma}}^{q_{1}} \left(\frac{\varrho}{\operatorname{Tr} \varrho} \left\| \frac{G_{\gamma}^{q_{0}}(\varrho\|\sigma)}{\operatorname{Tr} G_{\gamma}^{q_{0}}(\varrho\|\sigma)} \right) + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} G_{\gamma}^{q_{0}}(\varrho\|\sigma) \right] \\ &= \frac{1}{1-\gamma} D_{\frac{\alpha-\gamma}{1-\gamma}}^{q_{1}} \left(\frac{\varrho}{\operatorname{Tr} \varrho} \left\| \frac{G_{\gamma}^{q_{0}}(\varrho\|\sigma)}{\operatorname{Tr} G_{\gamma}^{q_{0}}(\varrho\|\sigma)} \right) + D_{\gamma}^{G_{\gamma}^{q_{0}}}(\varrho\|\sigma). \end{split}$$

Likewise, if D^{q_1} satisfies the scaling law (III.79), and $\operatorname{Tr} G^{q_0}_{\sim}(\varrho \| \sigma) \neq 0$, then (III.97) can be rewritten as

$$D^{q_1, G^{q_0}_{\gamma}}(\varrho \| \sigma) = \frac{\operatorname{Tr} \varrho}{1 - \gamma} \left[D^{q_1} \left(\frac{\varrho}{\operatorname{Tr} \varrho} \right) \left\| \frac{G^{q_0}_{\gamma}(\varrho \| \sigma)}{\operatorname{Tr} G^{q_0}_{\gamma}(\varrho \| \sigma)} \right) + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} G^{q_0}_{\gamma}(\varrho \| \sigma) \right]$$
(III.98)

$$= (\operatorname{Tr} \varrho) \left[\frac{1}{1 - \gamma} D^{q_1} \left(\frac{\varrho}{\operatorname{Tr} \varrho} \left\| \frac{G_{\gamma}^{q_0}(\varrho \| \sigma)}{\operatorname{Tr} G_{\gamma}^{q_0}(\varrho \| \sigma)} \right) + D_{\gamma}^{G_{\gamma}^{q_0}}(\varrho \| \sigma) \right].$$
 (III.99)

In particular, if $D_{\gamma}^{G_{\gamma}^{q_0}}$ is strictly positive and $D_{\overline{1-\gamma}}^{q_1}$ (resp. D^{q_1}) is non-negative, then $D_{\alpha}^{q_1,G_{\gamma}^{q_0}}$ (resp. $D^{q_1,G_{\gamma}^{q_0}}$) is strictly positive.

Another way of obtaining multi-variate quantum Rényi divergences using non-commutative geometric means is by starting with quantum relative entropies $D^{\mathbf{q}} := (D^{q_x})_{x \in \mathcal{X}}$ and non-commutative P-weighted geometric means $G_P^{\tilde{\mathbf{q}}} := (G_P^{\tilde{q}_x})_{x \in \mathcal{X}}$, and defining

$$-\log Q_P^{\tilde{\mathbf{q}},\mathbf{q}}(W) := \sum_{x \in \mathcal{X}} P(x) D^{q_x} \left(\frac{G_P^{\tilde{q}_x}(W)}{\operatorname{Tr} G_P^{\tilde{q}_x}(W)} \middle\| W_x \right). \tag{III.100}$$

One can easily verify that for a classical W this indeed reduces to the classical $-\log Q_P(W)$. It was shown in [63, Theorem 3.6] that for $D^{q_0} = D^{q_1} = D^{\mathrm{Um}}$ and $G_{\alpha}^{q_0} = G_{\alpha}^{q_1} = \widehat{G}_{\alpha,+\infty}$, the quantum Rényi α -divergence obtained by (III.100) is equal to $D_{\alpha,+\infty}$ in (III.59). We give an extension of this to the multivariate case in Proposition V.29. We also show in Lemma VI.14 below that in the case $D^{q_0} = D^{q_1} = D^{\max}$ and $G_{\alpha}^{q_0} = G_{\alpha,1}^{q_1} = \widehat{G}_{\alpha,1}$, the quantum Rényi α -divergence obtained by (III.100) is equal to D_{α}^{\max} .

Instead of writing normalized geometric means in the first arguments of the quantum relative entropies in (III.100), one may optimize over arbitrary states, which leads to

$$-\log Q_P^{\mathbf{b},\mathbf{q}}(W) := \inf_{\omega \in \mathcal{S}(\mathcal{H})} \sum_x P(x) D^{q_x} \left(\omega \| W_x \right). \tag{III.101}$$

(When the W_x may have different supports, we take the infimum on a restricted set of states; see Section V A for the precise definition.) This again gives a multivariate quantum Rényi divergence, which we call the *barycentric Rényi* α -divergence corresponding to $D^{\mathbf{q}}$. Moreover, as we show in Section V A, (III.101) can be equivalently written as

$$Q_P^{\mathbf{b},\mathbf{q}}(W) := \sup_{\tau \in \mathcal{B}(\mathcal{H})_{>0}} \left\{ \text{Tr } \tau - \sum_x P(x) D^{q_x} \left(\tau \| W_x \right) \right\}, \tag{III.102}$$

(the precise form again given in Section V A), and any optimal τ gives a multi-variate P-weighted geometric mean $G_P^{\mathbf{D}^{\mathbf{q}}}(W)$ of W, for which

$$Q_P^{\mathbf{b},\mathbf{q}}(W) = \operatorname{Tr} G_P^{D^{\mathbf{q}}}(W), \tag{III.103}$$

(see Section V B), whence this approach can be formally seen as a special case of (III.91). However, the initial data in the two approaches are different: a non-commutative P-weighted geometric mean in (III.91), and a collection of relative entropies in (III.101). Note also that while every non-commutative P-weighted geometric mean G_P^q defines a quantum extension of Q_P by (III.91), it is not clear whether every quantum Rényi quantity Q_P^q defines a non-commutative P-weighted geometric mean G_P^q for which (III.91) holds, and hence this is a non-trivial feature of the barycentric Rényi divergences.

The main goal of this paper is the detailed study of the barycentric Rényi divergences (in Sections V and VI.) We note that these give new quantum Rényi divergences even in the 2-variable case; indeed, we show in Section VI that under very mild conditions on D^{q_0}, D^{q_1} and $\varrho, \sigma, D_{\alpha, +\infty}(\varrho \| \sigma) < D_{\alpha}^{b, \mathbf{q}}(\varrho \| \sigma) < D_{\alpha}^{\max}(\varrho \| \sigma)$ holds for every $\alpha \in (0, 1)$.

Other approaches to non-commutative geometric means and multi-variate Rényi divergences are outlined in Figure 1.

IV. γ -WEIGHTED GEOMETRIC RELATIVE ENTROPIES

For positive definite operators $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$ and $\gamma \in \mathbb{R}$, let

$$\sigma \#_{\gamma} \varrho := \sigma^{1/2} \left(\sigma^{-1/2} \varrho \sigma^{-1/2} \right)^{\gamma} \sigma^{1/2} \tag{IV.104}$$

$$= P_{\mathrm{id}^{\gamma}}(\varrho, \sigma) = P_{\mathrm{id}^{1-\gamma}}(\sigma, \varrho)$$
 (IV.105)

$$= \varrho^{1/2} \left(\varrho^{-1/2} \sigma \varrho^{-1/2} \right)^{1-\gamma} \varrho^{1/2} = \varrho \#_{1-\gamma} \sigma, \tag{IV.106}$$

(see (II.13) for the equality in (IV.105)). The definition is extended to pairs of general PSD operators as

$$\sigma \#_{\gamma} \varrho := \lim_{\varepsilon \searrow 0} (\sigma + \varepsilon I) \#_{\gamma} (\varrho + \varepsilon I), \tag{IV.107}$$

whenever the limit exists. In particular, we have

$$\varrho^0 \le \sigma^0 \quad \Longrightarrow \quad \sigma \#_{\gamma} \varrho = \sigma^{1/2} \left(\sigma^{-1/2} \varrho \sigma^{-1/2} \right)^{\gamma} \sigma^{1/2}, \quad \alpha \in (0, 1) \cup (1, 2], \tag{IV.108}$$

$$\varrho^{0} \ge \sigma^{0} \implies \sigma \#_{\gamma} \varrho = \varrho^{1/2} \left(\varrho^{-1/2} \sigma \varrho^{-1/2} \right)^{1-\gamma} \varrho^{1/2}, \quad \alpha \in [-1, 0) \cup (0, 1), \tag{IV.109}$$

where the negative powers are taken on the support of the respective operators; see, e.g., [45] and [32, Proposition 3.26].

For $\gamma \in (0,1)$, $\#_{\gamma}$ is called the Kubo-Ando γ -weighted geometric mean [45]. For the rest, we assume that $\gamma \in (0,1)$. The following are well known [45] or easy to see:

$$(\sigma \#_{\gamma} \varrho)^0 = \sigma^0 \wedge \varrho^0; \tag{IV.110}$$

$$\sigma \#_{\gamma} \rho = \rho \#_{1-\gamma} \sigma; \tag{IV.111}$$

$$\varrho_1 \le \varrho_2, \, \sigma_1 \le \sigma_2 \implies \sigma_1 \#_{\gamma} \varrho_1 \le \sigma_2 \#_{\gamma} \varrho_2.$$
 (IV.112)

The γ -weighted geometric means are monotone continuous in the sense that for any functions $(0, +\infty) \ni \varepsilon \mapsto \varrho_{\varepsilon} \in \mathcal{B}(\mathcal{H})_{>0}$ and $(0, +\infty) \ni \varepsilon \mapsto \sigma_{\varepsilon} \in \mathcal{B}(\mathcal{H})_{>0}$ that are monotone decreasing in the PSD order,

$$(\sigma_{\varepsilon} \#_{\gamma} \varrho_{\varepsilon}) \searrow (\lim_{\varepsilon \searrow 0} \sigma_{\varepsilon}) \#_{\gamma} (\lim_{\varepsilon \searrow 0} \varrho_{\varepsilon}), \qquad \gamma \in (0, 1), \tag{IV.113}$$

as $\varepsilon \searrow 0$.

Note that for any $\varrho_k, \sigma_k \in \mathcal{B}(\mathcal{H}_k)_{\geq 0}$, k = 1, 2, and any sequence $\varepsilon_1 > \varepsilon_2 > \ldots \to 0$,

$$[(\sigma_1 + \varepsilon_n I) \otimes (\sigma_2 + \varepsilon_n I)] \#_{\gamma} [(\varrho_1 + \varepsilon_n I) \otimes (\varrho_2 + \varepsilon_n I)]$$

= $[(\sigma_1 + \varepsilon_n I) \#_{\gamma} (\rho_1 + \varepsilon_n I)] \otimes [(\sigma_2 + \varepsilon_n I) \#_{\gamma} (\rho_2 + \varepsilon_n I)],$

and taking the limit $n \to +\infty$ yields, by (IV.113), that

$$(\sigma_1 \otimes \sigma_2) \#_{\gamma}(\varrho_1 \otimes \varrho_2) = (\sigma_1 \#_{\gamma} \varrho_1) \otimes (\sigma_2 \#_{\gamma} \varrho_2). \tag{IV.114}$$

The following is well known; we state it for later use:

Lemma IV.1 For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any $\gamma \in (0,1)$,

$$\operatorname{Tr}(\sigma \#_{\gamma} \varrho) \le \operatorname{Tr} \varrho^{\gamma} \sigma^{1-\gamma} \le (\operatorname{Tr} \varrho)^{\gamma} (\operatorname{Tr} \sigma)^{1-\gamma},$$
 (IV.115)

and equality holds in the second inequality if and only if ϱ and σ are linearly dependent. In particular, for states $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$,

$$\operatorname{Tr}(\sigma \#_{\gamma} \rho) \le 1$$
, and $\operatorname{Tr}(\sigma \#_{\gamma} \rho) = 1 \iff \rho = \sigma$. (IV.116)

Proof The second inequality in (IV.115) and its equality condition is a simple consequence of Hölder's inequality; for the first inequality, see, e.g., [32, Example 4.5]. The assertion in (IV.116) follows immediately from (IV.115). \Box

The inequality between the first and the last terms in (IV.115) is a special case of the following lemma. For a proof of the lemma, see, e.g., [32, Proposition 3.30].

Lemma IV.2 For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any positive linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$,

$$\Phi\left(\varrho\#_{\gamma}\sigma\right)\leq\Phi(\varrho)\#_{\gamma}\Phi(\sigma).$$

The following is a special case of (III.97):

Definition IV.3 Let D^q be a quantum relative entropy. For every $\gamma \in [0,1)$, and every $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, let

$$D^{q,\#_{\gamma}}(\varrho\|\sigma) := \begin{cases} \frac{1}{1-\gamma} D^q(\varrho\|\sigma\#_{\gamma}\varrho), & \gamma \in (0,1), \\ D^q(\varrho\|\sigma), & \gamma = 0. \end{cases}$$

We call $D^{q,\#_{\gamma}}$ the γ -weighted geometric D^q .

Remark IV.4 Note that by (IV.110), $\sigma \#_{\gamma} \varrho = 0$ might happen even if both ϱ and σ are quantum states (thus, in particular, are non-zero); in this case the value of $D^{q,\#_{\gamma}}(\varrho \| \sigma)$ is $+\infty$, according to Remark III.26.

Remark IV.5 Assume that D^q satisfies the scaling property (III.79). Then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $\rho^0 \wedge \sigma^0 \neq 0$,

$$D^{q,\#_{\gamma}}(\varrho\|\sigma) = (\operatorname{Tr}\varrho) \left[\frac{1}{1-\gamma} D^{q} \left(\frac{\varrho}{\operatorname{Tr}\rho} \left\| \frac{\sigma \#_{\gamma}\varrho}{\operatorname{Tr}\sigma \#_{\gamma}\rho} \right) + D_{\gamma}^{\max}(\varrho\|\sigma) \right], \qquad \gamma \in (0,1).$$

This is a special case of (III.99).

It is clear that for any $\gamma \in (0,1)$, $D^{q,\#\gamma}$ is a quantum relative entropy. In the following we show how certain properties of D^q are inherited by $D^{q,\#\gamma}$.

- (i) If D_1^q is additive on tensor products then so is $D_1^{q, \#_{\gamma}}$, due to (IV.114).
- (ii) If D^q is block subadditive/superadditive/additive, then so are $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$. This follows immediately from the fact that for any sequence of projections P_1, \ldots, P_r summing to I, we have $(\sum_{i=1}^r P_i \varrho P_i) \#_{\gamma} (\sum_{i=1}^r P_i \sigma P_i) = \sum_{i=1}^r (P_i \varrho P_i) \#_{\gamma} (P_i \sigma P_i)$.
- (iii) If D^q satisfies the support condition

$$D^q(\rho \| \sigma) < +\infty \iff \rho^0 \le \sigma^0$$

then so do $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$. Indeed, by (IV.110), $(\sigma \#_{\gamma} \varrho)^0 = \varrho^0 \wedge \sigma^0$, and clearly, $\varrho^0 \leq \varrho^0 \wedge \sigma^0$ $\iff \varrho^0 \leq \sigma^0$, whence $D^{q,\#_{\gamma}}(\varrho \| \sigma) < +\infty \iff \varrho^0 \leq \sigma^0$.

- (iv) If D^q satisfies either of the scaling laws (III.80) or (III.81) then so do $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$. As a consequence, if D^q satisfies the scaling law (III.79) then so do $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$. These can be verified by straightforward computations, which we omit.
- (v) If D^q satisfies trace monotonicity (III.83) then $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$, also satisfy trace monotonicity (III.83); moreover, they are strictly positive. Indeed, if $\sigma \#_{\gamma} \varrho \neq 0$ then

$$D^{q,\#_{\gamma}}(\varrho\|\sigma) = \frac{1}{1-\gamma}D^{q}(\varrho\|\sigma\#_{\gamma}\varrho) \tag{IV.117}$$

$$\geq \frac{1}{1-\gamma} \left[(\operatorname{Tr} \varrho) \log \operatorname{Tr} \varrho - (\operatorname{Tr} \varrho) \log \underbrace{\operatorname{Tr} (\sigma \#_{\gamma} \varrho)}_{\leq (\operatorname{Tr} \varrho)^{\gamma} (\operatorname{Tr} \sigma)^{1-\gamma}} \right]$$
 (IV.118)

$$\geq \frac{1}{1-\gamma} [(\operatorname{Tr} \varrho) \log \operatorname{Tr} \varrho - (\operatorname{Tr} \varrho) \log ((\operatorname{Tr} \varrho)^{\gamma} (\operatorname{Tr} \sigma)^{1-\gamma})]$$
 (IV.119)

$$= (\operatorname{Tr} \varrho) \log \operatorname{Tr} \varrho - (\operatorname{Tr} \varrho) \log \operatorname{Tr} \sigma, \tag{IV.120}$$

where the first inequality follows from the assumed trace monotonicity of D^q , and the second inequality follows from Lemma IV.1. If $\sigma\#_{\gamma}\varrho=0$ then either $\varrho=0$ and thus $D^{q,\#_{\gamma}}(\varrho\|\sigma)=\frac{1}{1-\gamma}D^q(0\|0)=0=0$ ($D^q(0\|0)=D(Tr\,\varrho\|Tr\,\sigma)$), or $\varrho\neq0$, whence $D^{q,\#_{\gamma}}(\varrho\|\sigma)=\frac{1}{1-\gamma}D^q(\varrho\|0)=+\infty\geq D(Tr\,\varrho\|Tr\,\sigma)$

holds trivially. This shows that $D^{q,\#\gamma}$, $\gamma \in (0,1)$, satisfy trace monotonicity, and hence they are also non-negative, according to Remark III.37.

Assume now that $\operatorname{Tr} \varrho = \operatorname{Tr} \sigma = 1$ and $\varrho \neq \sigma$. If $\varrho^0 \wedge \sigma^0 = 0$ then $D^{q,\#_{\gamma}}(\varrho \| \sigma) = \frac{1}{1-\gamma}D^q(\varrho \| 0) = +\infty > 0$ holds trivially. Otherwise (IV.117)–(IV.120) hold, and the inequality in (IV.119) is strict, according to Lemma IV.1, whence $D^{q,\#_{\gamma}}(\varrho \| \sigma) > 0$ for every $\gamma \in (0,1)$. This proves that $D^{q,\#_{\gamma}}, \gamma \in (0,1)$, are strictly positive.

(vi) If D^q satisfies the scaling law (III.81) and it is non-negative then $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$, are strictly positive. Indeed, let $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$ be unequal states. If $\sigma \#_{\gamma} \varrho = 0$ then $D^{q,\#_{\gamma}}(\varrho \| \sigma) = \frac{1}{1-\gamma} D^q(\varrho \| 0) = +\infty > 0$ is obvious. Assume for the rest that $\sigma \#_{\gamma} \varrho \neq 0$. Then

$$(1-\gamma)D^{q,\#_{\gamma}}(\varrho\|\sigma) = D^{q}\left(\varrho\Big\|\frac{\sigma\#_{\gamma}\varrho}{\operatorname{Tr}\,\sigma\#_{\gamma}\varrho}\operatorname{Tr}\,\sigma\#_{\gamma}\varrho\right) = \underbrace{D^{q}\left(\varrho\Big\|\frac{\sigma\#_{\gamma}\varrho}{\operatorname{Tr}\,\sigma\#_{\gamma}\varrho}\right)}_{>0} - (\operatorname{Tr}\varrho)\log\underbrace{\operatorname{Tr}\,\sigma\#_{\gamma}\varrho}_{<1} > 0,$$

where the strict inequality is due to Lemma IV.1.

- (vii) If D^q is strongly regular then so are $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$. This follows immediately from the fact that if σ_n , $n \in \mathbb{N}$, is a sequence of PSD operators converging monotone decreasingly to σ then $\sigma_n \#_{\gamma} \varrho$ converges monotone decreasingly to $\sigma \#_{\gamma} \varrho$; see (IV.113).
- (viii) If D^q is anti-monotone in its second argument, then so are $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$. Indeed, this follows immediately from (IV.112).
- (ix) Assume that D^q is anti-monotone in its second argument. If $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a positive linear map such that

$$D^{q}(\Phi(\varrho)\|\Phi(\sigma)) \leq D^{q}(\varrho\|\sigma)$$
 then $D^{q,\#\gamma}(\Phi(\varrho)\|\Phi(\sigma)) \leq D^{q,\#\gamma}(\varrho\|\sigma), \ \gamma \in (0,1).$

Indeed,

$$D^{q,\#_{\gamma}}(\Phi(\varrho)\|\Phi(\sigma)) = \frac{1}{1-\gamma}D^{q}(\Phi(\varrho)\|\underbrace{\Phi(\sigma)\#_{\gamma}\Phi(\varrho)}_{\geq \Phi(\sigma\#_{\gamma}\varrho)}) \leq \frac{1}{1-\gamma}D^{q}(\Phi(\varrho)\|\Phi(\sigma\#_{\gamma}\varrho))$$
$$\leq \frac{1}{1-\gamma}D^{q}(\varrho\|\sigma\#_{\gamma}\varrho),$$

where the first inequality follows from Lemma IV.2, the second inequality from anti-monotonicity of D^q in its second argument, and the last inequality from the monotonicity of D^q under the given class of maps. In particular, $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$, are monotone under any class of positive maps under which D^q is monotone.

- (x) Since the γ -weighted geometric means are jointly concave in their arguments [45], it is easy to see that if D^q is anti-monotone in its second argument and it is jointly convex, then so are $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$.
- (xi) For any two quantum relative entropies D^{q_1}, D^{q_2} ,

$$D^{q_1} \le D^{q_2} \implies D^{q_1, \#_{\gamma}} \le D^{q_2, \#_{\gamma}}, \qquad \gamma \in (0, 1),$$
 (IV.121)

which follows immediately by definition. Moreover,

$$\forall \, \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}, \, \varrho \sigma \neq \sigma \varrho : \, D^{q_1}(\varrho \| \sigma) < D^{q_2}(\varrho \| \sigma)$$

$$\implies \quad \forall \, \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}, \, \varrho \sigma \neq \sigma \varrho : \, D^{q_1, \#_{\gamma}}(\varrho \| \sigma) < D^{q_2, \#_{\gamma}}(\varrho \| \sigma), \qquad \gamma \in (0, 1). \quad \text{(IV.122)}$$

Indeed, for this we only need that for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$, $\varrho\sigma \neq \sigma\varrho \implies \varrho(\sigma\#_{\gamma}\varrho) \neq (\sigma\#_{\gamma}\varrho)\varrho$. This is straightforward to verify; indeed, by (IV.111),

$$\begin{split} \varrho(\sigma\#_{\gamma}\varrho) &= (\sigma\#_{\gamma}\varrho)\varrho &\iff \varrho\varrho^{1/2} \left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right)^{1-\gamma}\varrho^{1/2} = \varrho^{1/2} \left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right)^{1-\gamma}\varrho^{1/2}\varrho \\ &\iff \varrho\left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right)^{1-\gamma} = \left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right)^{1-\gamma}\varrho \\ &\iff \varrho\left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right) = \left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right)\varrho \\ &\iff \varrho\sigma = \sigma\varrho. \end{split}$$

Remark IV.6 By Remark IV.5 and the strict positivity of D_{γ}^{\max} , if D^q satisfies the scaling law (III.79) and it is non-negative then $D^{q,\#_{\gamma}}$ is strictly positive. Note that (v) and (vi) above establish strict positivity of $D^{q,\#_{\gamma}}$ under slightly weaker conditions.

Remark IV.7 According to Remark III.34, if $D^{q,\#_{\gamma}}$ is monotone under CPTP maps then

$$D^{\text{meas}} \le D^{q, \#_{\gamma}} \le D^{\text{max}},$$

and if $D^{q,\#_{\gamma}}$ is also additive on tensor products then

$$D^{\mathrm{Um}} \le D^{q, \#_{\gamma}} \le D^{\mathrm{max}},\tag{IV.123}$$

for any $\gamma \in (0,1)$. In particular,

$$D^{\text{Um}} < D^{\text{Um}, \#_{\gamma}} < D^{\text{max}, \#_{\gamma}} < D^{\text{max}}, \qquad \gamma \in (0, 1),$$
 (IV.124)

where we also used (xi) above.

Lemma IV.8 Assume that D^q is anti-monotone in its second argument and regular. Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be the limits of monotone decreasing sequences $(\varrho_{\varepsilon})_{\varepsilon>0}$ and $(\sigma_{\varepsilon})_{\varepsilon>0}$, respectively. Then, for every $\gamma \in (0,1)$,

$$\frac{1}{1-\gamma}D^{q}(\varrho\|\sigma_{\varepsilon}\#_{\gamma}\varrho_{\varepsilon}) \nearrow D^{q,\#_{\gamma}}(\varrho\|\sigma) \quad as \quad \varepsilon \searrow 0, \tag{IV.125}$$

$$D^{q,\#_{\gamma}}(\varrho\|\sigma_{\varepsilon}) \nearrow D^{q,\#_{\gamma}}(\varrho\|\sigma) \quad as \quad \varepsilon \searrow 0.$$
 (IV.126)

Proof By the assumptions and Remark III.16, D^q is strongly regular. By (IV.113), both $\sigma_{\varepsilon} \#_{\gamma} \varrho_{\varepsilon}$ and $\sigma_{\varepsilon} \#_{\gamma} \varrho$ converge monotone decreasingly to $\sigma \#_{\gamma} \varrho$ as $\varepsilon \searrow 0$. From these, the assertions follow immediately.

Remark IV.9 (IV.126) is a special case of the anti-monotonicity and strong regularity stated in (viii) and (viii) above.

Corollary IV.10 If D^q is anti-monotone in its second argument, regular, and jointly lower semi-continuous in its arguments, then $D^{q,\#_{\gamma}}$ has the same properties for every $\gamma \in (0,1)$.

Proof Anti-monotonicity and (strong) regularity have been covered in (viii) and (vii) above (see also Remark III.16). For every $\varepsilon > 0$, $(\varrho, \sigma) \mapsto (\sigma + \varepsilon I) \#_{\gamma}(\varrho + \varepsilon I)$ is continuous (due to the continuity of functional calculus), and thus $(\varrho, \sigma) \mapsto \frac{1}{1-\gamma} D^q(\varrho \| (\sigma + \varepsilon I) \#_{\gamma}(\varrho + \varepsilon I))$ is lower semi-continuous. Thus, by (IV.125), $(\varrho, \sigma) \mapsto D^{q, \#_{\gamma}}(\varrho \| \sigma)$ is the supremum of lower semi-continuous functions, and therefore is itself lower semi-continuous.

Example IV.11 Combining Remark III.39 and Lemma IV.8 yields that (IV.125)–(IV.126) hold for $D^q = D^{\text{meas}}, D^{\text{Um}}, D^{\text{max}}$, and by Corollary IV.10, $D^{\text{meas}, \#_{\gamma}}$, $D^{\text{Um}, \#_{\gamma}}$, and $D^{\text{max}, \#_{\gamma}}$, are jointly lower semi-continuous in their arguments for every $\gamma \in (0, 1)$.

For a given relative entropy D^q , the γ -weighted relative entropies $D^{q,\#_{\gamma}}$, $\gamma \in (0,1)$, give (potentially) new quantum relative entropies. On the other hand, as Proposition IV.13 below shows, iterating this procedure does not give further quantum relative entropies. We will need the following simple lemma for the proof.

Lemma IV.12 For any $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any $\gamma_1, \gamma_2 \in (0, 1)$,

$$(\sigma \#_{\gamma_2} \varrho) \#_{\gamma_1} \varrho = \sigma \#_{1 - (1 - \gamma_1)(1 - \gamma_2)} \varrho. \tag{IV.127}$$

Proof First, assume that ρ, σ are positive definite. Then the statement follows as

$$(\sigma \#_{\gamma_2} \varrho) \#_{\gamma_1} \varrho = \varrho^{1/2} \left(\varrho^{-1/2} \left(\sigma \#_{\gamma_2} \varrho \right) \varrho^{-1/2} \right)^{1-\gamma_1} \varrho^{1/2}$$

$$= \varrho^{1/2} \left(\varrho^{-1/2} \left(\varrho^{1/2} \left(\varrho^{-1/2} \sigma \varrho^{-1/2} \right)^{1-\gamma_2} \varrho^{1/2} \right) \varrho^{-1/2} \right)^{1-\gamma_1} \varrho^{1/2}$$

$$= \varrho^{1/2} \left(\varrho^{-1/2} \sigma \varrho^{-1/2} \right)^{(1-\gamma_1)(1-\gamma_2)} \varrho^{1/2}$$

$$= \sigma \#_{1-(1-\gamma_1)(1-\gamma_2)} \varrho. \tag{IV.128}$$

Next, we consider the general case. By the joint concavity and homogeneity of the weighted geometric Kubo-Ando mean (see, e.g., [31, Corollary 3.2.3]),

$$(\sigma + \varepsilon I) \#_{\gamma}(\varrho + \varepsilon I) \ge \sigma \#_{\gamma} \varrho + (\varepsilon I) \#_{\gamma}(\varepsilon I) = \sigma \#_{\gamma} \varrho + \varepsilon I, \qquad \varepsilon > 0, \tag{IV.129}$$

holds for any $\gamma \in (0,1)$. On the other hand,

$$(\sigma + \varepsilon I) \#_{\gamma}(\varrho + \varepsilon I) = \sigma \#_{\gamma} \varrho + (\varrho + \varepsilon I) \#_{\gamma}(\sigma + \varepsilon I) - \sigma \#_{\gamma} \varrho$$

$$\leq \sigma \#_{\gamma} \varrho + I \underbrace{\|(\sigma + \varepsilon I) \#_{\gamma}(\varrho + \varepsilon I) - \sigma \#_{\gamma} \varrho\|_{\infty}}_{=:f(\varepsilon)} = \sigma \#_{\gamma} \varrho + f(\varepsilon)I, \quad (IV.130)$$

where $f: [0, +\infty) \to [0, +\infty)$ is monotone increasing and $0 = f(0) = \lim_{\varepsilon \searrow 0} f(\varepsilon)$, according to (IV.113). Thus,

$$\begin{split} \left(\sigma\#_{\gamma_2}\varrho\right)\#_{\gamma_1}\varrho &= \lim_{\varepsilon\searrow 0}\left(\sigma\#_{\gamma_2}\varrho+\varepsilon I\right)\#_{\gamma_1}(\varrho+\varepsilon I)\\ &\leq \lim_{\varepsilon\searrow 0}\left(\left(\sigma+\varepsilon I\right)\#_{\gamma_2}(\varrho+\varepsilon I)\right)\#_{\gamma_1}(\varrho+\varepsilon I)\\ &= \lim_{\varepsilon\searrow 0}(\sigma+\varepsilon I)\#_{1-(1-\gamma_1)(1-\gamma_2)}(\varrho+\varepsilon I)\\ &= \sigma\#_{1-(1-\gamma_1)(1-\gamma_2)}\varrho, \end{split}$$

where the first and the last equalities follow by the monotone continuity (IV.113) of $\#_{\gamma}$ for any $\gamma \in (0,1)$, the inequality is due to (IV.129), and the second equality is due to (IV.128). Similarly,

$$\begin{split} \left(\sigma\#_{\gamma_2}\varrho\right)\#_{\gamma_1}\varrho &= \lim_{\varepsilon\searrow 0}\left(\sigma\#_{\gamma_2}\varrho + f(\varepsilon)I\right)\#_{\gamma_1}(\varrho + \varepsilon I) \\ &\geq \lim_{\varepsilon\searrow 0}\left(\left(\sigma + \varepsilon I\right)\#_{\gamma_2}(\varrho + \varepsilon I)\right)\#_{\gamma_1}(\varrho + \varepsilon I) \\ &= \lim_{\varepsilon\searrow 0}(\sigma + \varepsilon I)\#_{1-(1-\gamma_1)(1-\gamma_2)}(\varrho + \varepsilon I) \\ &= \sigma\#_{1-(1-\gamma_1)(1-\gamma_2)}\varrho, \end{split}$$

where the inequality follows from (IV.130), and the rest follow as in the previous argument. This gives (IV.127). \Box

Proposition IV.13 For any quantum relative entropy D^q , and any $\gamma_1, \gamma_2 \in (0,1)$,

$$D^{(q,\#_{\gamma_1}),\#_{\gamma_2}} = D^{q,\#_{1-(1-\gamma_1)(1-\gamma_2)}}$$

Proof For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

$$\begin{split} D^{(q,\#_{\gamma_1}),\#_{\gamma_2}}(\varrho\|\sigma) &= \frac{1}{1-\gamma_2} D^{q,\#_{\gamma_1}}(\varrho\|\sigma\#_{\gamma_2}\varrho) \\ &= \frac{1}{1-\gamma_2} \cdot \frac{1}{1-\gamma_1} D^q(\varrho\|(\sigma\#_{\gamma_2}\varrho)\#_{\gamma_1}\varrho) \\ &= \frac{1}{1-\gamma_2} \cdot \frac{1}{1-\gamma_1} D^q(\varrho\|(\sigma\#_{1-(1-\gamma_1)(1-\gamma_2)}\varrho) \\ &= D^{q,\#_{1-(1-\gamma_1)(1-\gamma_2)}}(\varrho\|\sigma), \end{split}$$

where all equalities apart from the third one are by definition, and the third equality is due to Lemma IV.12. \Box

Next, we study the γ -weighted geometric relative entropies corresponding to the largest and the smallest additive and CPTP-monotone quantum relative entropies, i.e., to D^{\max} and D^{Um} . The first case turns out to be trivial, while in the second case we get that the γ -weighted geometric Umegaki relative entropies essentially interpolate between D^{Um} and D^{\max} in a monotone increasing way.

Proposition IV.14 For any $\gamma \in (0,1)$,

$$D^{\max, \#_{\gamma}} = D^{\max}$$

Proof We need to show that for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$\frac{1}{1-\gamma}D^{\max}(\varrho\|\sigma\#_{\gamma}\varrho) = D^{\max}(\varrho\|\sigma).$$

If $\varrho^0 \nleq \sigma^0$ then both sides are $+\infty$ (see (iii) above), and hence for the rest we assume that $\varrho^0 \leq \sigma^0$. Assume first that ϱ and σ are invertible. Then

$$\begin{split} \frac{1}{1-\gamma} D^{\max}(\varrho \| \sigma \#_{\gamma} \varrho) &= \frac{1}{1-\gamma} \operatorname{Tr} \varrho \log \left(\varrho^{1/2} (\sigma \#_{\gamma} \varrho)^{-1} \varrho^{1/2} \right) \\ &= \frac{1}{1-\gamma} \operatorname{Tr} \varrho \log \left(\varrho^{1/2} (\varrho^{-1/2} (\varrho^{1/2} \sigma^{-1} \varrho^{1/2})^{1-\gamma} \varrho^{-1/2}) \varrho^{1/2} \right) \\ &= \frac{1}{1-\gamma} \operatorname{Tr} \varrho \log (\varrho^{1/2} \sigma^{-1} \varrho^{1/2})^{1-\gamma} \\ &= \operatorname{Tr} \varrho \log (\varrho^{1/2} \sigma^{-1} \varrho^{1/2}) = D^{\max}(\varrho \| \sigma), \end{split} \tag{IV.131}$$

where the first equality is due to (III.68), the second equality follows from (IV.111), and the rest are obvious. If we only assume that $\rho^0 \leq \sigma^0$ then we have

$$\begin{split} D^{\max,\#_{\gamma}}(\varrho\|\sigma) &= \lim_{\varepsilon \searrow 0} \frac{1}{1-\gamma} D^{\max}(\varrho\|(\sigma+\varepsilon I)\#_{\gamma}(\varrho+\varepsilon I)) \\ &= \lim_{\varepsilon \searrow 0} \operatorname{Tr} \varrho \log \left((\varrho+\varepsilon I)^{1/2} (\sigma+\varepsilon I)^{-1} (\varrho+\varepsilon I)^{1/2} \right) \\ &= \operatorname{Tr} \varrho \log (\varrho^{1/2} \sigma^{-1} \varrho^{1/2}) = D^{\max}(\varrho\|\sigma), \end{split}$$

where the first equality was stated in Example IV.11, the second equality follows as in (IV.131), the third equality is easy to verify, and the last equality follows by (III.68).

Lemma IV.15 Assume that D^q is a quantum relative entropy such that $D^q \leq D^{q,\#_{\gamma}}$ for every $\gamma \in (0,1)$. Then $(0,1) \ni \gamma \mapsto D^{q,\#_{\gamma}}$ is monotone increasing.

Proof Let $0 < \gamma_1 \le \gamma_2 < 1$, and let $\gamma := \frac{\gamma_2 - \gamma_1}{1 - \gamma_1}$, so that $1 - \gamma = \frac{1 - \gamma_2}{1 - \gamma_1}$. For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\ge 0}$,

$$D^{q,\#_{\gamma_{1}}}(\varrho\|\sigma) = \frac{1}{1-\gamma_{1}}D^{q}(\varrho\|\sigma\#_{\gamma_{1}}\varrho)$$

$$\leq \frac{1}{1-\gamma_{1}}D^{q,\#_{\gamma}}(\varrho\|\sigma\#_{\gamma_{1}}\varrho)$$

$$= \frac{1}{1-\gamma_{1}}\frac{1}{1-\gamma}D^{q}(\varrho\|(\sigma\#_{\gamma_{1}}\varrho)\#_{\gamma}\varrho)$$

$$= \frac{1}{1-\gamma_{1}}\frac{1}{1-\gamma}D^{q}(\varrho\|(\sigma\#_{\gamma_{2}}\varrho)$$

$$= \frac{1}{1-\gamma_{2}}D^{q}(\varrho\|(\sigma\#_{\gamma_{2}}\varrho)$$

$$= D^{q,\#_{\gamma_{2}}}(\varrho\|\sigma).$$

where the first, the second, and the last equalities are by definition, the inequality is by assumption, while the third and the fourth equalities follow from Lemma IV.12 and the definition of γ .

Proposition IV.16 *Let* $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$. *Then*

$$(0,1) \ni \gamma \mapsto D^{\mathrm{Um},\#_{\gamma}}(\varrho \| \sigma)$$
 is monotone increasing, (IV.132)

and if ρ, σ are positive definite then

$$D^{\mathrm{Um},\#_{\gamma}}(\varrho\|\sigma) \searrow D^{\mathrm{Um}}(\varrho\|\sigma), \quad as \quad \gamma \searrow 0,$$
 (IV.133)

$$D^{\mathrm{Um}, \#_{\gamma}}(\varrho \| \sigma) \nearrow D^{\mathrm{max}}(\varrho \| \sigma), \quad as \quad \gamma \nearrow 1.$$
 (IV.134)

Proof Since D^{Um} is monotone under CPTP maps, so are $D^{\mathrm{Um},\#_{\gamma}}$, $\gamma \in (0,1)$, according to (ix) above, and hence, by (IV.124), $D^{\mathrm{Um}} \leq D^{\mathrm{Um},\#_{\gamma}}$, $\gamma \in (0,1)$. By Lemma IV.15, $(0,1) \ni \gamma \mapsto D^{\mathrm{Um},\#_{\gamma}}$ is monotone increasing.

Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$. Then (IV.133) follows simply by the continuity of functional calculus, so we only have to prove (IV.134). Note that the definition of $\sigma \#_{\gamma} \varrho$ in (IV.104), and hence also the definition of $D^{\mathrm{Um}}(\varrho \| \sigma \#_{\gamma} \varrho)$ make sense for any $\gamma \in \mathbb{R}$, and both are (infinitely many times) differentiable functions of γ . Moreover, $D^{\mathrm{Um}}(\varrho \| \sigma \#_{1} \varrho) = D^{\mathrm{Um}}(\varrho \| \varrho) = 0$. Hence,

$$\begin{split} \lim_{\gamma \nearrow 1} D^{\operatorname{Um}, \#_{\gamma}}(\varrho \| \sigma) &= -\lim_{\gamma \nearrow 1} \frac{D^{\operatorname{Um}}(\varrho \| \sigma \#_{\gamma} \varrho) - D^{\operatorname{Um}}(\varrho \| \sigma \#_{1} \varrho)}{\gamma - 1} \\ &= -\frac{d}{d\gamma} \left. D^{\operatorname{Um}}(\varrho \| \sigma \#_{\gamma} \varrho) \right|_{\gamma = 1} = -\frac{d}{d\gamma} \left(\operatorname{Tr} \varrho \log \varrho - \operatorname{Tr} \varrho \log(\sigma \#_{\gamma} \varrho) \right) \right|_{\gamma = 1} \\ &= -\operatorname{Tr} \varrho \left(D \log \right) [\sigma \#_{1} \varrho] \left(\frac{d}{d\gamma} (\sigma \#_{\gamma} \varrho) \right|_{\gamma = 1} \right) \\ &= -\sum_{i,j} \log^{[1]}(\lambda_{i}, \lambda_{j}) \operatorname{Tr} \varrho P_{i} \underbrace{\sigma^{1/2}(\sigma^{-1/2}\varrho \sigma^{-1/2})}_{=\varrho \sigma^{-1/2}} \log(\sigma^{-1/2}\varrho \sigma^{-1/2}) \sigma^{1/2} P_{j} \\ &= -\sum_{i,j} \log^{[1]}(\lambda_{i}, \lambda_{j}) \operatorname{Tr} \underbrace{P_{j}\varrho P_{i}\varrho}_{=\delta_{i,j}\lambda_{i}^{2}P_{i}} \sigma^{-1/2} \log(\sigma^{-1/2}\varrho \sigma^{-1/2}) \sigma^{1/2} \\ &= -\sum_{i} \underbrace{\log^{[1]}(\lambda_{i}, \lambda_{i})\lambda_{i}^{2}}_{=\lambda_{i}} \operatorname{Tr} P_{i} \sigma^{-1/2} \log(\sigma^{-1/2}\varrho \sigma^{-1/2}) \sigma^{1/2} \\ &= \operatorname{Tr} \sigma^{1/2} \varrho \sigma^{-1/2} \log(\sigma^{-1/2}\varrho \sigma^{-1/2}) \\ &= D^{\max}(\varrho \| \sigma), \end{split}$$

where $P_1, \ldots, P_r \in \mathbb{P}(\mathcal{H})$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ are such that $\sum_{i=1}^r P_i = I$, $\sum_{i=1}^r \lambda_i P_i = \varrho$, and in the fifth equality we used (II.10), while the rest of the steps are straightforward.

Corollary IV.17 Let D^q be a quantum relative entropy that is monotone under CPTP maps and additive on tensor products. For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

$$\lim_{\gamma \nearrow 1} D^{q, \#_{\gamma}}(\varrho \| \sigma) = D^{\max}(\varrho \| \sigma). \tag{IV.135}$$

If, moreover, D^q is continuous on $\mathcal{B}(\mathcal{H})_{>0} \times \mathcal{B}(\mathcal{H})_{>0}$ then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

$$\lim_{\gamma \searrow 0} D^{q, \#_{\gamma}}(\varrho \| \sigma) = D^{q}(\varrho \| \sigma). \tag{IV.136}$$

In particular, $(D^{q,\#_{\gamma}})_{\gamma \in (0,1)}$ continuously interpolates between D^q and D^{\max} when the arguments are restricted to be invertible.

Proof By (IV.123) and the preservation of ordering stated in (xi) above, we have

$$D^{\mathrm{Um},\#_{\gamma}}(\varrho\|\sigma) \leq D^{q,\#_{\gamma}}(\varrho\|\sigma) \leq D^{\mathrm{max},\#_{\gamma}}(\varrho\|\sigma) = D^{\mathrm{max}}(\varrho\|\sigma),$$

where the last equality follows from Proposition IV.14. Taking the limit $\gamma \nearrow 1$ and using (IV.134) yields (IV.135). The limit in (IV.136) is obvious from the assumed continuity and that $\lim_{\gamma\searrow 0} \varrho \#_{\gamma} \sigma = \sigma$ when ρ and σ are invertible.

Remark IV.18 Let $\sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $\psi \in \operatorname{ran} \sigma$ be a unit vector. Then

$$\sigma\#_{\gamma} \left| \psi \right\rangle \! \left\langle \psi \right| = \sigma^{1/2} \left(\sigma^{-1/2} \left| \psi \right\rangle \! \left\langle \psi \right| \sigma^{-1/2} \right)^{\gamma} \sigma^{1/2} = \left| \psi \right\rangle \! \left\langle \psi \right| \left\langle \psi, \sigma^{-1} \psi \right\rangle^{\gamma - 1},$$

whence

$$D^{\mathrm{Um},\#_{\gamma}}(|\psi\rangle\langle\psi|\,\|\sigma) = -\frac{1}{1-\gamma}\operatorname{Tr}|\psi\rangle\langle\psi|\log\left(|\psi\rangle\langle\psi|\,\langle\psi,\sigma^{-1}\psi\rangle^{\gamma-1}\right)$$
$$= \log\left\langle\psi,\sigma^{-1}\psi\right\rangle = \operatorname{Tr}|\psi\rangle\langle\psi|\log\left(|\psi\rangle\langle\psi|\,\sigma^{-1}|\psi\rangle\langle\psi|\right)$$
$$= D^{\mathrm{max}}(|\psi\rangle\langle\psi|\,\|\sigma)$$

for every $\gamma \in (0,1)$, while

$$D^{\mathrm{Um}}(|\psi\rangle\langle\psi|\,\|\sigma) = -\operatorname{Tr}|\psi\rangle\langle\psi|\log\sigma = \langle\psi,(\log\sigma^{-1})\psi\rangle.$$

Thus, we get that

$$D^{\mathrm{Um}}(|\psi\rangle\langle\psi|\,\|\sigma) \le D^{\mathrm{Um},\#_{\gamma}}(|\psi\rangle\langle\psi|\,\|\sigma) = D^{\mathrm{max}}(|\psi\rangle\langle\psi|\,\|\sigma), \qquad \gamma \in (0,1),$$

and the inequality is strict if ψ is not an eigenvector of σ . In particular, this shows that the condition that ϱ and σ are invertible cannot be completely omitted in (IV.133).

Remark IV.19 Obviously, $[0,1] \ni t \mapsto D^{(t)} := (1-t) D^{\mathrm{Um}} + t D^{\mathrm{max}}$ interpolates continuously and monotone increasingly between D^{Um} and D^{max} , and the same is true for $[0,1] \ni \gamma \mapsto D^{\mathrm{Um},\#_{\gamma}}$, according to Proposition IV.16. The two families, however, are different. Indeed, the example in Remark IV.18 shows that if a unit vector $\psi \in \mathrm{ran} \, \sigma$ is not an eigenvector of σ then

$$D^{(t)}(|\psi\rangle\langle\psi|\,\|\sigma) < D^{\max}(|\psi\rangle\langle\psi|\,\|\sigma) = D^{\mathrm{Um},\#_{\gamma}}(|\psi\rangle\langle\psi|\,\|\sigma), \qquad t, \gamma \in (0,1).$$

Since $D^q(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} D^q(\varrho + \varepsilon I\|\sigma + \varepsilon I)$ holds for both $D^q = D^{\operatorname{Um}}$ and $D^q = D^{\operatorname{max}}$ and any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geqslant 0}$, the above argument also shows that for any $t, \gamma \in (0,1)$ there exist invertible ϱ, σ such that $D^{(t)}(\varrho\|\sigma) < D^{\operatorname{Um}, \#_{\gamma}}(\varrho\|\sigma)$.

V. BARYCENTRIC RÉNYI DIVERGENCES

In the rest of the paper (i.e., in the present section and in Section VI) we use the term "quantum relative entropy" in a more restrictive (though still very general) sense than in the previous sections. Namely, a quantum divergence D^q will be called a quantum relative entropy if, on top of being a quantum extension of the classical relative entropy, it is also non-negative, it satisfies the scaling law (III.79), and the following support condition:

$$D^{q}(\varrho \| \sigma) < +\infty \quad \Longleftrightarrow \quad \varrho^{0} \le \sigma^{0}. \tag{V.137}$$

Note that by Remark III.37, any quantum relative entropy in the above sense is also trace-monotone. In particular, no quantum relative entropy can take the value $-\infty$.

Example V.1 It is easy to verify that that D^{Um} , D^{meas} and D^{max} are all quantum relative entropies in the above more restrictive sense.

A. Definitions

Definition V.2 Let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ be a gcq channel, let $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$ and

$$S_+ := \bigwedge_{x:\, P(x)>0} W^0_x, \qquad S_- := \bigwedge_{x:\, P(x)<0} W^0_x,$$

and for every $x \in \mathcal{X}$, let D^{q_x} be a quantum relative entropy. We define

$$Q_P^{b,q}(W) := \sup_{\tau \in \mathcal{B}(S_+ \mathcal{H})_{\geq 0}} \left\{ \text{Tr} \, \tau - \sum_{x \in \mathcal{X}} P(x) \, D^{q_x}(\tau \| W_x) \right\}, \tag{V.138}$$

$$\psi_P^{\mathbf{b},\mathbf{q}}(W) := \log Q_P^{\mathbf{b},\mathbf{q}}(W), \tag{V.139}$$

$$R_{D^{\mathbf{q}}, \text{left}}(W, P) := \inf_{\omega \in \mathcal{S}(S_{+}\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\omega \| W_x). \tag{V.140}$$

Here, $\mathbf{q} := (q_x)_{x \in \mathcal{X}}$, $D^{\mathbf{q}} := (D^{q_x})_{x \in \mathcal{X}}$, and $R_{D^{\mathbf{q}}, \mathrm{left}}(W, P)$ is the P-weighted left $D^{\mathbf{q}}$ -radius of W. We call any ω attaining the infimum in (V.140) a P-weighted left $D^{\mathbf{q}}$ -center for W.

When $P \notin \{\mathbf{1}_{\{x\}} : x \in \mathcal{X}\}$, we also define the P-weighted barycentric Rényi-divergence of W corresponding to $D^{\mathbf{q}}$ as

$$D_P^{\mathbf{b},\mathbf{q}}(W) := \frac{1}{\prod_{x \in \mathcal{X}} (1 - P(x))} \left(-\log Q_P^{\mathbf{b},\mathbf{q}} \left(\left(\frac{W_x}{\operatorname{Tr} W_x} \right)_{x \in \mathcal{X}} \right) \right).$$

Remark V.3 Since we almost exclusively consider only left divergence radii and left divergence centers in this paper, we will normally omit "left" from the terminology.

Remark V.4 Note that by definition,

$$P(x) \ge 0, x \in \mathcal{X} \implies S_{-} = I, \qquad P(x) \le 0, x \in \mathcal{X} \implies S_{+} = I.$$

Definition V.5 Let $D^{\mathbf{q}} = (D^{q_0}, D^{q_1})$ be quantum relative entropies. For any two non-zero PSD operators $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$, and any $\alpha \in [0, +\infty)$, let

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) := \sup_{\tau \in \mathcal{B}(\varrho^{0}\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \alpha D^{q_{0}}(\tau\|\varrho) - (1-\alpha) D^{q_{1}}(\tau\|\sigma) \right\}, \tag{V.141}$$

$$\psi_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) := \log Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma), \tag{V.142}$$

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) := \frac{\psi_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) - \psi_{1}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma)}{\alpha - 1},\tag{V.143}$$

where we define the last quantity only for $\alpha \in [0,1) \cup (1,+\infty)$. $D_{\alpha}^{b,\mathbf{q}}(\varrho \| \sigma)$ is called the barycentric Rényi α -divergence of ϱ and σ corresponding to $D^{\mathbf{q}}$.

Remark V.6 When $D^{q_0} = D^{q_1} = D^q$, we will use the simpler notation $D^{b,q}_{\alpha}$ instead of $D^{b,(q_0,q_1)}_{\alpha}$.

Remark V.7 Note that with the choice $\mathcal{X} = \{0,1\}$, $W_0 = \varrho$, $W_1 = \sigma$, and $P(0) = \alpha$, (V.141) and (V.142) are special cases of (V.138) and (V.139), respectively, when $\alpha \in (0,+\infty)$, and we will show in Lemma V.10 that also (V.143) is a special case of (V.140) in this case. When $\alpha = 0$, the restriction $\tau^0 \leq S_+$ in (V.138) would give $\tau^0 \leq \sigma^0$, while we use $\tau^0 \leq \varrho^0$ in (V.141). The reason for this is to guarantee the continuity of $D_{\alpha}^{\text{b,q}}$ at 0; see Proposition V.35.

Remark V.8 It is easy to see that when P is a probability measure, the supremum in (V.138) and the infimum in (V.140) can be equivalently taken over $\mathcal{B}(\mathcal{H})_{>0}$ and $\mathcal{S}(\mathcal{H})$, respectively, i.e.,

$$Q_P^{\mathbf{b},\mathbf{q}}(W) = \sup_{\tau \in \mathcal{B}(\mathcal{H})_{>0}} \left\{ \operatorname{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau \| W_x) \right\}, \tag{V.144}$$

$$R_{D^{\mathbf{q}}, \text{left}}(W, P) = \inf_{\omega \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x), \tag{V.145}$$

and in the 2-variable case we have

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \sup_{\tau \in \mathcal{B}(\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \alpha \, D^{q_0}(\tau\|\varrho) - (1-\alpha) \, D^{q_1}(\tau\|\sigma) \right\} \tag{V.146}$$

$$= \sup_{\tau \in \mathcal{B}((\varrho^0 \wedge \sigma^0)\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \alpha \, D^{q_0}(\tau \| \varrho) - (1 - \alpha) \, D^{q_1}(\tau \| \sigma) \right\}, \qquad \alpha \in (0, 1). \tag{V.147}$$

In the general case, the restriction $\tau^0 \leq S_+$ is introduced to avoid the appearance of infinities of opposite signs in $\sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x)$. In the 2-variable case (V.141), the restriction $\tau^0 \leq \varrho^0$ also serves to guarantee that $Q_{\alpha}^{\mathbf{b},\mathbf{q}}$ is a quantum extension of $Q_{\alpha}^{\mathbf{c}l}$ for $\alpha > 1$, which would not be true, for instance, if it was replaced by $\tau^0 \leq \varrho^0 \wedge \sigma^0$; see, e.g., Corollary V.40.

Remark V.9 Note that (V.141) can be seen as a 2-variable extension of the variational formula (III.85). In particular, we have

$$Q_1^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \max_{\tau \in \mathcal{B}(\varrho^0 \mathcal{H})} \{ \operatorname{Tr} \tau - D^{q_0}(\tau\|\varrho) \} = \operatorname{Tr} \varrho, \tag{V.148}$$

where the first equality is by definition (V.141), and the second equality is due to (III.85). Thus,

$$\psi_1^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \log \operatorname{Tr} \varrho, \tag{V.149}$$

and for every $\alpha \in [0,1) \cup (1,+\infty)$,

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \frac{1}{\alpha - 1}\log Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) - \frac{1}{\alpha - 1}\log \operatorname{Tr}\varrho. \tag{V.150}$$

By Remark III.38, the maximum in (V.148) is attained at τ if and only if $\operatorname{Tr} \tau = \operatorname{Tr} \varrho$ and $D^{q_0}(\tau \| \varrho) = 0$; in particular, $\tau = \varrho$ is the unique maximizer in (V.148) when D^{q_0} is strictly positive.

At $\alpha = 0$, (V.141) and (III.85) give

$$\sigma^0 \le \varrho^0 \quad \Longrightarrow \quad Q_0^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma) = \operatorname{Tr} \sigma \quad \Longrightarrow \quad D_0^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma) = \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma. \tag{V.151}$$

In general the above equalities do not hold; see Proposition V.41.

Lemma V.10 (i) In the setting of Definition V.2,

$$-\log Q_P^{\mathbf{b},\mathbf{q}}(W) = R_{D^{\mathbf{q}},\text{left}}(W,P) \tag{V.152}$$

Moreover, if $S_{+} \leq S_{-}$ then a $\tau \in \mathcal{B}(S_{+}\mathcal{H})_{\geq 0}$ is optimal in (V.138) if and only if

$$Q_P^{\mathrm{b},\mathbf{q}}(W) = \operatorname{Tr} \tau \qquad and \qquad \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x) = 0,$$
 (V.153)

and if $\tau \neq 0$ is optimal in (V.138) then $\omega := \tau/\operatorname{Tr} \tau$ is optimal in (V.152). Conversely, for any ω that is optimal in (V.152) $\tau := e^{-R_{D^{\mathbf{q}}, \operatorname{left}}(W, P)}\omega$ is optimal in (V.138).

(ii) In the setting of Definition V.5,

$$-\log Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \inf_{\omega \in \mathcal{S}(\rho^0\mathcal{H})} \{\alpha D^{q_0}(\omega\|\varrho) + (1-\alpha) D^{q_1}(\omega\|\sigma)\}, \quad \alpha \in [0,+\infty).$$
 (V.154)

Assume for the rest that $\alpha \in [0,1]$ or $\varrho^0 \leq \sigma^0$. Then τ is optimal in (V.141) if and only if

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \operatorname{Tr}\tau \qquad and \qquad \alpha D^{q_0}(\tau\|\varrho) + (1-\alpha)D^{q_1}(\tau\|\sigma) = 0, \tag{V.155}$$

and if $\tau \neq 0$ is optimal in (V.141) then $\omega := \tau / \operatorname{Tr} \tau$ is optimal in (V.154). Conversely, for any ω that is optimal in (V.154), $\tau := e^{\psi_{\alpha}^{b,\mathbf{q}}(\varrho \parallel \sigma)} \omega$ optimal in (V.141).

Proof (i) Assume first that $S_+ = 0$. Then the only admissible $\tau \in \mathcal{B}(\mathcal{S}_+\mathcal{H})_{\geq 0}$ in (V.138) is $\tau = 0$, whence $Q_P^{\mathrm{b},\mathbf{q}}(W) = 0$, according to (III.56), and thus $\psi_P^{\mathrm{b},\mathbf{q}}(W) = -\infty$. On the other hand, the infimum in (V.140) is taken over the empty set, and hence it is equal to $+\infty$. Thus, (V.152) and (V.153) hold.

Assume next that $S_+ \neq 0$. If there exists an $x \in \mathcal{X}$ such that P(x) < 0 and $S_+ \nleq W_x^0$ then taking $\tau := \omega := S_+ / \operatorname{Tr} S_+$ yields

$$Q_{P}^{b,q}(W) \ge \operatorname{Tr} \tau - \underbrace{\sum_{x:P(x)>0} P(x) D^{q_x}(\tau \| W_x)}_{\in \mathbb{R}} - \underbrace{\sum_{x:P(x)<0} P(x) D^{q_x}(\tau \| W_x)}_{=-\infty} = +\infty,$$

and

$$R_{D^{\mathbf{q}}, \operatorname{left}}(W, P) \leq \underbrace{\sum_{x: P(x) > 0} P(x) D^{q_x}(\omega \| W_x)}_{\in \mathbb{R}} + \underbrace{\sum_{x: P(x) < 0} P(x) D^{q_x}(\omega \| W_x)}_{= -\infty} = -\infty,$$

(where we used that D^{q_x} does not take the value $-\infty$), whence (V.152) holds.

Finally, if $0 \neq S_+ \leq S_-$ then the proof follows easily from representing a positive semi-definite operator $\tau \in \mathcal{B}(S_+\mathcal{H})_{\geq 0}$ as a pair $(\omega, t) \in \mathcal{S}(S_+\mathcal{H}) \times [0, +\infty)$. Indeed, we have

$$Q_{P}^{\mathbf{b},\mathbf{q}}(W) = \sup_{\omega \in \mathcal{S}(S_{+}\mathcal{H})} \sup_{t \in [0,+\infty)} \left\{ \operatorname{Tr} t\omega - \sum_{x \in \mathcal{X}} P(x) D^{q_{x}}(t\omega \| W_{x}) \right\}$$

$$= \sup_{\omega \in \mathcal{S}(S_{+}\mathcal{H})} \sup_{t \in [0,+\infty)} \left\{ t - t \log t - t \sum_{x \in \mathcal{X}} P(x) D^{q_{x}}(\omega \| W_{x}) \right\}, \tag{V.156}$$

where the first equality is by definition, and the second equality follows from the scaling property (III.80). Note that $c(\omega) \neq \pm \infty$ by assumption, and the inner supremum in (V.156) is equal to $e^{-c(\omega)}$, attained at $t = e^{-c(\omega)}$, according to Lemma II.1. From these, all the remaining assertions in (i) follow immediately.

The assertions in (ii) are special cases of the corresponding ones in (i) when $\alpha \in (0, +\infty)$ (also taking into account (V.147) when $\alpha \in (0, 1)$). The case $\alpha = 0$ can be verified analogously to the above; we omit the easy details.

Remark V.11 Clearly, when $\alpha > 1$ and $\varrho^0 \nleq \sigma^0$ then the set of optimal τ operators in (V.141) is exactly $\mathcal{B}(\varrho^0 \mathcal{H})_{>0}$, and the set of optimal ω states in (V.154) is exactly $\mathcal{S}(\varrho^0 \mathcal{H})$.

Corollary V.12 Assume that $S_{+} \leq S_{-}$. Then

$$Q_P^{\mathbf{b},\mathbf{q}}(W) = \max \left\{ \operatorname{Tr} \tau : \tau \in \mathcal{B}(S_+\mathcal{H})_{\geq 0}, \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x) = 0 \right\}.$$

Likewise, if $\alpha \in [0,1]$ or $\varrho^0 \leq \sigma^0$, then

$$Q_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) = \max\left\{\mathrm{Tr}\,\tau:\, \tau\in\mathcal{B}(\varrho^0\mathcal{H})_{\geq 0},\, \alpha\,D^{q_0}(\tau\|\varrho) + (1-\alpha)\,D^{q_1}(\tau\|\sigma) = 0\right\}.$$

Proof Immediate from the characterizations of the optimal τ in (V.153) and (V.155).

Remark V.13 Note that in the case $S_+ \leq S_-$, the condition $\sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x) = 0$ is necessary for the optimality of τ , but not sufficient. Indeed, it is easy to see from the scaling property (III.80) that

$$\left\{ \tau \in \mathcal{B}(S_{+}\mathcal{H}) : \sum_{x \in \mathcal{X}} P(x) D^{q_{x}}(\tau || W_{x}) = 0 \right\}$$

$$= \left\{ \exp \left(-\sum_{x \in \mathcal{X}} P(x) D^{q_{x}}(\tau || W_{x}) \right) \omega : \omega \in \mathcal{S}(S_{+}\mathcal{H}) \right\} \cup \{0\}.$$

On the other hand, each $\tau \in \mathcal{B}(S_+\mathcal{H}) \setminus \{0\}$ with $\sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x) = 0$ has the extremality property

$$\operatorname{Tr}(\lambda \tau) - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\lambda \tau || W_x) = (\lambda - \lambda \log \lambda) \operatorname{Tr} \tau$$

$$< \operatorname{Tr} \tau = \operatorname{Tr}(\tau) - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || W_x)$$

for every $\lambda \in (0,1) \cup (1,+\infty)$, where the first equality is again due to the scaling property (III.80).

Remark V.14 Under the conditions given in Lemma V.10, for the supremum in (V.138) to be a maximum, it is sufficient if the infimum in (V.140) is a minimum. For the latter, a natural sufficient condition is that each D^{q_x} with $x \in \text{supp } P$ is lower semi-continuous in its first argument (when P is a probability measure), or continuous in its first argument with its support dominated by the support of a fixed second argument (when P can take negative values), since the domain of optimization, namely, $S(\varrho^0 \mathcal{H})$, is a compact set.

Examples of quantum relative entropies that are lower semi-continuous in their first argument (in fact, in both of their arguments), include D^{meas} , D^{Um} , and their γ -weighted versions, as well D^{max} , and obviously, all possible convex combinations of these. D^{Um} and D^{max} are also clearly continuous in their first argument when its support is dominated by the support of a fixed second argument.

Remark V.15 Using (V.149) and the scaling law (III.81), (V.143) can be rewritten as

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \frac{1}{\alpha - 1} \sup_{\tau \in \mathcal{B}(\varrho^0 \mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \alpha \, D^{q_0} \left(\tau \left\| \frac{\varrho}{\operatorname{Tr} \varrho} \right) - (1 - \alpha) \, D^{q_1} \left(\tau \left\| \frac{\sigma}{\operatorname{Tr} \sigma} \right) \right\} + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma. \right\} \right\}$$

On the other hand, using Lemma V.10 we get that for every $\alpha \in [0,1) \cup (1,+\infty)$,

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \frac{1}{\alpha - 1}\log Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) - \frac{1}{\alpha - 1}\log \operatorname{Tr}\varrho \tag{V.157}$$

$$= \frac{1}{1-\alpha} \inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \{ \alpha D^{q_0}(\omega \| \varrho) + (1-\alpha) D^{q_1}(\omega \| \sigma) \} - \frac{1}{\alpha-1} \log \operatorname{Tr} \varrho$$
 (V.158)

$$= \frac{1}{1 - \alpha} \inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \left\{ \alpha D^{q_0} \left(\omega \left\| \frac{\varrho}{\operatorname{Tr} \varrho} \right) + (1 - \alpha) D^{q_1} \left(\omega \| \sigma \right) \right\} + \log \operatorname{Tr} \varrho \right\}$$
 (V.159)

$$= \frac{1}{1 - \alpha} \inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \left\{ \alpha D^{q_0} \left(\omega \left\| \frac{\varrho}{\operatorname{Tr} \varrho} \right) + (1 - \alpha) D^{q_1} \left(\omega \left\| \frac{\sigma}{\operatorname{Tr} \sigma} \right) \right\} + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma, \quad (V.160)$$

where the first equality is by (V.150), the second equality follows from (V.154), and the third and the fourth equalities from the scaling laws (III.80)–(III.81). Moreover, for $\alpha \in (0,1)$, the infimum can be taken over $S(\mathcal{H})$, i.e.,

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \frac{1}{1-\alpha} \inf_{\omega \in \mathcal{S}(\mathcal{H})} \{\alpha \, D^{q_0}(\omega\|\varrho) + (1-\alpha) \, D^{q_1}(\omega\|\sigma)\} - \frac{1}{\alpha-1} \log \operatorname{Tr} \varrho \tag{V.161}$$

$$= \frac{1}{1 - \alpha} \inf_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ \alpha D^{q_0} \left(\omega \left\| \frac{\varrho}{\text{Tr } \varrho} \right) + (1 - \alpha) D^{q_1} \left(\omega \left\| \frac{\sigma}{\text{Tr } \sigma} \right) \right\} + \log \text{Tr } \varrho - \log \text{Tr } \sigma, \quad (V.162)$$

because if $\omega^0 \nleq \varrho^0$ then $D^{q_0}(\omega \| \varrho) = D^{q_0}\left(\omega \| \frac{\varrho}{\operatorname{Tr} \varrho}\right) = +\infty$. The situation is different for $\alpha = 0$; see, e.g., (V.183).

The above formulas explain the term "barycentric Rényi divergence".

Definition V.16 For $\alpha \in (0,1)$, any ω attaining the infimum in (V.158) will be called an α -weighted (left) $D^{\mathbf{q}}$ -center for (ϱ, σ) .

B. Barycentric Rényi divergences are quantum Rényi divergences

In this section we show that the barycentric Rényi α -divergences are quantum Rényi divergences for every $\alpha \in (0,1)$, provided that the defining quantum relative entropies are monotone under pinchings. This latter condition does not pose a serious restriction; indeed, all the concrete quantum relative entropies that we consider in this paper (e.g., measured, Umegaki, maximal, and the γ -weighted versions of these) are monotone under PTP maps, and hence also under pinchings.

Isometric invariance holds even without this mild restriction, and also for $\alpha > 1$:

Lemma V.17 All the quantities in (V.138)–(V.143) are invariant under isometries, and hence they are all quantum divergences.

Proof We prove the statement only for $Q_P^{\mathrm{b},\mathbf{q}}$, as for the other quantities it either follows from that, or the proof goes the same way. Let $W \in \mathcal{B}(\mathcal{X},\mathcal{H})_{\geq 0}$ be a gcq channel, $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$, and $V : \mathcal{H} \to \mathcal{K}$ be an isometry. Obviosuly, $\tilde{S}_+ := \bigwedge_{x:P(x)>0} (VW_xV^*)^0 = V(\bigwedge_{x:P(x)>0} W_x^0)V^* = VS_+V^*$, and for any $\tau \in \mathcal{B}(\tilde{S}_+\mathcal{K})_{\geq 0}$ there exists a unique $\hat{\tau} \in \mathcal{B}(S_+\mathcal{H})_{\geq 0}$ such that $\tau = V\hat{\tau}V^*$. Thus,

$$Q_P^{\mathbf{b},\mathbf{q}}(VWV^*) = \sup_{\tau \in \mathcal{B}(\tilde{S}_+\mathcal{K})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau || VW_x V^*) \right\}$$
(V.163)

$$= \sup_{\hat{\tau} \in \mathcal{B}(S_{+}\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} V \hat{\tau} V^{*} - \sum_{x \in \mathcal{X}} P(x) D^{q_{x}} (V \hat{\tau} V^{*} || V W_{x} V^{*}) \right\}$$
(V.164)

$$= \sup_{\hat{\tau} \in \mathcal{B}(S_{+}\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \hat{\tau} - \sum_{x \in \mathcal{X}} P(x) D^{q_x} (\hat{\tau} || W_x) \right\}$$
 (V.165)

$$=Q_P^{\mathbf{b},\mathbf{q}}(W),\tag{V.166}$$

where the third equality follows by the isometric invariance of the relative entropies.

Recall that D^q is said to be monotone under pinchings if

$$D^{q}\left(\sum_{i=1}^{r} P_{i} \varrho P_{i} \middle\| \sum_{i=1}^{r} P_{i} \sigma P_{i}\right) \leq D^{q}(\varrho \lVert \sigma)$$

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $P_1, \ldots, P_r \in \mathbb{P}(\mathcal{H})$ such that $\sum_{i=1}^r P_i = I$.

Lemma V.18 Let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geqslant 0}$ be a gcq channel that is classical on the support of some $P \in \mathcal{P}_f(\mathcal{X})$, i.e., there exists an ONB $(e_i)_{i=0}^{d-1}$ in \mathcal{H} such that $W_x = \sum_{i=0}^{d-1} \widetilde{W}_x(i) |e_i\rangle\langle e_i|$, where $\widetilde{W}_x(i) := \langle e_i, W_x e_i\rangle$, $i \in [d]$, $x \in \operatorname{supp} P$. If all D^{q_x} , $x \in \operatorname{supp} P$, are monotone under pinchings then

$$Q_P^{\mathbf{b},\mathbf{q}}(W) = \sum_{i \in \widetilde{S}} \prod_{x \in \text{supp } P} \widetilde{W}_x(i)^{P(x)}, \tag{V.167}$$

where $\widetilde{S} := \bigcap_{x \in \operatorname{supp} P} \sup \widetilde{W}_x$ and $\sup \widetilde{W}_x = \{i \in [d] : \widetilde{W}_x(i) > 0\}$; moreover, there exists a unique optimal τ in (V.138), given by

$$\tau_P^{\mathbf{q}}(W) := \tau_P(\widetilde{W}) := \sum_{i \in S} |e_i\rangle \langle e_i| \prod_{x \in \text{supp } P} \widetilde{W}_x(i)^{P(x)} . \tag{V.168}$$

Proof If $S_+ = 0$ then $Q_P^{b,q}(W) = 0$, and the RHS of (V.167) is an empty sum, whence the equality in (V.167) holds trivially.

Thus, for the rest we assume that $S_+ \neq 0$. Let $\mathcal{E}(\cdot) := \sum_{i=0}^{d-1} |e_i\rangle\langle e_i| (\cdot) |e_i\rangle\langle e_i|$ be the pinching corresponding to the joint eigenbasis of the W_x , $x \in \operatorname{supp} P$, guaranteed by the classicality assumption. For any $\tau \in \mathcal{B}(S_+\mathcal{H})_{\geq 0}$,

$$\operatorname{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) \underbrace{D^{q_x}(\tau \| W_x)}_{\geq D^{q_x}(\mathcal{E}(\tau) \| \mathcal{E}(W_x))} \leq \underbrace{\operatorname{Tr} \tau}_{=\operatorname{Tr} \mathcal{E}(\tau)} - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\mathcal{E}(\tau) \| \underbrace{\mathcal{E}(W_x)}_{=W_x})$$

$$= \operatorname{Tr} \mathcal{E}(\tau) - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\mathcal{E}(\tau) \| W_x),$$

where the inequality follows from the monotonicity of the D^{q_x} under pinchings. Thus, the supremum in (V.138) can be restricted to τ operators that can be written as $\tau = \sum_{i=1}^d \tilde{\tau}(i) |e_i\rangle\langle e_i|$ with some $\tilde{\tau}(i) \in [0,+\infty), i \in [d]$. Clearly, $\tau^0 \leq S^0_+$ is equivalent to supp $\tilde{\tau} \subseteq \tilde{S}$. For any such τ ,

$$\operatorname{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau \| W_x) = \operatorname{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) \sum_{i \in \widetilde{S}} [\widetilde{\tau}(i) \log \widetilde{\tau}(i) - \widetilde{\tau}(i) \log W_x(i)]$$

$$= \sum_{i \in \widetilde{S}} \left[\widetilde{\tau}(i) - \widetilde{\tau}(i) \log \widetilde{\tau}(i) + \widetilde{\tau}(i) \sum_{x \in \operatorname{supp} P} P(x) \log \widetilde{W}_x(i) \right].$$

The supremum of this over all such τ is

$$\sum_{i \in \widetilde{S}} e^{\sum_{x \in \operatorname{supp} P} P(x) \log \widetilde{W}_x(i)} = \sum_{i \in \widetilde{S}} \prod_{x \in \operatorname{supp} P} \widetilde{W}_x(i)^{P(x)},$$

which is uniquely attained at the $\tau = \tau_P^{\mathbf{q}}(W)$ given in (V.168), according to Lemma II.1. This proves (V.167).

Corollary V.19 In the setting of Lemma V.18, the P-weighted left $D^{\mathbf{q}}$ -radius of W can be given explicitly as

$$R_{D^{\mathbf{q}}, \text{left}}(W, P) = -\log \sum_{i \in \widetilde{S}} \prod_{x \in \text{supp } P} \widetilde{W}_x(i)^{P(x)},$$

and if $\tilde{S} \neq \emptyset$ then there is a unique P-weighted left $D^{\mathbf{q}}$ -center for W, given by

$$\omega_P^{\mathbf{q}}(W) := \frac{\tau^{\mathbf{q}}(W, P)}{\operatorname{Tr} \tau^{\mathbf{q}}(W, P)} = \sum_{i \in \widetilde{S}} |e_i\rangle\langle e_i| \frac{\prod_{x \in \operatorname{supp} P} \widetilde{W}_x(i)^{P(x)}}{\sum_{j \in \widetilde{S}} \prod_{x \in \operatorname{supp} P} \widetilde{W}_x(j)^{P(x)}} =: \omega_P(\widetilde{W}). \tag{V.169}$$

Proof Immediate from Lemmas V.10 and V.18.

Lemma V.18 yields immediately the following:

Corollary V.20 Assume that $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ commute, and hence can be written as $\varrho = \sum_{i=1}^{d} \tilde{\varrho}(i) |e_i\rangle\langle e_i|$, $\sigma = \sum_{i=1}^{d} \tilde{\sigma}(i) |e_i\rangle\langle e_i|$, in some ONB $(e_i)_{i=1}^d$. If D^{q_0} and D^{q_1} are monotone under pinchings then

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = Q_{\alpha}(\tilde{\varrho}\|\tilde{\sigma}) = \sum_{i=1}^{d} \tilde{\varrho}(i)^{\alpha}\tilde{\sigma}(i)^{1-\alpha}, \qquad \alpha \in (0,1),$$

and there exists a unique optimal τ in (V.141), given by

$$\tau_{\alpha}^{\mathbf{q}}(\varrho\|\sigma) := \tau_{\alpha}(\tilde{\varrho}\|\tilde{\sigma}) := \sum_{i=1}^{d} |e_{i}\rangle\langle e_{i}|\,\tilde{\varrho}(i)^{\alpha}\tilde{\sigma}(i)^{1-\alpha}. \tag{V.170}$$

As a special case of Corollary V.19, we get the following:

Corollary V.21 In the setting of Corollary V.20, if $\varrho^0 \wedge \sigma^0 \neq 0$ then for every $\alpha \in (0,1)$ there exists a unique α -weighted $D^{\mathbf{q}}$ -center for (ϱ, σ) , given by

$$\omega_{\alpha}^{\mathbf{q}}(\varrho\|\sigma) := \frac{\tau_{\alpha}^{\mathbf{q}}(\varrho\|\sigma)}{\operatorname{Tr}\,\tau_{\alpha}^{\mathbf{q}}(\varrho\|\sigma)} = \sum_{i=1}^{d} |e_{i}\rangle\langle e_{i}| \frac{\tilde{\varrho}(i)^{\alpha}\tilde{\sigma}(i)^{1-\alpha}}{\sum_{j=1}^{d} \tilde{\varrho}(j)^{\alpha}\tilde{\sigma}(j)^{1-\alpha}} =: \omega_{\alpha}(\tilde{\varrho}\|\tilde{\sigma}). \tag{V.171}$$

Proof Immediate from Corollary V.20 and Lemma V.10.

Remark V.22 Note that $\tau_P^{\mathbf{q}}(W)$ in (V.168) and $\omega_P^{\mathbf{q}}(W)$ in (V.169) are independent of $D^{\mathbf{q}}$, as long as all D^{q_x} , $x \in \text{supp } P$, are monotone under pinchings. Likewise, $\tau_\alpha^{\mathbf{q}}(\varrho \| \sigma)$ in (V.170) and $\omega_\alpha^{\mathbf{q}}(\varrho \| \sigma)$ in (V.171) are independent of D^{q_0} and D^{q_1} , as long as both of them are monotone under pinchings.

Lemma V.17 and Corollary V.20 together give the following:

Proposition V.23 If D^{q_x} , $x \in \text{supp } P$, are quantum relative entropies that are monotone under pinchings then $Q_P^{b,\mathbf{q}}$ is a quantum extension (in the sense of Definition III.24) of the classical Q_P given in Definition III.19.

Likewise, if D^{q_0} and D^{q_1} are two quantum relative entropies that are monotone under pinchings then for every $\alpha \in (0,1)$ the corresponding barycentric Rényi α -divergence $D^{\mathrm{b},\mathbf{q}}_{\alpha}$ is a quantum Rényi α -divergence in the sense of Definition III.24.

Remark V.24 Note that in the classical case the barycentric Rényi α -divergence is equal to the unique classical Rényi α -divergence also for $\alpha > 1$; see (III.49). On the other hand, if $D^{q_0} \neq D^{q_1}$ then it may happen that $D_{\alpha}^{b,\mathbf{q}}$ is not a quantum Rényi α -divergence for some $\alpha > 1$; see Remark V.36.

Note that for a fixed $i \in \bigcap_{x \in \text{supp } P} \text{supp } \widetilde{W}_x$, the expression $\prod_{x \in \text{supp } P} \widetilde{W}_x(i)^{P(x)}$ in (V.167) is the weighted geometric mean of $(\widetilde{W}_x(i))_{x \in \text{supp } P}$ with weights $(P(x))_{x \in \text{supp } P}$. This motivates the following:

Definition V.25 If $D^{\mathbf{q}}$, $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$, and $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$ are such that there exists a unique optimizer $\tau =: \tau_P^{\mathbf{q}}(W)$ in (V.138) then this τ is called the P-weighted $D^{\mathbf{q}}$ -geometric mean of W, and is also denoted by $G_P^{\mathbf{pq}}(W) := \tau_P^{\mathbf{q}}(W)$.

Similarly, if there exists a unique optimizer $\tau =: \tau_{\alpha}^{\mathbf{q}}(\varrho \| \sigma)$ in (V.141) then it is called the α -weighted $D^{\mathbf{q}}$ -geometric mean of ϱ and σ , and it is also denoted by $G_{\alpha}^{D^{\mathbf{q}}}(\varrho \| \sigma) := \tau_{\alpha}^{\mathbf{q}}(\varrho \| \sigma)$.

Remark V.26 Note that if $G_P^{\mathbf{Dq}}(W)$ exists then by definition,

$$Q_P^{\mathrm{b},\mathbf{q}} = \operatorname{Tr} G_P^{D^{\mathbf{q}}}(W) = Q_P^{G_P^{D^{\mathbf{q}}}}(W)$$

(in particular, if $G_{\alpha}^{D^{\mathbf{q}}}(\varrho\|\sigma)$ exists then $Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \operatorname{Tr} G_{\alpha}^{D^{\mathbf{q}}}(\varrho\|\sigma)$), which can be seen as a special case of (III.91).

In classical statistics, the family of states $(\omega_{\alpha}(\tilde{\varrho}\|\tilde{\sigma}))_{\alpha\in(0,1)}$ given in (V.171) is called the *Hellinger arc*. (Note that if $\tilde{\varrho}$ and $\tilde{\sigma}$ are probability distributions with equal supports then the Hellinger arc connects them in the sense that $\lim_{\alpha\searrow 0} \omega_{\alpha}(\tilde{\varrho}\|\tilde{\sigma}) = \tilde{\varrho}$, $\lim_{\alpha\nearrow 1} \omega_{\alpha}(\tilde{\varrho}\|\tilde{\sigma}) = \tilde{\sigma}$.) This motivates the following:

Definition V.27 Assume that $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and D^{q_0} , D^{q_1} are such that for every $\alpha \in (0,1)$ there exists a unique α -weighted $D^{\mathbf{q}}$ -center $\omega_{\alpha}^{\mathbf{q}}(\varrho \| \sigma)$ for (ϱ, σ) . Then $(\omega_{\alpha}^{\mathbf{q}}(\varrho \| \sigma))_{\alpha \in (0,1)}$ is called the $D^{\mathbf{q}}$ -Hellinger arc for ϱ and σ .

More generally, if W and $D^{\mathbf{q}}$ are such that for every $P \in \mathcal{P}_f(\mathcal{X})$ there exists a unique P-weighted $D^{\mathbf{q}}$ -center $\omega_P^{\mathbf{q}}(W)$ for W then we call $(\omega_P^{\mathbf{q}}(W))_{P \in \mathcal{P}_f(\mathcal{X})}$ the $D^{\mathbf{q}}$ -Hellinger body for W.

Remark V.28 Note that by Lemma V.10, for given $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$, $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$, and $D^{\mathbf{q}}$, there exists a unique non-zero P-weighted $D^{\mathbf{q}}$ -geometric mean $G_P^{D^{\mathbf{q}}}(W) = \tau_P^{\mathbf{q}}(W)$ if and only if there exists a unique P-weighted $D^{\mathbf{q}}$ -center $\omega_P^{\mathbf{q}}(W)$ for W, and in this case we have

$$\begin{split} \omega_P^{\mathbf{q}}(W) &= \frac{G_P^{D^{\mathbf{q}}}(W)}{\operatorname{Tr} G_P^{D^{\mathbf{q}}}(W)}, \\ Q_P^{\mathbf{q}}(W) &= \operatorname{Tr} G_P^{D^{\mathbf{q}}}(W), \\ 0 &= \sum_{x \in \mathcal{X}} P(x) D^{q_x} (G_P^{D^{\mathbf{q}}}(W) \| W_x), \\ -\log Q_P^{\mathbf{q}}(W) &= \sum_{x \in \mathcal{X}} P(x) D^{q_x} (\omega_P^{\mathbf{q}}(W) \| W_x). \end{split}$$

The following is a multi-variate generalization of [64, Theorem 3.6]:

Proposition V.29 Let $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$, $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$, and for every $x \in \operatorname{supp} P$, let $D^{q_x} = D^{\operatorname{Um}}$. Assume that $0 \neq S_+ \leq S_-$. Then there exist a unique P-weighted D^{Um} -geometric mean $G_P^{\operatorname{D^{\operatorname{Um}}}}(W) := \tau_P^{\operatorname{Um}}(W)$ of W and a unique P-weighted D^{Um} -center $\omega_P^{\operatorname{Um}}$ for W, given by

$$G_P^{D^{\text{Um}}}(W) = \tau_P^{\text{Um}}(W) = S_+ e^{\sum_{x \in \text{supp } P} P(x)S_+(\widehat{\log}W_x)S_+},$$
 (V.172)

$$\omega_P^{\mathrm{Um}}(W) = \frac{G_P^{\mathrm{Um}}(W)}{\operatorname{Tr} G_P^{\mathrm{DUm}}(W)} = \frac{S_+ e^{\sum_{x \in \operatorname{supp} P} P(x) S_+(\widehat{\log} W_x) S_+}}{\operatorname{Tr} S_+ e^{\sum_{x \in \operatorname{supp} P} P(x) S_+(\widehat{\log} W_x) S_+}}, \qquad (V.173)$$

respectively, and

$$-\log Q_P(W) = -\log \operatorname{Tr} G_P^{D^{\operatorname{Um}}}(W) = \sum_{x \in \mathcal{X}} D^{\operatorname{Um}}(\omega_P^{\operatorname{Um}}(W) \| W_x).$$

Proof Note that for $\sigma := S_+ e^{\sum_{x \in \text{supp } P} P(x) S_+(\log W_x) S_+}$ and any $\tau \in \mathcal{B}(S_+ \mathcal{H})_{\geq 0}$, we have

$$D^{\mathrm{Um}}(\tau \| \sigma) = \operatorname{Tr} \tau \log \tau - \operatorname{Tr} \tau \log \left(S_{+} e^{\sum_{x \in \operatorname{supp} P} P(x) S_{+}(\widehat{\log} W_{x}) S_{+}} \right)$$

$$= \operatorname{Tr} \tau \log \tau - \operatorname{Tr} \tau \sum_{x \in \operatorname{supp} P} P(x) \log W_{x}$$

$$= \sum_{x \in \operatorname{supp} P} P(x) \underbrace{\left(\operatorname{Tr} \tau \log \tau - \operatorname{Tr} \tau \widehat{\log} W_{x} \right)}_{=D^{\mathrm{Um}}(\tau \| W_{x})}$$

$$(V.174)$$

Thus,

$$Q_P^{\mathbf{b},\mathbf{q}}(W) = \sup_{\tau \in \mathcal{B}(S_+\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \sum_{x \in \operatorname{supp} P} P(x) D^{\operatorname{Um}}(\tau \| W_x) \right\}$$

$$= \max_{\tau \in \mathcal{B}(S_+\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tau - D^{\operatorname{Um}}(\tau \| \sigma) \right\}$$

$$= \operatorname{Tr} \sigma,$$

$$(V.175)$$

where the first equality is by definition, the second equality is by (V.174), and last equality is due to (III.85). Moreover, since D^{Um} is strictly trace monotone (see, e.g., [33, Proposition A.4]), Remark III.38 yields that $\tau = \sigma$ is the unique state attaining the maximum in (V.175). This proves the assertion about the P-weighted D^{Um} -geometric mean, and the rest of the assertions follow from this according to Remark V.28.

Corollary V.30 Let $D^{q_x} = D^{\mathrm{Um}}$, $x \in \mathcal{X}$, and let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ be such that $\wedge_{x \in \mathcal{X}_0} W_x^0 \neq 0$ for any finite subset $W_0 \subseteq W$. Then the D^{Um} -Hellinger body for W exists.

Remark V.31 *Note that for* $P \in \mathcal{P}_f(\mathcal{X})$,

$$G_P^{D^{\mathrm{Um}}}(W) = \widehat{G}_{P,+\infty}(W)$$

where the latter was defined in (III.90).

C. Homogeneity and scaling

Note that the normalized relative entropies $D_1^{q_0}$ and $D_1^{q_1}$ satisfy the scaling property (III.78) by assumption. This property is inherited by all the corresponding barycentric Rényi divergences $D_{\alpha}^{\mathbf{q}}$. More generally, we have the following:

Lemma V.32 For any $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$, any gcq channel $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ and any $t \in (0, +\infty)^{\mathcal{X}}$,

$$Q_P^{\mathbf{b},\mathbf{q}}((t_x W_x)_{x \in \mathcal{X}}) = \left(\prod_{x \in \text{supp } P} t_x^{P(x)}\right) Q_P^{\mathbf{b},\mathbf{q}}(W), \tag{V.176}$$

$$-\log Q_P^{b,\mathbf{q}}((t_x W_x)_{x \in \mathcal{X}}) = -\log Q_P^{b,\mathbf{q}}(W) - \sum_x P(x) \log t_x.$$
 (V.177)

In particular, $Q_{\alpha}^{\mathbf{b},\mathbf{q}}$ is homogeneous.

Proof (V.177) is straightforward to verify from the definition (V.139), and the scaling law (III.79), and (V.176) follows immediately from it.

Corollary V.33 The barycentric Rényi divergences satisfy the scaling law (III.78), i.e.,

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(t\varrho\|s\sigma) = D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) + \log t - \log s, \tag{V.178}$$

for every $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, $t, s \in (0, +\infty)$, $\alpha \in [0, +\infty]$.

Proof Immediate from Lemma V.32, or alternatively, from (V.160).

Monotonicity in α and limiting values

Monotonicity in the parameter α is a characteristic property of the classical Rényi divergences, which is inherited by the measured, the regularized measured, and the maximal Rényi divergences, and it also holds for the Petz-type Rényi divergences. The representations in Lemma V.10 and Remark V.15 show that barycentric Rényi divergences have the same monotonicity property.

Proposition V.34 (i) For any $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$, the maps

$$P\mapsto Q_P^{\mathrm{b},\mathbf{q}}(W) \quad \ and \quad \ P\mapsto \log Q_P^{\mathrm{b},\mathbf{q}}(W)$$

are convex, and

$$P \mapsto R_{D^{\mathbf{q}}, \operatorname{left}}(W, P)$$

is concave, on $\mathcal{P}_f^{\pm}(\mathcal{X})$. (ii) For any fixed $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$\begin{array}{ll} \alpha \mapsto \log Q^{\mathrm{b},\mathbf{q}}_{\alpha}(\varrho\|\sigma) & \text{is convex on } [0,+\infty), \quad and \\ \alpha \mapsto D^{\mathrm{b},\mathbf{q}}_{\alpha}(\varrho\|\sigma) & \text{is monotone increasing on } [0,1) \cup (1,+\infty). \end{array}$$

Proof (i) By Definition V.140, $P \mapsto R_{D^{\mathbf{q}}, \text{left}}(W, P)$ is the infimum of affine functions in P, and hence it is concave. By (V.152), this implies the convexity of $P \mapsto \log Q_P^{b,q}(W)$, from which the convexity of $P \mapsto Q_P^{\mathbf{b},\mathbf{q}}(W)$ follows immediately.

(ii) By (V.154), $\alpha \mapsto \log Q_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho \| \sigma)$ is the supremum of affine functions, and hence convex. (The convexity on $(0, +\infty)$ also follows as a special case of the above.) The monotonicity of $\alpha \mapsto D_{\alpha}^{b,q}(\varrho \| \sigma)$ follows from this convexity by definition (V.143).

Let us introduce the following limiting values:

$$D_1^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) := \sup_{\alpha \in (0,1)} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\alpha \nearrow 1} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma), \tag{V.179}$$

$$D_{1+}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) := \inf_{\alpha>1} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\alpha\searrow1} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma), \tag{V.180}$$

$$D_{\infty}^{\mathbf{b},q}(\varrho\|\sigma) := \sup_{\alpha > 1} D_{\alpha}^{\mathbf{b},q}(\varrho\|\sigma) = \lim_{\alpha \nearrow +\infty} D_{\alpha}^{\mathbf{b},q}(\varrho\|\sigma), \tag{V.181}$$

where the equalities follow from the monotonicity established in Proposition V.34. Using the representations in Remark V.15, it is easy to show the following:

Proposition V.35 *Let* $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$.

(i) We have

$$D_0^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \inf_{\alpha \in (0,1)} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\alpha \searrow 0} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) \tag{V.182}$$

$$= \inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \left\{ D^{q_1} \left(\omega \left\| \frac{\sigma}{\operatorname{Tr} \sigma} \right) \right\} + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma, \right. \tag{V.183}$$

$$D_{\infty}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \sup_{\omega \in \mathcal{S}(\varrho^{0}\mathcal{H})} \left\{ D^{q_{1}}\left(\omega \left\| \frac{\sigma}{\operatorname{Tr}\sigma}\right) - D^{q_{0}}\left(\omega \left\| \frac{\varrho}{\operatorname{Tr}\varrho}\right) \right\} + \log \operatorname{Tr}\varrho - \log \operatorname{Tr}\sigma \right\}$$
(V.184)

$$= \sup_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \{ D^{q_1}(\omega \| \sigma) - D^{q_0}(\omega \| \varrho) \}. \tag{V.185}$$

(ii) If D^{q_0} is strictly positive and $D^{q_0}\left(\cdot \left\| \frac{\varrho}{\operatorname{Tr}\varrho}\right)\right)$ and $D^{q_1}\left(\cdot \left\| \sigma\right)$ are both lower semi-continuous on $\mathcal{S}(\varrho^0\mathcal{H})$ then

$$D_1^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \frac{1}{\operatorname{Tr}\rho} D^{q_1}(\varrho\|\sigma). \tag{V.186}$$

If, moreover, $D^{q_1}(\cdot || \sigma)$ is upper semi-continuous on $S(\varrho^0 \mathcal{H})$ then

$$D_1^{\mathbf{b},\mathbf{q}} = D_{1+}^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma).$$
 (V.187)

Proof (i) The second equality in (V.182) is obvious from the monotonicity established in Proposition V.34, and the rest of the equalities in (V.182)–(V.183) follow as

$$\inf_{\alpha \in (0,1)} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma) = \inf_{\alpha \in (0,1)} \inf_{\omega \in \mathcal{S}(\varrho^{0}\mathcal{H})} \left\{ \frac{\alpha}{1-\alpha} D^{q_{0}} \left(\omega \left\| \frac{\varrho}{\operatorname{Tr} \varrho} \right) + D^{q_{1}} \left(\omega \left\| \frac{\sigma}{\operatorname{Tr} \sigma} \right) \right\} + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma \right) \right\} \\
= \inf_{\omega \in \mathcal{S}(\varrho^{0}\mathcal{H})} \inf_{\alpha \in (0,1)} \left\{ \frac{\alpha}{1-\alpha} D^{q_{0}} \left(\omega \left\| \frac{\varrho}{\operatorname{Tr} \varrho} \right) + D^{q_{1}} \left(\omega \left\| \frac{\sigma}{\operatorname{Tr} \sigma} \right) \right\} + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma \right) \\
= \inf_{\omega \in \mathcal{S}(\varrho^{0}\mathcal{H})} \left\{ D^{q_{1}} \left(\omega \left\| \frac{\sigma}{\operatorname{Tr} \sigma} \right) \right\} + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma \right) \\
= \inf_{\omega \in \mathcal{S}(\varrho^{0}\mathcal{H})} \left\{ D^{q_{1}} \left(\omega \left\| \sigma \right) \right\} + \log \operatorname{Tr} \varrho \right) \\
= D_{0}^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma),$$

where the first equality is due to (V.160), the second equality is trivial, the third equality follows from the non-negativity of D^{q_0} , the fourth equality follows from the scaling law (III.78), and the last equality by definition.

The equalities in (V.184)–(V.185) follow as

$$\begin{split} \sup_{\alpha>1} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) &= \sup_{\alpha>1} \sup_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})} \left\{ \frac{\alpha}{1-\alpha} \, D^{q_0}\left(\omega\Big\|\frac{\varrho}{\mathrm{Tr}\,\varrho}\right) + D^{q_1}\left(\omega\Big\|\frac{\sigma}{\mathrm{Tr}\,\sigma}\right) \right\} + \log\mathrm{Tr}\,\varrho - \log\mathrm{Tr}\,\sigma \\ &= \sup_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})} \sup_{\alpha>1} \left\{ \frac{\alpha}{1-\alpha} \, D^{q_0}\left(\omega\Big\|\frac{\varrho}{\mathrm{Tr}\,\varrho}\right) + D^{q_1}\left(\omega\Big\|\frac{\sigma}{\mathrm{Tr}\,\sigma}\right) \right\} + \log\mathrm{Tr}\,\varrho - \log\mathrm{Tr}\,\sigma \\ &= \sup_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})} \left\{ D^{q_1}\left(\omega\Big\|\frac{\sigma}{\mathrm{Tr}\,\sigma}\right) - D^{q_0}\left(\omega\Big\|\frac{\varrho}{\mathrm{Tr}\,\varrho}\right) \right\} + \log\mathrm{Tr}\,\varrho - \log\mathrm{Tr}\,\sigma \\ &= \sup_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})} \left\{ D^{q_1}\left(\omega\|\sigma\right) - D^{q_0}\left(\omega\|\varrho\right) \right\}, \end{split}$$

where the first equality is due to (V.160), the second one is trivial, the third one follows from the non-negativity of $D^{q_0}\left(\omega \Big\|_{\overline{\text{Tr}\,\rho}}\right)$, and the last equality is due to the scaling property (III.81).

(ii) The equality in (V.186) follows as

$$\begin{split} \sup_{\alpha \in (0,1)} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma) &= \sup_{\alpha \in (0,1)} \inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \left\{ \frac{\alpha}{1-\alpha} \, D^{q_0} \left(\omega \Big\| \frac{\varrho}{\operatorname{Tr} \, \varrho} \right) + D^{q_1} \left(\omega \| \sigma \right) \right\} + \log \operatorname{Tr} \varrho \\ &= \inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} \sup_{\alpha \in (0,1)} \left\{ \frac{\alpha}{1-\alpha} \, D^{q_0} \left(\omega \Big\| \frac{\varrho}{\operatorname{Tr} \, \varrho} \right) + D^{q_1} \left(\omega \| \sigma \right) \right\} + \log \operatorname{Tr} \varrho \\ &= D^{q_1} \left(\frac{\varrho}{\operatorname{Tr} \, \varrho} \Big\| \sigma \right) + \log \operatorname{Tr} \varrho = \frac{1}{\operatorname{Tr} \, \varrho} \, D^{q_1} (\varrho \| \sigma), \end{split}$$

where the first equality is due to (V.159), and the second one follows from the minimax theorem in Lemma II.2, using the fact that $\alpha \mapsto \frac{\alpha}{1-\alpha}$ is monotone increasing on (0,1). The third equality follows from the fact that $\sup_{\alpha \in (0,1)} \frac{\alpha}{1-\alpha} = +\infty$, and that D^{q_0} is strictly positive, and the last equality is immediate from the scaling law (III.80).

Finally, to prove (V.187), first note that if $\varrho^0 \nleq \sigma^0$ then $D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma) = +\infty$ for every $\alpha > 1$; indeed, by (V.158),

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) \ge \frac{\alpha}{1-\alpha} \underbrace{D_{\alpha}^{q_0}(\varrho\|\varrho)}_{=0} + \underbrace{D_{\alpha}^{q_1}(\varrho\|\sigma)}_{=+\infty} - \frac{1}{\alpha-1} \log \operatorname{Tr} \varrho = +\infty. \tag{V.188}$$

(See also Corollary V.40.) Hence,

$$D_{1+}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \inf_{\alpha>1} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \inf_{\alpha>1} +\infty = +\infty = \frac{1}{\operatorname{Tr} \rho} D^{q_1}(\varrho\|\sigma) = D_{1}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma),$$

where the fourth equality is due to the support condition (V.137), and the last equality is (V.186). Hence, for the rest we assume that $\varrho^0 \leq \sigma^0$, so that $D^{q_0}\left(\omega \Big\| \frac{\varrho}{\text{Tr}\,\varrho}\right)$ and $D^{q_1}(\omega \| \sigma)$ are both finite for $\omega \in \mathcal{S}(\varrho^0 \mathcal{H})$. Moreover, by assumption, $\omega \mapsto \frac{\alpha}{1-\alpha} D^{q_0}\left(\omega \Big\| \frac{\varrho}{\text{Tr}\,\varrho}\right) + D^{q_1}\left(\omega \| \sigma\right)$ is upper semi-continous on $\mathcal{S}(\varrho^0 \mathcal{H})$ for every $\alpha > 1$. Then

$$\begin{split} D_{1^{+}}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) &= \inf_{\alpha>1} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) \\ &= \inf_{\alpha>1} \sup_{\omega\in\mathcal{S}(\varrho^{0}\mathcal{H})} \left\{ \frac{\alpha}{1-\alpha} D^{q_{0}}\left(\omega\Big\|\frac{\varrho}{\operatorname{Tr}\varrho}\right) + D^{q_{1}}\left(\omega\Big\|\sigma\right) \right\} + \log\operatorname{Tr}\varrho \\ &= \sup_{\omega\in\mathcal{S}(\varrho^{0}\mathcal{H})} \inf_{\alpha>1} \left\{ \frac{\alpha}{1-\alpha} D^{q_{0}}\left(\omega\Big\|\frac{\varrho}{\operatorname{Tr}\varrho}\right) + D^{q_{1}}\left(\omega\Big\|\sigma\right) \right\} + \log\operatorname{Tr}\varrho \\ &= D^{q_{1}}\left(\frac{\varrho}{\operatorname{Tr}\varrho}\Big\|\sigma\right) + \log\operatorname{Tr}\varrho = \frac{1}{\operatorname{Tr}\varrho} D^{q_{1}}(\varrho\|\sigma) = D_{1}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma), \end{split}$$

where the second equality is due to (V.159), the third one follows from the minimax theorem in Lemma II.2, the fourth equality follows from the strict positivity of D^{q_0} and the fact that $\inf_{\alpha>1} \alpha/(1-\alpha) = -\infty$, the fifth one is due to the scaling law (III.80), and the last equality is (V.186).

Remark V.36 Clearly, for any $\varrho \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any quantum Rényi α -divergence D^q_{α} , $D^q_{\alpha}(\varrho||\varrho) = 0$. On the other hand, if D^{q_0} and D^{q_1} are such that for any two invertible non-commuting $\omega_1, \omega_2 \in \mathcal{B}(\mathcal{H})_{>0}$, $D^{q_1}(\omega_1||\omega_2) > D^{q_0}(\omega_1||\omega_2)$, then by (V.185), $D^{\mathbf{b},\mathbf{q}}_{\infty}(\varrho||\varrho) > 0$, $\varrho \in \mathcal{B}(\mathcal{H})_{>0}$, and hence $D^{\mathbf{b},\mathbf{q}}_{\alpha}$ is not a quantum Rényi α -divergence for any large enough α . For instance, this is the case if $D^{q_0} = D^{\mathrm{Um}}$ and $D^{q_1} = D^{\mathrm{max}}$; see, e.g., [32, Proposition 4.7.].

We leave open the question whether in the above setting for every $\alpha > 1$ there exists a $\varrho \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $D^{b,q}_{\infty}(\varrho||\varrho) > 0$.

Remark V.37 It is well known and easy to verify that for commuting states ϱ and σ , the unique Rényi α -divergences satisfy

$$Q_{\alpha}(\varrho \| \sigma) = Q_{1-\alpha}(\sigma \| \varrho), \qquad \alpha \in (0,1).$$

As a consequence, if $D_{1-\alpha}^q$ is a quantum Rényi $(1-\alpha)$ -divergence for some $\alpha \in (0,1)$ then

$$\tilde{D}_{\alpha}^{q}(\varrho \| \sigma) := \frac{1}{1 - \alpha} \left[\alpha D_{1 - \alpha}^{q}(\sigma \| \varrho) + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma \right]$$

defines a quantum Rényi α -divergence. Given a collection $(D^q_{\alpha})_{\alpha \in (0,1)}$ of quantum Rényi α -divergences, we call $(\tilde{D}^q_{\alpha})_{\alpha \in (0,1)}$ the dual collection. The measured, the regularized measured and the maximal Rényi divergences are easily seen to be self-dual, as are the Petz-type (or standard) Rényi divergences (see Section III C for the definitions).

For the barycentric Rényi divergences, it is straightforward to verify from (V.161) that

$$\tilde{D}_{\alpha}^{\mathbf{b},(q_0,q_1)}(\varrho\|\sigma) = D_{\alpha}^{\mathbf{b},(q_1,q_0)}(\varrho\|\sigma).$$

In particular, $(D_{\alpha}^{\mathrm{b},(q_0,q_1)})_{\alpha\in(0,1)}$ is self-dual when $D^{q_0}=D^{q_1}$.

Combining this duality with (V.186) we obtain that if D^{q_1} is strictly positive and $D^{q_1}\left(\cdot \left\| \frac{\varrho}{\operatorname{Tr}\varrho}\right)\right)$ and $D^{q_0}\left(\cdot \left\| \sigma\right)$ are both lower semi-continuous on $\mathcal{S}(\rho^0\mathcal{H})$ then

$$\frac{1}{\operatorname{Tr} \varrho} D^{q_0}(\varrho \| \sigma) = \lim_{\alpha \nearrow 1} D^{b,(q_1,q_0)}_{\alpha}(\varrho \| \sigma) = \lim_{\alpha \nearrow 1} \tilde{D}^{b,(q_0,q_1)}_{\alpha}(\varrho \| \sigma)
= \lim_{\alpha \searrow 0} \frac{1}{\alpha} \left[(1-\alpha) D^{b,(q_0,q_1)}_{\alpha}(\sigma \| \varrho) + \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma \right].$$
(V.189)

Due to this and Proposition V.35, if both D^{q_0} and D^{q_1} are strictly positive and lower semi-continuous in their first variable, then they can both be recovered from $(D_{\alpha}^{\mathbf{b},\mathbf{q}})_{\alpha\in(0,1)}$ by taking limits at $\alpha\searrow 0$ and at $\alpha\nearrow 1$, respectively. In particular, if $(D^{q_0},D^{q_1})\neq (D^{\tilde{q}_0},D^{\tilde{q}_1})$ then $D_{\alpha}^{\mathbf{b},(q_0,q_1)}\neq D_{\alpha}^{\mathbf{b},(\tilde{q}_0,\tilde{q}_1)}$ for α close enough to 0 from above or to 1 from below.

E. Non-negativity and finiteness

Here we show that the barycentric Rényi divergences are pseudo-distances in the sense that $D_{\alpha}^{\mathbf{b},\mathbf{q}}$ is non-negative for every $\alpha \in [0,+\infty]$, and it is strictly positive for every $\alpha \in (0,+\infty]$ under some mild conditions on D^{q_0} and D^{q_1} .

We start more generally with giving bounds on the multi-variate Rényi quantities in Definition V.2, for which the following easy observation will be useful.

Lemma V.38 Let D^q be a quantum relative entropy. For any $\sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any state $\omega \in \mathcal{S}(\mathcal{H})$,

$$-\log \operatorname{Tr} \sigma \le D^q(\omega \| \sigma). \tag{V.190}$$

If, moreover, D^q is anti-monotone in its second argument, then

$$D^{q}(\omega \| \sigma) \begin{cases} \leq -\log \lambda_{\min}(\sigma) + \operatorname{Tr} \omega \log \omega \leq -\log \lambda_{\min}(\sigma), & \omega^{0} \leq \sigma^{0}, \\ = +\infty, & otherwise. \end{cases}$$
 (V.191)

Proof The inequality in (V.190) is immediate from the trace monotonicity of D^q , and $\omega^0 \nleq \sigma^0 \implies D^q(\omega \| \sigma) = +\infty$ in (V.191) follows from the support condition (V.137). Hence, we are left to prove the upper bound in (V.191).

Let $\lambda_{\min}(\sigma)$ denote the smallest non-zero eigenvalue of σ . By assumption, D^q is anti-monotone in its second argument, whence $\sigma \geq \lambda_{\min}(\sigma)\sigma^0$ implies that

$$D^{q}(\omega \| \sigma) \leq D^{q}(\omega \| \lambda_{\min}(\sigma) \sigma^{0})$$

$$= -\log \lambda_{\min}(\sigma) + \underbrace{D^{q}(\omega \| \sigma^{0})}_{=D(\omega \| \sigma^{0})}$$

$$= -\log \lambda_{\min}(\sigma) + \underbrace{D(\omega \| \sigma^{0})}_{=\operatorname{Tr} \omega \log \omega}$$

$$= -\log \lambda_{\min}(\sigma) + \operatorname{Tr} \omega \log \omega,$$

where D is the unique extension of the classical relative entropy to commuting pairs of operators (see Lemma III.6), the first equality follows from the scaling property (III.81), the second equality from the fact that ω and σ^0 commute due to the assumption that $\omega^0 \leq \sigma^0$, and the last equality is by the definition of D.

Proposition V.39 (i) Let $P \in \mathcal{P}_f(\mathcal{X})$. Then, for any $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$,

$$-\sum_{x} P(x) \log \operatorname{Tr} W_{x} \le -\psi_{P}^{\mathbf{b},\mathbf{q}}(W), \tag{V.192}$$

and if each D^{q_x} , $x \in \text{supp } P$, is anti-monotone in its second argument then

$$-\psi_P^{\mathbf{b},\mathbf{q}}(W) \begin{cases} \leq \sum_x P(x)(-\log \lambda_{\min}(W_x)) < +\infty, & S_+ \neq 0, \\ = +\infty, & otherwise. \end{cases}$$
 (V.193)

In particular,

$$-\psi_P^{\mathbf{b},\mathbf{q}}(W) = +\infty \quad \Longleftrightarrow \quad S_+ = 0 \quad \Longleftrightarrow \quad Q_P^{\mathbf{b},\mathbf{q}}(W) = 0. \tag{V.194}$$

(ii) Assume that $P \in \mathcal{P}_f^{\pm}(\mathcal{X})$ is such that P(x) < 0 for some $x \in \mathcal{X}$. If for each x such that P(x) > 0, D^{q_x} is anti-monotone in its second argument then

$$\sum_{x: P(x) > 0} P(x) \log \lambda_{\min}(W_x) + \sum_{x: P(x) < 0} P(x) \log \operatorname{Tr} W_x \le \psi_P^{b, \mathbf{q}}(W). \tag{V.195}$$

If for each x such that P(x) < 0, D^{q_x} is anti-monotone in its second argument then

$$\psi_P^{\mathbf{b},\mathbf{q}}(W) \begin{cases} \leq \sum_{x: P(x) > 0} P(x) \log \operatorname{Tr} W_x + \sum_{x: P(x) < 0} P(x) \log \lambda_{\min}(W_x), & S_+ \leq S_-, \\ = +\infty, & otherwise; \end{cases}$$
(V.196)

in particular,

$$\psi_P^{\mathbf{b},\mathbf{q}}(W) = +\infty \quad \Longleftrightarrow \quad S_+ \nleq S_- \quad \Longleftrightarrow \quad Q_P^{\mathbf{b},\mathbf{q}}(W) = +\infty. \tag{V.197}$$

Proof The inequalities in (V.192), (V.193), (V.195), and (V.196) are obvious from (V.152), (V.190), and (V.191), and (V.194) and (V.197) follow immediately.

As a special case, we get the following characterization of the finiteness of the 2-variable barycentric Rényi divergences. This gives an easy way to show that certain quantum Rényi divergences cannot be represented as barycentric Rényi divergences, as we show in Section V H.

Corollary V.40 We have

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = +\infty \begin{cases} \iff \varrho^{0} \wedge \sigma^{0} = 0, & when \ \alpha \in [0,1), \\ \iff \varrho^{0} \nleq \sigma^{0}, & when \ \alpha > 1. \end{cases}$$
(V.198)

If D^{q_1} is anti-monotone in its second argument then the one-sided implication above is also an equivalence.

Proof The case $\alpha \in (0, +\infty)$ is immediate from Proposition V.39. The case $\alpha = 0$ follows similarly; we leave the details to the reader.

Finally, we turn to the strict positivity of the 2-variable barycentric Rényi divergences.

Proposition V.41 Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$.

(i) For every $\alpha \in [0, +\infty]$,

$$D_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) \ge \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma.$$
 (V.199)

- (ii) If $\sigma^0 \leq \varrho^0$ then (V.199) holds with equality for $\alpha = 0$. If D^{q_1} is strictly positive and $D^{q_1}(\cdot \| \sigma)$ is lower semi-continuous on $\mathcal{S}(\varrho^0 \mathcal{H})$ then $\sigma^0 \leq \varrho^0$ is also necessary for equality in (V.199) for $\alpha = 0$.
- (iii) We have $(a) \Longrightarrow (b) \longleftarrow (c) \longleftarrow (d)$ and $(a) \Longrightarrow (c)$ in the following:
 - (a) Equality holds in (V.199) for every $\alpha \in [0, +\infty]$.
 - (b) Equality holds in (V.199) for some $\alpha \in (0, +\infty]$.
 - (c) Equality holds in (V.199) for every $\alpha \in [0, 1]$.

(d) $\rho / \operatorname{Tr} \rho = \sigma / \operatorname{Tr} \sigma$.

If D^{q_0} and D^{q_1} are strictly positive, and $D^{q_0}(\cdot \| \varrho / \operatorname{Tr} \varrho)$ and $D^{q_1}(\cdot \| \sigma / \operatorname{Tr} \sigma)$ are lower semi-continuous on $S(\varrho^0 \mathcal{H})$, then we have $(a) \Longrightarrow (b) \Longleftrightarrow (c) \Longleftrightarrow (d)$.

On the other hand, if $D^{q_0} = D^{q_1}$ then the implication $(d) \Longrightarrow (a)$ holds.

Proof (i) The inequality in (V.199) for $\alpha \in [0,1)$ is immediate from (V.160) and the non-negativity of D^{q_0} and D^{q_1} . From this (V.199) follows also for $\alpha \in [1, +\infty]$, using the monotonicity in Proposition V.34.

- (ii) Let $\alpha = 0$. If $\sigma^0 \leq \varrho^0$ then (V.199) holds with equality, according to (V.151). Assume now that $D^{q_1}(\cdot \| \sigma)$ is lower semi-continuous on $\mathcal{S}(\varrho^0 \mathcal{H})$, so that there exists an $\omega_0 \in \mathcal{S}(\varrho^0 \mathcal{H})$ such that $\inf_{\omega \in \mathcal{S}(\varrho^0 \mathcal{H})} D^{q_1} \left(\omega \| \frac{\sigma}{\operatorname{Tr} \sigma} \right) = D^{q_1} \left(\omega_0 \| \frac{\sigma}{\operatorname{Tr} \sigma} \right)$. Now, if D^{q_1} is strictly positive then, by (V.183), $D_0^{\text{b,q}}(\varrho \| \sigma) > \log \operatorname{Tr} \varrho \log \operatorname{Tr} \sigma$ unless $\omega_0 = \sigma / \operatorname{Tr} \sigma$, which in turn implies that $\sigma^0 = \omega_0^0 \leq \varrho^0$.
- (iii) The implications (a) \Longrightarrow (b) \Longleftrightarrow (c) and (a) \Longrightarrow (c) are obvious. If $\varrho/\operatorname{Tr}\varrho = \sigma/\operatorname{Tr}\sigma$ then choosing $\omega := \varrho/\operatorname{Tr}\varrho$ yields that the infimum in (V.160) is zero, and hence equality holds in (V.199), for every $\alpha \in [0, 1)$, and also for $\alpha = 1$, by taking the limit. This proves (d) \Longrightarrow (c).

Next, we prove (b) \Longrightarrow (d) under the assumption that D^{q_0} and D^{q_1} are strictly positive, and $D^{q_0}(\cdot \| \varrho / \operatorname{Tr} \varrho)$ and $D^{q_1}(\cdot \| \sigma / \operatorname{Tr} \sigma)$ are lower semi-continuous on $\mathcal{S}(\varrho^0 \mathcal{H})$. Note that equality in (V.199) for some $\alpha \in (0, +\infty]$ implies equality for every $\beta \in [0, \alpha]$, according to (i) and the monotonicity in Proposition V.34. In particular, the infimum in (V.160) is zero. By the semi-continuity assumptions, there exists an ω_0 where that infimum is attained. Strict positivity of D^{q_0} and D^{q_1} then implies $\omega_0 = \varrho / \operatorname{Tr} \varrho$ and $\omega_0 = \sigma / \operatorname{Tr} \sigma$, proving (d).

Finally, if $D^{q_0} = D^{q_1}$ and $\varrho / \operatorname{Tr} \varrho = \sigma / \operatorname{Tr} \sigma$ then (V.199), (V.184), and the monotonicity established in Proposition V.34, yield that for every $\alpha \in [0, +\infty]$,

$$\log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma \leq D_{\alpha}^{\mathrm{b},(q_0,q_1)}(\varrho \| \sigma) \leq D_{\infty}^{\mathrm{b},(q_0,q_1)}(\varrho \| \sigma) = \log \operatorname{Tr} \varrho - \log \operatorname{Tr} \sigma,$$

proving the implication $(d) \Longrightarrow (a)$.

Remark V.42 Note that by Remark V.36, the condition $D^{q_0} = D^{q_1}$ cannot be completely omitted for the implication $(d) \Longrightarrow (a)$ in (iii) to hold.

Proposition V.41 yields immediately the following:

Corollary V.43 For every $\alpha \in [0, +\infty]$, $D_{\alpha}^{b, \mathbf{q}}$ is non-negative. If D^{q_0} and D^{q_1} are both strictly positive and lower semi-continuous in their first arguments then $D_{\alpha}^{b, \mathbf{q}}$ is strictly positive for every $\alpha \in (0, +\infty]$.

Remark V.44 Using the scaling property (V.178), (V.199) can be rewritten equivalently as

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}\left(\frac{\varrho}{\operatorname{Tr}\rho}\left\|\frac{\sigma}{\operatorname{Tr}\sigma}\right)\geq 0, \qquad \alpha\in[0,+\infty].$$

Remark V.45 Note that (V.199) tells exactly that the trace-monotonicity of D^{q_0} and D^{q_1} is inherited by all the generated barycentric Rényi divergences $D_{\alpha}^{b,q}$, $\alpha \in [0,+\infty]$.

Since the barycentric Rényi divergences satisfy the scaling property (III.78) according to Corollary V.33, their non-negativity is in fact equivalent to trace-monotonicity, as noted in Remark III.37.

F. Monotonicity under CPTP maps and joint convexity

One of the most important properties of quantum divergences is monotonicity under quantum operations (i.e., CPTP maps). Many of the important quantum divergences are monotone under more general trace-preserving maps, e.g., dual Schwarz maps in the case of Petz-type Rényi divergences for $\alpha \in [0,2]$ [71], or PTP maps in the case of the sandwiched Rényi divergences for $\alpha \geq 1/2$ [8, 40, 67], and the measured as well as the maximal Rényi divergences for $\alpha \in [0,+\infty]$, by definition. It is easy to see that for $\alpha \in [0,1]$, the barycentric Rényi α -divergences are monotone under the same class of PTP maps as their generating quantum relative entropies. More generally, we have the following:

Proposition V.46 If all D^{q_x} , $x \in \mathcal{X}$, are monotone non-increasing under a trace non-decreasing positive map $\Phi \in P^+(\mathcal{H}, \mathcal{K})$ then $Q^{b,q}$ is monotone non-decreasing, and $R_{D^q,left}$ is monotone non-increasing under Φ , i.e., for every gcq channel $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ and every $P \in \mathcal{P}_f(\mathcal{X})$,

$$Q_P^{\mathbf{b},\mathbf{q}}\left(\Phi(W)\right) \ge Q_P^{\mathbf{b},\mathbf{q}}\left(W\right),\tag{V.200}$$

$$R_{D^{\mathbf{q}}, \text{left}} (\Phi(W), P) \le R_{D^{\mathbf{q}}, \text{left}} (W, P).$$
 (V.201)

Proof We have

$$Q_{P}^{\mathbf{b},\mathbf{q}}(\Phi(W)) = \sup_{\tau \in \mathcal{B}(\mathcal{K})_{\geq 0}} \left\{ \operatorname{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_{x}}(\tau \| \Phi(W_{x})) \right\}$$

$$\geq \sup_{\tilde{\tau} \in \mathcal{B}(\mathcal{H})_{\geq 0}} \left\{ \underbrace{\operatorname{Tr} \Phi(\tilde{\tau})}_{\geq \operatorname{Tr} \tilde{\tau}} - \sum_{x \in \mathcal{X}} P(x) \underbrace{D^{q_{x}}(\Phi(\tilde{\tau}) \| \Phi(W_{x}))}_{\leq D^{q_{x}}(\tilde{\tau} \| W_{x})} \right\}$$

$$\geq \sup_{\tilde{\tau} \in \mathcal{B}(\mathcal{H})_{\geq 0}} \left\{ \operatorname{Tr} \tilde{\tau} - \sum_{x \in \mathcal{X}} P(x) D^{q_{x}}(\tilde{\tau} \| W_{x}) \right\}$$

$$= Q_{P}^{\mathbf{b},\mathbf{q}}(W),$$

where the equalities are by definition (V.138) and by (V.144), the first inequality is obvious, and the second one follows from the assumptions. This proves (V.200), and (V.201) follows immediately by Lemma V.10. \Box

Proposition V.47 If D^{q_0} and D^{q_1} are monotone under a trace non-decreasing map $\Phi \in P^+(\mathcal{H}, \mathcal{K})$ then for every $\alpha \in [0, 1]$, $Q^{\mathbf{b}, \mathbf{q}}_{\alpha}$ is monotone non-decreasing under Φ , i.e., for every $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\Phi(\varrho)\|\Phi(\sigma)) \ge Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma). \tag{V.202}$$

If, moreover, Φ is trace-preserving, then for every $\alpha \in [0,1]$, $D_{\alpha}^{b,q}$ is monotone non-increasing under Φ , i.e., for every $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\Phi(\varrho)\|\Phi(\sigma)) \le D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma). \tag{V.203}$$

Vice versa, if D^{q_0} and D^{q_1} are strictly positive, and lower semi-continuous in their first variables, and $D^{b,q}_{\alpha}$, $\alpha \in (0,1)$, are monotone non-increasing under a trace-preserving positive map $\Phi \in P^+(\mathcal{H},\mathcal{K})$, then D^{q_0} and D^{q_1} are monotone non-increasing under the same map.

Proof Proposition V.46 yields (V.202) as a special case when $\alpha \in (0, 1]$, and the case $\alpha = 0$ follows by a trivial modification of the proof. From this, (V.203) follows immediately. The last assertion follows due to (V.186) and (V.189).

Remark V.48 By assumption, both D^{q_0} and D^{q_1} are trace-monotone, and hence so are $D^{\mathrm{b},\mathbf{q}}_{\alpha}$, $\alpha \in [0,1]$, according to Proposition V.47 above. This gives an alternative proof of (V.199).

Remark V.49 The above proof for Proposition V.46 only works when P is a probability measure (i.e., there is no x such that P(x) < 0), which translates to $\alpha \in [0,1]$ in Proposition V.47. These conditions cannot be removed in general; indeed, it was shown in [64, Lemma 3.17] that $D_{\alpha}^{\text{b,Um}}$ is not monotone under CPTP maps (in fact, not even under pinchings) for any $\alpha > 1$, even though D^{Um} is monotone.

Proposition V.50 Let $P \in \mathcal{P}_f(\mathcal{X})$, and assume that at least one of the following holds:

- (i) D^{q_x} , $x \in \text{supp } P$, are monotone under partial traces and are block subadditive.
- (ii) D^{q_x} , $x \in \text{supp } P$, are jointly convex in their variables.

Then $W \mapsto Q_P^{\mathbf{b},\mathbf{q}}(W)$ and $W \mapsto \psi_P^{\mathbf{b},\mathbf{q}}(W)$ are concave, and $W \mapsto R_{D^{\mathbf{q}},\mathrm{left}}(W,P)$ is convex on $\mathcal{B}(\mathcal{X},\mathcal{H})_{\geq 0}$.

Proof Since all D^{q_x} satisfy the scaling law (III.79), they are in particular homogeneous, and thus, by Lemma III.13, (i) implies (ii). Assume therefore (ii). Then

$$\mathcal{B}(\mathcal{H})_{\geq 0} \ni \tau \mapsto \operatorname{Tr} \tau - \sum_{x} P(x) D^{q_x}(\tau || W_x)$$

is jointly concave in τ and W, and hence its supremum over τ is concave in W. This proves the concavity of $W \mapsto Q_P^{\mathrm{b},\mathbf{q}}(W)$. The concavity of $W \mapsto \psi_P^{\mathrm{b},\mathbf{q}}(W)$ then follows immediately, which in turn implies the convexity of $W \mapsto R_{D^{\mathbf{q}},\mathrm{left}}(W,P)$ due to (V.152).

As a special case, we get the following:

Corollary V.51 Let $\alpha \in (0,1)$, and assume that at least one of the following holds:

- (i) D^{q_x} , $x \in \{0,1\}$, are monotone under partial traces and are block subadditive.
- (ii) D^{q_x} , $x \in \{0,1\}$, are jointly convex in their variables.

Then $(\varrho, \sigma) \mapsto Q_{\alpha}^{\mathbf{b}, \mathbf{q}}(\varrho, \sigma)$ and $(\varrho, \sigma) \mapsto \psi_{\alpha}^{\mathbf{b}, \mathbf{q}}(\varrho, \sigma)$ are concave, and $(\varrho, \sigma) \mapsto D_{\alpha}^{\mathbf{b}, \mathbf{q}}(\varrho, \sigma)$ is convex on $\mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$. The same conclusions hold if $\alpha = 0$ and the properties in (i) or (ii) are assumed for D^{q_1} , or if $\alpha = 1$ and the properties in (i) or (ii) are assumed for D^{q_0} .

G. Lower semi-continuity and regularity

Note that when \mathcal{X} is finite then $\mathcal{B}(\mathcal{X},\mathcal{H}) = \mathcal{B}(\mathcal{H})^{\mathcal{X}}$ is a finite-dimensional vector space, and hence there exists a unique norm topology on it, which is what we implicitly refer to in statements about (semi-)continuity on this space.

Proposition V.52 If $P \in \mathcal{P}_f(\mathcal{X})$ and D^{q_x} , $x \in \text{supp } P$, are all jointly lower semi-continuous in their variables then $W \mapsto Q_P^{b,\mathbf{q}}(W)$ is upper semi-continuous, and $W \mapsto R_{D^{\mathbf{q}},\text{left}}(W,P)$ is lower semi-continuous on $\mathcal{B}(\text{supp } P, \mathcal{H})_{\geq 0}$.

Proof By assumption,

$$\mathcal{B}(\operatorname{supp} P, \mathcal{H})_{\geq 0} \times \mathcal{S}(\mathcal{H}) \ni (W, \omega) \mapsto \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\omega || W_x)$$

is lower semi-continuous, and hence, by Lemma II.3, its infimum over $\omega \in \mathcal{S}(\mathcal{H})$ is lower semi-continuous, too, proving the assertion about $R_{D^{\mathbf{q}}, \text{left}}$. The assertion about $Q_P^{\mathbf{b}, \mathbf{q}}$ then follows immediately due to (V.152).

Corollary V.53 If D^{q_0} and D^{q_1} are jointly lower semi-continuous in their variables then for any $\alpha \in (0,1]$, $Q_{\alpha}^{\mathrm{b},\mathbf{q}}$ is jointly upper semi-continuous and $D_{\alpha}^{\mathrm{b},\mathbf{q}}$ is jointly lower semi-continuous in their arguments.

Proof The case $\alpha \in (0,1)$ follows as a special case of Proposition V.52. For $\alpha = 1$, $Q_1^{\mathbf{b},\mathbf{q}}(\varrho \| \sigma) = \operatorname{Tr} \varrho$ by (V.148), and continuity holds trivially, while $D_1^{\mathbf{b},\mathbf{q}}$ is the supremum of lower semi-continuous functions according to the above and (V.179), and hence is itself lower semi-continuous.

Proposition V.54 Let $P \in \mathcal{P}_f(\mathcal{X})$ and assume that D^{q_x} , $x \in \text{supp } P$, are weakly anti-monotone in their second arguments. Then

$$(0, +\infty) \ni \varepsilon \mapsto Q_P^{b,q}((W_x + \varepsilon I)_{x \in \mathcal{X}})$$
 is monotone increasing, (V.204)

$$(0, +\infty) \ni \varepsilon \mapsto R_{D^{\mathbf{q}}, \text{left}}^{\mathbf{b}} ((W_x + \varepsilon I)_{x \in \mathcal{X}})$$
 is monotone decreasing (V.205)

for any $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$. If, moreover, D^{q_x} , $x \in \text{supp } P$, are regular in the sense of (III.39), and lower semi-continuous in their first arguments, then for any $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$,

$$Q_P^{\mathbf{b},\mathbf{q}}(W) = \lim_{\varepsilon \searrow 0} Q_P^{\mathbf{b},\mathbf{q}}\left((W_x + \varepsilon I)_{x \in \mathcal{X}} \right) = \inf_{\varepsilon > 0} Q_P^{\mathbf{b},\mathbf{q}}\left((W_x + \varepsilon I)_{x \in \mathcal{X}} \right), \tag{V.206}$$

$$R_{D^{\mathbf{q}}, \text{left}}^{\mathbf{b}}(W) = \lim_{\varepsilon \searrow 0} R_{D^{\mathbf{q}}, \text{left}}^{\mathbf{b}}\left((W_x + \varepsilon I)_{x \in \mathcal{X}}\right) = \sup_{\varepsilon > 0} R_{D^{\mathbf{q}}, \text{left}}^{\mathbf{b}}\left((W_x + \varepsilon I)_{x \in \mathcal{X}}\right). \tag{V.207}$$

Proof The monotonicity assertions in (V.204)–(V.205) are obvious, as are the second equalities in (V.206)–(V.207). The first equality in (V.207) follows as

$$\sup_{\varepsilon>0} R_{D^{\mathbf{q}}, \text{left}}^{\mathbf{b}} ((W_x + \varepsilon I)_{x \in \mathcal{X}})$$

$$= \sup_{\varepsilon>0} \inf_{\omega \in \mathcal{S}(S_+ \mathcal{H})} \sum_{x} P(x) D^{q_x} (\omega \| W_x + \varepsilon I)$$

$$= \inf_{\omega \in \mathcal{S}(S_+ \mathcal{H})} \sup_{\varepsilon>0} \sum_{x} P(x) D^{q_x} (\omega \| W_x + \varepsilon I)$$

$$= \inf_{\omega \in \mathcal{S}(S_+ \mathcal{H})} \sum_{x} P(x) D^{q_x} (\omega \| W_x)$$

$$= R_{D^{\mathbf{q}}, \text{left}}^{\mathbf{b}} (W),$$

where the first and the last equalities are by definition, the second equality follows from the minimax theorem in Lemma II.2, and the third equality from the regularity of the D^{q_x} . From this, the first equality in (V.206) follows by (V.152).

In the 2-variable case we have the following:

Corollary V.55 Assume that D^{q_0} and D^{q_1} are weakly anti-monotone in their second arguments. Then

$$(0, +\infty) \ni \varepsilon \mapsto Q_{\alpha}^{b, \mathbf{q}}(\varrho + \varepsilon I \| \sigma + \varepsilon I)$$
 is monotone increasing (V.208)

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $\alpha \in [0,1]$. If, moreover, D^{q_0} and D^{q_1} are regular in the sense of (III.39), and lower semi-continuous in their first arguments, then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

$$Q_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} Q_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho + \varepsilon I\|\sigma + \varepsilon I) = \inf_{\varepsilon > 0} Q_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho + \varepsilon I\|\sigma + \varepsilon I), \qquad \quad \alpha \in (0,1], \tag{V.209}$$

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho + \varepsilon I\|\sigma + \varepsilon I), \qquad \alpha \in (0,1).$$
 (V.210)

Proof The monotonicity assertion in (V.208) is again obvious by definition, and the equalities in (V.209)– (V.210) follow as special cases of (V.206)–(V.207).

Remark V.56 Note that monotonicity of $(0, +\infty) \ni \varepsilon \mapsto D_{\alpha}^{\mathrm{b}, \mathbf{q}}(\varrho + \varepsilon I \| \sigma + \varepsilon I)$ is not true in general, as $\varepsilon \mapsto \frac{1}{\alpha - 1} \log Q_{\alpha}^{\mathrm{b}, \mathbf{q}}(\varrho + \varepsilon I \| \sigma + \varepsilon I)$ is monotone decreasing, while $\frac{1}{1 - \alpha} \log \mathrm{Tr}(\varrho + \varepsilon I)$ is monotone increasing.

Proposition V.57 Assume that D^{q_1} is weakly anti-monotone in its second argument. Then

$$(0, +\infty) \ni \varepsilon \mapsto Q_{\alpha}^{b, \mathbf{q}}(\varrho \| \sigma + \varepsilon I)$$
 is monotone increasing, (V.211)

$$(0, +\infty) \ni \varepsilon \mapsto D_{\alpha}^{b, \mathbf{q}}(\varrho \| \sigma + \varepsilon I)$$
 is monotone decreasing (V.212)

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $\alpha \in [0,1]$. If, moreover, D^{q_1} is regular in the sense of (III.39), and lower semi-continuous in its first argument, then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$,

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma + \varepsilon I) = \inf_{\varepsilon > 0} Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho + \varepsilon I\|\sigma + \varepsilon I), \qquad \alpha \in [0,1],$$
(V.213)

$$Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma + \varepsilon I) = \inf_{\varepsilon > 0} Q_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho + \varepsilon I\|\sigma + \varepsilon I), \qquad \alpha \in [0,1],$$

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0} D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma + \varepsilon I) \qquad \alpha \in [0,1].$$
(V.213)

Proof The proof is essentially the same as for Proposition V.54 and Corollary V.55, and hence we omit most of it, and only mention that the $\alpha = 1$ case in (V.212) and (V.214) follow from the respective statements for $\alpha \in [0, 1)$ using (V.179).

H. Finiteness and non-examples

Corollary V.40 gives an easily verifiable condition for a quantum Rényi α -divergence not being a barycentric Rényi α -divergence, as follows:

Proposition V.58 Let D^q_{α} be a quantum Rényi α -divergence for some $\alpha \in (0,1)$ with the property that $D^q_{\alpha}(\varrho \| \sigma) = +\infty \iff \varrho \perp \sigma$. Then there exist no quantum relative entropies D^{q_0} and D^{q_1} with which $D_{\alpha}^{q} = D_{\alpha}^{\mathrm{b},\mathbf{q}}.$

Proof Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be such that $\varrho^0 \wedge \sigma^0 = 0$ and $\varrho \not\perp \sigma$. Then

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\rho\|\sigma) = +\infty > D_{\alpha}^{q}(\rho\|\sigma) \tag{V.215}$$

for any quantum relative entropies D^{q_0} and D^{q_1} , according to Corollary V.40. Since such pairs exist in any dimension larger than 1, we get that $D_{\alpha}^{b,\mathbf{q}} \neq D_{\alpha}^{q}$.

Corollary V.59 $D_{\alpha,z}$ is not a barycentric Rényi α -divergence for any $\alpha \in (0,1)$ and $z \in (0,+\infty)$.

Proof It is obvious by definition that for any $\alpha \in (0,1)$ and $z \in (0,+\infty)$, and any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$, $D_{\alpha,z}(\varrho \| \sigma) = +\infty \iff \varrho \perp \sigma$, and hence the assertion follows immediately from Proposition V.58.

Corollary V.60 The measured Rényi α -divergence D_{α}^{meas} is not a barycentric Rényi α -divergence for any $\alpha \in (0, 1)$.

Proof According to Proposition V.58 we only need to prove that for any $\alpha \in (0,1)$ and any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, $D_{\alpha}^{\text{meas}}(\varrho \| \sigma) = +\infty \iff \varrho \perp \sigma$. This is well known and easy to verify, but we give the details for the readers' convenience. If $\varrho \perp \sigma$ then the measurement $M_0 := \varrho^0$, $M_1 := I - \varrho^0$ gives $D_{\alpha}^{\text{meas}}(\varrho \| \sigma) \geq D_{\alpha}(\mathcal{M}(\varrho) \| \mathcal{M}(\sigma)) = +\infty$. If $\varrho \not\perp \sigma$ then we have $D_{\alpha}^{\text{meas}}(\varrho \| \sigma) \leq D_{\alpha,1}(\varrho \| \sigma) < +\infty$, where the first inequality is due to the monotonicity of the Petz-type Rényi α -divergence under measurements [71]. \square

One might have the impression that the strict inequality in (V.215) is the result of some pathology, and would not happen if the operators had full support, and both Rényi divergences took finite values on them. This, however, is not the case, at least if we assume some mild and very natural continuity and regularity properties of D_{α}^q and the quantum relative entropies D^{q_0} and D^{q_1} .

Indeed, Proposition V.58 and Corollary V.53 yield the following:

Corollary V.61 Let D^q_{α} be a quantum Rényi α -divergence for some $\alpha \in (0,1)$, such that $D^q_{\alpha}(\varrho||\sigma) = +\infty$ $\iff \varrho \perp \sigma$, and assume that D^q_{α} is jointly continuous in its arguments. Let D^{q_0} and D^{q_1} be quantum relative entropies that are jointly lower semi-continuous in their arguments. Then for any two $\varrho_0, \sigma_0 \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $\varrho_0^0 \wedge \sigma_0^0 = 0$ and $\varrho_0 \not\perp \sigma_0$, and for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ in a neighbourhood of (ϱ_0, σ_0) ,

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) > D_{\alpha}^{q}(\varrho\|\sigma),\tag{V.216}$$

and $D_{\alpha}^{b,q}(\varrho \| \sigma) < +\infty$ for any pair of invertible elements in the neighbourhood. In particular, $D_{\alpha}^{q} \neq D_{\alpha}^{b,q}$.

Proof Let $M > D_{\alpha}^{q}(\varrho_{0}||\sigma_{0})$ be a finite number. By the (semi-)continuity assumptions, $\{(\varrho, \sigma) \in \mathcal{B}(\mathcal{H})_{\geq 0}^{2} : D_{\alpha}^{q}(\varrho||\sigma) < M < D_{\alpha}^{b,\mathbf{q}}(\varrho||\sigma)\}$ is an open subset of $\mathcal{B}(\mathcal{H})_{\geq 0}^{2}$ containing $(\varrho_{0}, \sigma_{0})$, and for any of its elements (ϱ, σ) , the inequality (V.216) holds. The assertion about the invertible pairs is obvious from Corollary V.40.

Likewise, Proposition V.58 and Corollary V.55 yield the following:

Corollary V.62 Let D^q_{α} be a quantum Rényi α -divergence for some $\alpha \in (0,1)$, such that $D^q_{\alpha}(\varrho\|\sigma) = +\infty$ $\iff \varrho \perp \sigma$, and such that D^q_{α} is regular in the sense that $D^q_{\alpha}(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} D^q_{\alpha}(\varrho + \varepsilon I \|\sigma + \varepsilon I)$ for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$. Let D^{q_0} and D^{q_1} be quantum relative entropies that are lower semi-continuous in their first arguments, weakly anti-monotone in their second arguments, and regular. Then for any $\varrho_0, \sigma_0 \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $\varrho_0^0 \wedge \sigma_0^0 = 0$ and $\varrho_0 \not \perp \sigma_0$, and for any $\varepsilon > 0$ small enough,

$$+\infty > D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho_0 + \varepsilon I \| \sigma_0 + \varepsilon I) > D_{\alpha}^q(\varrho_0 + \varepsilon I \| \sigma_0 + \varepsilon I). \tag{V.217}$$

In particular, $D_{\alpha}^q \neq D_{\alpha}^{b,\mathbf{q}}$.

Proof Let $M > D_{\alpha}^{q}(\varrho_{0} \| \sigma_{0})$ be a finite number. By Corollary V.40 and Corollary V.55, there exists some $\varepsilon_{1} > 0$ such that $+\infty > D_{\alpha}^{\text{b,q}}(\varrho_{0} + \varepsilon I \| \sigma_{0} + \varepsilon I) > M$ for every $0 < \varepsilon \leq \varepsilon_{1}$. By the assumed regularity of D_{α}^{q} , there exists some $\varepsilon_{2} > 0$ such that $D_{\alpha}^{q}(\varrho_{0} + \varepsilon I \| \sigma_{0} + \varepsilon I) < M$ for every $0 < \varepsilon < \varepsilon_{2}$. Hence (V.217) holds for every $0 < \varepsilon < \min\{\varepsilon_{1}, \varepsilon_{2}\}$.

Example V.63 For every $\alpha \in (0,1)$ and $z \in (0,+\infty)$, $D_{\alpha,z}$ satisfies the conditions in Corollaries V.61 and V.62, and hence both corollaries apply to it.

Example V.64 As discussed above, for every $\alpha \in (0,1)$, $D_{\alpha}^{\text{meas}}(\varrho \| \sigma) = +\infty \iff \varrho \perp \sigma$. Hence, Corollaries V.61 and V.62 can be applied to D_{α}^{meas} if it is continuous in its arguments. To show this, let us fix a finite-dimensional Hilbert space \mathcal{H} and an orthonormal system $(e_i)_{i=1}^d$ in it. Let $\mathbb{U}(\mathcal{H})$ denote the set of unitaries on \mathcal{H} . Continuity of the classical Rényi α -divergence D_{α} yields that the function

$$\mathcal{B}(\mathcal{H})_{\geq 0} \times \mathcal{B}(\mathcal{H})_{\geq 0} \times \mathbb{U}(\mathcal{H}) \ni (\varrho, \sigma, U) \mapsto D_{\alpha} \left((\langle e_i, U^* \varrho U e_i \rangle)_{i=1}^d \, \middle\| \, (\langle e_i, U^* \varrho U e_i \rangle)_{i=1}^d \right)$$

is continuous. Hence,

$$D_{\alpha}^{\text{meas}}(\varrho \| \sigma) = \sup_{U \in \mathbb{U}(\mathcal{H})} D_{\alpha} \left(\left(\langle e_i, U^* \varrho U e_i \rangle \right)_{i=1}^d \| \left(\langle e_i, U^* \varrho U e_i \rangle \right)_{i=1}^d \right)$$
 (V.218)

is continuous in ϱ and σ according to Lemma II.3. (For the equality in (V.218), see [10, Theorem 4].)

VI. EXAMPLES OF BARYCENTRIC RÉNYI DIVERGENCES

In this section we consider the relations among various known quantum Rényi α -divergences and barycentric Rényi α -divergences obtained from specific quantum relative entropies. Our main results in this respect are Corollary VI.12 and Theorem VI.20; we illustrate these in Figure 2.

A. General relations

Recall the definition of the ordering of quantum divergences given in Definition III.7, and the definition of strict ordering of 2-variable quantum divergences defined on pairs of non-zero PSD operators from Definition III.8.

We start with two simple observations. The first one is trivial by definition.

Lemma VI.1 If $P \in \mathcal{P}_f(\mathcal{X})$ and $D^{q_x} \leq D^{\tilde{q}_x}$, $x \in \text{supp } P$, then

$$-\log Q_P^{\mathbf{b},\mathbf{q}} \le -\log Q_P^{\mathbf{b},\tilde{\mathbf{q}}}.\tag{VI.219}$$

In particular, if $\mathcal{X} = \{0, 1\}$ then

$$D^{q_0} \le D^{\tilde{q}_0}, \ D^{q_1} \le D^{\tilde{q}_1} \implies D^{\mathbf{b}, \mathbf{q}}_{\alpha} \le D^{\mathbf{b}, \tilde{\mathbf{q}}}_{\alpha}, \qquad \alpha \in [0, 1].$$
 (VI.220)

The following is also easy to verify:

Proposition VI.2 Let $P \in \mathcal{P}_f(\mathcal{X})$, and let D^{q_x} , $x \in \text{supp } P$, be monotone under CPTP maps. Then

$$-\log Q_P^{\text{meas}} \le -\log Q_P^{\text{b,meas}} \le -\log Q_P^{\text{b,q}} \le -\log Q_P^{\text{b,max}} \le -\log Q_P^{\text{max}}. \tag{VI.221}$$

In particular, if D^{q_0} and D^{q_1} are quantum relative entropies that are monotone under CPTP maps then

$$D_{\alpha}^{\text{meas}} \leq D_{\alpha}^{\text{b,meas}} \leq D_{\alpha}^{\text{b,q}} \leq D_{\alpha}^{\text{b,max}} \leq D_{\alpha}^{\text{max}}, \qquad \alpha \in [0,1]. \tag{VI.222}$$

Proof The second and the third inequalities in (VI.221) are immediate from (VI.219) and (III.75). Since D^{meas} and D^{max} are monotone under CPTP maps, so are $-\log Q_P^{\text{b,meas}}$ and $-\log Q_P^{\text{b,max}}$ as well, according to Proposition V.46. Hence, the first and the last inequalities in (VI.221) follow immediately from Example III.41. The inequalities in (VI.222) follow the same way.

We have seen in Corollary V.60 that the first inequality in (VI.222) is not an equality, i.e., for any $\alpha \in (0,1)$, the smallest barycentric Rényi α -divergence generated by CPTP-monotone quantum relative entropies is above and not equal to the smallest CPTP-monotone quantum Rényi α -divergence. We will show in Section VID that the same happens "at the top of the spectrum", i.e., the last inequality in (VI.222) is not an equality, either.

One might expect that the strict ordering of relative entropies yields a strict ordering of the corresponding variational Rényi divergences. This, however, is not true in complete generality, as Example VI.13 below shows. On the other hand, $D_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) < D_{\alpha}^{\mathrm{b},\tilde{\mathbf{q}}}(\varrho\|\sigma)$ might nevertheless hold if some extra conditions are imposed on the inputs, as we show in Sections VIB–VID below. Here we make the following general observation:

Lemma VI.3 Let $P \in \mathcal{P}_f(\mathcal{X}) \setminus \{\mathbf{1}_{\{x\}} : x \in \mathcal{X}\}$ and $D^{\mathbf{q}}, D^{\tilde{\mathbf{q}}}$ be such that $D^{q_x} \leq D^{\tilde{q}_x}$, $x \in \operatorname{supp} P$, and $D^{q_y} < D^{\tilde{q}_y}$ for some $y \in \operatorname{supp} P$. Let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$, and assume that one of the following holds:

- (i) There exists a P-weighted $D^{\tilde{\mathbf{q}}}$ -center for W that does not commute with W_y .
- (ii) There exists a P-weighted $D^{\tilde{\mathbf{q}}}$ -center for W, and no state that commutes with W_y is a P-weighted $D^{\mathbf{q}}$ -center for W.

Then
$$-\log Q_P^{b,\mathbf{q}}(W) < -\log Q_P^{b,\tilde{\mathbf{q}}}(W).$$

Proof Assume (i), and let ω be a P-weighted $D^{\tilde{\mathbf{q}}}$ -center for W that does not commute with W_{u} . Then

$$-\log Q_P^{\mathbf{b},\tilde{\mathbf{q}}}(W) = \sum_{x \in \mathcal{X}} P(x) D^{\tilde{q}_x}(\omega \| W_x)$$

$$= \underbrace{D^{\tilde{q}_y}(\omega \| W_y)}_{> D^{q_y}(\omega \| W_y)} + \sum_{x \in \mathcal{X} \setminus \{y\}} P(x) \underbrace{D^{\tilde{q}_x}(\omega \| W_x)}_{\geq D^{q_x}(\omega \| W_x)}$$

$$> \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\omega \| W_x)$$

$$\geq -\log Q_P^{\mathbf{b},\mathbf{q}}(W),$$

where the first equality and the last inequality are by definition, and the strict inequality follows from the assumption that $\omega W_y \neq W_y \omega$ and that $D^{q_y} < D^{\tilde{q}_y}$.

Assume now (ii), and let $\tilde{\omega}$ be a P-weighted $D^{\tilde{\mathbf{q}}}$ -center for W. If $\tilde{\omega}$ does not commute with W_y then $-\log Q_P^{\mathbf{b},\mathbf{q}}(W)<-\log Q_P^{\mathbf{b},\tilde{\mathbf{q}}}(W)$ by the previous point. If $\tilde{\omega}$ commutes with W_y then it cannot be a P-weighted $D^{\mathbf{q}}$ -center for W by assumption, and hence there exists an $\omega \in \mathcal{S}(S_+\mathcal{H})$ such that

$$\sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tilde{\omega} || W_x) > \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\omega || W_x).$$

Thus,

$$-\log Q_P^{\mathbf{b},\tilde{\mathbf{q}}}(W) = \sum_{x \in \mathcal{X}} P(x) D^{\tilde{q}_x}(\tilde{\omega} || W_x)$$

$$\geq \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tilde{\omega} || W_x)$$

$$\geq \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\omega || W_x)$$

$$\geq -\log Q_P^{\mathbf{b},\mathbf{q}}(W),$$

proving the assertion.

Remark VI.4 The condition in (ii) of Lemma VI.3 that there exists a P-weighted $D^{\tilde{\mathbf{q}}}$ -center for W is very mild; indeed, it is satisfied whenever for every $x \in \operatorname{supp} P$, D^{q_x} is lower semi-continuous in its first variable.

B. Umegaki relative entropy and smaller/larger relative entropies

Proposition VI.5 Let $D^{q_0}, D^{q_1} \leq D^{\text{Um}}$, and assume that at least one of them is strictly smaller than D^{Um} . Then for any two non-commuting invertible positive operators $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

$$D_{\alpha}^{\mathrm{b},\mathbf{q}}(\rho\|\sigma) < D_{\alpha}^{\mathrm{b},\mathrm{Um}}(\rho\|\sigma) \quad (= D_{\alpha+\infty}(\rho\|\sigma)), \qquad \alpha \in (0,1).$$
 (VI.223)

In particular, if $D^{\text{meas}} \leq D^{q_0}$, $D^{q_1} \leq D^{\text{Um}}$ then for any two non-commuting invertible positive operators

$$D_{\alpha}^{\text{b,meas}}(\varrho\|\sigma) \le \left\{ \begin{array}{l} D_{\alpha}^{\text{b,(meas},q_1)}(\varrho\|\sigma) \\ D_{\alpha}^{\text{b,(q_0,meas)}}(\varrho\|\sigma) \end{array} \right\} < D_{\alpha}^{\text{b,Um}}(\varrho\|\sigma), \qquad \alpha \in (0,1).$$
 (VI.224)

Proof Recall the form of the α -weighted D^{Um} -center $\omega_{\alpha} := \omega_{\alpha}^{\mathrm{Um}}(\varrho \| \sigma)$ in (V.173). It is easy to see that if it commutes with ϱ or σ then ϱ and σ have to commute with each other. Indeed, assume that ω_{α} commutes with ϱ ; then ϱ also commutes with any function of ω_{α} , in particular, with $(1-\alpha)\log\varrho + \alpha\log\sigma$, and hence it also commutes with σ . The same argument works if ω_{α} commutes with σ . The strict inequality in (VI.223) follows from this by Lemma VI.3 (using condition (i)). The strict inequality in (VI.224) follows from (VI.223) and (III.77). Finally, the first inequality in (VI.224) is obvious from Lemma VI.1 and (III.77).

Remark VI.6 Note that due to the assumption that $\varrho, \sigma > 0$ and that in this case $\omega_{\alpha}^{\mathrm{Um}}(\varrho \| \sigma) > 0$, the strict inequality condition $D^{q_0} < D^{\mathrm{Um}}$ or $D^{q_1} < D^{\mathrm{Um}}$ in Proposition VI.5 can be weakened to the assumption that for any strictly positive $\omega \in \mathcal{S}(\mathcal{H})$ not commuting with ϱ and σ , $D^{q_0}(\omega \| \varrho) < D^{\mathrm{Um}}(\omega \| \varrho)$ or $D^{q_1}(\omega \| \sigma) < D^{\mathrm{Um}}(\omega \| \sigma)$ holds. In particular, it is sufficient to assume that $D^{q_0}(\omega_1 \| \omega_2) < D^{\mathrm{Um}}(\omega_1 \| \omega_2)$ for every non-commuting invertible ω_1, ω_2 , or $D^{q_1}(\omega_1 \| \omega_2) < D^{\mathrm{Um}}(\omega_1 \| \omega_2)$ for every non-commuting invertible ω_1, ω_2 .

Proposition VI.7 Let $D^{q_0}, D^{q_1} \geq D^{\mathrm{Um}}$, and assume that at least one of them is strictly larger than D^{Um} . Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$ be non-commuting invertible positive operators and $\alpha \in (0,1)$ be such that there exists an α -weighted $D^{\mathbf{q}}$ -center for (ϱ, σ) . Then

$$D_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) > D_{\alpha}^{\mathrm{b},\mathrm{Um}}(\varrho\|\sigma) \quad (= D_{\alpha,+\infty}(\varrho\|\sigma)).$$
 (VI.225)

Proof Follows immediately from Lemma VI.3 (using condition (ii)) and the fact that $\omega_{\alpha}^{\text{Um}}$ commutes with neither ϱ nor σ , as we have seen in the proof of Proposition VI.5.

Corollary VI.8 Let $D^{\mathrm{Um}} \leq D^{q_0}, D^{q_1} \leq D^{\mathrm{max}}$, and let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be non-commuting, and $\alpha \in (0,1)$. If there exists an α -weighted $D^{(q_0, \mathrm{max})}$ -center and an α -weighted $D^{(\mathrm{max}, q_1)}$ -center for (ϱ, σ) then

$$D_{\alpha}^{\mathrm{b,max}}(\varrho\|\sigma) \ge \left\{ \begin{array}{l} D_{\alpha}^{\mathrm{b},(q_0,\max)}(\varrho\|\sigma) \\ D_{\alpha}^{\mathrm{b},(\max,q_1)}(\varrho\|\sigma) \end{array} \right\} > D_{\alpha}^{\mathrm{b,Um}}(\varrho\|\sigma). \tag{VI.226}$$

Proof The first inequality in (VI.226) is obvious from Lemma VI.1 and (III.77). The strict inequality in (VI.226) follows from Proposition VI.7 and (III.77).

Corollary VI.9 For any two non-commuting invertible (ϱ, σ) ,

$$D_{\alpha}^{\mathrm{b,Um}}(\varrho \| \sigma) < D_{\alpha}^{\mathrm{b,max}}(\varrho \| \sigma), \qquad \alpha \in (0,1).$$

Proof Note that D^{\max} is jointly lower semi-continuous in its arguments, and hence for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ with $\varrho^0 \wedge \sigma^0 \neq 0$ and any $\alpha \in (0,1)$, there exists an α -weighted D^{\max} -center for (ϱ,σ) . Thus, the assertion follows from Corollary VI.8.

C. Maximal relative entropy and a smaller relative entropy

Lemma VI.10 Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be such that $\varrho^0 \wedge \sigma^0 \neq \{0\}$, and ϱ and σ do not have a common eigenvector. Then for any $\alpha \in (0,1)$, there exists an α -weighted D^{\max} -center for (ϱ, σ) such that it commutes with neither ϱ nor σ .

Proof Let $\mathcal{A} := \{\varrho\}' \cap \{\sigma\}'$ be the *-subalgebra of operators commuting with both ϱ and σ , and let P_1, \ldots, P_r , be a sequence of minimal projections in \mathcal{A} summing to I (in particular, $P_i \perp_{i \neq j} P_j$). Let $\Phi(\cdot) = \sum_{i=1}^r P_i(\cdot)P_i$ be the corresponding pinching operation. By the lower semi-continuity of D^{\max} in its first argument, there exists at least one state $\omega \in \mathcal{S}(\varrho^0\mathcal{H})$ attaining the infimum in (V.158). Moreover, for any $\omega \in \mathcal{S}(\varrho^0\mathcal{H})$, we have

$$\begin{split} &\frac{\alpha}{1-\alpha}D^{\max}(\omega\|\varrho) + D^{\max}(\omega\|\sigma) \\ &\geq \frac{\alpha}{1-\alpha}D^{\max}(\Phi(\omega)\|\underbrace{\Phi(\varrho)}) + D^{\max}(\Phi(\omega_{\alpha})\|\underbrace{\Phi(\sigma)}) \\ &= \frac{\alpha}{1-\alpha}D^{\max}(\Phi(\omega)\|\varrho) + D^{\max}(\Phi(\omega)\|\sigma), \end{split}$$

where the inequality follows by the monotonicity of D^{\max} under CPTP maps, and the equality is due to $\Phi(\varrho) = \varrho$ and $\Phi(\sigma) = \sigma$. Hence, there exists an optimal ω that also satisfies $\omega = \Phi(\omega) = \sum_i P_i \omega P_i$. Let ω_{α} be such an optimal state. Using then the block additivity of D^{\max} , we get

$$D_{\alpha}^{\mathrm{b,max}}(\varrho \| \sigma) = \sum_{i=1}^{r} \frac{\alpha}{1-\alpha} D^{\mathrm{max}}(P_{i}\omega_{\alpha}P_{i} \| P_{i}\varrho P_{i}) + D^{\mathrm{max}}(P_{i}\omega_{\alpha}P_{i} \| P_{i}\sigma P_{i}) - \frac{1}{\alpha-1}\log \operatorname{Tr} \varrho.$$

Assume now that ω_{α} commutes with both ϱ and σ , or equivalently, that $P_i\omega_{\alpha}P_i$ commutes with both $P_i\varrho P_i$ and $P_i\sigma P_i$ for every $i=1,\ldots,r$. Note that, by the definition of the P_i , the only operators of the form P_iXP_i that commute with both $P_i\varrho P_i$ and $P_i\sigma P_i$ are constant multiples of P_i . Therefore our assumption yields that there exists an i such that $P_i\omega_{\alpha}P_i=cP_i$ with some $c\in(0,+\infty)$. For the rest we may restrict the Hilbert space to ran P_i , and use the notations I for P_i , $\tilde{\varrho}$ for $P_i\varrho P_i$, and $\tilde{\sigma}$ for $P_i\sigma P_i$. Note that the assumption that $\varrho^0\wedge\sigma^0\neq 0$ yields that $D_{\alpha}^{\rm b,max}(\varrho\|\sigma)<+\infty$, according to (V.198). Thus, $D^{\rm max}(cI\|\tilde{\varrho})<+\infty$, $D^{\rm max}(cI\|\tilde{\varrho})<+\infty$, and therefore $\tilde{\varrho}^0=I=\tilde{\sigma}^0$, according to (III.68). By the definition of ω_{α} , for any self-adjoint traceless operator $X\in\mathcal{B}({\rm ran}\,P_i)$, and any $t\in(-c/\|X\|,c/\|X\|)$,

$$f_X(t) := \frac{\alpha}{1 - \alpha} D^{\max}(cI + tX \| \tilde{\varrho}) + D^{\max}(cI + tX \| \tilde{\varrho}) \ge \frac{\alpha}{1 - \alpha} D^{\max}(cI \| \tilde{\varrho}) + D^{\max}(cI \| \tilde{\varrho}). \tag{VI.227}$$

By the joint convexity of D^{\max} [56], $f_X(\cdot)$ is a convex function, and hence, by the above, it has a global minimum at t=0. Since it is also differentiable at t=0, as we show below, we get that

$$f'_X(0) = 0,$$
 $X \in \mathcal{B}(\operatorname{ran} P_i)_{\operatorname{sa}},$ $\operatorname{Tr} X = 0.$

We have

$$\begin{split} & \left. \frac{d}{dt} D^{\max}(cI + tX \| \tilde{\varrho}) \right|_{t=0} \\ & = \left. \frac{d}{dt} \operatorname{Tr} \, \tilde{\varrho}^{1/2}(cI + tX) \tilde{\varrho}^{-1/2} \log \left(\tilde{\varrho}^{-1/2}(cI + tX) \tilde{\varrho}^{-1/2} \right) \right|_{t=0} \\ & = \operatorname{Tr} \, \tilde{\varrho}^{1/2} X \tilde{\varrho}^{-1/2} \log \left(\underbrace{\tilde{\varrho}^{-1/2} cI \tilde{\varrho}^{-1/2}}_{= c\tilde{\varrho}^{-1}} \right) + \operatorname{Tr} \underbrace{\tilde{\varrho}^{1/2} cI \tilde{\varrho}^{-1/2}}_{= cI} \underbrace{\frac{d}{dt} \log \left(\tilde{\varrho}^{-1/2}(cI + tX) \tilde{\varrho}^{-1/2} \right) \right|_{t=0}}_{t=0}. \end{split}$$

Let $\tilde{\varrho} = \sum_{i=1}^r \lambda_i Q_i$ be the spectral decomposition of $\tilde{\varrho}$. Then the Fréchet derivative of log at $c\tilde{\varrho}^{-1}$ is the linear operator

$$D\log(c\tilde{\varrho}^{-1}): A \mapsto \sum_{i,j=1}^{r} \log^{[1]}(c/\lambda_i, c/\lambda_j) Q_i A Q_j, \qquad A \in \mathcal{B}(\operatorname{ran} P_i),$$

where

$$\log^{[1]}(x,y) = \begin{cases} \frac{\log x - \log y}{x - y}, & x \neq y, \\ 1/x, & x = y, \end{cases}$$

is the first divided difference function of log (see (II.10)). Thus,

$$\operatorname{Tr} cI \frac{d}{dt} \log \left(\tilde{\varrho}^{-1/2} (cI + tX) \tilde{\varrho}^{-1/2} \right) \Big|_{t=0}$$

$$= \operatorname{Tr} cI \sum_{i,j=1}^{r} \log^{[1]} \left(c/\lambda_{i}, c/\lambda_{j} \right) \underbrace{Q_{i} \tilde{\varrho}^{-1/2} X \tilde{\varrho}^{-1/2} Q_{j}}_{= \lambda_{i}^{-1/2} \lambda_{j}^{-1/2} Q_{i} X Q_{j}}$$

$$= c \sum_{i,j=1}^{r} \log^{[1]} \left(c/\lambda_{i}, c/\lambda_{j} \right) \lambda_{i}^{-1/2} \lambda_{j}^{-1/2} \underbrace{\operatorname{Tr} Q_{i} X Q_{j}}_{= \delta_{i,j} \operatorname{Tr} Q_{i} X}$$

$$= c \sum_{i=1}^{r} \frac{\lambda_{i}}{c} \frac{1}{\lambda_{i}} \operatorname{Tr} Q_{i} X = \operatorname{Tr} X.$$

By an exactly analogous computation for $\frac{d}{dt}D^{\max}(cI + tX \|\tilde{\sigma})\Big|_{t=0}$, we finally get

$$0 = f_X'(0) = \operatorname{Tr} X \left[\frac{\alpha}{1 - \alpha} \left(I + \log(c\tilde{\varrho}^{-1}) \right) + I + \log(c\tilde{\sigma}^{-1}) \right] = -\operatorname{Tr} X \left[\frac{\alpha}{1 - \alpha} \log \tilde{\varrho} + \log \tilde{\sigma} \right],$$

for any $X \in \mathcal{B}(\operatorname{ran} P_i)_{\operatorname{sa}}$ with $\operatorname{Tr} X = 0$. This is equivalent to the existence of some $\kappa \in \mathbb{R}$ such that

$$\frac{\alpha}{1-\alpha}\log\tilde{\varrho} + \log\tilde{\sigma} = \kappa I,$$

i.e.,

$$\tilde{\sigma} = e^{\kappa} \tilde{\rho}^{\frac{\alpha}{\alpha-1}}.$$

In particular, $\tilde{\varrho}$ and $\tilde{\sigma}$ have a common eigenvector $\psi \in \operatorname{ran} P_i$, which is also a common eigenvector of ϱ and σ , contradicting our initial assumptions.

Theorem VI.11 Let D^{q_0} and D^{q_1} be quantum relative entropies such that $D^{q_0} \leq D^{\max}$, $D^{q_1} \leq D^{\max}$, and at least one of the inequalities is strict. Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be such that $\varrho^0 \wedge \sigma^0 \neq \{0\}$, and ϱ and σ do not have a common eigenvector. Then

$$D_{\alpha}^{\mathbf{b},\mathbf{q}}(\varrho\|\sigma) < D_{\alpha}^{\mathbf{b},\max}(\varrho\|\sigma), \qquad \alpha \in (0,1).$$
 (VI.228)

Proof Immediate from Lemmas VI.3 and VI.10.

Corollary VI.12 Let D^{q_0} , D^{q_1} be lower semi-continuous in their first arguments, and assume that $D^{\mathrm{Um}} \leq D^{q_0}$, $D^{q_1} \leq D^{\mathrm{max}}$. Assume, moreover, that $D^{\mathrm{Um}} < D^{q_0}$ or $D^{\mathrm{Um}} < D^{q_1}$, and $D^{q_0} < D^{\mathrm{max}}$ or $D^{q_1} < D^{\mathrm{max}}$. Then for any two non-commuting ϱ , σ such that $\varrho^0 \wedge \sigma^0 \neq 0$ and ϱ and σ do not have a common eigenvector,

$$D_{\alpha}^{\mathrm{b},\mathrm{Um}}(\varrho\|\sigma) < D_{\alpha}^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) < D_{\alpha}^{\mathrm{b},\mathrm{max}}(\varrho\|\sigma), \qquad \alpha \in (0,1).$$

Proof Immediate from Proposition VI.7 and Theorem VI.11.

Example VI.13 Let D^{q_0} and D^{q_1} be quantum relative entropies as in Theorem VI.11. Let

$$\varrho := p \oplus (1-p) |\psi_1\rangle\langle\psi_1|$$
 and $\sigma := q \oplus (1-q) |\psi_2\rangle\langle\psi_2|$

be PSD operators on $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}^d$ for some d > 1, where $p, q \in (0, 1)$, and $\psi_1, \psi_2 \in \mathbb{C}^d$ are unit vectors that are neither parallel nor orthogonal. Then $\varrho^0 \wedge \sigma^0 = 1 \oplus 0$, and hence the unique optimal ω_α for any $\alpha \in (0, 1)$ and any barycentric Rényi α -divergence is $\omega_\alpha = 1 \oplus 0$. Thus,

$$\begin{split} D_{\alpha}^{\mathrm{b,max}}(\varrho \| \sigma) &= \frac{\alpha}{1 - \alpha} D^{\mathrm{max}}(\omega_{\alpha} \| \varrho) + D^{\mathrm{max}}(\omega_{\alpha} \| \sigma) + \kappa_{\varrho,\alpha} \\ &= \frac{\alpha}{1 - \alpha} (-\log p) - \log q + \kappa_{\varrho,\alpha} \\ &= \frac{\alpha}{1 - \alpha} D^{q_0}(\omega_{\alpha} \| \varrho) + D^{q_1}(\omega_{\alpha} \| \sigma) + \kappa_{\varrho,\alpha} \\ &= D_{\alpha}^{\mathrm{b,q}}(\varrho \| \sigma). \end{split}$$

This shows that the assumption that ϱ and σ do not have a common eigenvector cannot be completely omitted in Lemma VI.10 or in Theorem VI.11.

D. Maximal Rényi divergences vs. the barycentric maximal Rényi divergences

By Proposition VI.2, for every $\alpha \in (0,1)$, $D_{\alpha}^{\text{b,max}} \leq D_{\alpha}^{\text{max}}$. Our aim in this section is to show that equality does not hold. In fact, we conjecture the stronger relation that for non-commuting invertible PSD operators ϱ, σ , $D_{\alpha}^{\text{b,max}}(\varrho \| \sigma) < D_{\alpha}^{\text{max}}(\varrho \| \sigma)$, $\alpha \in (0,1)$, which is supported by numerical examples. We will prove this below in the special case where the inputs are 2-dimensional. Of course, this already gives at least that

$$D_{\alpha}^{\mathrm{b,max}} \lneq D_{\alpha}^{\mathrm{max}}, \qquad \alpha \in (0,1).$$

Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be such that $\varrho^0 \wedge \sigma^0 \neq 0$. Recall the definition of the maximal Rényi α -divergence and the optimal reverse test $(\hat{p}, \hat{q}, \hat{\Gamma})$ from Example III.30. Let $\omega_{\alpha} := \omega_{\alpha}(\hat{p} \| \hat{q}) = \sum_{x \in \text{supp } \hat{q}} \hat{p}(x)^{\alpha} \hat{q}(x)^{1-\alpha} \mathbf{1}_{\{x\}}$ be as in (V.171). Then for any $\alpha \in (0, 1)$,

$$\frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho + D_{\alpha}^{\max}(\varrho \| \sigma) = \frac{1}{\alpha - 1} \log \operatorname{Tr} \hat{p} + D_{\alpha}(\hat{p} \| \hat{q})$$
 (VI.229)

$$= \frac{\alpha}{1 - \alpha} D(\omega_{\alpha} || \hat{p}) + D(\omega_{\alpha} || \hat{q})$$
 (VI.230)

$$\geq \frac{\alpha}{1-\alpha} D^{\max}(\hat{\Gamma}(\omega_{\alpha}) \| \underbrace{\hat{\Gamma}(\hat{p})}_{=\varrho}) + D^{\max}(\hat{\Gamma}(\omega_{\alpha}) \| \underbrace{\hat{\Gamma}(\hat{q})}_{=\sigma}) \tag{VI.231}$$

$$= \frac{\alpha}{1 - \alpha} D^{\max}(\hat{\Gamma}(\omega_{\alpha}) \| \varrho) + D^{\max}(\hat{\Gamma}(\omega_{\alpha}) \| \sigma), \tag{VI.232}$$

where the first two equalities follow from Example III.30 and (III.49)–(III.50), the inequality is due to the monotonicity of D^{max} under positive trace-preserving maps, and the third equality is by the definition of $\hat{\Gamma}$. By (III.50) and (III.63)–(III.66),

$$\omega_{\alpha} = \sum_{i=1}^{r} \frac{\lambda_{i}^{\alpha} \operatorname{Tr} \sigma P_{i}}{Q_{\alpha}^{\max}(\varrho \| \sigma)} \mathbf{1}_{\{i\}}, \tag{VI.233}$$

$$Q_{\alpha}^{\max}(\varrho\|\sigma) = Q_{\alpha}(\hat{p}\|\hat{q}) = \sum_{i=1}^{r} \lambda_{i}^{\alpha} \operatorname{Tr} \sigma P_{i} = \operatorname{Tr} \sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha} = \operatorname{Tr} \sigma \#_{\alpha}\varrho,$$
(VI.234)

$$\hat{\Gamma}(\omega_{\alpha}) = \frac{1}{Q_{\alpha}^{\max}(\varrho\|\sigma)} \sum_{i} \lambda_{i}^{\alpha} \sigma^{1/2} P_{i} \sigma^{1/2} = \frac{1}{Q_{\alpha}^{\max}(\varrho\|\sigma)} \underbrace{\sigma^{1/2}(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha} \sigma^{1/2}}_{=\sigma\#_{\alpha}\rho} =: \widehat{\sigma\#_{\alpha}\varrho}, \quad (\text{VI}.235)$$

where $\sigma \#_{\alpha} \varrho$ is the α -weighted Kubo-Ando geometric mean of ϱ and σ (see Section IV).

The inequality in (VI.231) is in fact an equality, according to the following:

Lemma VI.14 Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be such that $\varrho^0 \wedge \sigma^0 \neq 0$, and assume that $\alpha \in (0,1)$, or that $\alpha \in (1,2]$ and $\varrho^0 \leq \sigma^0$. Then

$$D_{\alpha}^{\max}(\varrho\|\sigma) = \frac{\alpha}{1-\alpha} D^{\max}\left(\frac{\sigma\#_{\alpha}\varrho}{Q_{\alpha}^{\max}(\varrho\|\sigma)}\|\varrho\right) + D^{\max}\left(\frac{\sigma\#_{\alpha}\varrho}{Q_{\alpha}^{\max}(\varrho\|\sigma)}\|\sigma\right) - \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho. \tag{VI.236}$$

Proof Let $Q_{\alpha}^{\max} := Q_{\alpha}^{\max}(\varrho \| \sigma)$. Assume first that ϱ and σ are invertible, and recall that in this case,

$$\sigma \#_{\alpha} \varrho = \varrho \#_{1-\alpha} \sigma = \varrho^{1/2} (\varrho^{-1/2} \sigma \varrho^{-1/2})^{1-\alpha} \varrho^{1/2}; \tag{VI.237}$$

see (IV.104)–(IV.106). Thus, by (III.67),

$$D^{\max}\left(\frac{\sigma\#_{\alpha}\varrho}{Q_{\alpha}^{\max}(\varrho\|\sigma)}\|\varrho\right) = \frac{1}{Q_{\alpha}^{\max}}\operatorname{Tr}\varrho(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha}\log\frac{(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha}}{Q_{\alpha}^{\max}}$$

$$= -\frac{\log Q_{\alpha}^{\max}}{Q_{\alpha}^{\max}}\underbrace{\operatorname{Tr}\varrho(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha}}_{=\operatorname{Tr}\sigma\#_{\alpha}\varrho=Q_{\alpha}^{\max}}$$

$$+\frac{1-\alpha}{Q_{\alpha}^{\max}}\operatorname{Tr}\varrho(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha}\log(\varrho^{-1/2}\sigma\varrho^{-1/2})$$

$$= -\log Q_{\alpha}^{\max} - \frac{1-\alpha}{Q_{\alpha}^{\max}}\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\log(\sigma^{-1/2}\varrho\sigma^{-1/2}), \qquad (VI.238)$$

where in the last equality we used that the transpose function of $f(\cdot) := (\cdot)^{1-\alpha} \log(\cdot)$ is $\tilde{f}(\cdot) := -(\cdot)^{\alpha} \log(\cdot)$, whence

$$\operatorname{Tr} \varrho(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha}\log(\varrho^{-1/2}\sigma\varrho^{-1/2}) = -\operatorname{Tr} \sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\log(\sigma^{-1/2}\varrho\sigma^{-1/2}), \tag{VI.239}$$

according to (II.13). Similarly,

$$D^{\max}\left(\frac{\sigma\#_{\alpha}\varrho}{Q_{\alpha}^{\max}(\varrho\|\sigma)}\|\sigma\right) = \frac{1}{Q_{\alpha}^{\max}}\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\log\frac{(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}{Q_{\alpha}^{\max}}$$

$$= -\frac{\log Q_{\alpha}^{\max}}{Q_{\alpha}^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{=Q_{\alpha}^{\max}}$$

$$+ \frac{\alpha}{Q_{\alpha}^{\max}}\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\log(\sigma^{-1/2}\varrho\sigma^{-1/2}). \tag{VI.240}$$

From (VI.238) and (VI.240) we obtain that the RHS of (VI.236) is

$$\frac{1}{\alpha - 1} \log Q_{\alpha}^{\max} - \frac{1}{\alpha - 1} \log \operatorname{Tr} \varrho = D_{\alpha}^{\max}(\varrho \| \sigma),$$

where the equality is by definition. This proves (VI.236) for invertible ϱ and σ .

Assume now that $\alpha \in (0,1)$, or that $\alpha \in (1,2]$ and $\varrho^0 \leq \sigma^0$. By the above, for any $\varepsilon > 0$, (VI.236) holds with $\varrho_{\varepsilon} := \varrho + \varepsilon I$ and $\sigma_{\varepsilon} := \sigma + \varepsilon I$ in place of ϱ and σ , respectively, and taking the limit $\varepsilon \searrow 0$ yields (VI.236), according to (IV.107), (IV.108), and (III.70).

Remark VI.15 Using (VI.238)-(VI.240), a straightforward computation gives

$$D^{\max}(\hat{\Gamma}(\omega_{\alpha})\|\varrho) = D(\omega_{\alpha}\|\hat{p}) = -\log Q_{\alpha}^{\max}(\varrho\|\sigma) + (1-\alpha) \sum_{i} \omega_{\alpha}(i) \log \frac{\hat{q}(i)}{\hat{p}(i)},$$

$$D^{\max}(\hat{\Gamma}(\omega_{\alpha})\|\sigma) = D(\omega_{\alpha}\|\hat{q}) = -\log Q_{\alpha}^{\max}(\varrho\|\sigma) - \alpha \sum_{i} \omega_{\alpha}(i) \log \frac{\hat{q}(i)}{\hat{p}(i)}.$$

From these, the equality in (VI.231) can also be verified directly.

Remark VI.16 A different way of proving (VI.236) in the case $\alpha \in (0,1)$ is by noting that

$$\begin{split} \hat{\Gamma}\left(\omega_{\alpha}^{2}/\hat{p}\right) &= Q_{\alpha}^{\max}(\varrho\|\sigma)^{-2} \sum_{i} \lambda_{i}^{2\alpha-1} \sigma^{1/2} P_{i} \sigma^{1/2} = Q_{\alpha}^{\max}(\varrho\|\sigma)^{-2} \sigma^{1/2} (\sigma^{-1/2} \varrho \sigma^{-1/2})^{2\alpha-1} \sigma^{1/2}, \\ \hat{\Gamma}\left(\omega_{\alpha}\right) \hat{\Gamma}\left(\hat{p}\right)^{-1} \hat{\Gamma}\left(\omega_{\alpha}\right) &= Q_{\alpha}^{\max}(\varrho\|\sigma)^{-2} \sum_{i,j} \lambda_{i}^{\alpha} \lambda_{j}^{\alpha} \sigma^{1/2} P_{i} \underbrace{\sigma^{1/2} \varrho^{-1} \sigma^{1/2}}_{=\sum_{k} \lambda_{k}^{-1} P_{k}} P_{j} \sigma^{1/2} \\ &\underbrace{-\sum_{k} \lambda_{k}^{-1} P_{k}}_{=\delta_{i,j} \lambda_{i}^{-1} P_{i}} \\ &= Q_{\alpha}^{\max}(\varrho\|\sigma)^{-2} \sum_{i} \lambda_{i}^{2\alpha-1} \sigma^{1/2} P_{i} \sigma^{1/2} = \hat{\Gamma}\left(\omega_{\alpha}^{2}/p\right). \end{split}$$

Hence, by [32, Theorem 3.34], $D(\omega_{\alpha}\|\hat{p}) = D^{\max}(\hat{\Gamma}(\omega_{\alpha})\|\varrho)$. A completely analogous computation yields $D(\omega_{\alpha}\|\hat{q}) = D^{\max}(\hat{\Gamma}(\omega_{\alpha})\|\sigma)$. Thus, the inequality in (VI.231) is an equality.

Our aim now is to prove that $\widehat{\sigma\#_{\alpha}\varrho}$ is not an optimal ω in the variational formula (V.158) for $D_{\alpha}^{\mathrm{b,max}}$ when $\alpha \in (0,1)$. We prove this (at least in the 2-dimensional case) by showing that any state ω on the line segment connecting $\widehat{\sigma\#_{\alpha}\varrho}$ and the maximally mixed state $\pi_{\mathcal{H}}:=I/d, d:=\dim \mathcal{H}$, that is close enough to $\widehat{\sigma\#_{\alpha}\varrho}$ but is not equal to it, gives a strictly lower value than the RHS of (VI.236) when substituted into $\frac{\alpha}{1-\alpha}D^{\mathrm{max}}(\cdot\|\varrho)+D^{\mathrm{max}}(\cdot\|\sigma)-\frac{1}{\alpha-1}\log\mathrm{Tr}\,\varrho.$

Lemma VI.17 Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$, and let $P_1, \ldots, P_r \in \mathbb{P}(\mathcal{H})$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, be such that $\sum_{i=1}^r P_i = I$, and

$$\sigma^{-1/2}\varrho\sigma^{-1/2} = \sum_{i=1}^{r} \lambda_i P_i.$$

Then

$$\partial_{\pi_{\mathcal{H}}} := \frac{d}{dt} \left[\alpha D^{\max} \left((1 - t) \widehat{\sigma \#_{\alpha} \varrho} + t \pi_{\mathcal{H}} \| \varrho \right) + (1 - \alpha) D^{\max} \left((1 - t) \widehat{\sigma \#_{\alpha} \varrho} + t \pi_{\mathcal{H}} \| \sigma \right) \right] \Big|_{t=0}$$

$$= -1 + \frac{1}{d} \sum_{i,j} \operatorname{Tr} P_{i} \sigma P_{j} \sigma^{-1} \underbrace{\left[\alpha \log^{[1]} \left(\lambda_{i}^{\alpha - 1}, \lambda_{j}^{\alpha - 1} \right) \lambda_{i}^{\alpha - 1} + (1 - \alpha) \log^{[1]} \left(\lambda_{i}^{\alpha}, \lambda_{j}^{\alpha} \right) \lambda_{i}^{\alpha} \right]}_{=: \Lambda_{\alpha, i, i}}, \quad (VI.241)$$

where

$$\Lambda_{\alpha,i,j} = \begin{cases}
\alpha(1-\alpha)\left(\log \lambda_i - \log \lambda_j\right) \frac{(\lambda_i - \lambda_j)}{(\lambda_i^{\alpha} - \lambda_j^{\alpha})(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha})}, & \lambda_i \neq \lambda_j, \\
1, & \lambda_i = \lambda_j.
\end{cases}$$
(VI.242)

Proof We defer the slightly lengthy proof to Appendix B.

Our aim is therefore to prove that $\partial_{\pi_{\mathcal{H}}} < 0$. For this, we will need the following:

Lemma VI.18 The following equivalent inequalities are true: for every $\alpha \in (0,1)$,

$$\frac{\log \lambda - \log \eta}{\lambda - \eta} > \frac{1}{\alpha} \frac{\lambda^{\alpha} - \eta^{\alpha}}{\lambda - \eta} \cdot \frac{1}{1 - \alpha} \frac{\lambda^{1 - \alpha} - \eta^{1 - \alpha}}{\lambda - \eta}, \qquad \lambda, \eta \in (0, +\infty), \lambda \neq \eta, \qquad (VI.243)$$

$$\frac{\log x}{x-1} > \frac{1}{\alpha} \frac{x^{\alpha} - 1}{x-1} \frac{1}{1-\alpha} \frac{x^{1-\alpha} - 1}{x-1}, \qquad x \in (0, +\infty) \setminus \{1\}, \qquad (\text{VI}.244)$$

$$\int_{0}^{1} \frac{1}{tx+1-t} dt > \int_{0}^{1} \frac{1}{(tx+1-t)^{\alpha}} dt \int_{0}^{1} \frac{1}{(tx+1-t)^{1-\alpha}} dt, \qquad x \in (0,+\infty) \setminus \{1\}.$$
 (VI.245)

Proof It is straightforward to verify that the above inequalities are equivalent to each other. The inequality in (VI.245) follows from the strict concavity of the power functions, as

$$\int_0^1 \frac{1}{(tx+1-t)^{\gamma}} dt < \left(\int_0^1 \frac{1}{tx+1-t} dt \right)^{\gamma}, \qquad \gamma \in (0,1).$$

Corollary VI.19 In the setting of Lemma VI.17,

$$\Lambda_{\alpha,i,j} > 1, \qquad i \neq j.$$
 (VI.246)

Proof Immediate from (VI.243).

Note that we may take the P_i in Lemma VI.17 to be rank 1, i.e., $P_i = |e_i\rangle\langle e_i|$, $i = 1, \ldots, d$, for some orthonormal eigenbasis of $\sigma^{-1/2}\varrho\sigma^{-1/2}$. Then (VI.241) can be rewritten as

$$\partial_{\pi_{\mathcal{H}}} = -1 + \frac{1}{d} \sum_{i,j} \underbrace{\langle e_i, \sigma e_j \rangle}_{=:S_{i,j}} \underbrace{\langle e_j, \sigma^{-1} e_i \rangle}_{=(S^{-1})_{i,j}^{\mathrm{T}}} \cdot \Lambda_{\alpha,i,j}$$
 (VI.247)

$$= -1 + \langle u, (S \star (S^{-1})^{\mathrm{T}} \star \Lambda_{\alpha}) u \rangle, \qquad (VI.248)$$

where $u = \frac{1}{\sqrt{d}}(1, 1, \dots, 1)$ and $A \star B$ denotes the component-wise (also called Hadamard, or Schur) product of two matrices A and B.

Next, note that

$$(S^{-1})_{j,i} = (-1)^{i+j} \frac{\det([S]_{i,j})}{\det S},$$

where $[S]_{i,j}$ is the matrix that we get by omitting the *i*-th row and *j*-th column of S. Thus, (VI.247) can be rewritten as

$$\partial_{\pi_{\mathcal{H}}} = -1 + \frac{1}{d} \sum_{i=1}^{d} \frac{1}{\det S} \sum_{j=1}^{d} (-1)^{i+j} S_{i,j} \det([S]_{i,j}) \Lambda_{\alpha,i,j}.$$

Note that for every i,

$$\frac{1}{\det S} \sum_{i=1}^{d} (-1)^{i+j} S_{i,j} \det([S]_{i,j}) = (SS^{-1})_{i,i} = 1.$$

Theorem VI.20 Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$, where dim $\mathcal{H} = 2$, and assume that $\varrho\sigma \neq \sigma\varrho$. Then

$$D_{\alpha}^{\mathrm{b,max}}(\varrho \| \sigma) < D_{\alpha}^{\mathrm{max}}(\varrho \| \sigma), \quad \alpha \in (0,1).$$

Proof By Corollary V.33, we may assume that $\text{Tr } \varrho = \text{Tr } \sigma = 1$. By the above, it is sufficient to prove that $\partial_{\pi_{\mathcal{H}}} < 0$. Let (e_1, e_2) be an orthonormal eigenbasis of $\sigma^{-1/2}\varrho\sigma^{-1/2}$ with corresponding eigenvalues λ_1, λ_2 . By assumption, $\varrho\sigma \neq \sigma\varrho$, which implies that $\lambda_1 \neq \lambda_2$. (In fact, $\lambda_1 = \lambda_2 \iff \varrho = c\sigma$ for some c > 0, in which case $c = \lambda_1 = \lambda_2$.) Writing out everything in the ONB (e_1, e_2) , we have

$$S = \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

with some $r := (x, y, z) \in \mathbb{R}^3$ such that $||r||^2 = x^2 + y^2 + z^2 < 1$, and

$$(S^{-1})^{\mathrm{T}} = \frac{4}{1 - ||r||^2} \cdot \frac{1}{2} \begin{bmatrix} 1 - z & -x + iy \\ -x - iy & 1 + z \end{bmatrix},$$

whence

$$S \star (S^{-1})^{\mathrm{T}} = \frac{1}{1 - \|r\|^2} \begin{bmatrix} 1 - z^2 & -(x^2 + y^2) \\ -(x^2 + y^2) & 1 - z^2 \end{bmatrix}.$$

Hence, by (VI.247) and the symmetry $\Lambda_{\alpha,1,2} = \Lambda_{\alpha,2,1}$,

$$\partial_{\pi_{\mathcal{H}}} = -1 + \frac{1}{1 - \|r\|^2} \left[1 - z^2 - (x^2 + y^2) \Lambda_{\alpha, 1, 2} \right]. \tag{VI.249}$$

Since σ is not diagonal in the given ONB (otherwise it would commute with ϱ), we have $(x^2 + y^2) > 0$. Combining this with Corollary VI.19, we get $\partial_{\pi_{\mathcal{H}}} < 0$, as required.

VII. CONCLUSION AND OUTLOOK

We defined a new family of additive and monotone quantum relative entropies, and gave a general procedure of defining monotone multi-variate quantum Rényi divergences from monotone quantum relative entropies via a variational formula. For the latter, probably the biggest open question is additivity. While it is clear from the definition that if all the generating relative entropies are additive on tensor products then $D_P^{b,\mathbf{q}}$ is subadditive for any $P \in \mathcal{P}_f(\mathcal{X})$, but it is not clear whether superadditivity holds, and at the moment we see no general argument to establish it. Of course, additivity may be easily established when an explicit expression for $D_P^{b,\mathbf{q}}$ is available, as is the case when all the generating relative entropies coincide with the Umegaki relative entropy. This leads to another open question: to find explicit expressions for $D_P^{b,\mathbf{q}}$ at least for the most important quantum relative entropies; e.g., when $D^{q_x} = D^{\max}$ for all x, or more generally, when $D^{q_x} = D^{\text{Um},\#_{\gamma}}$ for some $\gamma \in (0,1)$. Of course, it might turn out that additivity does not hold in general; in this case the natural continuation would be the study of the regularized quantity

$$\overline{D}_P^{\mathbf{b},\mathbf{q}}(W) := \inf_{n \in \mathbb{N}} \frac{1}{n} D_P^{\mathbf{b},\mathbf{q}}(W^{\hat{\otimes}n}) = \lim_{n \to +\infty} \frac{1}{n} D_P^{\mathbf{b},\mathbf{q}}(W^{\hat{\otimes}n}),$$

where the equality follows from subadditivity. While this quantity is weakly additive by definition, establishing its additivity is still a non-trivial problem. We remark that there are a number of notable 2-variable quantum Rényi divergences that are not additive, but have very important regularizations; for instance, $D_{\alpha}^{\#}$ from [24], or the Rényi divergences considered very recently in [26, 39].

The study of multivariate quantum Rényi divergences seems to be a new initiative; the only other paper that we are aware of dealing with the subject is [27], and partly [16]. There are of course a host of interesting open problems in this direction, the most important probably being finding multi-variate extensions of the (2-variable) Petz-type and the sandwiched Rényi divergences. These have great operational significance in quantum information theory as quantifiers of the trade-off between the operational quantities in problems characterized by two competing operational quantities, and the goal would be to find multi-variate extensions of these Rényi divergences that play a similar role in problems characterized by multiple competing operational quantities. Such problems include multi-state conversion problems and state exclusion, where definitive results in terms of multi-variate Rényi divergences have been obtained very recently in the classical case [23, 57]. As for the multi-variate extension of the sandwiched Rényi divergences, probably the most natural candidate is the regularized measured Rényi divergence

$$\overline{D}_{P}^{\text{meas}}(W) := \lim_{n \to +\infty} \frac{1}{n} D_{P}^{\text{meas}}(W^{\hat{\otimes}n}) = \inf_{n \in \mathbb{N}} \frac{1}{n} D_{P}^{\text{meas}}(W^{\hat{\otimes}n}). \tag{VII.250}$$

The question of course is whether this has a closed-form expression, as is the case for two variables, and whether this quantity has analogous operational interpretations to the 2-variable version. An alternative approach could be to extend the definition given in [24] to the multi-variate case and then take regularization; here the choice of a multi-variate extension of the Kubo-Ando weighted geometric means emerges as a question at the first step.

Another possible approach seems to be the extension of a variational formula for the Rényi (α, z) -divergences. Assume for simplicity that $\mathcal{X} = \{0, \dots, r\}$ and that supp $P = \mathcal{X}$. When P(0) > 1, we define

$$Q_{P,z}(W) := \sup_{H \in \mathcal{B}(\mathcal{H})_{\geq 0}} \left\{ P(0) \operatorname{Tr} \left(H^{\frac{1}{2}} W_0^{\frac{P(0)}{z}} H^{\frac{1}{2}} \right)^{\frac{z}{P(0)}} + \sum_{k=1}^r P(k) \operatorname{Tr} \left(H^{\frac{1}{2}} W_0^{-\frac{P(k)}{z}} H^{\frac{1}{2}} \right)^{-\frac{z}{P(k)}} \right\}.$$

This coincides with $Q_{\alpha,z}(\varrho||\sigma)$ when r=1, $P(0)=\alpha$, and $W_0=\varrho$, $W_1=\sigma$; see [25, 59, 81]. Note that for any $B\in\mathcal{B}(\mathcal{H}),\,\mathcal{B}(\mathcal{H})_{\geq 0}\ni A\mapsto \mathrm{Tr}\,(B^*A^pB)^{\frac{1}{p}}$ is concave if $0\leq p\leq 1$, and convex if $1\leq p\leq 2$ [17, 25]. Thus, $Q_{P,z}$ is convex in W whenever $P(0)/2\leq z\leq P(0)$ and $-P(k)\leq z,\,k=1,\ldots,r$. According to

Lemma III.13, $D_{P,z}$ is monotone under CPTP maps if and only if $Q_{P,z}(W)$ is convex in W. In particular, the natural extension of the sandwiched Rényi divergences seems to be the choice when z = P(0), in which case

$$1 = P(0) + P(k) + \sum_{\substack{i \neq k \\ \leq 0}} P(i) \implies 1 \leq P(0) + P(k) \implies -P(k) \leq P(0) = z, \qquad k = 1, \dots, r,$$

i.e., $Q_{P,P(0)}$ is jointly convex for any P with P(0) > 1. The natural extension of the Petz-type Rényi divergences in this case is probably given by the choice z = 1, in which case convexity holds whenever $P(0) \in (1,2]$ (since $P(k) \ge -1$, $k = 1, \ldots, r$, holds due to $P(0) + \sum_{k=1}^{r} P(k) = 1$). One of the obvious questions here is whether $Q_{P,z}$ has a closed expression, at least in the multi-variate sandwiched and Petz-type versions. Another natural question is whether

$$D_{P,P(0)}(W) = \overline{D}_P^{\text{meas}}(W),$$

which is the case when r = 1 [63]. Yet another question is whether

$$\log Q_{P,z}(W) = \sup_{H \in \mathcal{B}(\mathcal{H})_{\geq 0}} \left\{ P(0) \log \operatorname{Tr} \left(H^{\frac{1}{2}} W_0^{\frac{P(0)}{z}} H^{\frac{1}{2}} \right)^{\frac{z}{P(0)}} + \sum_{k=1}^r P(k) \log \operatorname{Tr} \left(H^{\frac{1}{2}} W_0^{-\frac{P(k)}{z}} H^{\frac{1}{2}} \right)^{-\frac{z}{P(k)}} \right\},$$

analogously to the 2-variable case given in [59, Lemma 3.23]. Note that this could be an alternative definition of the multi-variate Rényi (α, z) -divergences.

While the maximal Rényi divergences have no known direct operational interpretation, they nevertheless have a distinguished role as the largest monotone Rényi divergences, as well as because of their close connection to the famous Kubo-Ando weighted geometric means. While their definition is straightforward also in the multi-variate case, it is an open question whether they can be given by an explicit formula (at least when the weights are non-negative), similarly to the 2-variable case. Here it may be expected that a relation of the form $Q_P^{\max}(W) = \operatorname{Tr} G_P^q(W)$ holds, where G_P^q is some multi-variate extension of the Kubo-Ando weighted geometric means; one natural candidate would be the Karcher mean. A closely related, but not completely identical question is whether a multi-variate extension of (VI.236) holds as

$$-\log Q_P^{\max}(W) = \sum_{x \in \mathcal{X}} P(x) D^{\max}\left(\frac{G_P^q(W)}{\operatorname{Tr} G_P^q(W)} \middle\| W_x\right)$$

with some multi-variate non-commutative weighted geometric mean G_P^q .

ACKNOWLEDGMENTS

This work was partially funded by the National Research, Development and Innovation Office of Hungary via the research grants K124152 and KH129601, and by the Ministry of Culture and Innovation and the National Research, Development and Innovation Office within the Quantum Information National Laboratory of Hungary (Grant No. 2022-2.1.1-NL-2022-00004). The work of Péter Vrana was further supported by a Bolyai fellowship of the Hungarian Academy of Sciences, and by the ÚNKP-21-5 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund. We thank Mark Wilde for comments on a previous version of the paper.

Appendix A: Proof of Lemma III.13

Proof Let $W^{(i)} \in \mathcal{D}_{\mathcal{H}}(\Delta)$, $i \in [r]$, let $(t_i)_{i \in [r]}$ be a probability distribution, and let $(|i\rangle)_{i \in [r]}$ be an orthonormal system in some Hilbert space \mathcal{K} . Then $\left(\sum_{i=0}^{r-1} t_i W_x^{(i)} \otimes |i\rangle\langle i|\right)_{x \in \mathcal{X}} \in \mathcal{D}_{\mathcal{H} \otimes \mathcal{K}}(\Delta)$ due to the

isometric invariance of Δ and the convexity of $\mathcal{D}_{\mathcal{H}\otimes\mathcal{K}}(\Delta)$, and

$$\Delta\left(\left(\sum_{i=0}^{r-1} t_i W_x^{(i)}\right)_{x \in \mathcal{X}}\right) = \Delta\left(\left(\operatorname{Tr}_{\mathcal{K}} \sum_{i=0}^{r-1} t_i W_x^{(i)} \otimes |i\rangle\langle i|\right)_{x \in \mathcal{X}}\right)$$

$$\geq \Delta\left(\left(\sum_{i=0}^{r-1} t_i W_x^{(i)} \otimes |i\rangle\langle i|\right)_{x \in \mathcal{X}}\right)$$

$$\geq \sum_{i=0}^{r-1} \Delta\left(\left(t_i W_x^{(i)} \otimes |i\rangle\langle i|\right)_{x \in \mathcal{X}}\right)$$

$$= \sum_{i=0}^{r-1} t_i \Delta\left(\left(W_x^{(i)} \otimes |i\rangle\langle i|\right)_{x \in \mathcal{X}}\right)$$

$$= \sum_{i=0}^{r-1} t_i \Delta\left(\left(W_x^{(i)} \otimes |i\rangle\langle i|\right)_{x \in \mathcal{X}}\right),$$

where the first equality is obvious, the first inequality is by the assumption that Δ is monotone nondecreasing under partial traces, the second inequality is due to the block superadditivity of Δ , the second equality follows from homogeneity, and the last equality is due to the isometric invariance of Δ . This proves joint concavity, and joint superadditivity follows from it immediately due to homogeneity.

Assume now that Δ is jointly concave and it is stable under tensoring with the maximally mixed state. Let $W \in \mathcal{D}_{\mathcal{H}}(\Delta)$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a CPTP map such that $\Phi(W) \in \mathcal{D}_{\mathcal{K}}(\Delta)$. Let $\Phi(.) = \operatorname{Tr}_E V(.)V^*$ be a Stinespring representation of Φ , where $V : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{K}$ is an isometry. Let $(U_{ab})_{a,b=0}^{d_E-1}$ be the discrete Weyl unitaries in some ONB of \mathcal{H}_E , so that $(1/d_E^2) \sum_{a,b=0}^{d_E-1} U_{a,b}(.)U_{a,b}^* = (1/d_E)I_E$ Tr(.) Then

$$\Delta(\Phi(W)) = \Delta \left(\operatorname{Tr}_{E} VWV^{*} \right)$$

$$= \Delta \left((1/d_{E})I_{E} \otimes \operatorname{Tr}_{E} VWV^{*} \right)$$

$$= \Delta \left(\frac{1}{d_{E}^{2}} \sum_{a,b=0}^{d_{E}-1} (U_{a,b} \otimes I_{K})W(U_{a,b} \otimes I_{K})^{*} \right)$$

$$\geq \frac{1}{d_{E}^{2}} \sum_{a,b=0}^{d_{E}-1} \Delta \left((U_{a,b} \otimes I_{K})W(U_{a,b} \otimes I_{K})^{*} \right)$$

$$= \frac{1}{d_{E}^{2}} \sum_{a,b=0}^{d_{E}-1} \Delta \left(W \right) = \Delta \left(W \right),$$

where the second equality is due to stability, the inequality is due to the joint concavity of Δ , and the fourth equality is due to isomeric invariance.

Appendix B: Proof of Lemma VI.17.

Proof Let us introduce the notation

$$(\varrho/\sigma) := \sigma^{-1/2}\varrho\sigma^{-1/2} = \sum_{i} \lambda_{i} P_{i}.$$

Let

$$X := \varrho^{1/2} \sigma^{-1/2}, \quad \text{and} \quad X = U|X| = U \sum_{i} \lambda_i^{1/2} P_i$$
 (B.1)

be its polar decomposition. Then

$$\varrho^{-1/2}\sigma\varrho^{-1/2} = (X^{-1})^*(X^{-1}) = U\left(\sum_{i} \lambda_i^{-1} P_i\right) U^* = \sum_{i} \lambda_i^{-1} \underbrace{UP_i U^*}_{=:B_i}$$
(B.2)

is a spectral decomposition of $\varrho^{-1/2}\sigma\varrho^{-1/2}$. Recall from (VI.235) that

$$\widehat{\sigma\#_{\alpha}\varrho}:=\frac{1}{Q_{\alpha}^{\max}}\sigma\#_{\alpha}\varrho=\frac{1}{Q_{\alpha}^{\max}}\varrho\#_{1-\alpha}\sigma,$$

where $Q_{\alpha}^{\max} := Q_{\alpha}^{\max}(\varrho \| \sigma)$, and note the following identities:

$$\begin{split} \sigma^{-1/2}\widehat{\sigma\#_{\alpha}\varrho}\,\sigma^{-1/2} &= \frac{1}{Q_{\alpha}^{\max}}(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha} = \sum_{i} \frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}}P_{i}, \\ \varrho^{-1/2}\widehat{\sigma\#_{\alpha}\varrho}\,\varrho^{-1/2} &= \frac{1}{Q_{\alpha}^{\max}}(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha} = \sum_{i} \frac{\lambda_{i}^{\alpha-1}}{Q_{\alpha}^{\max}}R_{i}, \end{split}$$

where in the last line we used (VI.237).

Recall that $\pi_{\mathcal{H}} = I/d$ denotes the maximally mixed state on \mathcal{H} . We have

$$\begin{split} &\frac{d}{dt}D^{\max}\left((1-t)\widehat{\sigma\#_{\alpha}\varrho}+t\pi_{\mathcal{H}}\|\sigma\right)\Big|_{t=0} \\ &=\operatorname{Tr}\underbrace{\sigma^{1/2}\left(\pi_{\mathcal{H}}-\widehat{\sigma\#_{\alpha}\varrho}\right)\sigma^{-1/2}\log\underbrace{\sigma^{-1/2}\widehat{\sigma\#_{\alpha}\varrho}\sigma^{-1/2}}_{=\overline{Q_{\max}^{\max}}(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}} = \underbrace{\frac{1}{Q_{\max}^{\max}}(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{=\overline{Q_{\max}^{\max}}(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}} \\ &+\operatorname{Tr}\underbrace{\sigma^{1/2}\widehat{\sigma\#_{\alpha}\varrho}\sigma^{-1/2}}_{=\overline{Q_{\max}^{\max}}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}} \sum_{i,j}\log^{[1]}\left(\frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}},\frac{\lambda_{j}^{\alpha}}{Q_{\alpha}^{\max}}\right)\underbrace{P_{i}\sigma^{-1/2}\left(\pi_{\mathcal{H}}-\widehat{\sigma\#_{\alpha}\varrho}\right)\sigma^{-1/2}P_{j}}_{=d^{-1}P_{i}\sigma^{-1}P_{j}-\frac{1}{Q_{\alpha}^{\max}}\delta_{i,j}\lambda_{i}^{\alpha}P_{i}} \\ &=-\log Q_{\alpha}^{\max}+\frac{\alpha}{d}\operatorname{Tr}\log(\sigma^{-1/2}\varrho\sigma^{-1/2}) \\ &+\frac{1}{Q_{\alpha}^{\max}}(\log Q_{\alpha}^{\max})\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{=Q_{\alpha}^{\max}}-\frac{\alpha}{Q_{\alpha}^{\max}}\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\log(\sigma^{-1/2}\varrho\sigma^{-1/2}) \\ &+\frac{1}{dQ_{\alpha}^{\max}}\sum_{i,j}\log^{[1]}\left(\frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}},\frac{\lambda_{j}^{\alpha}}{Q_{\alpha}^{\max}}\right)\operatorname{Tr}\sigma\underbrace{(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}P_{i}\sigma^{-1}P_{j}}_{=\lambda_{i}^{\alpha}P_{i}\sigma^{-1}P_{j}} \\ &-\frac{1}{Q_{\alpha}^{\max}}\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\sum_{i}\log^{[1]}\left(\frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}},\frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}}\right)\frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}}P_{i} \\ &=\frac{\alpha}{d}\operatorname{Tr}\log(\sigma^{-1/2}\varrho\sigma^{-1/2})-\frac{\alpha}{Q_{\alpha}^{\max}}\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}\log(\sigma^{-1/2}\varrho\sigma^{-1/2}) \\ &+\frac{1}{dQ_{\alpha}^{\max}}\sum_{i,j}\log^{[1]}\left(\frac{\lambda_{i}^{\alpha}}{Q_{\alpha}^{\max}},\frac{\lambda_{j}^{\alpha}}{Q_{\alpha}^{\max}}\right)\lambda_{i}^{\alpha}\operatorname{Tr}\sigma P_{i}\sigma^{-1}P_{j}-\frac{1}{Q_{\alpha}^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma(\sigma^{-1/2}\varrho\sigma^{-1/2})^{\alpha}}_{-\Omega^{\max}}\underbrace{\operatorname{Tr}\sigma$$

An exactly analogous calculation yields

$$\begin{split} &\frac{d}{dt}D^{\max}\left((1-t)\widehat{\sigma\#_{\alpha}\varrho}+t\pi_{\mathcal{H}}\|\varrho\right)\Big|_{t=0} \\ &=\frac{1-\alpha}{d}\operatorname{Tr}\log(\varrho^{-1/2}\sigma\varrho^{-1/2})-\frac{1-\alpha}{Q_{\alpha}^{\max}}\operatorname{Tr}\varrho(\varrho^{-1/2}\sigma\varrho^{-1/2})^{1-\alpha}\log(\varrho^{-1/2}\sigma\varrho^{-1/2}) \\ &+\frac{1}{dQ_{\alpha}^{\max}}\sum_{i,j}\log^{[1]}\left(\frac{\lambda_{i}^{\alpha-1}}{Q_{\alpha}^{\max}},\frac{\lambda_{j}^{\alpha-1}}{Q_{\alpha}^{\max}}\right)\lambda_{i}^{\alpha-1}\operatorname{Tr}\varrho R_{i}\varrho^{-1}R_{j}-1. \end{split}$$

Thus,

$$\partial_{\pi_{\mathcal{H}}} = \frac{d}{dt} \left[\alpha D^{\max} \left((1 - t) \widehat{\sigma_{\#\alpha} \varrho} + t \pi_{\mathcal{H}} \| \varrho \right) + (1 - \alpha) D^{\max} \left((1 - t) \widehat{\sigma_{\#\alpha} \varrho} + t \pi_{\mathcal{H}} \| \sigma \right) \right] \Big|_{t=0}$$

$$= \frac{\alpha (1 - \alpha)}{d} \left[\underbrace{\operatorname{Tr} \log(\varrho^{-1/2} \sigma \varrho^{-1/2})}_{= \sum_{i} \log \lambda_{i}^{-1}} + \underbrace{\operatorname{Tr} \log(\sigma^{-1/2} \varrho \sigma^{-1/2})}_{= \sum_{i} \log \lambda_{i}} \right]$$

$$- \frac{\alpha (1 - \alpha)}{Q_{\alpha}^{\max}} \left[\underbrace{\operatorname{Tr} \varrho(\varrho^{-1/2} \sigma \varrho^{-1/2})^{1 - \alpha} \log(\varrho^{-1/2} \sigma \varrho^{-1/2})}_{= 0} + \operatorname{Tr} \sigma(\sigma^{-1/2} \varrho \sigma^{-1/2})^{\alpha} \log(\sigma^{-1/2} \varrho \sigma^{-1/2}) \right]$$

$$+ \frac{\alpha}{d} \sum_{i,j} \log^{[1]} \left(\lambda_{i}^{\alpha - 1}, \lambda_{j}^{\alpha - 1} \right) \lambda_{i}^{\alpha - 1} \operatorname{Tr} \varrho R_{i} \varrho^{-1} R_{j} + \frac{1 - \alpha}{d} \sum_{i,j} \log^{[1]} \left(\lambda_{i}^{\alpha}, \lambda_{j}^{\alpha} \right) \lambda_{i}^{\alpha} \operatorname{Tr} \sigma P_{i} \sigma^{-1} P_{j} - 1$$

$$= \frac{\alpha}{d} \sum_{i,j} \log^{[1]} \left(\lambda_{i}^{\alpha - 1}, \lambda_{j}^{\alpha - 1} \right) \lambda_{i}^{\alpha - 1} \operatorname{Tr} \varrho R_{i} \varrho^{-1} R_{j} + \frac{1 - \alpha}{d} \sum_{i,j} \log^{[1]} \left(\lambda_{i}^{\alpha}, \lambda_{j}^{\alpha} \right) \lambda_{i}^{\alpha} \operatorname{Tr} \sigma P_{i} \sigma^{-1} P_{j} - 1,$$

$$(B.3)$$

where the first expression above is equal to 0 due to (B.2), and the second expression is equal to 0 according to (VI.239).

Note that by (B.1),

$$U = X|X|^{-1} = \varrho^{1/2}\sigma^{-1/2}(\varrho/\sigma)^{-1/2},$$

whence

$$\begin{array}{rcl} U^* & = & (\varrho/\sigma)^{-1/2}\sigma^{-1/2}\varrho^{1/2} \\ \parallel & & \\ U^{-1} & = & (\varrho/\sigma)^{1/2}\sigma^{1/2}\varrho^{-1/2}, \end{array}$$

which in turn yields

$$U = (U^{-1})^* = \varrho^{-1/2} \sigma^{1/2} (\varrho/\sigma)^{1/2}.$$

Thus,

$$\operatorname{Tr} \varrho R_{i} \varrho^{-1} R_{j} = \operatorname{Tr} \varrho (U P_{i} U^{*}) \varrho^{-1} (U P_{j} U^{*})$$

$$= \operatorname{Tr} \varrho \underbrace{\varrho^{-1/2} \sigma^{1/2} (\varrho / \sigma)^{1/2}}_{=U} P_{i} \underbrace{(\varrho / \sigma)^{-1/2} \sigma^{-1/2} \varrho^{1/2}}_{=U^{*}} \varrho^{-1} \underbrace{\varrho^{1/2} \sigma^{-1/2} (\varrho / \sigma)^{-1/2}}_{=U} P_{j} \underbrace{(\varrho / \sigma)^{1/2} \sigma^{1/2} \varrho^{-1/2}}_{=U^{*}}$$

$$= \operatorname{Tr} \sigma^{1/2} \underbrace{(\varrho / \sigma)^{1/2} P_{i} (\varrho / \sigma)^{-1/2}}_{=P_{i}} \sigma^{-1} \underbrace{(\varrho / \sigma)^{-1/2} P_{j} (\varrho / \sigma)^{1/2}}_{=P_{j}} \sigma^{1/2}$$

$$= \operatorname{Tr} \sigma P_{i} \sigma^{-1} P_{j}.$$

Writing this back into (B.3), we get

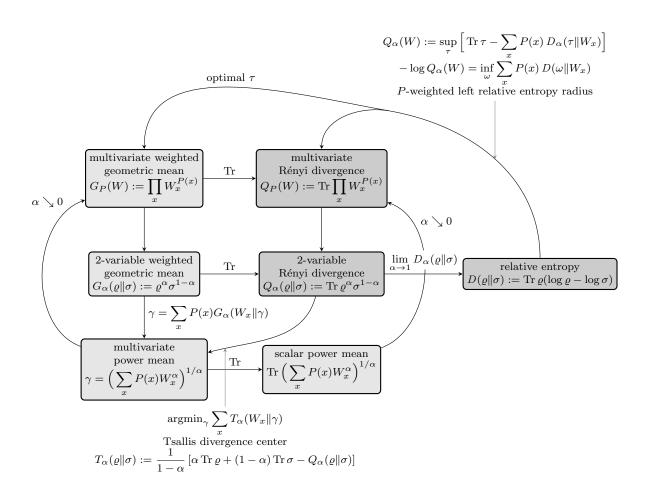
$$\partial_{\pi_{\mathcal{H}}} = -1 + \frac{1}{d} \sum_{i,j} \operatorname{Tr} \sigma P_i \sigma^{-1} P_j \underbrace{\left[\alpha \log^{[1]} \left(\lambda_i^{\alpha - 1}, \lambda_j^{\alpha - 1} \right) \lambda_i^{\alpha - 1} + (1 - \alpha) \log^{[1]} \left(\lambda_i^{\alpha}, \lambda_j^{\alpha} \right) \lambda_i^{\alpha} \right]}_{=:\Lambda_{\alpha,i,j}}. \tag{B.4}$$

It follows by a straightforward computation that $\Lambda_{\alpha,i,j}$ can be written as in (VI.242). Note that Λ_{α} is symmetric, i.e., $\Lambda_{\alpha,i,j} = \Lambda_{\alpha,j,i}$. Exchanging the indices i and j in (B.4) yields (VI.241).

Appendix C: Different approaches to multi-variate Rényi divergences

Figure 1 below shows some relations between 2-variable and multi-variate weighted geometric means, Rényi divergences, weighted power means, divergence radii and centers, and fixpoint equations, an arrow indicating that one quantity can be obtained from the other. These work when all operators are commuting; non-commutative extensions of any of these quantities may by obtained by extending any quantity in the diagram to non-commutative variables, and trying to follow the arrows to obtain non-commutative extensions of other quantities. For instance, a non-commutative multi-variate weighted geometric mean (the so-called Karcher mean) was obtained in [54] by taking the $\alpha \to 0$ limit of the solution of the fixed point equation $\gamma = \sum_x P(x) \gamma \#_\alpha W_x$. In Sections V–VI of this paper we focused on the top right corner of the diagram. One might try the same procedure starting with a different 2-variable weighted geometric mean; for instance, Proposition A.24 and Remark A.27 in [65] show that the Tsallis divergence center and the solution of the fixed point equation corresponding to the Petz-type weighted geometric mean $G_{\alpha,1}(\varrho||\sigma) := \sigma^{\frac{1-\alpha}{2}} \varrho^\alpha \sigma^{\frac{1-\alpha}{2}}$ give the same result $(\sum_x P(x)W_x^\alpha)^{1/\alpha}$, and it is easy to see that (at least when $W_x^0 = I$, $x \in \text{supp } P$) its limit at $\alpha \to 1$ is $G_P^{\text{DUm}}(W) = \exp(\sum_x P(x)\log W_x)$. This shows, in particular, that the diagram is not commutative in the quantum case. Another example of this is that the solution of the fixpoint equation might differ from the Tsallis divergence center corresponding to the same non-commutative 2-variable weighted geometric mean; see, e.g., [73] for the case $G_\alpha = \#_\alpha$.

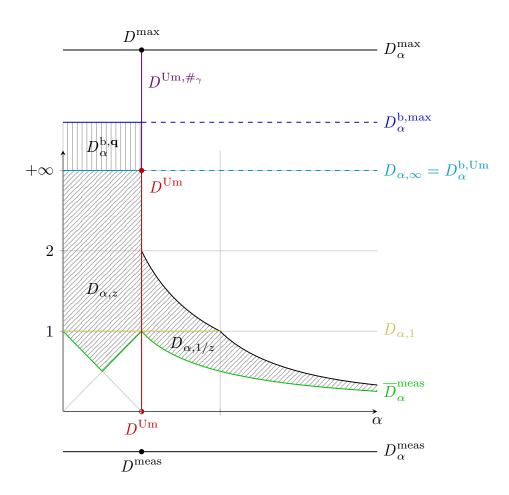
FIG. 1. Different approaches to multi-variate Rényi divergences



Appendix D: Order of relative entropies and Rényi divergences

Figure 2 below illustrates the known relations between the most relevant relative entropies and 2-variable Rényi divergences studied previously in the literature, and the new relative entropies and Rényi divergences introduced in this paper. A divergence higher in the picture gives a higher value when evaluated on a given pair of inputs than a divergence lower in the picture with the same α coordinate. It is easy to see from the Araki-Lieb-Thirring inequality [5, 53] that the Rényi (α, z) -divergences are monotone increasing in the z parameter when $\alpha \in (0,1)$, and monotone decreasing when $\alpha > 1$, (see, e.g., [55]), which is why the z coordinate is transformed into 1/z for $\alpha > 1$ in the representation of the Rényi (α, z) -divergences. The diagonally shaded area shows the region in the (α, z) -plane where the Rényi (α, z) -divergences are monotone [81]. The vertically shaded area is an illustration of the position of the barycentric Rényi divergences compared to the Rényi (α, z) -divergences when the generating relative entropies are monotone and additive, and hence are between $D^{\rm Um}$ and $D^{\rm max}$.

FIG. 2. Order of 2-variable quantum Rényi divergences



- [1] Martial Agueh and Guillaume Carlier. Barycenters in the Wasserstein space. SIAM Journal on Mathematical Analysis, 43(2):904–924, 2011.
- [2] W. N. Anderson, Jr. and G. E. Trapp. Shorted operators. II. SIAM Journal on Applied Mathematics, 28(1):60-71, 1975.
- [3] T. Ando, Chi-Kwong Li, and Roy Mathias. Geometric means. Linear Algebra and its Applications, 385:305—334, 2004.
- [4] Tsuyoshi Ando and Fumio Hiai. Log majorization and complementary Golden-Thompson type inequalities. Linear Algebra and its Applications, 197-198:113-131, 1994.
- [5] H. Araki. On an inequality of Lieb and Thirring. Letters in Mathematical Physics, 19:167–170, 1990.
- [6] K. M. R. Audenaert, M. Nussbaum, A. Szkola, and F. Verstraete. Asymptotic error rates in quantum hypothesis testing. Communications in Mathematical Physics, 279:251–283, 2008. arXiv:0708.4282.
- [7] Koenraad M. R. Audenaert and Nilanjana Datta. α-z-relative Renyi entropies. J. Math. Phys., 56:022202, 2015. arXiv:1310.7178.
- [8] Salman Beigi. Sandwiched Rényi divergence satisfies data processing inequality. Journal of Mathematical Physics, 54(12):122202, December 2013. arXiv:1306.5920.
- V. P. Belavkin and P. Staszewski. C*-algebraic generalization of relative entropy and entropy. Ann. Inst. H. Poincaré Phys. Théor., 37:51–58, 1982.
- [10] Mario Berta, Omar Fawzi, and Marco Tomamichel. On variational expressions for quantum relative entropies. Letters in Mathematical Physics, 107(12):2239–2265, 2017. arXiv:1512.02615.
- [11] Rajendra Bhatia. Matrix Analysis. Number 169 in Graduate Texts in Mathematics. Springer, 1997.
- [12] Rajendra Bhatia, Stephane Gaubert, and Tanvi Jain. Matrix versions of the Hellinger distance. Letters in Mathematical Physics, 2019.
- [13] Rajendra Bhatia and John Holbrook. Riemannian geometry and matrix geometric means. *Linear Algebra and its Applications*, 413(2):594–618, 2006. Special Issue on the 11th Conference of the International Linear Algebra Society, Coimbra, 2004.
- [14] Rajendra Bhatia, Tanvi Jain, and Yongdo Lim. On the Bures-Wasserstein distance between positive definite matrices. Expositiones Mathematicae, 2018.
- [15] F. G. S. L. Brandao, M. Horodecki, N. H. Y. Ng, J. Oppenheim, and S. Wehner. The second laws of quantum thermodynamics. Proceedings of the National Academy of Sciences USA, 112(1):3275–3279, 2014.
- [16] Gergely Bunth and Péter Vrana. Equivariant relative submajorization. IEEE Transactions on Information Theory, 2022. arXiv:2108.13217.
- [17] Eric A. Carlen and Elliott H. Lieb. A Minkowski type trace inequality and strong subadditivity of quantum entropy II: Convexity and concavity. Letters in Mathematical Physics, 83:107–126, 2008.
- [18] Imre Csiszár. Generalized cutoff rates and Rényi's information measures. *IEEE Transactions on Information Theory*, 41(1):26–34, January 1995.
- [19] Imre Csiszár and František Matúš. Information projections revisited. IEEE Transactions on Information Theory, 49(6):1474–1490, 2003.
- [20] Nilanjana Datta. Min- and max-relative entropies and a new entanglement monotone. IEEE Transactions on Information Theory, 55(6):2816–2826, 2009.
- [21] A. Ebadian, I. Nikoufar, and M. Eshaghi Gordji. Perspectives of matrix convex functions. Proc. Natl. Acad. Sci. USA, 108(18):7313-7314, 2011.
- [22] E. Effros. A matrix convexity approach to some celebrated quantum inequalities. *Proc. Natl. Acad. Sci. USA*, 106(4):1006–1008, 2009.
- [23] Muhammad Usman Farooq, Tobias Fritz, Erkka Haapasalo, and Marco Tomamichel. Asymptotic and catalytic matrix majorization. arXiv:2301.07353, 2023.
- [24] Hamza Fawzi and Omar Fawzi. Defining quantum divergences via convex optimization. Quantum, 5:387, January 2021.
- [25] Rupert L. Frank and Elliott H. Lieb. Monotonicity of a relative Rényi entropy. Journal of Mathematical Physics, 54(12):122201, December 2013. arXiv:1306.5358.
- [26] Péter E. Frenkel. Integral formula for quantum relative entropy implies data processing inequality. Quantum, 7:1102, 2023. arXiv:2208.12194.
- [27] Keiichiro Furuya, Nima Lashkari, and Shoy Ouseph. Monotonic multi-state quantum f-divergences. J. Math. Phys., 64:042203, 2023.
- [28] Masahito Hayashi. Error exponent in asymmetric quantum hypothesis testing and its application to classical-quantum channel coding. *Physical Review A*, 76(6):062301, December 2007. arXiv:quant-ph/0611013.
- [29] Masahito Hayashi, Masato Koashi, Keiji Matsumoto, Fumiaki Morikoshi, and Andreas Winter. Error exponents for entanglement concentration. Journal of Physics A: Mathematical and General, 36(2):527, 2002.
- [30] Masahito Hayashi and Marco Tomamichel. Correlation detection and an operational interpretation of the Rényi mutual information. *Journal of Mathematical Physics*, 57:102201, 2016.
- [31] F. Hiai. Matrix analysis: Matrix monotone functions, matrix means, and majorization. Interdisciplinary Information Sciences, 16:139–248, 2010.
- [32] F. Hiai and M. Mosonyi. Different quantum f-divergences and the reversibility of quantum operations. Rev. Math. Phys., 29:1750023, 2017.

- [33] F. Hiai, M. Mosonyi, D. Petz, and C. Bény. Quantum f-divergences and error correction. Rev. Math. Phys., 23:691-747, 2011.
- [34] Fumio Hiai. Concavity of certain matrix trace and norm functions. Linear Algebra Appl., 439:1568–1589, 2013.
- [35] Fumio Hiai. Quantum f-divergences in von Neumann algebras. II. Maximal f-divergences. J. Math. Phys., 50:012203, 2019.
- [36] Fumio Hiai and Yongdo Lim. Operator means of probability measures. Advances in Mathematics, 365:107038, 2020.
- [37] Fumio Hiai and Dénes Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Communications in Mathematical Physics*, 143(1):99–114, December 1991.
- [38] Fumio Hiai and Dénes Petz. The Golden-Thompson trace inequality is complemented. Linear Algebra Appl., 181:153–185, 1993.
- [39] Christoph Hirche and Marco Tomamichel. Quantum rényi and f-divergences from integral representations. arXiv:2306.12343, 2023.
- [40] A. Jenčová. Rényi relative entropies and noncommutative L_p -spaces II. Annales Henri Poincaré, 22:3235–3254, 2021. arXiv:1707.00047.
- [41] Asger Kjærulff Jensen. Asymptotic majorization of finite probability distributions. *IEEE Transactions on Information Theory*, 65(12):8131–8139, 2019. arXiv:1808.05157.
- [42] Asger Kjærulff Jensen and Péter Vrana. The asymptotic spectrum of LOCC transformations. *IEEE Transactions on Information Theory*, 66(1):155–166, January 2019. arXiv:1807.05130.
- [43] Sejong Kim and Hosoo Lee. The power mean and the least squares mean of probability measures on the space of positive definite matrices. *Linear Algebra and its Applications*, 465:325–346, 2015.
- [44] M. Klimesh. Inequalities that collectively completely characterize the catalytic majorization relation. arXiv:0709.3680, 2007.
- [45] F. Kubo and T. Ando. Means of positive linear operators. Math. Ann., 246:205–224, 1980.
- [46] J. Lawson and Y. Lim. Monotonic properties of the least squares mean. Math. Ann., 351:267–279, 2011.
- [47] J. Lawson and Y. Lim. Karcher means and Karcher equations of positive definite operators. *Trans. Amer. Math. Soc. Series B*, 1:1–22, 2014.
- [48] Ke Li and Yongsheng Yao. Reliability function of quantum information decoupling via the sandwiched Rényi divergence. arXiv:2111.06343, 2021.
- [49] Ke Li and Yongsheng Yao. Reliable simulation of quantum channels. arXiv:2112.04475, 2021.
- [50] Ke Li and Yongsheng Yao. Operational interpretation of the sandwiched Rényi divergence of order 1/2 to 1 as strong converse exponents. arXiv:2209.00554, 2022.
- [51] Ke Li and Yongsheng Yao. Strong converse exponent for entanglement-assisted communication. arXiv:2209.00555, 2022.
- [52] Ke Li, Yongsheng Yao, and Masahito Hayashi. Tight exponential analysis for smoothing the max-relative entropy and for quantum privacy amplification. *IEEE Transactions on Information Theory*, 69(3):1680–1694, 2023.
- [53] E.H. Lieb and W. Thirring. Studies in mathematical physics. University Press, Princeton, 1976.
- [54] Yongdo Lim and Miklós Pálfia. Matrix power means and the karcher mean. Journal of Functional Analysis, 262(4):1498–1514, February 2012.
- [55] Mingyan Simon Lin and Marco Tomamichel. Investigating properties of a family of quantum renyi divergences. Quantum Information Processing, 14(4):1501–1512, 2015.
- [56] K. Matsumoto. A new quantum version of f-divergence. In Nagoya Winter Workshop 2015: Reality and Measurement in Algebraic Quantum Theory, pages 229–273, 2018.
- [57] H.K. Mishra and M.M. Wilde M. Nussbaum. On the optimal error exponents for classical and quantum antidistinguishability. arXiv:2309.03723, 2023.
- [58] Maher Moakher. A differential geometric approach to the geometric mean of symmetric positive-definite matrices. SIAM Journal on Matrix Analysis and Applications, 26(3):735–747, 2005.
- [59] Milán Mosonyi. The strong converse exponent of discriminating infinite-dimensional quantum states. Communications in Mathematical Physics, 2023. arXiv:2107.08036.
- [60] Milán Mosonyi and Fumio Hiai. On the quantum Rényi relative entropies and related capacity formulas. *IEEE Transactions on Information Theory*, 57(4):2474–2487, April 2011.
- [61] Milán Mosonyi and Fumio Hiai. Some continuity properties of quantum Rényi divergences. arXiv:2209.00646, 2023.
- [62] Milán Mosonyi and Fumio Hiai. Test-measured Rényi divergences. IEEE Transactions on Information Theory, 69(2):1074–1092, 2023. arXiv:2201.05477.
- [63] Milán Mosonyi and Tomohiro Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. Communications in Mathematical Physics, 334(3):1617–1648, 2015. arXiv:1309.3228.
- [64] Milán Mosonyi and Tomohiro Ogawa. Strong converse exponent for classical-quantum channel coding. Communications in Mathematical Physics, 355(1):373–426, June 2017. arXiv:1409.3562.
- [65] Milán Mosonyi and Tomohiro Ogawa. Divergence radii and the strong converse exponent of classical-quantum channel coding with constant compositions. *IEEE Transactions on Information Theory*, 67(3):1668–1698, 2021. arXiv:1811.10599.

- [66] X. Mu, L. Pomatto, P. Strack, and O. Tamuz. From Blackwell dominance in large samples to Rényi divergences and back again. *Econometrica*, 89(1):475–506, 2020.
- [67] Alexander Müller-Hermes and David Reeb. Monotonicity of the quantum relative entropy under positive maps. Ann. Henri Poincaré, 18:1777–1788, 2017.
- [68] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum Rényi entropies: A new generalization and some properties. *Journal of Mathematical Physics*, 54(12):122203, December 2013. arXiv:1306.3142.
- [69] Hiroshi Nagaoka. The converse part of the theorem for quantum Hoeffding bound. arXiv:quant-ph/0611289, November 2006.
- [70] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.
- [71] Dénes Petz. Quasi-entropies for finite quantum systems. Reports in Mathematical Physics, 23:57–65, 1986.
- [72] Dénes Petz and Róbert Temesi. Means of positive numbers and matrices. SIAM Journal on Matrix Analysis and Applications, 27(3):712–720, 2005.
- [73] J. Pitrik and D. Virosztek. Quantum Hellinger distances revisited. Letters in Mathematical Physics, 110:2039–2052, 2020.
- [74] Alfréd Rényi. On measures of entropy and information. In Proc. 4th Berkeley Sympos. Math. Statist. and Prob., volume I, pages 547–561. Univ. California Press, Berkeley, California, 1961.
- [75] Helmut Strasser. Mathematical Theory of Statistics. Walter de Gruyter, Berlin, New York, 1985.
- [76] M. Tomamichel. Quantum Information Processing with Finite Resources, volume 5 of Mathematical Foundations, SpringerBriefs in Math. Phys. Springer, 2016.
- [77] Joel A. Tropp. From joint convexity of quantum relative entropy to a concavity theorem of Lieb. *Proceedings* of the American Mathematical Society, 140(5):1757–1760, 2011.
- [78] Sadi Turgut. Catalytic transformations for bipartite pure states. Journal of Physics A: Mathematical and Theoretical, 40(40):12185, 2007. arXiv:0707.0444.
- [79] H. Umegaki. Conditional expectation in an operator algebra, IV: Entropy and information. Kōdai Math. Sem. Rep., 14:59–85, 1962.
- [80] Mark M. Wilde, Andreas Winter, and Dong Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Communications in Mathematical Physics*, 331(2):593–622, October 2014. arXiv:1306.1586.
- [81] Haonan Zhang. From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture. Advances in Mathematics, 365:107053, 2020. arXiv:1811.01205.