

An upper bound on the dimension of the Rauzy gasket

Mark Pollicott* and Benedict Sewell†

May 23, 2023

1 Introduction

The Rauzy Gasket \mathcal{G} is a compact subset of the standard 2-simplex, $\Delta = \{(x, y, z) : x, y, z \geq 0, x + y + z = 1\}$. It plays the role of an exceptional set in the theory of interval exchange transformations and other settings, and is the limit set of the iterated function scheme for the three weak projectivised linear maps $T_1, T_2, T_3 : \Delta \rightarrow \Delta$, defined by

$$\begin{aligned} T_1(x, y, z) &= \left(\frac{1}{2-x}, \frac{y}{2-x}, \frac{z}{2-x} \right), \\ T_2(x, y, z) &= \left(\frac{x}{2-y}, \frac{1}{2-y}, \frac{z}{2-y} \right), \\ T_3(x, y, z) &= \left(\frac{x}{2-z}, \frac{y}{2-z}, \frac{1}{2-z} \right); \end{aligned}$$

i.e., \mathcal{G} is the smallest non-trivial closed set such that $\mathcal{G} = \bigcup_{j=1}^3 T_j(\mathcal{G})$.

*Supported by ERC grant 833802-resonances and EPSRC grant EP/T001674/1.

†Supported by the Alfréd Rényi Institute Young Researcher Fund.

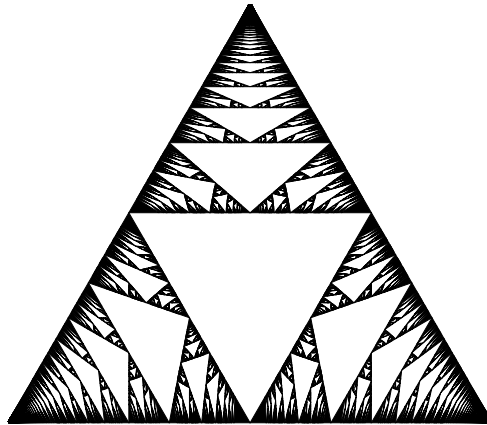


Figure 1: The Rauzy gasket

The gasket has an interesting history, appearing for the first time in 1991 in the work of Arnoux and Rauzy [1], in the context of interval exchange transformations, where it was conjectured that $\text{Leb}(\mathcal{G}) = 0$. The gasket was rediscovered by Levitt in 1993 [13], in a paper which also included a proof (due to Yoccoz) that $\text{Leb}(\mathcal{G}) = 0$. The gasket \mathcal{G} emerged for a third time in the work of De Leo and Dynnikov [5], this time in the context of Novikov's theory of magnetic induction on monocrystals (see [6] for the dichotomy between this and [1]). They gave an alternative proof that $\text{Leb}(\mathcal{G}) = 0$ and proposed the stronger result $\dim_H(\mathcal{G}) < 2$. Novikov and Maltsev [14] also conjectured the stronger bound $\dim_H(\mathcal{G}) < 2$, which was rigorously established by Avila, Hubert and Skripchenko [4]. Empirical estimates in [5] suggest $\dim_H(\mathcal{G}) \approx 1.72$, and a lower bound was shown in [11]. Lastly, Fougeron used semiflows and thermodynamic techniques to show $\dim_H(\mathcal{G}) < 1.825$ [9]. Using completely elementary methods, we show the following improved upper bound.

Theorem 1.1. $\dim_H(\mathcal{G}) \leq 1.7415$.

The Rauzy Gasket has a number of interesting recent applications. Gamburd, Magee and Ronan [10] showed asymptotic estimates for integer solutions of the Markov-Hurwitz equations featuring $\dim_H(\mathcal{G})$. Hubert and Paris-Romaskevich in [12] considered triangular tiling billiards, modelling refraction in crystals. The gasket \mathcal{G} parameterises triangles admitting trajectories which escape non-linearly to infinity and closed orbits which approximate fractal-like sets.

In section 2 we give the technical result which leads to the bound in Theorem 1.1. This is formulated in terms of certain infinite matrices. In section 3 we give elementary preliminary bounds on the area and diameter of small triangles given as the images of Δ under compositions of the maps T_1 , T_2 and T_3 . In section 4 we use these to obtain a bound for the dimension, provided an associated sequence of real numbers X_n converges to zero. In sections 5 and 6 we present the core of the proof. In section 5, we use the estimates from section 3 to bound X_n in terms of expressions satisfying an iterative relation. In section 6 we use the renewal theorem to deduce that $X_n \rightarrow 0$ under the hypotheses of Theorem 2.8. Finally, in section 7 we apply Theorem 2.8 empirically to deduce the bound in Theorem 1.1.

A fuller account appears in [15].

2 A formal statement

The bound in Theorem 1.1 is a special case of a decreasing sequence of upper bounds, indexed by a parameter $m \in \mathbb{N}_{\geq 2}$. Each bound can be described using powers of an infinite matrix.

Definition 2.1 (An index set). *Let $\mathcal{V} = \bigcup_{k=1}^{m-1} \mathcal{V}_k$ denote the finite set where, for $k < m$,*

$$\mathcal{V}_k := \{1\}^k \times \{2\} \times \{1, 2, 3\}^{m-k} = \{(1^k, 2, v_{k+2}, \dots, v_{m+1}) : v_{k+2}, \dots, v_{m+1} \in \{1, 2, 3\}\},$$

where we denote, e.g., $1^k = \overbrace{1, \dots, 1}^k$. i.e., \mathcal{V} is the family of strings of length $m+1$ beginning with a sequence of 1s of length k , followed by a 2, then a sequence of 1s, 2s and 3s of length $m-k$.

Remark 2.2. The elements of \mathcal{V} represent orbits in $\{1, 2, 3\}^{m+1}$ under the natural action of the dihedral group, excluding the orbits of $(1^m, 2)$ and (1^{m+1}) .

More specifically, any two words $\underline{i}, \underline{v} \in \{1, 2, 3\}^{m+1}$ will be considered equivalent (written $\underline{i} \sim \underline{v}$) if there is some permutation π of $\{1, 2, 3\}$ such that $\pi(i_j) = v_j$ for all $j = 1, \dots, m+1$. For example, if $i_1 \neq i_2$, (i_1, i_2^m) is equivalent to $(1, 2^m) =: \otimes$, which we consider as a distinguished element of \mathcal{V} .

For each $\underline{v} \in \mathcal{V}$ and $n \in \mathbb{N}$, there will be $2 \times 3^{n-m}$ strings $\underline{i} \in \{1, 2, 3\}^n$ for which the truncation $(i_1, \dots, i_{m+1}) \in \{1, 2, 3\}^{m+1}$ is equivalent to \underline{v} .

For $n \geq 1$ and $\underline{i} = (i_1, \dots, i_n) \in \{1, 2, 3\}^n$ we denote the composition $T_{\underline{i}} = T_{i_1} \circ \dots \circ T_{i_n}$ and its image $\Delta_{\underline{i}} = T_{\underline{i}}(\Delta) \subset \Delta$. The following is easily seen [15].

Lemma 2.3. *Each $\Delta_{\underline{i}}$, $\underline{i} \in \{1, 2, 3\}^n$, $n \geq 1$ is again a triangle.*

As we shall see later, these 3^n small triangles provide a useful family of covers for \mathcal{G} .

We now formally define a finite matrix B , whose size depends on m and whose entries depend on δ , as follows. Since B acts as a weighted adjacency matrix, we first need a condition for adjacency.

Definition 2.4 (Adjacency). *Given $\underline{v} = (1^k, 2, v_{k+2}, \dots, v_{m+1}) \in \mathcal{V}$, we write $\underline{v} \mapsto_1 \underline{v}'$, where $\underline{v}' \in \mathcal{V} \cup \{m\}$ is as follows:*

- a) *If $k \leq m-2$, then $\underline{v}' = (1^{k+1}, 2, v_{k+2}, \dots, v_m)$.*
- b) *If $k = m-1$, then $\underline{v}' = m$.*

Moreover, for $j = 2, 3$, we write $\underline{v} \mapsto_j \underline{v}'$, where $\underline{v}' = (1, 2^k, \dots) \in \mathcal{V}_1$ is equivalent to $(j, 1^k, 2, v_{k+2}, \dots, v_m)$.

This definition describes the non-zero indices of the matrix B , as follows.

Definition 2.5 (A finite matrix). *Fix the value $\lambda := \frac{3}{2} - \frac{1}{\sqrt{3}} = 0.9226\dots$ and $\delta > 0$. Then, we can consider the square matrix B indexed by $\mathcal{V} \cup \{m\}$, defined by*

$$B_{\underline{v}, \underline{w}} = \begin{cases} \max_{x \in \Delta_{\underline{w}}} (2 - x_j)^{-3\delta - \lambda(1-\delta)}, & \text{if } \underline{w} \mapsto_j \underline{v}; \\ 0, & \text{otherwise.} \end{cases}$$

(Note that $B_{\otimes, \underline{v}} = B_{\underline{v}, m} = 0$ for all $\underline{v} \in \mathcal{V} \cup \{m\}$.)

Definition 2.6. *To define the infinite matrix, we introduce the following values, for $k \in \mathbb{N}$.*

$$a_k = \left(\frac{k+1}{2k+1} \right)^{3\delta + \lambda(1-\delta)} + \frac{1}{8\delta} \left(\frac{k+1}{2k+1} \right)^{\lambda(1-\delta)} \quad \text{and} \quad b_k = \left(\frac{k+2}{k+3} \right)^{3\delta + \lambda(1-\delta)},$$

We now extend the finite matrix B to give an infinite matrix.

Definition 2.7 (An infinite matrix). *Fixing $\delta > 0$, we define the infinite matrix D indexed by $\mathcal{V} \cup \{m, m+1, \dots\}$ as follows.¹*

$$D = \begin{pmatrix} B_{1,1} & \cdots & B_{1,N-1} & B_{1,N} + a_m & a_{m+1} & a_{m+2} & \cdots \\ B_{2,1} & \cdots & B_{2,N-1} & B_{2,N} & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\ B_{N,1} & \cdots & B_{N,N-1} & B_{N,N} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & b_m & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & b_{m+1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & b_{m+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ = \begin{pmatrix} 0 & \cdots & 0 & a_m & a_{m+1} & a_{m+2} & \cdots \\ B_{2,1} & \cdots & B_{2,N-1} & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\ B_{N,1} & \cdots & B_{N,N-1} & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & b_m & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & b_{m+1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & b_{m+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $(D^k)_{i,j}$ be the entry corresponding to the (i, j) th entry of the k th power of D . Our main technical result is the following, which we will rephrase later in a more explicit way.

Theorem 2.8. *If $\delta > \frac{1-\lambda}{3-\lambda} = 0.037\dots$ satisfies $\sum_{k=1}^{\infty} (D^k)_{1,1} \leq 1$, then $\dim_H(\mathcal{G}) \leq 1 + \delta$.*

In particular, letting $m = 9$ and estimating numerically the value of $\delta \approx 0.7415$ (rounding up to four decimal places) giving equality in the hypothesis of Theorem 2.8 gives the bound in Theorem 1.1 in the introduction as a corollary (see §7).

3 Triangle estimates

In this section we collect together elementary but useful estimates for the triangles $\Delta_{\underline{i}}$.

Lemma 3.1 (Area estimate). *If $\underline{i} = (i_1, \dots, i_n) \in \{1, 2, 3\}^n$ and $j \in \{1, 2, 3\}$ then, writing $j\underline{i} = (j, i_1, \dots, i_n)$, we have that*

$$\frac{\text{area}(\Delta_{j\underline{i}})}{\text{area}(\Delta_{\underline{i}})} \leq \max_{x \in \Delta_{\underline{i}}} (2 - x_j)^{-3}.$$

Proof. By a change of variables,

$$\text{area}(\Delta_{j\underline{i}}) = \text{area}(T_j \Delta_{\underline{i}}) = \int_{\Delta_{\underline{i}}} \text{Jac } T_j(x) \, dx \leq \max_{x \in \Delta_{\underline{i}}} (\text{Jac } T_j(x)) \text{area}(\Delta_{\underline{i}}).$$

¹Here, $N - 1$ is the cardinality of \mathcal{V} , and in the ordering of $\mathcal{V} \cup \{m\}$, we take \otimes to be first and m last.

To complete the proof, we now show that $\text{Jac } T_j(x) = (2 - x_j)^{-3}$. If $j = 1$, with respect to the orthogonal basis $(\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}), \frac{1}{\sqrt{6}}(2\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}))$, the derivative map $DT_j(x)$ takes the following form:

$$DT_1(x) = \frac{1}{(2 - x_1)^2} \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{3}}(x_2 - x_3) & 2 - x_1 \end{pmatrix}. \quad (1)$$

Thus, $\text{Jac } T_1(x) = (2 - x_1)^{-3}$. The other cases ($j \neq 1$) follow by symmetry. \square

Remark 3.2. In fact, one can deduce a simple formula for $\text{area}(\Delta_{\underline{i}})$ from the precise form of the above jacobian, using matrix products (see [15]).

We can apply a similar reasoning to the associated diameters.

Lemma 3.3 (Diameter estimate). *If $\underline{i} = (i_1, \dots, i_n) \in \{1, 2, 3\}^n$ and $j \in \{1, 2, 3\}$ then*

$$\frac{\text{diam}(\Delta_{j\underline{i}})}{\text{diam}(\Delta_{\underline{i}})} \leq \max_{x \in \Delta_{\underline{i}}} (2 - x_j)^{-\lambda}.$$

Remark 3.4. The constant λ used as the exponent in Lemma 3.3 was suggested by noticing empirically that the contours of the operator norm $\|DT_1(x)\|$ of $DT_1(x)$ were very similar to those of its main eigenvalue, $(2 - x_1)^{-1}$. Hence it was natural to consider the ratio

$$\frac{\log \|DT_1(x)\|}{\log(2 - x_1)},$$

whose supremal value is obtained in the limit as $x \rightarrow (1, 0, 0)$, and may be computed using L'Hôpital's rule.

Proof. Consider the operator norm, $\|DT_j(x)\|$, of the derivatives $T_j(x)$ (i.e., with respect to the ambient metric in \mathbb{R}^3). Considering $j = 1$, the matrix in (1) satisfies

$$\|DT_1(x)\|^2 = \frac{1}{2b^2} \left(1 + a + b + \sqrt{(1 + a + b)^2 - 4b} \right),$$

where $a = \frac{1}{3}(x_2 - x_3)^2$ and $b = (2 - x_1)^2$. Using the simple bound $a \leq \frac{1}{3}(1 - x_1)^2$, one has that, for all $x \in \Delta$,

$$\|DT_1(x)\|^2 \leq f(x_1) := \frac{2x_1^2 - 7x_1 + 8 + 2(1 - x_1)\sqrt{x_1^2 - 5x_1 + 7}}{3(2 - x_1)^4}.$$

We claim that $g : t \mapsto f(t)(2 - t)^{2\lambda}$ is increasing on $[0, 1]$, and hence has maximum $g(1) = 1$ (we defer the proof of this statement to the appendix), i.e., $\|DT_1(x)\| \leq \sqrt{f(x_1)} \leq (2 - x_1)^{-\lambda}$. The result for $j = 1$ follows simply from the mean value theorem and for $j \neq 1$ by symmetry. \square

Remark 3.5. In the context of higher-dimensional Rauzy gaskets (see [2] for the definition), an effective version of Lemma 3.3 certainly applies, since the derivative maps there have a very similar structure to that observed above.

In view of the previous lemmas, it is useful to have more explicit bounds involving $(2 - x_j)^{-1}$ depending on where the point $x \in \Delta$ approximately lies. We henceforth denote

$$A_{n,k} = \{\underline{i} \in \{1, 2, 3\}^n : i_1 = \dots = i_k \neq i_{k+1}\} \quad (1 \leq k < n)$$

and work with the following simple bounds.

Lemma 3.6. *If $\underline{i} \in A_{n,k}$ and $i_1 = i, j, l \in \{1, 2, 3\}$ are distinct then, for any $\delta > 0$,*

$$\begin{aligned} \max_{x \in \Delta_{\underline{i}}} (2 - x_i)^{-1} &\leq \frac{k+2}{k+3}, \\ \max_{x \in \Delta_{\underline{i}}} (2 - x_j)^{-1} &\leq \frac{k+1}{2k+1}, \quad \text{and} \\ \max_{x \in \Delta_{\underline{i}}} \{(2 - x_j)^{-3\delta} + (2 - x_l)^{-3\delta}\} &\leq \left(\frac{k+1}{2k+1}\right)^{3\delta} + \frac{1}{8^\delta}. \end{aligned}$$

Proof. By symmetry, it suffices to consider $\underline{i} = (i_1, \dots, i_n)$ with $i_1 = \dots = i_k = 1$ and $i_{k+1} \neq 1$. Then $\Delta_{\underline{i}}$ is contained in

$$\text{cl}(T_1^k(\Delta) \setminus T_1^{k+1}(\Delta)) = \left\{x \in \Delta : \frac{k}{k+1} \leq x_1 \leq \frac{k+1}{k+2}\right\} \quad (2)$$

(cl denoting the topological closure), where we have used that $T_1^k(\Delta) = \{x \in \Delta : x_1 \geq \frac{k}{k+1}\}$. The first two bounds follow directly:

$$(2 - x_1)^{-1} \leq \left(2 - \frac{k+1}{k+2}\right)^{-1} = \frac{k+2}{k+3} \quad \text{and} \quad (2 - x_j)^{-1} \leq \left(2 - \frac{1}{k+1}\right)^{-1} = \frac{k+1}{2k+1}.$$

For the third bound, we observe that the function $f(x) := (2 - x_2)^{-3\delta} + (2 - x_3)^{-3\delta}$ is convex on Δ and thus takes its maximum on $T_1^k(\Delta)$ at one of its vertices:

$$\max_{\Delta_{\underline{i}}} f \leq \max_{T_1^k(\Delta)} f = \max \{f \circ T_1^k(1, 0, 0), f \circ T_1^k(0, 1, 0), f \circ T_1^k(0, 0, 1)\}.$$

More explicitly, noting that $f \circ T_1^k(1, 0, 0) = f(1, 0, 0) \leq f \circ T_1^k(0, 1, 0) = f\left(\frac{k}{k+1}, \frac{1}{k+1}, 0\right) = f\left(\frac{k}{k+1}, 0, \frac{1}{k+1}\right) = f \circ T_1^k(0, 0, 1)$, one has

$$\max_{\Delta_{\underline{i}}} f \leq \left(2 - \frac{1}{k+1}\right)^{-3\delta} + 2^{-3\delta} = \left(\frac{k+1}{2k+1}\right)^{3\delta} + \frac{1}{8^\delta},$$

as required. \square

4 Cover estimates

The upper bound on the dimension in Theorem 2.8 is based on finding a value $\delta \in (0, 1)$ so that Lemma 4.1 below applies.

Its proof is simple and based on a simple sequence of open covers of $\mathcal{G}, \mathcal{U}_n$, each obtained by covering the set of n th level triangles, $\{\Delta_{\underline{i}} : |\underline{i}| = n\}$.

Lemma 4.1. Assume $\delta > 0$ and that the sequence

$$X_n := \sum_{|\underline{i}|=n} \text{area}(\Delta_{\underline{i}})^\delta \text{diam}(\Delta_{\underline{i}})^{1-\delta} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then $\dim_H(\mathcal{G}) \leq 1 + \delta$.

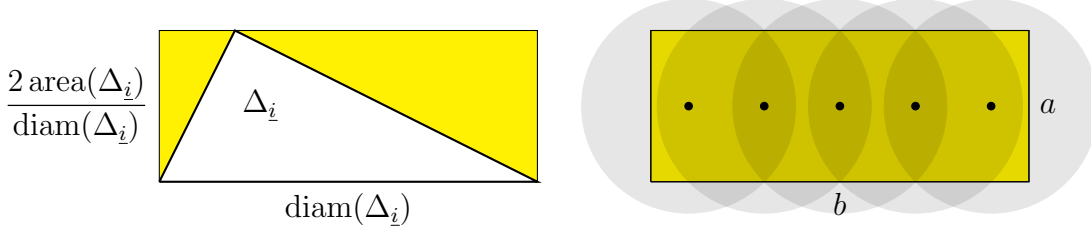


Figure 2: Left: covering $\Delta_{\underline{i}}$ by a rectangle. Right: covering a rectangle by open disks.

Proof. As illustrated in Figure 2, each triangle $\Delta_{\underline{i}}$ is contained in a rectangle with side lengths $b = \text{diam}(\Delta_{\underline{i}})$ and $a = 2 \text{area}(\Delta_{\underline{i}}) / \text{diam}(\Delta_{\underline{i}})$, which can in turn be covered by $\lceil 2b/a \rceil$ disks of diameter $2a$. That is, $\Delta_{\underline{i}}$ can be covered by $\text{diam}(\Delta_{\underline{i}})^2 / \text{area}(\Delta_{\underline{i}})$ disks of diameter $4 \text{area}(\Delta_{\underline{i}}) / \text{diam}(\Delta_{\underline{i}})$. Denoting by \mathcal{U}_n the union of the disks covering all of the triangles $\Delta_{\underline{i}}$, this gives

$$\sum_{U \in \mathcal{U}_n} \text{diam}(U)^{1+\delta} \leq 4^{1+\delta} \sum_{|\underline{i}|=n} \left(\frac{\text{diam}(\Delta_{\underline{i}})^2}{\text{area}(\Delta_{\underline{i}})} \right) \left(\frac{\text{area}(\Delta_{\underline{i}})}{\text{diam}(\Delta_{\underline{i}})} \right)^{1+\delta} = 4^{1+\delta} X_n.$$

The result follows from the standard definition of the Hausdorff dimension (see, e.g., [7]). \square

Remark 4.2. Perhaps surprisingly, this lemma appears to give a significant improvement on working exclusively with either $\text{diam}(\Delta_{\underline{i}})$ or $\text{area}(\Delta_{\underline{i}})$ (as in [9] and [1]).

To apply Lemma 4.1 we want to bound X_n from above using a partition of $\{1, 2, 3\}^n$. We then apply different bounds to the triangles $T_j(\Delta_{\underline{i}})$ according to which element of the partition the index \underline{i} lies in.

Definition 4.3 (Partitioning up the sequences). *For each $n > m$, we can partition*

$$\{1, 2, 3\}^n := \bigcup_{\underline{v} \in \mathcal{V}} A_{n,\underline{v}} \cup \bigcup_{k=m}^{n-1} A_{n,k} \cup \bigcup_{j=1}^3 \{(j^n)\},$$

where $A_{n,k}$ is as above and $A_{n,\underline{v}} = \{\underline{i} \in \{1, 2, 3\}^n : (i_1, \dots, i_{m+1}) \sim \underline{v}\}$, recalling the equivalence relation \sim on page 3.

This partition naturally gives the following components of X_n , for $n > m$.

Definition 4.4. *For $n > m$ and any $\alpha \in \mathcal{V} \cup \{m, \dots, n-1\}$, we write*

$$X_{n,\alpha} := \sum_{\underline{i} \in A_{n,\alpha}} \text{area}(\Delta_{\underline{i}})^\delta \text{diam}(\Delta_{\underline{i}})^{1-\delta}.$$

We now derive bounds on X_{n+1} using estimates on $X_{n,\underline{v}}$ and $X_{n,k}$.

5 Inductive bounds on $X_{n,\underline{v}}$ and $X_{n,k}$

We now turn to the problem of showing that $X_n \rightarrow 0$ as $n \rightarrow \infty$, which will enable us to apply Lemma 4.1. Our method, broadly speaking, is based on obtaining inductive bounds bounding

1. $X_{n+1,k+1}$ in terms of $X_{n,k}$, and
2. $X_{n+1,\underline{v}}$ in terms of $X_{n,\underline{w}}$ and $X_{n,k}$.

These will prove useful in applying the renewal theorem in the next section.

5.1 Bounds of terms indexed by numbers

The next simple lemma is independent of m , and features the constants b_k .

Lemma 5.1 (Number Lemma). *Suppose $\delta \in (0, 1)$. Then, for all $n > k$,*

$$X_{n+1,k+1} \leq b_k X_{n,k}.$$

Proof. Since $j\underline{i} \in A_{n+1,k+1}$ if and only if $\underline{i} \in A_{n,k}$ and $j = i_1$, we may write

$$\begin{aligned} X_{n+1,k+1} &= \sum_{\underline{i} \in A_{n,k}} \text{area}(\Delta_{i_1 \underline{i}})^\delta \text{diam}(\Delta_{i_1 \underline{i}})^{1-\delta} \\ &\leq \sum_{\underline{i} \in A_{n,k}} \left(\frac{k+2}{k+3} \right)^{3\delta} \left(\frac{k+2}{k+3} \right)^{\lambda(1-\delta)} \text{area}(\Delta_{\underline{i}})^\delta \text{diam}(\Delta_{\underline{i}})^{1-\delta} \\ &= b_k X_{n,k}, \end{aligned}$$

where we have applied Lemmas 3.1 and 3.3. □

5.2 Bounds of terms indexed by words

We can bound $X_{n+1,\underline{v}}$, $n > m$, with $\underline{v} \in \mathcal{V} \cup \{m\}$, using the matrix B and the constants a_k , as formulated in the next lemma.

Lemma 5.2 (Word Lemma). *Let $\delta \in (0, 1)$, $n > m$ and $\underline{v} \in \mathcal{V} \cup \{m\}$. Then, recalling $\oplus = (1, 2^m)$, there exists a positive sequence c_n such that $\sum_n c_n < \infty$ and*

$$\begin{aligned} X_{n+1,\oplus} &\leq c_n + \sum_{k=m}^{n-1} a_k X_{n,k} \quad \text{and} \\ X_{n+1,\underline{v}} &\leq \sum_{\underline{w} \in \mathcal{V}} B_{\underline{v},\underline{w}} X_{n,\underline{w}} \quad \text{if } \underline{v} \neq \oplus. \end{aligned}$$

Proof. For the first inequality, note that $j\underline{i} \in A_{n+1,\otimes}$ if and only if $j \neq i_1$ and $\underline{i} \in \bigcup_{k=m}^{n-1} A_{n,k} \cup \{(1^n), (2^n), (3^n)\}$. Therefore, applying Lemmas 3.3 and 3.1 in turn and using symmetry, one has that

$$\begin{aligned}
X_{n+1,\otimes} &= 6 \operatorname{area}(\Delta_{(2,1^n)})^\delta \operatorname{diam}(\Delta_{(2,1^n)})^{1-\delta} + \sum_{k=m}^{n-1} \sum_{\underline{i} \in A_{n,k}} \sum_{j \neq i_1} \operatorname{area}(\Delta_{j\underline{i}})^\delta \operatorname{diam}(\Delta_{j\underline{i}})^{1-\delta} \\
&\leq c_n + \sum_{k=m}^{n-1} \sum_{\underline{i} \in A_{n,k}} \left(\frac{k+1}{2k+1} \right)^{\lambda(1-\delta)} \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta} \sum_{j \neq i_1} \operatorname{area}(\Delta_{j\underline{i}})^\delta \\
&\leq c_n + \sum_{k=m}^{n-1} \sum_{\underline{i} \in A_{n,k}} \left(\frac{k+1}{2k+1} \right)^{\lambda(1-\delta)} \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta} \left(\left(\frac{k+1}{2k+1} \right)^{3\delta} + \frac{1}{8^\delta} \right) \operatorname{area}(\Delta_{\underline{i}})^\delta \\
&= c_n + \sum_{k=m}^{n-1} a_k X_{n,k},
\end{aligned}$$

where $c_n := 6 \operatorname{area}(\Delta_{(2,1^n)})^\delta \operatorname{diam}(\Delta_{(2,1^n)})^{1-\delta}$, as required.

Regarding the second inequality: The combinatorics above and in the previous proof show that $\underline{i} \in \bigcup_{k=m}^{n-1} A_{n,k} \cup \{(1^n), (2^n), (3^n)\}$ implies $j\underline{i} \notin A_{n+1,\underline{v}}$ for every $\underline{v} \in \mathcal{V} \cup \{m\} \setminus \{\otimes\}$. Fixing such a \underline{v} and considering the contrapositive, one sees that $j\underline{i} \in A_{n+1,\underline{v}}$ implies that $\underline{i} \in A_{n,\underline{w}}$ and $\underline{w} \mapsto_{j'} \underline{v}$ for some \underline{w}, j' . Assuming $(i_1, \dots, i_{m+1}) = \underline{w}$ for simplicity, we have that $j' = j$ and $\Delta_{\underline{i}} \subset \Delta_{\underline{w}}$. Hence, by Lemmas 3.1 and 3.3,

$$\frac{\operatorname{area}(\Delta_{j\underline{i}})^\delta \operatorname{diam}(\Delta_{j\underline{i}})^{1-\delta}}{\operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta}} \leq \max_{\underline{x} \in \Delta_{\underline{w}}} (2 - x_j)^{-3\delta - \lambda(1-\delta)} =: B_{\underline{v},\underline{w}}.$$

Thus, we may write the following.

$$\begin{aligned}
X_{n+1,\underline{v}} &= \sum_{j=1}^3 \sum_{\substack{\underline{w} \in \mathcal{V}: \\ \underline{w} \mapsto_j \underline{v}}} \sum_{\underline{i} \in A_{n,\underline{w}}} \operatorname{area}(\Delta_{j\underline{i}})^\delta \operatorname{diam}(\Delta_{j\underline{i}})^{1-\delta} \leq \sum_{\substack{\underline{w} \in \mathcal{V}: \\ B_{\underline{v},\underline{w}} \neq 0}} \sum_{\underline{i} \in A_{n,\underline{w}}} B_{\underline{v},\underline{w}} \operatorname{area}(\Delta_{\underline{i}})^\delta \operatorname{diam}(\Delta_{\underline{i}})^{1-\delta} \\
&= \sum_{\underline{w} \in \mathcal{V}} B_{\underline{v},\underline{w}} X_{n,\underline{w}},
\end{aligned}$$

as required.

Finally, a direct computation of the vertices of $\Delta_{(2,1^n)} = T_2 T_1^n(\Delta)$ shows that its diameter and area are proportional to n^{-1} and n^{-2} respectively, as $n \rightarrow \infty$. Hence c_n is proportional to $n^{-1-\delta}$ as $n \rightarrow \infty$, and is thus summable for every $\delta > 0$, completing the proof. \square

6 Renewal Theorem

We now use the iterative bounds of the last section to show that $X_n \rightarrow 0$ as $n \rightarrow \infty$, under the hypotheses of Theorem 2.8. This uses the following mild adaptation of the classical renewal theorem of Feller [8, p.330].

Theorem 6.1. Suppose that the sequences $(Y_n)_{n=0}^\infty$, $(\mu_n)_{n=1}^\infty$ and $(\nu_n)_{n=1}^\infty$ are non-negative and satisfy $\sum_{n=1}^\infty \mu_n < 1$, $\sum_{n=1}^\infty \nu_n < \infty$ and

$$Y_n \leq \nu_n + \sum_{k=1}^n \mu_k Y_{n-k}$$

for each $n \in \mathbb{N}$. Then $Y_n \rightarrow 0$ as $n \rightarrow \infty$.

We apply this theorem with $Y_n = X_{n+m+1, \otimes}$ to give Theorem 2.8, which can be rewritten as follows.

Theorem 6.2. If $\delta > \frac{1-\lambda}{3-\lambda} = 0.037 \dots$ satisfies

$$\sum_{k=1}^\infty (B^k)_{m, \otimes} \cdot \sum_{k=m}^\infty a_k \prod_{i=m}^{k-1} b_i \leq 1, \quad (3)$$

then $\dim_H(\mathcal{G}) \leq 1 + \delta$.

Proof. Assume that (3) holds with a strict inequality (the case of equality follows in the limit, since the LHS is decreasing in δ), and that $\delta < 1$ (the conclusion being otherwise trivial). Applying Lemma 5.1 repeatedly in the first estimate of Lemma 5.2 gives

$$X_{n+1, \otimes} \leq c_n + \sum_{k=m}^{n-1} a_k \prod_{i=m}^{k-1} b_i X_{n+m-k, m}.$$

Moreover, the second estimate of Lemma 5.2 extends inductively to give, for any $k \in \mathbb{N}$,

$$X_{n+m+k, m} \leq \sum_{j=1}^{n+m-1} (B^j)_{m, \otimes} X_{n+m, \otimes} + \sum_{\substack{\underline{v} \in \mathcal{V}: \\ \underline{v} \neq \otimes}} (B^{n+k-1})_{m, \underline{v}} X_{m+1, \underline{v}}.$$

Putting these two together gives the renewal-style inequality

$$X_{n+1, \otimes} \leq \nu_n + \sum_{k=1}^{n-m} \mu_k X_{n+1-k, \otimes},$$

with coefficients

$$\mu_k = \sum_{i+j=k} (B^i)_{m, \otimes} a_{m+j} \prod_{l=m}^{m+j-1} b_l$$

and remainder term

$$\nu_n = c_n + \sum_{k=m}^{n-1} a_k \prod_{i=m}^{k-1} b_i \cdot \sum_{\underline{v} \in \mathcal{V} \setminus \{\otimes\}} (B^{n-k-1})_{m, \underline{v}} X_{m+1, \underline{v}}.$$

Consider the other hypotheses of the renewal theorem. The hypothesis on the coefficients is precisely our assumption that strengthens (3):

$$\sum_{k=1}^{\infty} \mu_k = \sum_{k=0}^{\infty} (B^k)_{m,\otimes} \cdot \sum_{k=m}^{\infty} a_k \prod_{i=m}^{k-1} b_i < 1.$$

Regarding the summability of the remainder terms ν_n : By elementary combinatorics, B^{m+1} has only one zero row and zero column, so this inequality also ensures that the spectral radius $\rho = \rho(B) < 1$. In any case, by the Perron-Frobenius theorem there exists $C > 0$ such that

$$\sum_{n=1}^{\infty} \nu_n - c_n \leq C \sum_{n=1}^{\infty} \sum_{k=m}^{n-1} a_k \prod_{i=m}^{k-1} b_i \rho^{n-k} = C \sum_{k=m}^{\infty} a_k \prod_{i=m}^{k-1} b_i \sum_{n=1}^{\infty} \rho^n.$$

Thus, since $\rho \in (0, 1)$ and

$$a_k \prod_{i=m}^{k-1} b_i = \mathcal{O}(k^{-3\delta - \lambda(1-\delta)}) \quad (k \rightarrow \infty),$$

we see that the right hand side is summable if $\delta > \frac{1-\lambda}{3-\lambda}$. That is, $\sum_n \nu_n < \infty$, and the renewal theorem gives $X_{n,\otimes} \rightarrow 0$ as $n \rightarrow \infty$.

Finally, we use this to bound X_n via the following lemma.

Lemma 6.3. *Given $m \in \mathbb{N}$ and $\delta \in (0, 1)$, there exists $C > 0$ such that $X_n \leq C X_{n+m+1,\otimes}$.*

Proof. For each $\underline{i} \in \{1, 2, 3\}^n$, the word $\underline{i}' = (1, 2^m, i_1, \dots, i_n) \in A_{n+m+1,\otimes}$. This word satisfies

$$\frac{\text{area}(\Delta_{\underline{i}'}^\delta \text{diam}(\Delta_{\underline{i}'}^{1-\delta})}{\text{area}(\Delta_{\underline{i}}^\delta \text{diam}(\Delta_{\underline{i}}^{1-\delta})} = \frac{\text{area}(T_1 T_2^m \Delta_{\underline{i}})^\delta \text{diam}(T_1 T_2^m \Delta_{\underline{i}})^{1-\delta}}{\text{area}(\Delta_{\underline{i}})^\delta \text{diam}(\Delta_{\underline{i}})^{1-\delta}} \geq K^{(1+\delta)(m+1)},$$

where K is obtained by bounding the *minimum* singular value of $DT_j(x)$ uniformly over $x \in \Delta$: namely, applying the estimates of section 3,

$$\|D(T_j^{-1})(T_j(x))\| = \frac{\text{Jac } T_j(x)}{\|DT_j(x)\|} \geq (2 - x_j)^{\lambda-3} \geq 2^{\lambda-3} =: K.$$

In particular, with $C = K^{(1+\delta)(m+1)}$

$$X_{n+m+1,\otimes} \geq \sum_{|\underline{i}|=n} \text{area}(\Delta_{\underline{i}'}^\delta \text{diam}(\Delta_{\underline{i}'}^{1-\delta}) \geq C X_n,$$

as required. \square

Thus, $X_{n,\otimes} \rightarrow 0$ as $n \rightarrow \infty$ implies that $X_n \rightarrow 0$ as $n \rightarrow \infty$, and applying Lemma 4.1 completes the proof of Theorem 2.8. \square

m	$\delta_m + 1$	m	$\delta_m + 1$
2	1.8285	6	1.7545
3	1.7982	7	1.7485
4	1.7771	8	1.7444
5	1.7635	9	1.7415

Table 1: Upper bounds $\delta_m + 1$ on $\dim_H(\mathcal{G})$ for different choices of m , rounded upwards to four decimal places.

7 Numerical estimates

The usefulness of Theorem 2.8 lies in our ability to check (3) for given values δ and m . Since the left hand side of (3) is decreasing and continuous in each of its two factors, the infimal such δ for which the theorem holds is the unique value δ_m yielding equality.

The values of these δ_m for m up to 9 are given in Table 1. To check the condition for given δ and m , we used Wolfram Mathematica to codify B as a matrix and to compute the first factor of (3),

$$\sum_{k=0}^{\infty} (B^k)_{m,\otimes} = \left((I - B)^{-1} \right)_{m,\otimes},$$

as a ratio of two determinants (via Cramer's Rule); and the second factor,

$$\begin{aligned} \sum_{k=m}^{\infty} a_k \prod_{i=m}^{k-1} b_i &= \sum_{k=m}^{\infty} \left[\left(\frac{k+1}{2k+1} \right)^{3\delta+\lambda(1-\delta)} + \frac{1}{8^\delta} \left(\frac{k+1}{2k+1} \right)^{\lambda(1-\delta)} \right] \prod_{i=m}^{k-1} \left(\frac{i+2}{i+3} \right)^{3\delta+\lambda(1-\delta)} \\ &= \sum_{k=m}^{\infty} \left(\frac{(k+1)(m+2)}{(2k+1)(k+2)} \right)^{3\delta+\lambda(1-\delta)} + \frac{1}{8^\delta} \left(\frac{(k+1)(m+2)}{(2k+1)(k+2)} \right)^{\lambda(1-\delta)} (k+2)^{-3\delta}, \end{aligned}$$

using the inbuilt `NSum` function. For the precise code, explanation and a breakdown of the confirmation, see [16] and the documentation therein. In particular, we rely on the closed-source Wolfram Mathematica programme (with its inbuilt error bounds) to accurately

- calculate the determinant of large matrices, and
- estimate the polynomially decaying sum.

Both calculations can be rigorised (see [16] for details) and we encourage all efforts to do so. However, any changes we observed were too insignificant to affect the final values given here: accuracy to three decimal places in each factor appears to suffice.

That δ_m is computed only up to $m = 9$ owes to the exponential size of B , which places a commensurable computing burden on the hardware. We note that slightly larger choices of m should be possible with more efficient compiled programming languages.

Remark 7.1. The elementary nature of this method suggests a similar approach might be applied to other related examples of sets to give bounds on their dimension; though likely with different limiting returns.

A Proof that g is increasing.

Here, we give an elementary but delicate proof of the following technical fact, used in the proof of Lemma 3.3:

Lemma A.1. *The function $g : [0, 1] \rightarrow \mathbb{R}$,*

$$g(t) := f(t)(2-t)^{2\lambda} = \frac{1}{3}(2-t)^{-1-2/\sqrt{3}} \left(2t^2 - 7t + 8 + 2(1-t)\sqrt{t^2 - 5t + 7} \right)$$

is increasing. In particular, $g(t) \leq g(1) = 1$ for all t .

Proof. We simply show that $g' > 0$ on $(0, 1)$. Writing $\alpha(t) = \sqrt{t^2 - 5t + 7} > 0$, we have

$$g'(t) = \frac{h(t)}{\alpha(t)} \cdot \frac{1}{9} \cdot (2-t)^{-2-2/\sqrt{3}},$$

where $h(t) \in \mathbb{Q}[t, \alpha(t), \sqrt{3}]$ is given by

$$\begin{aligned} h(t) := & \left(6 - 4\sqrt{3} \right) t^3 + \left(\left(4\sqrt{3} - 6 \right) \alpha(t) + 24\sqrt{3} - 39 \right) t^2 \\ & + \left(\left(24 - 14\sqrt{3} \right) \alpha(t) - 48\sqrt{3} + 87 \right) t + \left(16\sqrt{3} - 18 \right) \alpha(t) + 28\sqrt{3} - 72. \end{aligned}$$

It remains to show h is positive, using Taylor approximants for α based at 1. Namely, since

$$\alpha^{(3)}(t) = \frac{9}{8} \frac{5-2t}{\alpha(t)^5}, \quad \alpha^{(4)}(t) = \frac{9}{16} \frac{16t^2 - 80t + 97}{\alpha(t)^7}$$

are positive for all $t \in (0, 1)$, the remainder term given by Taylor's Theorem for the degree two and degree three Taylor polynomials are negative and positive, respectively, hence these polynomials are upper and lower bounds, respectively, for α . More explicitly, for $t \in [0, 1]$,

$$\frac{t^3 - t^2 - 25t + 73}{16\sqrt{3}} \leq \alpha(t) \leq \frac{t^2 - 14t + 37}{8\sqrt{3}}.$$

Applying these bounds everywhere in the definition of h (according to the signs of the coefficients) reduces to the concrete polynomial bound

$$h(t) \geq \left(5\sqrt{3} + 8 + \left(2 - \sqrt{3} \right) t^2(t+1) - 3 \left(\sqrt{3} + 2 \right) t \right) (1-t)^2.$$

It is easy to see that the right hand side is everywhere positive, hence so too are h and g' . \square

References

- [1] P. Arnoux and F. Rauzy. Représentation géométrique de suites de complexité $2n+1$. *Bull. Soc. Math. France* **119** (1991), no. 2, 199–215.

- [2] P. Arnoux and Š. Starosta. *The Rauzy gasket*. In: J. Barral and S. Seuret (eds) Further developments in fractals and related fields, 1–23. Birkhäuser/Springer: New York, 2013.
- [3] P. Arnoux and J.-C. Yoccoz. Construction de difféomorphismes pseudo-Anosov. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 1, 75–78.
- [4] A. Avila, P. Hubert and A. Skripchenko. On the Hausdorff dimension of the Rauzy gasket. *Bull. Soc. Math. France* **144** (2016), no. 3, 539–568.
- [5] R. De Leo and I. Dynnikov. Geometry of plane sections of the infinite regular skew polyhedron, *Geom. Dedicata* **138** (2009), 51–67.
- [6] I. A. Dynnikov, P. Hubert and A. Skripchenko. Dynamical systems around the Rauzy gasket and their ergodic properties. *Int. Math. Research Notices* **rnac440** (2022).
- [7] K. Falconer. *Fractal Geometry*, Wiley: New York, 1990.
- [8] W. Feller. *An Introduction to Probability Theory and Its Applications*, vol 2, Wiley: New York, 1971.
- [9] C. Fougeron. Dynamical properties of simplicial systems and continued fraction algorithms. [arXiv:2001.01367](https://arxiv.org/abs/2001.01367)
- [10] A. Gamburd, M. Magee and R. Ronan. An asymptotic formula for integer points on Markoff-Hurwitz varieties. *Ann. of Math. (2)* **190** (2019), no. 3, 751–809.
- [11] R. Gutiérrez-Romo and C. Matheus. Lower bounds on the dimension of the Rauzy gasket. *Bull. Soc. Math. France* **148** (2020), no. 2, 321–327.
- [12] P. Hubert and O. Paris-Romaskevich. Triangle tiling billiards and the exceptional family of their escaping trajectories: circumcenters and Rauzy gasket. *Exp. Math.* (2019).
- [13] G. Levitt. La dynamique des pseudogroupes de rotations. *Invent. Math.* **113** (1993), no. 3, 633–670.
- [14] A. Maltsev and S. P. Novikov. Dynamical systems, topology, and conductivity in normal metals. *J. Statist. Phys.* **115** (2004), no. 1–2, 31–46.
- [15] B. Sewell. *Equidistribution of infinite interval substitution schemes, explicit resonances of Anosov toral maps, and the Hausdorff dimension of the Rauzy gasket*. PhD thesis: https://wrap.warwick.ac.uk/view/author_id/347116.html
- [16] B. Sewell. Github repository: <https://github.com/sewell-math/rauzy-gasket-upper>