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On the weighted Bojanov-Chebyshev problem and the sum of translates method of Fenton

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Abstract. Minimax and maximin problems are investigated for a special class of functions on the interval $[0,1]$. These functions are sums of translates of positive multiples of one kernel function and a very general external field function. Due to our very general setting the minimax, equioscillation and characterization results obtained extend those of Bojanov, Fenton, Hardin, Kendall, Saff, Ambrus, Ball and Erdélyi. Moreover, we discover a surprising intertwining phenomenon of interval maxima, which provides new information even in the most classical extremal problem of Chebyshev.

Bibliography: 25 titles.

Keywords: minimax problem, Chebyshev polynomial, weighted Bojanov problem, kernel function, sum of translates function.

§ 1. Introduction

Our starting point is the following theorem of Bojanov (see Theorem 1 in [6]).

Theorem 1.1. *For $n \in \mathbb{N}$ let $\nu_1, \nu_2, \dots, \nu_n > 0$ be positive integers. Given an interval $[a,b]$ $(a<b)$, there exists a unique set of points $x_1^* \leqslant \dots \leqslant x_n^*$ such that*

$$\|(x-x_1^*)^{\nu_1}\cdots(x-x_n^*)^{\nu_n}\| = \inf_{a\leqslant x_1 \leqslant \dots \leqslant x_n \leqslant b} \|(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n}\|,$$

where $\|\cdot\|$ is the sup norm over $[a,b]$. Moreover, $a < x_1^* < \dots < x_n^* < b$, and the extremal polynomial $T(x) := (x-x_1^*)^{\nu_1}\cdots(x-x_n^*)^{\nu_n}$ is uniquely determined by the equioscillation property: there is $(t_k)_{k=0}^n$ with $a = t_0 < t_1 < \dots < t_n = b$ such that

$$T(t_k) = (-1)^{\nu_{k+1}+\dots+\nu_n}\|T\|, \qquad k = 0, 1, \dots, n.$$

This contains the classical and well-investigated Chebyshev problem — where all the ν_j are equal to 1. Generalizations of the original Chebyshev problem were extensively studied for systems $\{\varphi_j\}_{j=1}^n$ with the so-called Chebyshev or Haar property [17] that all generalized 'polynomials' (linear combinations) $\sum_{j=1}^n \alpha_j\varphi_j$ have at most n zeros (on the interval $[a,b]$), but direct adaptations of this approach abort very early, as here the polynomials occurring for the Bojanov problem do not even form a vector space, not to speak of the fact that the number of zeros can be as large as $\nu_1 + \dots + \nu_n$.

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As in the Chebyshev problem, it is natural to consider weighted maximum norms. Note that $\|wf\| = C \Leftrightarrow -C \leqslant f(x)w(x) \leqslant C$, $x \in [a,b]$, with equality at some points. We set $W(x) := 1/w(x)$, consider $-CW(x) \leqslant f(x) \leqslant CW(x)$, $x \in [a,b]$, and minimize C needed for the validity of these bounds. If $\nu_1 = \cdots = \nu_n = 1$, then very general results are known, even for nonsymmetric weights satisfying $-CU(x) \leqslant f(x) \leqslant CV(x)$. Extremal polynomials — called 'snake polynomials' — equioscillate between these bounds [19]. For more on this we refer the reader, for example, to [9], [8], [20] and [23].

We consider only equal lower and upper bounds. This is natural from the point of view of our approach, which can deal with absolute values, but not with signs. However, in the case of algebraic polynomials as above it is easy to trace back the signs taken on various intervals between zeros, and Bojanov's original results can easily be recovered.[1] Let us only note here that our discussion will reveal that in the above Bojanov theorem we can as well say that the unique system of optimizing nodes is characterized by the attainment, in all the intervals $[a,x_1^*], [x_1^*,x_2^*], \ldots, [x_{n-1}^*,x_n^*]$ and $[x_n^*,b]$, at some points $t_0, t_1, \ldots, t_n$, of the norm: $|T(t_i)| = \|T\|$, $i = 0,1,\ldots,n$, — so for characterization there is no need to assume signed equioscillation, but the only thing that really matters is the attainment of the norm. Note that equioscillation properties are important in approximation theory (see the papers by Bojanov and Naidenov [3], [4], Bojanov and Rahman [5] and Nikolov and Shadrin [21], [22]).

Our approach allows us to consider arbitrary positive real exponents ν_i, and therefore to address the Bojanov-type extremal problem for so-called *generalized algebraic polynomials* (cf. [7]). In fact, the setting of this paper is more general. Let us point out that Bojanov used in his papers the classical, Chebyshev-Markov style approximation theoretic approach, with fine tracing of zeros, monotonicity observations, interlacing properties, zero counting and so on. Bojanov's result was never extended to the weighted case or to trigonometric polynomials.

In [11] we used an approach similar to the one presented here to address general minimax problems on the torus, and obtained (generalized) trigonometric polynomial and also generalized algebraic polynomial versions of Bojanov's theorem. However, the approach there involved a pull-back of the interval to the torus, with nodes on the interval corresponding to a *pair of symmetric trigonometric nodes* on the torus. If the extremal situation were not symmetric, then the backward transfer would not work. For this reason the method can at best be generalized to the weighted case only if we also assume the evenness of the weight on the interval (normalizing it to $[-1,1]$). However, our aim here is to obtain results valid also for not necessarily even weights, and the setting is thus different from the one in [11]. The motivation for considering the torus case in [11] was the polarization problem (also see [1] and [18]). Here our applications are different.

The Bojanov-type minimization problem can immediately be rephrased as a minimax problem by taking the logarithm:

$$\log\left|(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n}\right| = \sum_{j=1}^{n} \nu_j \log|x-x_j|.$$

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[1] We leave this to the reader throughout, though.

Thus the original multiplicative extremal problem is reformulated as an additive one, namely, to minimize (with respect to $x_1,\dots,x_n$ subject to $x_1 \leqslant \cdots \leqslant x_n$) the quantity $\max_{[a,b]} \sum_{j=1}^n \nu_j \log|x-x_j|$. The weighted norm version is:

$$\text{minimize} \quad \left\|(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n} w(x)\right\|_{C([a,b])} \quad \text{in} \quad x_1 \leqslant \cdots \leqslant x_n.$$

After taking the logarithm (and assuming that $w(x) \geqslant 0$) the problem becomes:

$$\text{minimize} \quad \max_{[a,b]} \left(\log w(x) + \sum_{i=1}^n \nu_i \log|x-x_i| \right) \quad \text{in} \quad x_1 \leqslant \cdots \leqslant x_n.$$

We formulate just one model result of our investigations in a concrete situation.

Theorem 1.2. *Let $n \in \mathbb{N}$, and let $r_1, r_2, \dots, r_n$ be positive numbers, $[a,b]$ be a non-degenerate compact interval, and w be an upper semicontinuous, nonnegative weight function on $[a,b]$ assuming nonzero values at at least n interior points plus at least one more point in $[a,b]$. Then there exists a unique extremizer set of points $a \leqslant x_1^* \leqslant \cdots \leqslant x_n^* \leqslant b$ such that*

$$\left\|w(x)|x-x_1^*|^{r_1}\cdots|x-x_n^*|^{r_n}\right\| = \inf_{a\leqslant x_1\leqslant\cdots\leqslant x_n\leqslant b} \left\|w(x)|x-x_1|^{r_1}\cdots|x-x_n|^{r_n}\right\|$$

(for $\|\cdot\|$ being the sup norm over $[a,b]$) and, in fact, $a < x_1^* < \cdots < x_n^* < b$.

Moreover, the extremal generalized polynomial $T(x) := \prod_{i=1}^n |x-x_i^*|^{r_i}$ is uniquely determined by the following equioscillation property: there exists $(t_i)_{i=0}^n$ interlacing with the x_i^*, that is, $a \leqslant t_0 < x_1^* < t_1 < x_2^* < \cdots < x_n^* < t_n \leqslant b$, such that

$$w(t_k)T(t_k) = \|wT\|, \qquad k = 0, 1, \dots, n.$$

The real origins of the sum of translates approach go back to Fenton (see [13]), whose original aim was to prove a conjecture due to Barry from 1962 about the growth of entire functions, in which he succeeded in [12]. Even if it turned out that Barry's original problem had been already solved by Goldberg [16] slightly earlier, later on Fenton showed further fruitful applications of his approach (see [14] and [15]). Our main results extend and develop Fenton's original work on the sum of translates function and related minimax problems. Here we do not discuss possible further applications to the theory of entire functions. What we do explore somewhat are a few applications to approximation theory. Apart from the Bojanov-direction sketched above, we show that some seemingly unrelated questions, like, for example, Chebyshev constants of the union of k intervals, can also be dealt with. Here it bears a crucial relevance that the weights we allow here are only assumed to be upper semicontinuous, in contrast to [11], where logarithmic concavity was a fundamental assumption.

The most interesting findings are in Theorem 4.1, because the phenomena described there do not seem to have been observed so far, not even in the most classical case of the original Chebyshev problem. It is well known that Chebyshev nodes have the property that for any other node system some of the arising interval maxima stay below, while some of the interval maxima become larger than the maximum of the Chebyshev polynomial (which, by equioscillation, is attained as the interval maximum on all the $n+1$ intervals between neighbouring nodes and the endpoints). But we will prove that this 'intertwining property' of interval maxima is not at all the unique feature of the extremal Chebyshev nodes — in fact, *any two node systems* exhibit this intertwining property with each other.

§ 2. The basic setting

A function $K\colon (-1,0)\cup(0,1)\to\mathbb{R}$ is called a *kernel function*[2] if it is concave on $(-1,0)$ and on $(0,1)$ and satisfies

$$\lim_{t\downarrow 0} K(t) = \lim_{t\uparrow 0} K(t). \tag{2.1}$$

These limits exist by the concavity assumption, and a kernel function also has one-sided limits at -1 and 1. We set

$$K(0) := \lim_{t\to 0} K(t), \qquad K(-1) := \lim_{t\downarrow -1} K(t) \quad \text{and} \quad K(1) := \lim_{t\uparrow 1} K(t).$$

Note explicitly that we thus obtain the extended continuous function $K\colon [-1,1]\to \mathbb{R}\cup\{-\infty\} =: \underline{\mathbb{R}}$, and that we still have $\sup K < \infty$. Note that a kernel function is almost everywhere differentiable.

We say that the kernel function K is *strictly concave* if it is strictly concave on both the intervals $(-1,0)$ and $(0,1)$.

Further, we call it *monotone*[3] if

$$K \text{ is monotonically decreasing on } (-1,0) \text{ and monotonically increasing on } (0,1). \tag{M}$$

By concavity, under the monotonicity condition (M) the values $K(-1)$ and $K(1)$ are also finite. If K is strictly concave, then (M) implies *strict monotonicity*:

$$K \text{ is strictly decreasing on } [-1,0) \text{ and strictly increasing on } (0,1]. \tag{SM}$$

A kernel function K is called *singular* if

$$K(0) = -\infty. \tag{∞}$$

This condition is fundamental for the intended applications, so in this paper we confine ourselves in general to singular kernels.

Let $n\in\mathbb{N}=\{1,2,\ldots,\}$ be fixed. We call a function $J\colon[0,1]\to\underline{\mathbb{R}}$ an *external n-field function*[4] or — if the value of n is unambiguous from the context — simply a *field function*, if it is bounded above on $[0,1]$ and takes finite values at more than

[2] The terminology used by Fenton in [13] is that K is a *cusp*, perhaps better fitting to his settings where K is not assumed to satisfy the singularity condition (∞) below, but rather the 'derivative singularity' condition $\lim_{t\to 0\pm 0} K'(t) = \pm\infty$.

[3] These conditions — and more, like C^2-smoothness and strictly negative second derivatives — were imposed on the kernel functions in the ground-breaking paper of Fenton [13].

[4] Again, the terminology of kernels and fields came to our mind by analogy: in the case of the logarithmic kernel $K(t) := \log|t|$ and an external field $J(t)$ arising from a weight $w(t) := \exp(J(t))$ they are indeed discussed in logarithmic potential theory. However, in our analysis no further potential-theoretic notions or tools are used.

n different points, where we count the points 0 and 1 with weight[5] 1/2 only, while the points in $(0,1)$ are accounted for with weight 1. Therefore, for a field function[6] J the set $(0,1) \setminus J^{-1}(\{-\infty\})$ has at least n elements, and if it has precisely n elements, then either $J(0)$ or $J(1)$ is finite.

Further, we consider the *open simplex*

$$S := S_n := \big\{\mathbf{y}\colon \mathbf{y} = (y_1, \dots, y_n) \in (0,1)^n,\ 0 < y_1 < \dots < y_n < 1\big\}$$

and its closure, the *closed simplex*

$$\overline{S} := \big\{\mathbf{y}\colon \mathbf{y} \in [0,1]^n,\ 0 \leqslant y_1 \leqslant \dots \leqslant y_n \leqslant 1\big\}.$$

Given $n \in \mathbb{N}$, a kernel function K, constants $r_j > 0$, $j = 1, \dots, n$, and a field function J, we consider the *pure sum of translates function*

$$f(\mathbf{y}, t) := \sum_{j=1}^{n} r_j K(t - y_j), \qquad \mathbf{y} \in \overline{S}, \quad t \in [0,1], \tag{2.2}$$

and also the (*weighted*) *sum of translates function*

$$F(\mathbf{y}, t) := J(t) + \sum_{j=1}^{n} r_j K(t - y_j), \qquad \mathbf{y} \in \overline{S}, \quad t \in [0,1]. \tag{2.3}$$

Note that the functions J and K can take the value $-\infty$, but not $+\infty$, and therefore the sum of translates functions can be defined consistently. Furthermore, if $g, h\colon A \to \underline{\mathbb{R}}$ are extended continuous functions on some topological space A, then their sum is extended continuous, too; therefore, $f\colon \overline{S} \times [0,1] \to \underline{\mathbb{R}}$ is extended continuous. Note that for any $\mathbf{y} \in \overline{S}$ the function $f(\mathbf{y}, \cdot\,)$ is finite-valued on $(0,1) \setminus \{y_1, \dots, y_n\}$. Moreover, $f(\mathbf{y}, 0) = -\infty$ can occur only when some $y_j = 0$ (hence also $y_1 = 0$) and $K(0) = -\infty$ or if some $y_j = 1$ (and so certainly $y_n = 1$) and $K_j(-1) = -\infty$. An analogous statement can be made about the equality $f(\mathbf{y}, 1) = -\infty$. By the assumption[7] on J, we have $F(\mathbf{y}, \cdot\,) \not\equiv -\infty$, that is, $\sup_{t \in [0,1]} F(\mathbf{y}, t) > -\infty$.

Further, for any fixed $\mathbf{y} \in \overline{S}$ and $t \neq y_1, \dots, y_n$ there exists a relative (with respect to $\overline{S}$) open neighbourhood of $\mathbf{y} \in \overline{S}$ where $f(\,\cdot\,, t)$ is concave (hence continuous). Indeed, such a neighbourhood is $B(\mathbf{y}, \delta) := \{\mathbf{x} \in \overline{S}\colon \|\mathbf{x} - \mathbf{y}\| < \delta\}$ for $\delta := \min_{j=1,\dots,n} |t - y_j|$, where $\|\mathbf{v}\| := \max_{j=1,\dots,n} |v_j|$.

We introduce the *singularity set* of the field function J by

$$X := X_J := \{t \in [0,1]\colon J(t) = -\infty\}, \tag{2.4}$$

and recall that $X^c := [0,1] \setminus X$ has cardinality exceeding n (in the weighted sense described above); in particular, $X \neq [0,1]$.

[5]Weighted counting makes a difference only for the case when $J^{-1}(\{-\infty\})$ contains the two endpoints; with only $n-1$ further interior points in $(0,1)$ the weights in this configuration add up to n only, hence the node system is considered inadmissible.

[6]To keep coherence with our companion paper [10], we do not assume *a priori* that a field function must be upper semicontinuous. However, in this paper this mild extra assumption is needed throughout — hence it is signalled in the formulation of all assertions relying also on this additional assumption.

[7]Note that our somewhat complicated-looking assumptions on the weighted counting of points of finiteness of J is *the exact condition* to ensure this independently of the concrete choice of the kernels in general.

Writing $y_0 := 0$ and $y_{n+1} := 1$, for each $\mathbf{y} \in \overline{S}$ and $j \in \{0, 1, \ldots, n\}$ we also set

$$I_j(\mathbf{y}) := [y_j, y_{j+1}], \qquad m_j(\mathbf{y}) := \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t),$$

$$\overline{m}(\mathbf{y}) := \max_{j=0,\ldots,n} m_j(\mathbf{y}) = \sup_{t \in [0,1]} F(\mathbf{y}, t) \quad \text{and} \quad \underline{m}(\mathbf{y}) := \min_{j=0,\ldots,n} m_j(\mathbf{y}).$$

As said above, for each $\mathbf{y} \in \overline{S}$ the quantity $\overline{m}(\mathbf{y}) = \sup_{t \in [0,1]} F(\mathbf{y}, t) \in \mathbb{R}$ is finite. On the other hand $m_j(\mathbf{y}) = -\infty$ if and only if $F(\mathbf{y}, \cdot)|_{I_j(\mathbf{y})} \equiv -\infty$, in which case $I_j(\mathbf{y})$ is called a singular interval (for the given node system). If there is $j \in \{0, 1, \ldots, n\}$ such that $m_j(\mathbf{y}) = -\infty$, then $\mathbf{y}$ is called a *singular* node system. A node system $\mathbf{y} \subset \partial S = \overline{S} \setminus S$ is called *degenerate*. If the kernels are singular, then each degenerate node system is singular, and, furthermore, for a nondegenerate node system $\mathbf{y}$ we have $m_j(\mathbf{y}) = -\infty$ if and only if $\operatorname{rint} I_j(\mathbf{y}) \subseteq X$, where rint denotes the relative interior of a set with respect to $[0, 1]$.

If K is singular, then the functions m_j are extended continuous. This is not immediately obvious because of the arbitrariness of J, but it was proved in [10] as Lemma 3.3. We record this fact here explicitly for later reference.

Proposition 2.1. *Let K be a singular kernel function and J be an arbitrary n-field function. Then for each $j \in \{0, 1, \ldots, n\}$ the function*

$$m_j \colon \overline{S} \to \underline{\mathbb{R}}$$

is continuous (in the extended sense). Moreover, $\overline{m}, \underline{m} \colon \overline{S} \to \mathbb{R}$ are extended continuous and $\overline{m}$ is finite valued and continuous in the usual sense.

Remark 2.1. For a singular kernel K, an arbitrary n-field function J, any node system $\mathbf{w} \in \overline{S}$ and $k \in \{0, \ldots, n\}$, if $I_k(\mathbf{w})$ is degenerate or singular, then $m_k(\mathbf{w}) = -\infty$, hence $m_k(\mathbf{w}) < \overline{m}(\mathbf{w})$.

We will primarily be interested in the minimax and maximin expressions

$$M(S) := \inf_{\mathbf{x} \in S} \overline{m}(\mathbf{x}) = \inf_{\mathbf{x} \in S} \max_{j=0,1,\ldots,n} m_j(\mathbf{x}) \tag{2.5}$$

and

$$m(S) := \sup_{\mathbf{x} \in S} \underline{m}(\mathbf{x}) = \sup_{\mathbf{x} \in S} \min_{j=0,1,\ldots,n} m_j(\mathbf{x}). \tag{2.6}$$

In this respect an essential role is played by the *regularity set*

$$Y := Y_n = \big\{\mathbf{y} \in S \colon m_j(\mathbf{y}) \neq -\infty,\ j = 0, 1, \ldots, n\big\}. \tag{2.7}$$

Since J is an n-field function we necessarily have $Y \neq \varnothing$, and we trivially have

$$m(Y) := \sup_{\mathbf{x} \in Y} \underline{m}(\mathbf{x}) = \sup_{\mathbf{x} \in S} \underline{m}(\mathbf{x}) = m(S) = \sup_{\mathbf{x} \in \overline{S}} \underline{m}(\mathbf{x}) =: m(\overline{S}).$$

It will turn out as a byproduct of our results (see Theorem 3.1) that also

$$M(Y) := \inf_{\mathbf{x} \in Y} \overline{m}(\mathbf{x}) = \inf_{\mathbf{x} \in S} \overline{m}(\mathbf{x}) = M(S) = \inf_{\mathbf{x} \in \overline{S}} \overline{m}(\mathbf{x}) =: M(\overline{S}).$$

As a matter of fact, for a singular, strictly concave kernel function K and an upper semicontinuous n-field function J there is a unique system $\mathbf{w} \in \overline{S}$ such that $\overline{m}(\mathbf{w}) = M(S)$; in fact, $\mathbf{w} \in Y$ and it is the unique node system in $\overline{S}$ such that $\underline{m}(\mathbf{w}) = m(S)$, see Corollary 3.2. Before we can prove these facts, some further preparation is needed.

If the kernel K is singular (a main assumption in this paper), then we have

$$Y = \{\mathbf{y} \in S\colon \operatorname{rint} I_j(\mathbf{y}) \not\subseteq X,\ j = 0, 1, \dots, n\}. \tag{2.8}$$

In case the kernel K is singular, it follows from (2.8) that the regularity set is an open subset of S, and we have $S = Y$ if and only if X has an empty interior.

We also introduce the *interval maxima vector function*

$$\mathbf{m}(\mathbf{w}) := (m_0(\mathbf{w}), m_1(\mathbf{w}), \dots, m_n(\mathbf{w})) \in \underline{\mathbb{R}}^{n+1}, \qquad \mathbf{w} \in \overline{S},$$

and the *interval maxima difference function* or simply *difference function*

$$\begin{aligned}\Phi(\mathbf{w}) &:= (m_1(\mathbf{w}) - m_0(\mathbf{w}), m_2(\mathbf{w}) - m_1(\mathbf{w}), \dots, m_n(\mathbf{w}) - m_{n-1}(\mathbf{w})) \\ &=: (\Phi_1(\mathbf{w}), \dots, \Phi_n(\mathbf{w})) \in [-\infty, \infty]^n, \qquad \mathbf{w} \in Y.\end{aligned} \tag{2.9}$$

Note that $\mathbf{m}\colon \overline{S} \to \underline{\mathbb{R}}^{n+1}$ is extended continuous, hence also $\Phi\colon Y \to \mathbb{R}^n$ is a continuous function by Proposition 2.1.

A key foothold for our investigation below is the following (very special case of the main) result of [10] (see Corollary 2.2 therein).

Theorem 2.1. *For $n \in \mathbb{N}$ let $r_1, \dots, r_n > 0$, suppose that the singular kernel function K is strictly monotone (see (SM)) and take an arbitrary n-field function J. Consider the sum of translates function F as in (2.3).*

Then the corresponding difference function (2.9), as restricted to Y, is a locally bi-Lipschitz homeomorphism between Y and $\mathbb{R}^n$.

Note that this theorem contains, among other things, the already nontrivial fact that $Y \subseteq S$ must be a (simply) connected domain.

§ 3. Perturbation lemmas

A variant of the next lemma is contained in [11] (see Lemma 11.5; but also [25], Lemma 10 on p. 1069). A similar but slightly simpler form was given by Fenton in [13] (though it was not formulated there explicitly, see around formula (15) in [13]).

Lemma 3.1 (Interval Perturbation Lemma). *Let K be a kernel function. Let $0 < \alpha < a < b < \beta < 1$ and $p, q > 0$. Set*

$$\mu := \frac{p(a-\alpha)}{q(\beta - b)}. \tag{3.1}$$

a) *If K satisfies (M) and $\mu \geqslant 1$, then for every $t \in [0, \alpha]$ we have*

$$pK(t-\alpha) + qK(t-\beta) \leqslant pK(t-a) + qK(t-b). \tag{3.2}$$

b) *If K satisfies* (M) *and $\mu \leqslant 1$, then* (3.2) *holds for every $t \in [\beta, 1]$.*

c) *If $\mu = 1$, then even when K does not satisfy* (M), (3.2) *holds for every $t \in [0, \alpha] \cup [\beta, 1]$.*

d) *In the case of a strictly concave kernel function* a), b) *and* c) *hold with strict inequality in* (3.2).

e) *If K satisfies* (M), *then for every $t \in [a, b]$*

$$pK(t-\alpha) + qK(t-\beta) \geqslant pK(t-a) + qK(t-b), \tag{3.3}$$

with strict inequality if K is strictly monotone.

Proof. Rearranging (3.2) and dividing by $q(\beta - b)$ yields the equivalent assertion

$$\frac{p(a-\alpha)}{q(\beta-b)} \frac{K(t-\alpha) - K(t-a)}{a-\alpha} \leqslant \frac{K(t-b) - K(t-\beta)}{\beta - b}.$$

This expresses the inequality $\mu c \leqslant C$, where μ is defined in (3.1) and

$$c := \frac{K(t-\alpha) - K(t-a)}{a-\alpha} \quad \text{and} \quad C := \frac{K(t-b) - K(t-\beta)}{\beta - b}$$

equal to the slopes of the chords of the graph of the kernel function K drawn over the points $t-a, t-\alpha$ and $t-\beta, t-b$, respectively. Note that $t-\beta < t-b < t-a < t-\alpha$, and all of them are in $[0,1]$ if $t \in [\beta, 1]$, while all of them are in $[-1, 0]$ if $t \in [0, \alpha]$. It follows that these points lie in the same interval of concavity of K, and the slope of the left-hand chord exceeds that of the chord to the right of it: that is, we have $c \leqslant C$. In particular, for $\mu = 1$ assertion c) follows immediately and even with strict inequality, provided that K is strictly concave, as then even the inequality $c < C$ holds.

It remains to see when we can have $\mu c \leqslant C$, for which it suffices to have $\mu c \leqslant c$.

Now if $c \leqslant 0$ then this holds for $\mu \geqslant 1$, and if $c \geqslant 0$ then it holds for $\mu \leqslant 1$. In view of monotonicity, however, we surely have $c \leqslant 0$ whenever $t-a, t-\alpha \leqslant 0$, that is, for $t \in [0, \alpha]$, so that in this case $\mu \geqslant 1$ suffices; and in the same way. For $t \in [\beta, 1]$ we have $t-a, t-\alpha > 0$ and $c \geqslant 0$, so that $\mu \leqslant 1$ suffices. Altogether, we obtain both assertions a) and b).

Similar arguments yield the strict inequalities in assertions a) and b) whenever K is strictly concave. Altogether, assertion d) is proved.

Assertion e) is immediate, since under the condition (M) for $t \in [a, b]$ we have $K(t-\alpha) \geqslant K(t-a)$ and also $K(t-\beta) \geqslant K(t-b)$, with strict inequality whenever K is strictly monotone.

The lemma is proved.

We record the following, trivial but extremely useful, fact as a separate lemma.

Lemma 3.2 (Trivial Lemma). *Let $f, g, h \colon D \to \mathbb{R}$ be upper semicontinuous functions on a Hausdorff topological space D, and let $\varnothing \neq A \subseteq B \subseteq D$ be arbitrary. Assume that*

$$f(t) < g(t) \quad \textit{for all } t \in A. \tag{3.4}$$

If $A \subseteq B$ is a compact set, then

$$\max_A (f+h) < \sup_B (g+h) \quad \textit{unless } h \equiv -\infty \quad \textit{on } A. \tag{3.5}$$

Proof. It is obvious that $\sup_A(f+h) \leqslant \sup_B(g+h)$. If A is compact, then $f+h$ attains its supremum at some point $a \in A$. If $h(a) = -\infty$, then also $\max_A(f+h) = f(a)+h(a) = -\infty$, and the strict inequality in (3.5) follows unless $h+g \equiv -\infty$ all over B, hence, in particular, all over A. In this case, however, we must have $h \equiv -\infty$ all over A, since strict inequality in condition (3.4) entails that $g > -\infty$ on A. Therefore, the statement (3.5) is proved whenever $h(a) = -\infty$. Now, if $h(a) > -\infty$ is finite, then we necessarily have $\max_A(f+h) = f(a)+h(a) < g(a)+h(a) \leqslant \sup_B(g+h)$, and we are also done in this case. The proof is complete.

Remark 3.1. The upper semicontinuity of the field function J is needed for the validity of Lemma 3.2, on which our arguments rely heavily. This is why we need to assume this upper semicontinuity in the main results.

Theorem 3.1 (Minimax Equioscillation). *Let $n \in \mathbb{N}$, let K be a singular, strictly concave and strictly monotone (see (SM)) kernel function, and let $J\colon [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous n-field function. For $j = 1, \dots, n$ consider $r_j > 0$ and the sum of translates function as in (2.3).*

Then there is a minimum point $\mathbf{w} \in \overline{S}$ of $\overline{m}$ in $\overline{S}$ (a minimax point), that is,

$$M(\overline{S}) = \inf_{\mathbf{x}\in\overline{S}} \overline{m}(\mathbf{x}) = \inf_{\mathbf{x}\in\overline{S}} \max_{j=0,1,\dots,n} m_j(\mathbf{x}) = \overline{m}(\mathbf{w}) := \max_{j=0,1,\dots,n} m_j(\mathbf{w}). \tag{3.6}$$

Each such minimum point $\mathbf{w} \in \overline{S}$ is an equioscillation point, that is, it satisfies

$$m_0(\mathbf{w}) = m_1(\mathbf{w}) = \cdots = m_n(\mathbf{w}). \tag{3.7}$$

Furthermore, the point $\mathbf{w}$ is nonsingular, that is, it belongs to the regularity set Y and, in particular, to the open simplex S.

Proof. By the continuity of $\overline{m}$ (see Proposition 2.1) some minimum point must exist on the compact set $\overline{S}$. Let $\mathbf{w} \in \overline{S}$ be any such minimum point. In what follows, first we set to prove that $\mathbf{w}$ is an equioscillation point, that is, $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$ for $j = 0,1,\dots,n$. Assume for a contradiction that $m_j(\mathbf{w}) < \overline{m}(\mathbf{w})$ for some $j \in \{0,1,\dots,n\}$.

Case 1. First we consider the case when $I_j = [w_j, w_{j+1}] \subseteq (0,1)$, and note that then $0 < j < n$ since $w_0 = 0 < w_j \leqslant w_{j+1} < 1 = w_{n+1}$. Consider the following sets of indices with positions of the w_i at the left-hand (respectively, right-hand) endpoint of I_j:

$$\mathcal{L} := \{i \leqslant j \colon w_i = w_j\} \quad \text{and} \quad \mathcal{R} := \{i \geqslant j+1 \colon w_i = w_{j+1}\}.$$

Note that, in principle, I_j can be degenerate, that is, $w_j = w_{j+1}$ can hold, but we have defined the index sets $\mathcal{L}$ and $\mathcal{R}$ to be disjoint. Further, we set

$$L := \sum_{i\in\mathcal{L}} r_i, \qquad R := \sum_{i\in\mathcal{R}} r_i \quad \text{and} \quad \mathcal{I} := \{0,\dots,n+1\} \setminus (\mathcal{L}\cup\mathcal{R}).$$

We apply Lemma 3.1 for $\alpha := w_j - Rh$, $a := w_j$, $b := w_{j+1}$, $\beta := w_{j+1} + Lh$, $p := L$ and $q := R$, with a small but positive h to be specified suitably below. As now the value of μ in (3.1) is exactly 1, Lemma 3.1 yields the strict inequality

$$LK(t-(w_j - Rh)) + RK(t-(w_{j+1}+Lh)) < LK(t-w_j) + RK(t-w_{j+1}) \tag{3.8}$$

for all points $t \in A := [0, w_j - Rh] \cup [w_{j+1} + Lh, 1]$.

Next, we define a new node system $\mathbf{w}'$ by $w'_i := w_i - Rh = w_j - Rh = \alpha$ for all $i \in \mathcal{L}$, $w'_i := w_i + Lh = w_{j+1} + Lh = \beta$ for all $i \in \mathcal{R}$, and the other points unchanged: $w'_i := w_i$ for $i \in \mathcal{I}$. Note that if h is smaller than the distance $\rho > 0$ between the sets $\{w_i : i \in \mathcal{I}\}$ and I_j, then $0 = w'_0 \leqslant \cdots \leqslant w'_j < w'_{j+1} \leqslant \cdots \leqslant 1$, and therefore $\mathbf{w}' \in \overline{S}$. So from now on assume that $0 < h < \rho$ and also that h is chosen sufficiently small so that $m_j(\mathbf{w}') < \overline{m}(\mathbf{w})$ remains in effect (the continuity of m_j; see Proposition 2.1).

For the new node system $\mathbf{w}'$ we have $A = [0,1] \setminus \operatorname{int} I_j(\mathbf{w}')$ and, moreover, inequality (3.8) can be rewritten as

$$LK(t - w'_j) + RK(t - w'_{j+1}) < LK(t - w_j) + RK(t - w_{j+1}) \quad \text{for } t \in A.$$

Note that after adding $J(t) + \sum_{i\in\mathcal{I}} r_i K(t - w_i)$ to both sides, the left-hand side becomes $F(\mathbf{w}', t)$ and the right-hand side becomes $F(\mathbf{w}, t)$.

Applying Lemma 3.2 for $A = [0,1] \setminus \operatorname{int} I_j(\mathbf{w}')$, $B := [0,1] \setminus \operatorname{int} I_j(\mathbf{w})$ and $D = [0,1]$ we obtain $\max_A F(\mathbf{w}', \cdot\,) < \max_B F(\mathbf{w}, \cdot\,) \leqslant \overline{m}(\mathbf{w})$, unless the added expression $J(t) + \sum_{i\in\mathcal{I}} r_i K(t - w_i)$ is identically equal to $-\infty$ on A, in which case the left-hand side $\max_A F(\mathbf{w}', \cdot\,)$ is also $-\infty$. In either case we arrive at the relation $\max_A F(\mathbf{w}', \cdot\,) < \overline{m}(\mathbf{w})$.

Taking into account that $\max_{I_j(\mathbf{w}')} F(\mathbf{w}', \cdot\,) = m_j(\mathbf{w}') < \overline{m}(\mathbf{w})$, we thus infer that $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, which contradicts the choice of $\mathbf{w}$ as a minimum point of $\overline{m}$. This contradiction proves that $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$ must hold.

Case 2. Now let $0 = w_j$. Then $I_0 = [0, w_1] \subseteq [0, w_{j+1}] = I_j$, hence $m_0(\mathbf{w}) \leqslant m_j(\mathbf{w}) < \overline{m}(\mathbf{w})$. This implies that $I_0(\mathbf{w}) \subseteq [0,1)$ that is, $w_1 < 1$, so that there is a maximum index $k \leqslant n$ such that $w_1 = w_k$ and $w_k < w_{k+1}$. (Note that $w_1 = \cdots = w_k$ may or may not be in the position 0—this does not matter.)

We consider the new node system $\mathbf{w}'$ such that $w'_1 = \cdots = w'_k = w_1 + h$ (and the other points unchanged). Since $0 < h < w_{k+1} - w_k$, the new node system $\mathbf{w}'$ also belongs to $\overline{S}$. As above, for h small enough continuity (Proposition 2.1) furnishes the inequality $m_0(\mathbf{w}') < \overline{m}(\mathbf{w})$.

Now let $t \in A := [w_1 + h, 1] = [0,1] \setminus \operatorname{rint} I_0(\mathbf{w}')$. Taking into account the strict monotonicity condition (SM) and $h > 0$ we must have

$$\sum_{i=1}^{k} r_i K(t - w_1 - h) < \sum_{i=1}^{k} r_i K(t - w_1) \quad \text{for all } t \in A.$$

Note that the left-hand side can attain $-\infty$ (at $t = w_1 + h$), but since $h > 0$ the right-hand side cannot. If we add $J(t) + \sum_{i=k+1}^{n} r_i K(t - w_i)$ to both sides, then the left-hand side becomes $F(\mathbf{w}', t)$, and the right-hand side becomes $F(\mathbf{w}, t)$. Putting $B := [w_1, 1] = [0,1] \setminus \operatorname{rint} I_1(\mathbf{w})$, an application of Lemma 3.2 furnishes $\max_A F(\mathbf{w}', \cdot\,) < \max_B F(\mathbf{w}, \cdot\,) \leqslant \overline{m}(\mathbf{w})$, unless the left-hand side is $-\infty$; in particular, as $\overline{m}(\mathbf{w}) > -\infty$, in either case we obtain $\max_A F(\mathbf{w}', \cdot\,) < \overline{m}(\mathbf{w})$.

As above, taking the relation $\max_{I_1(\mathbf{w}')} F(\mathbf{w}', \cdot\,) = m_1(\mathbf{w}') < \overline{m}(\mathbf{w})$ into account, we thus infer that $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, which is a contradiction with the minimality of $\overline{m}(\mathbf{w})$. Therefore, $m_0(\mathbf{w}) = m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$ follows.

Case 3: $w_{j+1} = 1$. This is completely analogous to Case 2.

Cases 1–3 altogether yield that $\mathbf{w}$ is an equioscillation point, as claimed in (3.7).

Furthermore, if $I_k(\mathbf{w})$ is degenerate or singular, then by Remark 2.1 $m_k(\mathbf{w}) = -\infty < \overline{m}(\mathbf{w})$ must hold. This contradicts equioscillation, hence is excluded by the above. That is, $\mathbf{w} \in Y$, so that no interval $I_k(\mathbf{w})$ can be singular.

Theorem 3.1 is proved.

There exist some maximum points of $\underline{m}$ on $\overline{S}$ by continuity and compactness. Quite analogously to the above, we can as well prove the following about these.

Theorem 3.2 (Maximin Equioscillation). *Let $n \in \mathbb{N}$, let K be a singular, strictly concave and strictly monotone (see (SM)) kernel function, and let $J\colon [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous n-field function. For $j = 1, \dots, n$ let $r_j > 0$ and consider the sum of translates function as in (2.3). If $\mathbf{y} \in \overline{S}$ is a maximum point of $\underline{m}$ (a maximin point), that is, $\underline{m}(\mathbf{y}) = m(\overline{S}) := \max_{\overline{S}} \underline{m}$, then $\mathbf{y} \in Y \subseteq S$ and $\mathbf{y}$ is an equioscillation point.*

Proof. By Theorem 3.1 we have a minimax point $\mathbf{w} \in Y$. Thus, for any maximin point $\mathbf{y} \in \overline{S}$ we have $\underline{m}(\mathbf{y}) = m(S) \geqslant \underline{m}(\mathbf{w}) = \overline{m}(\mathbf{w}) = \max_{[0,1]} F(\mathbf{w}, \cdot) > -\infty$, hence for all $j = 0, 1, \dots, n$ we also have $m_j(\mathbf{y}) > -\infty$. We conclude that $\mathbf{y} \in Y$. By Remark 2.1 no degenerate intervals can exist among the $I_j(\mathbf{y})$. This somewhat simplifies our considerations in comparison with the proof of Theorem 3.1.

It remains to prove that once $\mathbf{y} \in \overline{S}$ is a maximin point, we necessarily have that it is an equioscillation point, that is, $m_j(\mathbf{y}) = \underline{m}(\mathbf{y})$ for $j = 0, 1, \dots, n$. For the proof we assume for a contradiction that there exists some $j \in \{0, 1, \dots, n\}$ such that $m_j(\mathbf{y}) > \underline{m}(\mathbf{y})$.

Case 1. First let $I_j(\mathbf{y}) = [y_j, y_{j+1}] \subseteq (0,1)$. Note that then $0 < j < n$, as $y_0 = 0 < y_j < y_{j+1} < 1 = y_{n+1}$ (the inequality $y_j < y_{j+1}$ has been clarified above).

We apply Lemma 3.1, c), for $\alpha := y_j$, $a := y_j + h/r_j$, $b := y_{j+1} - h/r_{j+1}$, $\beta := y_{j+1}$, $p := r_j$ and $q := r_{j+1}$, where $h > 0$ is sufficiently small so that $a < b$. Then for all $t \in A := [0,1] \setminus \operatorname{rint} I_j(\mathbf{y}) = \bigcup_{i=0, i\neq j}^{n} I_i(\mathbf{y})$ we obtain

$$r_j K(t - y_j) + r_{j+1} K(t - y_{j+1}) < r_j K\left(t - \left(y_j + \frac{h}{r_j}\right)\right) + r_{j+1} K\left(t - \left(y_j + \frac{h}{r_{j+1}}\right)\right). \tag{3.9}$$

We define a new node system $\mathbf{y}'$ by setting $y'_j := y_j + h/r_j$ and $y'_{j+1} := y_{j+1} - h/r_{j+1}$ and keeping the rest of the nodes unchanged: $y'_i := y_i$ for $i \neq j, j+1$. By the choice of $h > 0$ we have $\mathbf{y}' \in S$. Taking smaller $h > 0$ if necessary, we can ensure that $m_j(\mathbf{y}') > \underline{m}(\mathbf{y})$ (the continuity of m_j: see Proposition 2.1). Now adding $J(t) + \sum_{i \neq j, j+1} r_i K(t - y_i)$ to both sides of (3.9), the left-hand side becomes $F(\mathbf{y}, t)$ and the right-hand side becomes $F(\mathbf{y}', t)$.

Now let $i \in \{0, 1, \dots, n\} \setminus \{j\}$ be any index and consider $m_i(\mathbf{y}')$ and $m_i(\mathbf{y}) > -\infty$ (recall that $\mathbf{y}$ is nonsingular). Lemma 3.2 for $A := I_i(\mathbf{y}) \subseteq B := I_i(\mathbf{y}')$ yields the relations $-\infty < \underline{m}(\mathbf{y}) \leqslant m_i(\mathbf{y}) = \max_A F(\mathbf{y}, t) < \max_B F(\mathbf{y}', \cdot) = m_i(\mathbf{y}')$. As we have already ensured that $\underline{m}(\mathbf{y}) < m_j(\mathbf{y}')$, we find that $\underline{m}(\mathbf{y}) < m_i(\mathbf{y}')$ for all $i \in \{0, 1, \dots, n\}$, whence we conclude that $\underline{m}(\mathbf{y}) < \underline{m}(\mathbf{y}')$, which is a contradiction with the maximality of $\underline{m}(\mathbf{y})$. We arrive at the equality $m_j(\mathbf{y}) = \underline{m}(\mathbf{y})$ in this case.

Case 2. Suppose $y_j = 0$. As $\mathbf{y}$ is a nondegenerate node system, we must have $j = 0$ and $I_j(\mathbf{y}) = I_0(\mathbf{y}) = [0, y_1]$, $y_1 > 0$. We consider the new node system $\mathbf{y}'$, where $y'_1 = y_1 - h$ and the other points remain unchanged: $y'_i = y_i$ for $i = 2, \dots, n$.

Since $0 < h < y_1$, the new node system $\mathbf{y}'$ also belongs to S. As above, for $h > 0$ small enough the continuity of m_0 (see Proposition 2.1) furnishes $m_0(\mathbf{y}') > \underline{m}(\mathbf{y})$.

Let $i \in \{1, 2, \ldots, n\}$ be arbitrary and consider $I_i(\mathbf{y})$ and $I_i(\mathbf{y}')$. Obviously, $I_i(\mathbf{y}) \subseteq I_i(\mathbf{y}')$. Further, $-\infty < \underline{m}(\mathbf{y}) \leqslant m_i(\mathbf{y})$. In view of the strict monotonicity of K we obviously have $r_1 K(t - y_1) < r_1 K(t - y_1 + h)$ for every $t \in I_i(\mathbf{y})$. Adding $J(t) + \sum_{i=2}^{n} r_i K(t - y_i)$ to this inequality, the left-hand side becomes $F(\mathbf{y}, t)$ and the right-hand side becomes $F(\mathbf{y}', t)$. Applying Lemma 3.2 for $A := I_i(\mathbf{y})$ and $B := I_i(\mathbf{y}')$ we obtain $-\infty < \underline{m}(\mathbf{y}) \leqslant m_i(\mathbf{y}) < m_i(\mathbf{y}')$ for each $i = 1, 2, \ldots, n$. In fact, also the inequality $-\infty < \underline{m}(\mathbf{y}) < m_0(\mathbf{y}')$ was guaranteed above, which then furnishes $\underline{m}(\mathbf{y}) < \min_i m_i(\mathbf{y}') = \underline{m}(\mathbf{y}')$, contradicting the maximality of $\underline{m}(\mathbf{y})$. Therefore, $m_0(\mathbf{y}) = \underline{m}(\mathbf{y})$.

Case 3. The case $y_{j+1} = 1$ is completely analogous to Case 2.

Cases 1–3 taken together yield that $\mathbf{y}$ is an equioscillation point, as claimed.

Theorem 3.2 is proved.

Corollary 3.1. *Let K be a singular (see (∞)), strictly concave and (strictly) monotone (see (SM)) kernel function, and let J be an upper semicontinuous field function.*

Then $M(S) = m(S)$ and there exists a unique equioscillation point $\mathbf{w} \in \overline{S}$, which, in fact, belongs to $Y \subseteq S$. This point $\mathbf{w}$ is the unique minimax point in $\overline{S}$, that is, $\overline{m}(\mathbf{w}) = M(S)$, and it is the unique maximin point in $\overline{S}$, that is, $\underline{m}(\mathbf{w}) = m(S)$. In particular, the so-called Sandwich Property holds:

$$\underline{m}(\mathbf{x}) \leqslant M(S) = m(S) \leqslant \overline{m}(\mathbf{x}) \quad \textit{for any node system } \mathbf{x} \in S.$$

Proof. The previous two theorems give that both minimax and maximin points must be equioscillation node systems. Now, points in $\overline{S} \setminus Y$ cannot be equioscillation points, as degenerate or singular points $\mathbf{x}$ satisfy $m_i(\mathbf{x}) = -\infty$ for some $i \in \{0, 1, \ldots, n\}$ while $\overline{m}(\mathbf{x}) > -\infty$. By Theorem 2.1 the difference mapping Φ is a homeomorphism between Y and $\mathbb{R}^n$. In particular, there exists exactly one pre-image of $\mathbf{0}$, that is, only one equioscillation point in Y, and therefore also in $\overline{S}$ in general. As a result, both the maximin and minimax points must coincide with this unique equioscillation point (Theorems 3.1 and 3.2). The proof of Corollary 3.1 is complete.

Corollary 3.2. *Let K be a singular (see (∞)) and monotone (see (M)) kernel function, and let J be an upper semicontinuous field function.*

Then $M(S) = m(S)$ and there exists some node system $\mathbf{w} \in \overline{S}$, also belonging to Y, such that it is an equioscillation point and $\underline{m}(\mathbf{w}) = m(S) = M(S) = \overline{m}(\mathbf{w})$.

In particular, the so-called Sandwich Property holds: for any node system $\mathbf{x} \in S$ we have $\underline{m}(\mathbf{x}) \leqslant M(S) = m(S) \leqslant \overline{m}(\mathbf{x})$, and $M(S) = m(S)$ is the unique equioscillation value.[8]

If, in addition, the kernel K satisfies (SM), then $\mathbf{w}$ is the unique equioscillation point.

Proof. For the proof, first we apply the previous corollary to the situation with the same field function J and the modified kernel functions $K^{(\eta)}(t) := K(t) + \eta\sqrt{|t|}$. If $\eta > 0$, then $K^{(\eta)}$ is strictly concave and strictly monotone, thus Corollary 3.1

[8] By definition the value of a generalized alternance is the unique value taken by the function $\overline{m}$ at a point of generalized alternance.

applies and provides node systems $\mathbf{w}_\eta$ with the three properties asserted: $m^{(\eta)}(S) = M^{(\eta)}(S) = \underline{m}^{(\eta)}(\mathbf{w}_\eta) = \overline{m}^{(\eta)}(\mathbf{w}_\eta)$, where the notation refers to the corresponding quantities defined with the use of the kernel $K^{(\eta)}$. Using similar notation for the sum of translates function and putting $R := \sum_{i=1}^n r_i$, it is obvious that $F(\mathbf{x},t) \leqslant F^{(\eta)}(\mathbf{x},t) \leqslant F(\mathbf{x},t) + \eta R$ for all $\mathbf{x} \in \overline{S}$ and $t \in [0,1]$. Therefore, also $m_i(\mathbf{x}) \leqslant m_i^{(\eta)}(\mathbf{x}) \leqslant m_i(\mathbf{x}) + \eta R$ and so $m_i^{(\eta)}(\mathbf{x}) \to m_i(\mathbf{x})$ for every $i = 0,1,\dots,n$ and $\mathbf{x} \in \overline{S}$. By the compactness of $\overline{S}$ we can take a convergent subsequence $(\mathbf{w}_{1/k_\ell})$ of $(\mathbf{w}_{1/k})$ with limit $\mathbf{w} := \lim_{\ell\to\infty} \mathbf{w}_{1/k_\ell}$. Moreover, by the continuity of m_i (see Proposition 2.1) we obtain

$$
\begin{aligned}
m_i(\mathbf{w}) &= \lim_{\ell\to\infty} m_i(\mathbf{w}_{1/k_\ell}) \leqslant \liminf_{\ell\to\infty} m_i^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) \leqslant \limsup_{\ell\to\infty} m_i^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) \\
&\leqslant \limsup_{\ell\to\infty} \biggl(m_i(\mathbf{w}_{1/k_\ell}) + \frac{R}{k_\ell} \biggr) = m_i(\mathbf{w}),
\end{aligned}
$$

that is, $\lim_{\ell\to\infty} m_i^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) = m_i(\mathbf{w})$. Therefore, $\mathbf{w}$ is an equioscillation point in the case of the kernel K, hence $\mathbf{w} \in Y$. Let $\mathbf{x}$ be a minimum point of $\overline{m}$ on $\overline{S}$. Then

$$
M(S) = \overline{m}(\mathbf{x}) = \lim_{\eta\to 0} \overline{m}^{\,(\eta)}(\mathbf{x}) \geqslant \limsup_{\eta\to 0} M^{(\eta)}(S) \geqslant \liminf_{\eta\to 0} M^{(\eta)}(S) \geqslant M(S),
$$

where the last inequality obviously follows from $K^{(\eta)} \geqslant K$. Therefore, $M(S) = \lim_{\eta\to 0} M^{(\eta)}(S)$, whence we can also conclude that

$$
M(S) = \lim_{\ell\to\infty} M^{(1/k_\ell)}(S) = \lim_{\ell\to\infty} \overline{m}^{\,(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) = \overline{m}(\mathbf{w}),
$$

that is, $\mathbf{w}$ is a minimum point of $\overline{m}$. Since for the η-perturbed kernel we have $M^{(\eta)}(S) = m^{(\eta)}(S)$, we infer that

$$
\lim_{\eta\to 0} \underline{m}^{(\eta)}(\mathbf{w}_\eta) = \lim_{\eta\to 0} m^{(\eta)}(S) = \lim_{\eta\to 0} M^{(\eta)}(S)
$$

exists and equals $M(S)$.

On the other hand $m(S) \leqslant m^{(\eta)}(S) = \underline{m}^{(\eta)}(\mathbf{w}_\eta)$, so

$$
m(S) \leqslant \lim_{\eta\to 0} m^{(\eta)}(S) = \lim_{\eta\to 0} \underline{m}^{(\eta)}(\mathbf{w}_\eta) = \lim_{\ell\to\infty} \underline{m}^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) = \underline{m}(\mathbf{w}) \leqslant m(S).
$$

Hence $\lim_{\eta\to 0} m^{(\eta)}(S) = m(S)$ and $\underline{m}(\mathbf{w}) = m(S)$, that is, $\mathbf{w}$ is a maximum point of $\underline{m}$. Finally, the uniqueness of the equioscillation point under condition (SM) follows from Theorem 2.1. Corollary 3.2 is proved.

§ 4. Intertwining

The main result of this section, Theorem 4.1, shows that for different node systems $\mathbf{x},\mathbf{y} \in Y$ it is not possible to have $m_j(\mathbf{x}) \leqslant m_j(\mathbf{y})$ for every $j \in \{0,1,\dots,n\}$, that is, majorization cannot occur (cf. [11] for the terminology). In other words, for two different node systems $\mathbf{x},\mathbf{y} \in Y$ both $m_j(\mathbf{x}) < m_j(\mathbf{y})$ and $m_i(\mathbf{x}) > m_i(\mathbf{y})$ hold for some $i,j \in \{0,1,\dots,n\}$, a property which it is natural to call *intertwining*. For the proof, we need the following perturbation-type lemma, which is interesting on its own.

Lemma 4.1 (General Maximum Perturbation Lemma). *Let $n \in \mathbb{N}$ be a natural number, let $r_1, \ldots, r_n > 0$, let $J : [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous n-field function, and let K be a kernel function satisfying the monotonicity condition* (M). *Consider the sum of translates function F as in* (2.3).

Let $\mathbf{w} \in S$ be a nondegenerate node system, and let $\mathcal{I} \cup \mathcal{J} = \{0, 1, \ldots, n\}$ be a nontrivial partition. Then there exists $\mathbf{w}' \in S \setminus \{\mathbf{w}\}$ arbitrarily close to $\mathbf{w}$ such that

$$F(\mathbf{w}', t) \leqslant F(\mathbf{w}, t) \quad \text{for all } t \in I_i(\mathbf{w}'), \qquad I_i(\mathbf{w}') \subseteq I_i(\mathbf{w}) \quad \text{for all } i \in \mathcal{I}, \tag{4.1}$$

and

$$F(\mathbf{w}', t) \geqslant F(\mathbf{w}, t) \quad \text{for all } t \in I_j(\mathbf{w}), \qquad I_j(\mathbf{w}') \supseteq I_j(\mathbf{w}) \quad \text{for all } j \in \mathcal{J}. \tag{4.2}$$

As a result, we also have

$$m_i(\mathbf{w}') \leqslant m_i(\mathbf{w}) \quad \text{for } i \in \mathcal{I} \quad \text{and} \quad m_j(\mathbf{w}') \geqslant m_j(\mathbf{w}) \quad \text{for } j \in \mathcal{J} \tag{4.3}$$

for the corresponding interval maxima.

Moreover, if K is strictly concave (and therefore also strictly monotone by condition (M)*), then the inequalities in* (4.1) *and* (4.2) *are strict for all points in the corresponding intervals where $J(t) \neq -\infty$.*

Furthermore, the inequalities in (4.3) *are also strict for all indices k with nonsingular $I_k(\mathbf{w})$; in particular, for all indices if $\mathbf{w} \in Y$.*

Proof. Before the main argument, we observe that the assertion in (4.3) is indeed a trivial consequence of the previous inequalities (4.2) and (4.1), so we need not give a separate proof of it.

A second important observation is as follows. Using the pure sum of translates function f we write $F(\mathbf{w}, t) = f(\mathbf{w}, t) + J(t)$, and so (4.1) and (4.2) follow from the inequalities

$$f(\mathbf{w}', t) \leqslant f(\mathbf{w}, t) \quad \forall\, t \in I_i(\mathbf{w}'), \qquad I_i(\mathbf{w}') \subseteq I_i(\mathbf{w}) \quad \text{for all } i \in \mathcal{I}, \tag{4.4}$$

and

$$f(\mathbf{w}', t) \geqslant f(\mathbf{w}, t) \quad \forall\, t \in I_j(\mathbf{w}) \quad \text{and} \quad I_j(\mathbf{w}') \supseteq I_j(\mathbf{w}) \quad \text{for all } j \in \mathcal{J}. \tag{4.5}$$

Moreover, strict inequalities at all points t such that $J(t) \neq -\infty$ will follow in (4.1) and (4.2) if we can prove strict inequalities in (4.4) and (4.5) for all values of t in the said intervals.

Furthermore, in case we have strict inequalities in (4.4) and (4.5) for all points t, then for nonsingular $I_k(\mathbf{w})$ this also entails strict inequalities in (4.3) (for the corresponding k; and for all k if $\mathbf{w} \in Y$). To see this, one may refer back to Lemma 3.2 for $\{f, g\} = \{f(\mathbf{w}, \cdot), f(\mathbf{w}', \cdot)\}$, $h = J$ and $\{A, B\} = \{I_k(\mathbf{w}), I_k(\mathbf{w}')\}$.

In the next, main part of the argument we prove (4.1), (4.2), (4.4) and (4.5) by induction on n for any n-field function and any kernel function.

If $n = 1$ and $\mathcal{I} = \{0\}$, $\mathcal{J} = \{1\}$, then $\mathbf{w}' = (w_1') = (w_1 + h)$, and if $\mathcal{J} = \{0\}$ and $\mathcal{I} = \{1\}$, then $\mathbf{w}' = (w_1') = (w_1 - h)$ works for any $0 < h < \min(w_1, 1 - w_1)$. For this only monotonicity or strict monotonicity, respectively, of the kernel is needed, and therefore (4.4) and (4.5) follow readily, while (4.1) and (4.2) follow by the preliminary observations made above.

Now let $n > 1$ and assume, as the inductive hypothesis, the validity of the assertions for $\widetilde{n} := n-1$, for any choice of kernel and $\widetilde{n}$-field functions.

Case 1. If some of the partition sets $\mathcal{I}$ and $\mathcal{J}$ contain neighbouring indices k and $k+1$, then we consider the kernel function $\widetilde{K} := K$ and the $\widetilde{n}$-field function $\widetilde{J} := K(\cdot - w_k)$ where now the sum of translates function $\widetilde{F}$ are formed by using $\widetilde{n} = n-1$ translates with respect to the node system

$$\widetilde{\mathbf{w}} := (w_1, w_2, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n).$$

Formally, the indices change: $\widetilde{w}_\ell = w_\ell$ for $\ell = 1, \ldots, k-1$, but $\widetilde{w}_\ell = w_{\ell+1}$ for $\ell = k, \ldots, n$, the kth coordinate being left out.

We use the same change of indices in the partition: k is dropped out (but the corresponding index set $\mathcal{I}$ or $\mathcal{J}$ does not become empty, for it contains $k+1$); and then we shift the indices by one to the left for $\ell > k$ so that

$$\widetilde{\mathcal{I}} := \{i \in \mathcal{I} \colon i < k\} \cup \{i-1 \colon i \in \mathcal{I},\ i > k\}$$

and

$$\widetilde{\mathcal{J}} := \{j \in \mathcal{J} \colon j < k\} \cup \{j-1 \colon j \in \mathcal{J},\ j > k\}.$$

Observe that $\widetilde{F}(\widetilde{\mathbf{w}}, t) = f(\mathbf{w}, t)$ for all $t \in [0,1]$, while

$$I_\ell(\widetilde{\mathbf{w}}) = \begin{cases} I_\ell(\mathbf{w}) & \text{if } \ell < k, \\ I_k(\mathbf{w}) \cup I_{k+1}(\mathbf{w}) & \text{if } \ell = k, \\ I_{\ell+1}(\mathbf{w}) & \text{if } \ell > k. \end{cases}$$

If $\mathbf{w}'$ is close enough to $\mathbf{w}$, then a similar correspondence holds for $\mathbf{w}' \in S^{(n)}$ and $\widetilde{\mathbf{w}}' \in S^{(\widetilde{n})}$ (where $S^{(n)}$ and $S^{(\widetilde{n})}$ denote simplices of the corresponding dimension). We use this only for $w'_k = w_k$ remaining the same. In this case, using that k and $k+1$ belong to the same index set $\mathcal{I}$ or $\mathcal{J}$, it is easy to check that the inclusion $I_i(\widetilde{\mathbf{w}}') \subseteq I_i(\widetilde{\mathbf{w}})$ for all $i \in \widetilde{\mathcal{I}}$ is equivalent to $I_i(\mathbf{w}') \subseteq I_i(\mathbf{w})$ for all $i \in \mathcal{I}$, and the inclusion $I_j(\widetilde{\mathbf{w}}') \supseteq I_j(\widetilde{\mathbf{w}})$ for all $j \in \widetilde{\mathcal{J}}$ is equivalent to $I_j(\mathbf{w}') \supseteq I_j(\mathbf{w})$ for all $j \in \mathcal{J}$. Therefore, an application of the inductive hypothesis yields the assertions (4.4) and (4.5) in this case. Hence, by the preliminary observations also (4.1) and (4.2) follow.

Case 2. It remains to prove the assertion when $\mathcal{I}$ and $\mathcal{J}$ contain no neighbouring indices: so $\mathcal{I}$ and $\mathcal{J}$ partition $\{0, 1, \ldots, n\}$ into the subsets of odd and even natural numbers up to n. We can assume that $\mathcal{I} = (2\mathbb{N}_0 + 1) \cap \{0, 1, \ldots, n\}$ and $\mathcal{J} = 2\mathbb{N}_0 \cap \{0, 1, \ldots, n\}$ (the other case can be handled analogously).

We emphasize here that it is important that $\mathbf{w} \in S$ is nondegenerate. This allows us, for sufficiently small $\delta > 0$, to move any w_ℓ within a distance $\delta > 0$ still keeping the perturbed node system $\mathbf{w}'$ in S. We fix such $\delta > 0$ at the outset and consider perturbations $\mathbf{w}'$ of $\mathbf{w}$ only within distance δ from now on. Our new perturbed node system $\mathbf{w}'$ is, for an arbitrary h, $0 < h < \delta / \max\{r_1, \ldots, r_n\}$, the system

$$\mathbf{w}' := (w'_1, \ldots, w'_n), \quad \text{where } w'_\ell := w_\ell - (-1)^\ell \frac{1}{r_\ell} h, \quad \ell = 1, 2, \ldots, n. \tag{4.6}$$

Obviously, $I_j(\mathbf{w}') \supseteq I_j(\mathbf{w})$ for all $j \in \mathcal{J}$, and $I_i(\mathbf{w}') \subseteq I_i(\mathbf{w})$ for all $i \in \mathcal{I}$.

Now take an even indexed interval $I_{2k}(\mathbf{w}) = [w_{2k}, w_{2k+1}]$, so that $2k \in \mathcal{J}$. Our change of nodes can now be grouped as *pairs of changing nodes* $w_{2\ell-1}, w_{2\ell}$ among $w_1, \ldots, w_{2k}$, and then again among $w_{2k+1}, \ldots, w_{2\lfloor n/2\rfloor}$, plus a left-over change of w_n in case n is odd. Now, the pairs are always changed so that the intervals in between shrink, and shrink exactly as it is described in Lemma 3.1. We apply this lemma to each pair of such nodes for $a = w'_{2\ell-1}$, $b = w'_{2\ell}$, $\alpha = w_{2\ell-1}$, $\beta = w_{2\ell}$, $p = r_{2\ell-1}$ and $q = r_{2\ell}$. This gives us that for each such pair of changes, for t outside of the enclosed interval $(w_{2\ell-1}, w_{2\ell})$ we have

$$r_{2\ell-1}K(t - w'_{2\ell-1}) + r_{2\ell}K(t - w'_{2\ell}) \geqslant r_{2\ell-1}K(t - w_{2\ell-1}) + r_{2\ell}K(t - w_{2\ell}). \quad (4.7)$$

Note that $I_{2k}(\mathbf{w})$, hence any $t \in I_{2k}(\mathbf{w})$ is always outside these intervals, therefore (4.7) holds. If there is a left-over, unpaired change, then n is odd, the corresponding node w_n is increased, and now from monotonicity we conclude for $t \in I_{2k}(\mathbf{w})$ that $K(t - w'_n) = K(t - w_n - h/r_n) \geqslant K(t - w_n)$. Altogether, taking $\eta := 1$ for n odd and $\eta := 0$ for n even we find that

$$\begin{aligned} f(\mathbf{w}, t) &= \sum_{\ell=1}^{\lfloor n/2\rfloor} \big(r_{2\ell-1}K(t - w_{2\ell-1}) + r_{2\ell}K(t - w_{2\ell})\big) + \eta K(t - w_n) \\ &\leqslant \sum_{\ell=1}^{\lfloor n/2\rfloor} \big(r_{2\ell-1}K(t - w'_{2\ell-1}) + r_{2\ell}K(t - w'_{2\ell})\big) + \eta K(t - w'_n) = f(\mathbf{w}', t). \end{aligned} \quad (4.8)$$

Furthermore, all the appearing inequalities are strict in case K is strictly monotone (and therefore strictly concave). We have proved (4.5), and even with strict inequality under appropriate assumptions.

The proof of (4.4) runs analogously by grouping the change of nodes as a change of the singleton w_1, and then of pairs $w_{2\ell}, w_{2\ell+1}$ for $\ell = 1, \ldots, \lfloor (n-1)/2\rfloor$, and of another singleton w_n if n is even. Lemma 4.1 is proved.

Theorem 4.1 (Intertwining Theorem). *Let $n \in \mathbb{N}$, let $r_1, \ldots, r_n > 0$, let K be a singular (see (∞)), strictly concave and (strictly) monotone (see (SM)) kernel function, and let $J\colon [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous n-field function.*

Then for nodes $\mathbf{x}, \mathbf{y} \in Y$ majorization cannot hold, that is, the coordinatewise inequality $\mathbf{m}(\mathbf{x}) \leqslant \mathbf{m}(\mathbf{y})$ can only hold for $\mathbf{x} = \mathbf{y}$.

Proof. Take two node systems $\mathbf{x}, \mathbf{y} \in Y$ and assume that majorization holds between them: say, $\mathbf{m}(\mathbf{x}) \leqslant \mathbf{m}(\mathbf{y})$ in the sense of the inequality $m_i(\mathbf{x}) \leqslant m_i(\mathbf{y})$ for $i = 0, 1, \ldots, n$. We need to show that, in fact, $\mathbf{x} = \mathbf{y}$.

First, if $\mathbf{m}(\mathbf{x}) = \mathbf{m}(\mathbf{y})$, then, of course, $\Phi(\mathbf{x}) = \Phi(\mathbf{y})$, hence in view of the homeomorphism theorem (Theorem 2.1, which requires condition (∞)) only $\mathbf{x} = \mathbf{y}$ is possible.

So assume that $\mathbf{m}(\mathbf{x}) \neq \mathbf{m}(\mathbf{y})$, and so there exists i such that $m_i(\mathbf{x}) < m_i(\mathbf{y})$. We introduce the following two ('maximal' and 'minimal') distance functions:

$$d(\mathbf{z}, \mathbf{y}) := \max_{i=0,1,\ldots,n} (m_i(\mathbf{y}) - m_i(\mathbf{z}))$$

and

$$\rho(\mathbf{z},\mathbf{y}) := \min_{i=0,1,\dots,n} (m_i(\mathbf{y}) - m_i(\mathbf{z})).$$

Then $\rho(\mathbf{z},\mathbf{y}) \leqslant d(\mathbf{z},\mathbf{y})$. Moreover, $0 \leqslant \rho(\mathbf{z},\mathbf{y})$ if and only if $\mathbf{m}(\mathbf{z}) \leqslant \mathbf{m}(\mathbf{y})$, and for $d_0 := d(\mathbf{x},\mathbf{y})$ we have $d_0 > 0$. Consider the set

$$Z := \{\mathbf{z} \in \overline{S}\colon \mathbf{m}(\mathbf{z}) \leqslant \mathbf{m}(\mathbf{y}),\ d(\mathbf{z},\mathbf{y}) \leqslant d_0\}.$$

Obviously, $\mathbf{x} \in Z$, hence $Z \neq \varnothing$. By Proposition 2.1 the distance functions $d(\,\cdot\,,\mathbf{y})$ and $\rho(\,\cdot\,,\mathbf{y})$ are (extended) continuous on $\overline{S}$ and continuous on Y. In fact, $\mathbf{z} \in Z$ cannot be singular, thus $Z \subseteq Y$, where the distance functions are continuous. Further, as an intersection of $\leqslant$ sublevel sets of continuous functions (see Proposition 2.1), Z is closed and therefore compact.

Here we arrive at the key to our argument. Now we maximize $\rho(\,\cdot\,,\mathbf{y})$ on the compact set Z. Of course, $\rho(\,\cdot\,,\mathbf{y})$ can be at most d_0 on Z. Let $\mathbf{z}_0 \in Z$ be a maximum point of $\rho(\,\cdot\,,\mathbf{y})$, and set $\rho_0 := \rho(\mathbf{z}_0,\mathbf{y}) \leqslant d_0$. We claim that the difference $m_i(\mathbf{y}) - m_i(\mathbf{z}_0)$ is the constant ρ_0 for all $i = 0,1,\dots,n$.

Indeed, if this is not the case, then by means of Lemma 4.1 we can perturb $\mathbf{z}_0$ to another node system $\mathbf{w}$ with a larger value ρ. In detail: assume for a contradiction that $\mathbf{m}(\mathbf{y}) - \mathbf{m}(\mathbf{z}_0) \neq \rho_0\mathbf{1}$. Note that then we also have $\rho_0 < d_0$, for in case $\rho_0 = d_0$ we must have $\mathbf{m}(\mathbf{z}_0) = \mathbf{m}(\mathbf{y}) - d_0\mathbf{1} = \mathbf{m}(\mathbf{y}) - \rho_0\mathbf{1}$, contradicting the assumption.

Now let us define the index sets

$$\mathcal{I} := \{i \in \{0,1,\dots,n\}\colon m_i(\mathbf{z}_0) = m_i(\mathbf{y}) - \rho_0\}$$

and

$$\mathcal{J} := \{j \in \{0,1,\dots,n\}\colon m_j(\mathbf{y}) - d_0 \leqslant m_j(\mathbf{z}_0) < m_j(\mathbf{y}) - \rho_0\}.$$

As $m_i(\mathbf{y}) - m_i(\mathbf{z}_0)$ is not constant in i, we certainly have indices in both index sets $\mathcal{I}$ and $\mathcal{J}$. Moreover, in view of $\mathbf{z}_0 \in Z$ we have $m_k(\mathbf{z}_0) \in [m_k(\mathbf{y}) - d_0, m_k(\mathbf{y}) - \rho_0]$ for all $k = 0,1,\dots,n$. Therefore, $\mathcal{I} \cup \mathcal{J}$ is in fact a nontrivial partition of $\{0,1,\dots,n\}$. Thus, Lemma 4.1 applies to these indices and the nonsingular nondegenerate point $\mathbf{z}_0$, resulting in another node system $\mathbf{w} \in Y \setminus \{\mathbf{z}_0\}$, which is arbitrarily close to $\mathbf{z}_0$ and satisfies $m_i(\mathbf{w}) < m_i(\mathbf{z}_0)$ for $i \in \mathcal{I}$ and $m_j(\mathbf{w}) > m_j(\mathbf{z}_0)$ for $j \in \mathcal{J}$. Since for $j \in \mathcal{J}$ we have $m_j(\mathbf{z}_0) < m_j(\mathbf{y}) - \rho_0$, by the continuity of the functions m_i (Proposition 2.1), if $\mathbf{w}$ is sufficiently close to $\mathbf{z}_0$, then we have $m_j(\mathbf{w}) < m_j(\mathbf{y}) - \rho_0$ for all $j \in \mathcal{J}$. Of course, for these indices $j \in \mathcal{J}$ the inequality $m_j(\mathbf{w}) \geqslant m_j(\mathbf{y}) - d_0$ also remains valid, since $m_j(\mathbf{w}) > m_j(\mathbf{z}_0) \geqslant m_j(\mathbf{y}) - d_0$ for $j \in \mathcal{J}$.

Similarly, after a perturbation we find that $m_i(\mathbf{w}) < m_i(\mathbf{z}_0) = m_i(\mathbf{y}) - \rho_0$ for all $i \in \mathcal{I}$, and, by continuity, in a sufficiently small neighbourhood of $\mathbf{z}_0$ the inequality $m_i(\mathbf{w}) \geqslant m_i(\mathbf{y}) - d_0$ also holds. (Here we need, of course, that $0 < d_0 - \rho_0$ and use continuity.)

Altogether, we find $\mathbf{w} \in Z$ such that $m_k(\mathbf{w}) < m_k(\mathbf{y}) - \rho_0$ for all $k = 0,1,\dots,n$, hence $\rho(\mathbf{w},\mathbf{y}) > \rho_0$, which is a contradiction with the choice of $\mathbf{z}_0$ as maximizing $\rho(\,\cdot\,,\mathbf{y})$ on Z. This proves that $\mathbf{z}_0 \in Z$ can only be a point such that the coordinates of $\mathbf{m}(\mathbf{z}_0)$ have a constant distance ρ_0 from the corresponding coordinates of $\mathbf{m}(\mathbf{y})$: $m_k(\mathbf{z}_0) = m_k(\mathbf{y}) - \rho_0$, for $k = 0,1,\dots,n$.

It follows that $\mathbf{y}$ and $\mathbf{z}_0$ are two points of Y with equal difference vectors: $\Phi(\mathbf{y}) = \Phi(\mathbf{z}_0)$. By Theorem 2.1 Φ is, in particular, injective, hence $\mathbf{z}_0 = \mathbf{y}$. It follows that $\rho_0 = 0$, and the maximum of the ρ-distances between points in Z and $\mathbf{y}$ — and therefore the ρ-distances of any node system $\mathbf{z} \in Z$ to $\mathbf{y}$ — can only be zero (since $\rho(\,\cdot\,, \mathbf{y}) \geqslant 0$ on Z). That is, all $\mathbf{z} \in Z$ are maximum points for the ρ-distance: $\rho(\mathbf{z}, \mathbf{y}) = \rho_0 = 0$ for all $\mathbf{z} \in Z$. It follows that to any $\mathbf{z} \in Z$ the same applies as to $\mathbf{z}_0$ selected above, and we conclude that $Z = \{\mathbf{y}\}$. Since $\mathbf{x} \in Z$, it follows that $\mathbf{x} = \mathbf{y}$, as required.

Theorem 4.1 is proved.

Remark 4.1. Similar nonmajorization results are rare, but we can compare the above, for instance, with Theorem 1 on p. 17 of [24]. If the kernel function K and also the external field function J are smooth, then we can consider the Jacobi matrix of Φ (the vector-valued interval maxima difference function) and the proof of the homeomorphism theorem (Theorem 2.1) — that is, the proof of Theorem 2.1 in [10] — where it was shown that this Jacobi matrix is diagonally dominant. It is known that diagonally dominant matrices are so-called P-matrices (for more information, we refer to pp. 134–137 in [2]), hence the conditions of Theorem 1 in [24] are satisfied. However, the conclusion of that result is far weaker than ours: it excludes majorization only in case the nodes $\mathbf{x}$ and $\mathbf{y}$ are ordered similarly coordinatewise: $x_i \leqslant y_i$, $i = 1, \dots, n$. The generality that we obtain nonmajorization for all node systems $\mathbf{x}, \mathbf{y} \in Y$ can be attributed to the special setup, where Φ is formed by the differences of interval maxima of sum of translates functions satisfying our assumptions.

Corollary 4.1. *Consider the (almost) two-century-old classical Chebyshev problem, where in our terminology* $K(t) := \log|t|$ *and* $J(t) \equiv 0$, *and so strict concavity and monotonicity hold. Then for any two node systems* $\mathbf{x}, \mathbf{y} \in S$ *there necessarily exist indices* $0 \leqslant i \neq j \leqslant n$ *such that*

$$\max_{t \in I_i(\mathbf{x})} \left| \prod_{k=1}^{n} (t - x_k) \right| < \max_{t \in I_i(\mathbf{y})} \left| \prod_{k=1}^{n} (t - y_k) \right|$$

and

$$\max_{t \in I_j(\mathbf{x})} \left| \prod_{k=1}^{n} (t - x_k) \right| > \max_{t \in I_j(\mathbf{y})} \left| \prod_{k=1}^{n} (t - y_k) \right|$$

(*see Figure* 1).

Remark 4.2. It seems that even in this classical situation the above general statement has not been observed thus far. The special case when one of the node systems, say, $\mathbf{x}$ is an extremal (equioscillating) node system $\mathbf{w}$, is well known and seems to be folklore. However, a comparison of two arbitrary node systems looks more complicated and nothing was written about it in the literature that we could page through.

Corollary 4.2 (Nonmajorization Theorem). *Let* K *be a singular (see* (∞)*) and monotone (see* (M)*) kernel function, and let* J *be an upper semicontinuous field function. Then strict majorization* $m_i(\mathbf{x}) > m_i(\mathbf{y})$ *for every* $i = 0, 1, \dots, n$ *cannot hold between two arbitrary node systems* $\mathbf{x}, \mathbf{y} \in Y$.

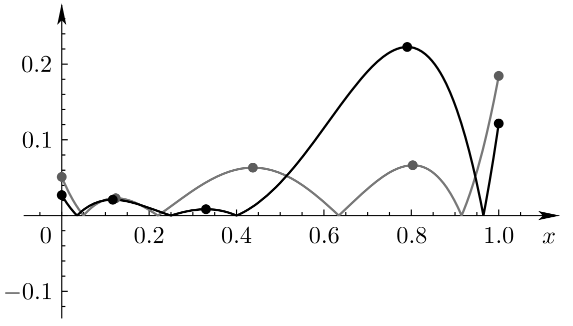

Figure 1. The graphs of $|(x - 0.915)(x - 0.634)(x - 0.22)(x - 0.05)|$ and $|(x-0.965)(x-0.4)(x-0.25)(x-0.035)|$ with dots at local maxima on the interval $[0, 1]$ in grey and black, respectively (cf. Corollary 4.1).

Proof. As in the proof of Corollary 3.2, consider the modified kernel functions $K^{(\eta)}(t) := K(t) + \eta\sqrt{|t|}$, which are strictly concave and strictly monotone kernel functions. Let $\mathbf{x}, \mathbf{y} \in \overline{S}$. Then, as in that proof, as $\eta \downarrow 0$, we have $\mathbf{m}^{(\eta)}(\mathbf{x}) \to \mathbf{m}(\mathbf{x})$ and $\mathbf{m}^{(\eta)}(\mathbf{y}) \to \mathbf{m}(\mathbf{y})$. This implies that once $\mathbf{m}(\mathbf{x}) > \mathbf{m}(\mathbf{y})$, we must have $\mathbf{m}^{(\eta)}(\mathbf{x}) > \mathbf{m}^{(\eta)}(\mathbf{y})$ for every sufficiently small $\eta > 0$, which is impossible by Theorem 4.1, given that $\mathbf{x} \neq \mathbf{y}$ by condition. Hence we conclude that the inequality $\mathbf{m}(\mathbf{x}) > \mathbf{m}(\mathbf{y})$ is impossible for $\mathbf{x}, \mathbf{y} \in Y$. Corollary 4.2 is proved

Corollary 4.3. *Let K be a singular (see (∞)) and monotone (see (M)) kernel function, and let J be an upper semicontinuous field function. Let $\mathbf{x}, \mathbf{y} \in Y$ satisfy $\underline{m}(\mathbf{x}) = m(S) = M(S) = \overline{m}(\mathbf{y})$.*

Then there is $j \in \{0, 1, \dots, n\}$ such that

$$m_j(\mathbf{x}) = m(S) = M(S) = m_j(\mathbf{y}).$$

§ 5. A discussion of conditions

In this section we show in examples that dropping conditions from our results entails that the conclusions may not hold true any more. This justifies the conditions imposed, even if at first glance assuming, for example, monotonicity or strict concavity may not seem to be natural or necessary. The examples here also highlight the generality of our statements, where, for example, further smoothness conditions were not supposed. In our results only conditions which are shown here not to be simply dispensable were assumed throughout. Below we talk about the ‘necessity’ of conditions in this, logically weaker sense of indispensability.

Example 5.1 (Necessity of singularity). Let $n = 2$, $J(t) := 8\sqrt{1-t}$, and let $K(t) := \sqrt{t+4}$ for $t \in [0, 1]$ and $K(t) := K(-t)$ for $t \in [-1, 0)$. Then J is a concave field function, $J \in C^\infty([0, 1])$, and $K \in C^\infty([-1, 1] \setminus \{0\})$ is a strictly concave kernel function; furthermore, K is monotone as in (M) and J and K do not satisfy condition (∞).

We have $M(S) = m(S) = -4$, there is a unique equioscillation, a unique minimax and a unique maximin node system and all of these are $(0, 0) \in \partial S$. In other words, almost all conclusions of the above theorems hold true, except that this point of extrema and equioscillation is not in S, but on the boundary ∂S.

The key observation is that $\frac{d}{dt}F(\mathbf{y},t) < 0$ for any $\mathbf{y} \in S$, at every point $t \in [0,1]$. Hence

$$m_0(\mathbf{y}) = \max\{F(\mathbf{y},t)\colon 0 \leqslant t \leqslant y_1\} = F(\mathbf{y},0) = 8 + \sqrt{4+y_1} + \sqrt{4+y_2},$$
$$m_1(\mathbf{y}) = F(\mathbf{y},y_1) = 8\sqrt{1-y_1} + 2 + \sqrt{4+y_2-y_1}$$

and

$$m_2(\mathbf{y}) = F(\mathbf{y},y_2) = 8\sqrt{1-y_2} + \sqrt{4+y_2-y_1} + 2.$$

By this observation $m_0(\mathbf{y}) \geqslant m_1(\mathbf{y})$, with equality if and only if $y_1 = 0$ and similarly, $m_1(\mathbf{y}) \geqslant m_2(\mathbf{y})$ with equality if and only if $y_1 = y_2$. Also, $\overline{m}(\mathbf{y}) = m_0(\mathbf{y})$ and $\underline{m}(\mathbf{y}) = m_2(\mathbf{y})$. Obviously, $m_0(\mathbf{y})$, $\mathbf{y} \in \overline{S}$, is minimal if and only if $\mathbf{y} = (0,0)$, and $m_2(\mathbf{y})$ is maximal if and only if $\mathbf{y} = (0,0)$. Therefore, $M(\overline{S}) = m(\overline{S}) = -4$ and these are attained at $\mathbf{y} = (0,0)$ only, and there is a unique equioscillation configuration in $\overline{S}$, namely, $\mathbf{y} = (0,0)$.

For convenience we introduce the kernel function $L_a(t) := \min(0, \log|t/a|)$.

Example 5.2 (Necessity of monotonicity). Let $n = 1$, $J(t) := \sqrt{t}$ and $K(t) := L_{0.1}(t) + 1 - 2t^2$. Note that J is a strictly concave field function, K is a strictly concave kernel function, and K is singular, but it does not satisfy any of the monotonicity conditions (M) and (SM). Then the global minimum $M(S)$ of $\overline{m}$ is attained only at $\mathbf{y} = (y_1) = (0)$, so I_0 is degenerate and $\mathbf{y} \in \partial S$. Also, $m_0(\mathbf{y}) = -\infty$ and $m_1(\mathbf{y}) = F(0, 1/4) = 11/8$. Obviously, $F(\mathbf{y}, \cdot\,)$ does not equioscillate.

Indeed, if $0 < y_1 \leqslant 1/2$, then $\overline{m}(\mathbf{y}) \geqslant F(y_1, y_1 + 1/4) = \sqrt{y_1 + 1/4} + 7/8 > 1/2 + 7/8 = 11/8$, and if $1/2 < y_1 \leqslant 1$, then $\overline{m}(\mathbf{y}) \geqslant F(y_1, y_1 - 1/4) = \sqrt{y_1 - 1/4} + 7/8 > 11/8$. So $\overline{m}(\mathbf{y}) = 11/8$ is attained at $\mathbf{y} = (y_1) = (0)$ only.

Example 5.3 (Necessity of strict monotonicity and concavity). Let $0 < a < e/(1+e)$ be arbitrary, and let $K(t) := L_a(t)$. Set $n = 1$. Let $b \in (0,1)$ satisfy $(a <)1 - a/e < b < 1$, and let J be the characteristic function of the interval $[b, 1]$. Then

a) $\mathbf{m}(\mathbf{x}) = (0,0)$ if and only if $\mathbf{x} = (x)$, where $x = 1 - a/e$, and this is the unique equioscillation point of F, moreover, $m(S) = M(S) = 0$;

b) however, $\underline{m}(\mathbf{x}) = 0$ for all $a \leqslant x \leqslant 1 - a/e$ and thus $\underline{m}(\mathbf{x}) = m(S) = 0$ is attained not only for the equioscillating node, but also for several other, nonequioscillating ones;

c) in particular, majorization occurs on Y.

Note that K is a concave and monotone kernel function, but it is not strictly concave or strictly monotone.

In Figure 2 we show the graphs of sum of translates functions.

Indeed,[9] $F(x, \cdot\,)$ cannot attain positive values on $[0, x]$. If $x \leqslant b$, then $J(x) = 0$ and $L_a(x) \leqslant 0$. If $b \leqslant x \leqslant 1$, then by the monotonicity of L_a, $F(x, \cdot\,)$ is maximal on $[0, x]$ either at b or at 0. The value at b is $L_a(b - x) + 1 \leqslant L_a(b - 1) + 1 < \log|(a/e)/a| + 1 = 0$. The value at 0 is $F(x, 0) \leqslant 0$; moreover, taking into account that $x \geqslant a$ also holds, we have $L_a(0 - x) = 0$, so that $F(x, 0) = 0$.

[9]For convenience we write $F(x,t)$ and $m_j(x)$ in place of $F(\mathbf{x},t)$ and $m_j(\mathbf{x})$, respectively, and so on.

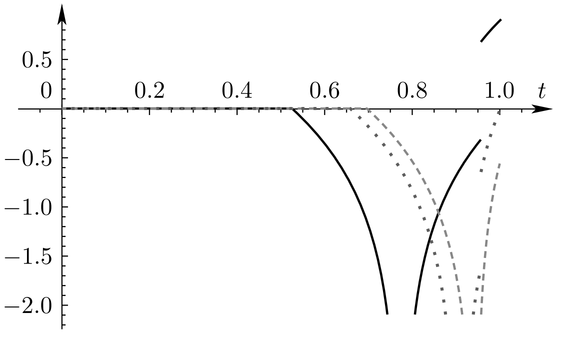

Figure 2. The graphs of the sum of translates functions $J(t) + K(t - x)$ for $a = 1/4$, $b = 0.955671$ and $x = 0.775$ (black), $x = 0.907$ (dotted) and $x = 0.946$ (dashed); see Example 5.3.

It follows that $m_0(x) \leqslant 0$. Note that $m_0(0) = -\infty$ and m_0 is strictly increasing on $[0, a]$. If $a \leqslant x \leqslant b$, then $F(x, t)$ is monotonically decreasing on $t \in [0, x]$ and $F(x, 0) = 0$, so $m_0(x) = 0$. If $b \leqslant x \leqslant 1$, then $F(x, t)$ is monotonically decreasing on $t \in [0, b]$ and is also monotonically decreasing on $t \in [b, x]$. Moreover, $F(x, 0) = 0$ and $F(x, b) = 1 + L_a(b - x) \leqslant 1 + L_a(b - 1) \leqslant 1 + \log|(1 - b)/a| \leqslant 0$, so that $m_0(x) = F(x, 0) = 0$ in this case, too. Altogether, $m_0(x) \leqslant 0$ for all x, and $m_0(x) = 0$ precisely for $a \leqslant x \leqslant 1$.

Regarding m_1, note that J is monotonically increasing and $L_a(\cdot - x)$ is also monotonically increasing on $[x, 1]$, so $m_1(x) = F(x, 1) = 1 + L_a(1 - x)$. If $0 \leqslant x \leqslant 1 - a$, then $L_a(1 - x) = 0$, so that $m_1(x) = 1$; and if $1 - a < x \leqslant 1$, then $L_a(1 - x) = \log((1 - x)/a)$, so that $m_1(x) = \log((1 - x)/a) + 1 < 1$. Altogether, $m_1(x) = 1$ for $x \in [0, 1 - a]$, and then it is strictly decreasing from 1 to $-\infty$ on $[1 - a, 1]$, attaining 0 exactly for $x = 1 - a/e$.

Comparing these cases for different ranges of x, we see that $m_0(x) = m_1(x)$ holds if and only if $m_0(x) = 0$ and $m_1(x) = 0$, which occurs precisely for $x = 1 - a/e$ (see Figure 3).

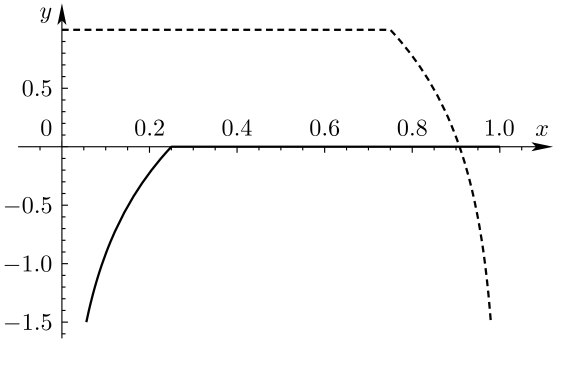

Figure 3. The graphs of the interval maxima functions $m_0(x)$ (a solid line) and $m_1(x)$ (a dashed line); see Example 5.3.

Therefore, there is a unique equioscillation point. Also, if $0 \leqslant x \leqslant 1 - a/e$, then $\underline{m}(x) = 0$.

Example 5.4 (Necessity of monotonicity). Let

$$K(t) := \min\left(\log|10t|, \log\left(\frac{10}{9}(1-|t|)\right)\right)$$

and $J(t) := 0$. Then K is strictly concave, but not monotone. Set $n = 2$. Then

a) there are several equioscillating node systems, but the corresponding values of $\overline{m}$ are the same;
b) there is a unique minimax node system;
c) there are several maximin node systems;
d) strict majorization occurs.

First observe that $K(t) \leqslant 0$ for $t \in [-1, 1]$. This immediately follows from the fact that $\log|10t| = \log(10(1-|t|)/9)$ holds if and only if $t = 1/10$, and $K(1/10) = 0$ and K is strictly monotonically increasing on $[0, 1/10]$ and strictly monotonically decreasing on $[1/10, 1]$.

We write nodes as $\mathbf{x} = (a-\delta, a+\delta)$, where $a := (x_1+x_2)/2$ and $\delta := (x_2-x_1)/2$. By condition $0 \leqslant a - \delta \leqslant a + \delta \leqslant 1$; in other words, $0 \leqslant a \leqslant 1$ and $0 \leqslant \delta \leqslant a, 1-a$. First observe that

$$m_1(\mathbf{x}) = \sup\{F(\mathbf{x}, t)\colon a - \delta \leqslant t \leqslant a + \delta\} = F(\mathbf{x}, a),$$

and so

$$m_1(\mathbf{x}) = \begin{cases} 2\log(10\delta) & \text{if } 0 \leqslant \delta \leqslant \dfrac{1}{10}, \\[2ex] 2\log\dfrac{10(1-\delta)}{9} & \text{if } \dfrac{1}{10} \leqslant \delta. \end{cases}$$

Also, $m_1(\mathbf{x}) \leqslant 0$, and $m_1(\mathbf{x}) = 0$ if and only if $\delta = 1/10$.

If $t \in [a+\delta, 1]$, then we have the following three cases depending on t (the interval $a + \delta \leqslant t \leqslant a - \delta + 0.1$ can as well be empty), where $u = t - a$, $\delta \leqslant u \leqslant 1 - a$:

$$F(\mathbf{x}, t) = \begin{cases} \log\big(100(u-\delta)(u+\delta)\big) & \text{if } \delta \leqslant \dfrac{1}{20}, \quad \delta \leqslant u \leqslant \dfrac{1}{10} - \delta, \\[2ex] \log\left(\dfrac{100}{9}(1-(u+\delta))(u-\delta)\right) & \text{if } \dfrac{1}{10} - \delta \leqslant u \leqslant \delta + \dfrac{1}{10}, \\[2ex] \log\left(\dfrac{100}{81}(1-(u+\delta))(1-(u-\delta))\right) & \text{if } \delta + \dfrac{1}{10} \leqslant u \leqslant 1 - a \end{cases}$$

(see Figure 4). Here it is clear that the first expression is strictly increasing in u and the third expression is strictly decreasing in u, so that $m_2(\mathbf{x})$ equals the maximum of the second expression. Now elementary calculus shows that the second expression is strictly increasing in u if $\delta + 0.1 \leqslant 1/2$, while if $\delta + 0.1 > 1/2$, then it has a strict local maximum at $u = 1/2$, that is, for $t = a + 1/2$. Therefore,

$$m_2(\mathbf{x}) = \begin{cases} F\left(\mathbf{x}, a + \delta + \dfrac{1}{10}\right) = \log\left(1 - \dfrac{20}{9}\delta\right) & \text{if } \delta + \dfrac{1}{10} \leqslant \dfrac{1}{2}, \\[2ex] F\left(\mathbf{x}, \dfrac{1}{2} + a\right) = \log\left(\dfrac{100}{9}\left(\dfrac{1}{2} - \delta\right)^2\right) & \text{if } \dfrac{4}{10} \leqslant \delta \leqslant \dfrac{1}{2}. \end{cases}$$

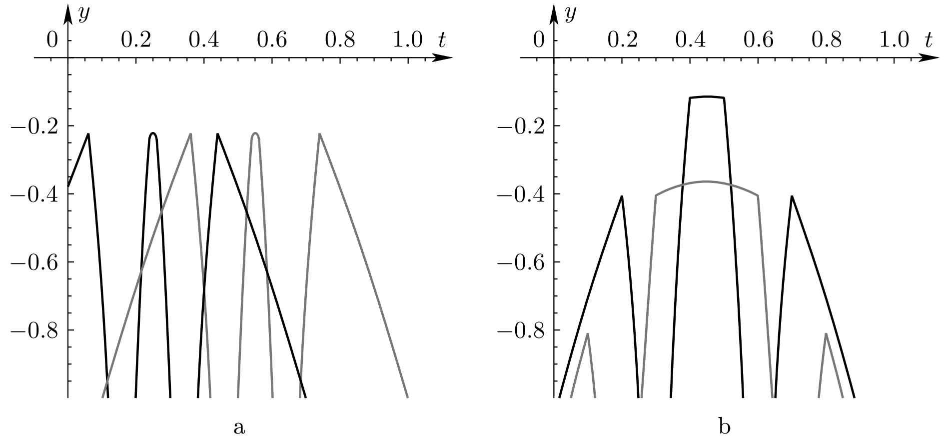

Figure 4. The graphs of $F(\mathbf{x},t)$ for $\mathbf{x}=(a-\delta,a+\delta)$: (a) for $a=1/4$, $\delta=(\sqrt{82}-1)/90$ (black) and for $a=55/100$ (grey); (b) for $a=45/100$ and $\delta=15/100$ (black) and $\delta=25/100$ (grey); see Example 5.4.

To determine the equioscillating configurations, we compare the values of $m_1(\mathbf{x})$ and $m_2(\mathbf{x})$ in three cases depending on δ. They can be equal if and only if $\delta=\delta_0:=(\sqrt{82}-1)/90\approx 0.0895$. Therefore, any $\mathbf{x}=(a-\delta_0,a+\delta_0)$, $a\in[\delta_0+1/10,1-\delta_0-1/0]$ is an equioscillating configuration.

Also, $\overline{m}(\mathbf{x})\leqslant 0$ and $\overline{m}(\mathbf{x})=0$ if and only if $\mathbf{x}=(a-\delta,a+\delta)$ and $\delta=0$ or $\delta=1/10$.

Moreover, both $m_1(\mathbf{x})$ and $m_2(\mathbf{x})$ are strictly decreasing if $\delta\geqslant 1/10$ (and δ is not too large), which shows that strict majorization holds (for some configurations).

§ 6. Applications

6.1. The Bojanov problem on an interval. Consider now the set of *monic* 'generalized algebraic polynomials' (cf. Appendix A4 on p. 392 in [7]) of given degree $\mathbf{r}:=(r_1,\dots,r_n)$, where $r_1,\dots,r_n>0$ are given positive exponents:

$$\mathcal{P}_{\mathbf{r}}[a,b]:=\biggl\{P\colon P(t)=\prod_{j=1}^{n}|t-x_j|^{r_j},\ t\in[a,b],\ a\leqslant x_1\leqslant\dots\leqslant x_n\leqslant b\biggr\}.$$

Take an upper semicontinuous weight function $w: I\to[0,\infty)$ satisfying the condition that it is nonzero at at least $n+1$ points on the interval $I:=[a,b]$ (the endpoints a and b are counted with weight $1/2$). Consider the w-weighted uniform norm $\|\cdot\|_w$ defined by $\|f\|_w:=\|fw\|_\infty:=\sup_I|f|w$. Then the Bojanov Extremal Problem, is to find the generalized algebraic polynomial $P\in\mathcal{P}_{\mathbf{r}}[a,b]$ with the least possible norm $\|P\|_w$, so that $\|P\|_w=\min_{Q\in\mathcal{P}_{\mathbf{r}}[a,b]}\|Q\|_w$. If such an extremal polynomial exists, then it is called a *Bojanov-Chebyshev polynomial.*

Actually, similarly to the classical case there are two possible formulations of the extremal problem, since there is an 'unrestricted' version, where we do not assume that the zeros of the polynomial belong to $[a,b]$. However, here it is of importance that the order of the zero factors arising follows the order of the prescribed exponents r_j, so for arbitrary complex zeros the correct interpretation is that we take

$$\mathcal{P}_{\mathbf{r}} := \left\{ \prod_{j=1}^{n} |t-(x_j+iy_j)|^{r_j} : \ -\infty < x_1 \leqslant \cdots \leqslant x_n < \infty,\ y_1,\dots,y_n \in \mathbb{R} \right\}.$$

Thus the (*restricted*) Bojanov-Chebyshev constant is $R_{\mathbf{r}}^w[a,b] := \min_{Q\in\mathcal{P}_{\mathbf{r}}[a,b]} \|Q\|_w$, and the *unrestricted* Bojanov-Chebyshev constant is $C_{\mathbf{r}}^w[a,b] := \min_{Q\in\mathcal{P}_{\mathbf{r}}} \|Q\|_w$. As in the classical case, we easily see that although $C_{\mathbf{r}}^w[a,b]$ is formally the infimum over a larger set, we still have $C_{\mathbf{r}}^w[a,b] = R_{\mathbf{r}}^w[a,b]$; furthermore, extremizers exist only in $\mathcal{P}_{\mathbf{r}}[a,b]$ (if anywhere).

We can now give a slightly more precise statement than Theorem 1.2.

Theorem 6.1. *Let $n \in \mathbb{N}$, let $r_1, r_2, \dots, r_n$ be positive numbers, let $[a,b]$ a nondegenerate compact interval and w be an upper semicontinuous, nonnegative weight function on $[a,b]$ assuming nonzero values at more than n points of $[a,b]$ (counted with weights). Then $C_{\mathbf{r}}^w[a,b] = R_{\mathbf{r}}^w[a,b]$, and there exists a unique Chebyshev-Bojanov extremal generalized polynomial P belonging to $\mathcal{P}_{\mathbf{r}}[a,b]$. This generalized algebraic polynomial has the form*

$$P(t) = \prod_{j=1}^{n} |t - x_j^*|^{r_j}, \tag{6.1}$$

where the node system $\mathbf{x}^* := (x_1^*, \dots, x_n^*)$ satisfies $a < x_1^* < \cdots < x_n^* < b$ and is uniquely determined by the following equioscillation property: there exists an array of $n+1$ points $a \leqslant t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \leqslant b$ interlacing with the x_i^*, that is, $a \leqslant t_0 < x_1^* < t_1 < x_2^* < \cdots < x_n^* < t_n \leqslant b$, such that

$$P(t_k)w(t_k) = \|P\|_w, \qquad k = 0, 1, \dots, n. \tag{6.2}$$

Furthermore, if w is, in addition, log-concave, then the unique Chebyshev-Bojanov extremal generalized polynomial P is determined by the property that there exists an array of $n+1$ points $a \leqslant t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \leqslant b$ such that (6.2) holds.

Remark 6.1. Note that Theorem 13.7 in [11] is the unweighted case. Also note that here we depart from considering signatures, but in case $r_j \in \mathbb{N}$ the analogous signed version can be seen easily. Also, in fact, one can assign signs to factors of type $|t|^r$ arbitrarily, for example, by considering $|t|^r \operatorname{sign} t$ or, in case $r \in \mathbb{N}$, $|t|^r (\operatorname{sign} t)^r$. Then the arising signed problem can easily be seen to become equivalent to the version with absolute values. This shows that not sign changes, but attaining the minimum norm, are the decisive properties of an extremizer.

Proof of Theorem 6.1. Making a simple linear substitution, it suffices to consider the case when $[a,b] = [0,1]$. Let $K := \log|\cdot|$ and $J := \log w$. It is clear that K is a singular, strictly concave, strictly monotone kernel function and J is an n-field function. By taking the logarithms, the original extremal problem of the minimization of $\|P\|_w$ is equivalent to the minimization problem $M(S)$ for $\sup F(\mathbf{x}, \cdot)$ with positive constants r_j $(j = 1, \dots, n)$ as the ones fixed in $\mathbf{r}$.

We have already discussed that $C^w_{\mathbf{r}}[0,1] = R^w_{\mathbf{r}}[0,1]$. For the latter quantity we know that $R^w_{\mathbf{r}}[0,1] = \exp(M(S))$. Therefore, an application of Corollary 3.1 furnishes a characterization of extremal generalized polynomials. To see the last assertion note that by assumption J is concave, so that $F(\mathbf{x}^*, \cdot\,)$ is strictly concave on each $I_j(\mathbf{x}^*)$. By this the nodes and the (unique by strict concavity) maximum points $t_0, \dots, t_n$ interlace, so the already proved characterization of extremal generalized polynomials applies. The theorem is proved.

Consider generalized algebraic polynomials $Q_{\mathbf{x}}(t) := \prod_{j=1}^n |t - x_j|^{r_j}$. According to the above there is an even more precise understanding of the Bojanov-Chebyshev problem. Setting $M_j(\mathbf{x}) := \max_{I_j(\mathbf{x})} |Qw|$, we have the following intertwining property: for *any two admissible node systems* $\mathbf{x}, \mathbf{y} \in Y$ there exist indices i and k such that $M_i(\mathbf{x}) < M_i(\mathbf{y})$ and $M_k(\mathbf{x}) > M_k(\mathbf{y})$; in particular, for any node system $\mathbf{x} \neq \mathbf{x}^*$ there exist indices i and k such that $M_i(\mathbf{x}) < C^w_{\mathbf{r}}[a,b]$ and $M_k(\mathbf{x}) > C^w_{\mathbf{r}}[a,b]$, so that these Chebyshev constants are bounded on both sides by the interval maxima of an arbitrary node system: $\min_{i=0,\dots,n} M_i(\mathbf{x}) < C^w_{\mathbf{r}}[a,b] < \max_{k=0,\dots,n} M_k(\mathbf{x})$.

Let us emphasize that the above discussion does not only generalize the Bojanov-Chebyshev problem to weighted generalized algebraic polynomials, but it is much more general, given that we can take any log-concave, monotone factors $G(t)$ in place of $|t|$ (which corresponds to $K(t) := \log G(t)$ being more general than $\log|t|$).

6.2. A comparison of Chebyshev constants of a union of intervals. The above discussion of various versions of the Chebyshev constant might have been considered trivial, but if we move to nonconvex sets, then the distinction between restricted and nonrestricted Chebyshev constants becomes essential.

Let $E \subset \mathbb{R}$ be a compact set and $w \geqslant 0$ be a weight. As above, we define the *restricted Bojanov-Chebyshev constant* $R^w_{\mathbf{r}}(E) := \min_{Q \in \mathcal{P}_{\mathbf{r}}(E)} \|Q\|_w$, where

$$\mathcal{P}_{\mathbf{r}}(E) := \left\{ P \colon P(t) = \prod_{j=1}^n |t - x_j|^{r_j},\ t \in E,\ x_1, \dots, x_n \in E \right\},$$

and the *unrestricted Bojanov-Chebyshev constant* $C^w_{\mathbf{r}}(E) := \min_{Q \in \mathcal{P}_{\mathbf{r}}} \|Q\|_w$.

What we can do here is as follows.

Theorem 6.2. *Let $k, n \in \mathbb{N}$, and let $a_1 < b_1 < a_2 < b_2 < \dots < a_k < b_k$ be arbitrary real numbers, let $E := \cup_{\ell=1}^k [a_\ell, b_\ell]$, and let $\mathbf{r} \in (0,\infty)^n$ be an arbitrary system of exponents. Then the restricted and unrestricted Chebyshev constants satisfy the inequality*

$$C^w_{\mathbf{r}}(E) \leqslant R^w_{\mathbf{r}}(E) \leqslant C(k, \mathbf{r}) C^w_{\mathbf{r}}(E), \tag{6.3}$$

where $C(k,\mathbf{r}) := 2^{\max\{r_{i_1} + \dots + r_{i_{k-1}} \,:\, 1 \leqslant i_1 < \dots < i_{k-1} \leqslant n\}}$. In particular, if $\mathbf{r} := \mathbf{1}$, that is, $\mathcal{P}_{\mathbf{r}} = \mathcal{P}^1_n$ is the family of (absolute values of) ordinary monic degree n algebraic polynomials, then

$$C^w_n(E) \leqslant R^w_n(E) \leqslant 2^{k-1} C^w_n(E),$$

independently of the value of n.

Proof. As above, it is easy to see that for the unrestricted Chebyshev constant it suffices to consider polynomials with all roots x_j lying in the closed convex hull $E^* := \operatorname{con} E = [a_1, b_k]$ of E. So by the compactness of E^* and the upper semicontinuity of w there exists an extremizer, and for this $P \in \mathcal{P}_{\mathbf{r}}(E^*)$ we have $C^w_{\mathbf{r}}(E) = \|P\|_w$ ($:= \max_E |Pw|$). We will construct $Q \in \mathcal{P}_{\mathbf{r}}(E)$ such that $\|Q\|_w \leqslant C(k, \mathbf{r})\|P\|_w$, so that minimizing over $\mathcal{P}_{\mathbf{r}}(E)$ will be seen to yield $R^w_{\mathbf{r}}(E) \leqslant C(k, \mathbf{r})\|P\|_w = C(k, \mathbf{r})C^w_{\mathbf{r}}(E)$, as required.

For convenience assume that $E \subset [0, 1]$ or even, for technical ease, that $a_1 = 0$ and $b_k = 1$. Take $K(t) := \log |t|$ and $J(t) := \log(w\chi_E)$, where w is understood as defined on the whole of $\mathbb{R}$ and χ_E is the indicator function of E (which is upper semicontinuous). Then for any $Q_{\mathbf{x}}(t) \in \mathcal{P}_{\mathbf{r}}(E^*)$ with root system $\mathbf{x} \in E^* = [0, 1]$ we obviously have

$$\log \|Q_{\mathbf{x}}\|_w = \max_{t\in[0,1]} \left(J(t) + \sum_{j=1}^{n} r_j K(t - x_j) \right) = \max_{t\in[0,1]} F(\mathbf{x}, t) = \overline{m}(\mathbf{x}).$$

So, in particular, the minimality of $\|P\|_w$ over choices of zeros $\mathbf{x} \in E^* = [0, 1]$ translates into the statement that $P = Q_{\mathbf{w}}$ for some $\mathbf{w} \in \overline{S}$ which is a minimax point of F. As shown in Theorem 3.1 above, for the strictly concave and singular kernel function K satisfying strict monotonicity (see (SM)) we have $\mathbf{w} \in Y$. As the indicator function χ_E vanishes on the complementary intervals $J_\ell := (b_\ell, a_{\ell+1})$, $\ell = 1, k-1$, we also have $J = -\infty$, and subintervals of these complementary intervals are singular. So no $I_i(\mathbf{w})$ can be a subinterval of these complementary intervals, because $\mathbf{w}$ is nonsingular. In other words, in such a complementary interval there is at most one $w_i \in J_\ell$.

To construct our $Q_{\mathbf{x}}$, that is, the corresponding $\mathbf{x}$ (with all $x_i \in E$) and $F(\mathbf{x}, \cdot)$, we choose a node system $\mathbf{x}$ such that $x_i = w_i$ whenever $w_i \in E$, and $x_i = b_\ell$ or $x_i = a_{\ell+1}$, whichever is closer to w_i, for $w_i \in J_\ell$ (and, say, $x_i := b_\ell$ if they are of equal distance, that is, $w_i = (b_\ell + a_{\ell+1})/2$).

Let us compare the pure sum of translates functions for $\mathbf{w}$ and $\mathbf{x}$. We obtain

$$f(\mathbf{x}, t) - f(\mathbf{w}, t) = \sum_{i \ w_i \notin E} \big(r_i K(t - x_i) - r_i K(t - w_i) \big).$$

If i is such that $w_i \notin E$, then $w_i \in J_\ell = (b_\ell, a_{\ell+1})$ for some $1 \leqslant \ell \leqslant k - 1$), while for $t \in E$ we have either $t \leqslant b_\ell$, or $t \geqslant a_{\ell+1}$. In case $x_i = b_\ell$, for all $0 \leqslant t \leqslant b_\ell$ we have $K(t - x_i) < K(t - w_i)$ by monotonicity. Now let $a_{\ell+1} \leqslant t \leqslant 1$. Then $K(t-x_i) = \log(t-x_i) = \log(t-w_i + (w_i - x_i)) \leqslant \log(2(t-w_i)) = \log 2 + K(t-w_i)$ for $t - w_i \geqslant a_{\ell+1} - w_i \geqslant \frac{a_{\ell+1} - b_\ell}{2} \geqslant w_i - x_i$ by construction. It follows that $K(t-x_i) \leqslant \log 2 + K(t-w_i)$ for all $t \in E$. Similarly, it is easy to see that the same holds when $x_i = a_{\ell+1}$. Adding these relations for all indices i such that $w_i \notin E$ we find that

$$f(\mathbf{x}, t) - f(\mathbf{w}, t) = \sum_{i \ w_i \notin E} r_i \log 2 \leqslant \log C(k, \mathbf{r}) \quad \forall\, t \in E.$$

Therefore, we also have $F(\mathbf{x}, t) \leqslant \log C(k, \mathbf{r}) + F(\mathbf{w}, t)$ for all points $t \in [0, 1]$ such that $J(t) \neq -\infty$. However, if $t \notin E$, then adding $J(t) = -\infty$ makes both sides infinite: $F(\mathbf{x}, t) = F(\mathbf{w}, t) = -\infty$, so the same inequality reads as $-\infty \leqslant -\infty$, and it remains valid in fact. Finally, taking maxima we obtain $\overline{m}(\mathbf{x}) \leqslant \overline{m}(\mathbf{w}) + \log C(k, \mathbf{r})$. Theorem 6.2 is proved.

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