

How to use motion as a problem-solving tool?

Problems from the Pósa camps

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Abstract

During the recent decades a successful Hungarian talent nurturing programme has been developing. After a brief introduction of Hungarian talent nurturing tradition, we introduce the Pósa camps. Pósa camps are talent nurturing camps for student from grade 7 to grade 12. One of the main characteristics of the method is the structure of its problems, called problem-threads (Gosztonyi, 2014, 2017). The structure of the problem-threads can be a powerful tool to teaching problem-solving skills. In this article we introduce the motion problem-thread, in which motion serves as a problem-solving tool, it is also creating a unique connection between seemingly unrelated problems. Through this problem-thread we describe how problem-threads are used in the camps.

Keywords: Pósa method, Hungarian tradition, problem solving, problem-threads, motion as problem solving tool,

Introduction

Hungary has a rich tradition of teaching problem-solving skills. In this article we would like to introduce the Pósa method, an extracurricular program for students from grade 7 to grade 12.

In the first part of this paper, there is a brief overview of the Hungarian problem-solving culture to better understand the background of the camps. The origin of the camps, the general background and the structure of its problems are discussed as well. In the second part, problems are introduced and their role in the camp's curriculum. Problems from other problem-threads are also described and mentioned, to better understand how problem-threads are working in the camps. In the end, some of the general characteristics of the method are summarized. The future of the method in a wider context is also discussed.

Brief overview of the tradition of the Hungarian gifted education

In the following section important milestones of Hungarian mathematics education for the gifted are summarized.

Before the nineteenth century mathematics, there were outstanding individuals working mostly in isolation since scientists were not really organized at the time in Hungary. Such a talented individual was János Bolyai, who created a non-Euclidean geometry (his discovery is independent but simultaneous of Gauss and Lobachevsky), "I created a new, different world out of nothing" - he wrote to his father. (Győri et al., 2020) After the foundation of the Hungarian Academy of Sciences in 1831, the foundation of the (still existing nowadays) Society of Dissemination of Scientific Knowledge (Tudományos Ismeretterjesztő Társaság in Hungarian) in 1840 and the cultural and socio-economic changes of Hungary within the Austro-Hungarian Empire cultural and scientific activities started to flourish. Several educational reforms took place. To read

more about the historical background, about notable figures of Hungarian mathematics culture see Gosztonyi's article (2016) and Györi and his peers' work (2020).

Important to note the emergence of one of the first problem-solving competitions organized for graduating high-school students by the Hungarian Society of Physics and Mathematics in 1894. The competition is still organized annually. According to Koichu and Andžāns (2009), this competition had an impact on the International Mathematical Olympiad (IMO) movement.

The society also launched two journals, the Journal of Mathematics and Physics for teachers in 1892, and Mathematics Journal for Secondary Schools (Középiskolai Matematika Lapok or KöMaL in short in Hungarian) in 1894. In the latter journal there are several problems to submit their solutions in each month. Solutions from the students are also published in later issues. KöMaL still exists and it plays an important role in Hungarian talent-care.

After World War II the first large-scale research-oriented competition (according to Koichuo and Andžāns, 2009) was the Miklós Schweitzer Memorial Mathematical Competition for undergraduate and high-school students. In this competition 8-12 problems are proposed from diverse topics on advanced level, and students can work on them for 7-10 days, they are encouraged to use the literature.

Hungarian heuristic tradition

The Hungarian "heuristic" tradition can be traced back to the start of the 20th century. Well-known figures of the tradition are László Rátz, George Pólya, Alfréd Rényi, Imre Lakatos, who was a philosopher of mathematics and science. Their work (for example see Pólya (1954, 1990), Rényi (1967,1972) Lakatos (1976)) influenced thinkers beyond Hungary. According to Gosztonyi (2016), the following are the characteristics of the Hungarian "heuristic" tradition. We believe the Pósa method is also part of this tradition and incorporates many of its characteristics.

1. Mathematics is not static and everlasting but is rather a constantly developing and changing creation of the human mind. Students should be accompanied through the same process.
2. The source of mathematics is intuition and experience. (Not constrained to actual physical observations.) Without it, neither mathematical creation, nor real understanding can be achieved, so it is important to develop intuition with the help of a handful of experiences in every level of education.
3. Mathematical activity is basically dialogical; it is a sequence of questions, problems, and the attempts to answer them. Teaching mathematics is not a one-sided passing on of knowledge; it is more a joint activity of the student and the teacher. The teacher acts as an aid in the students' rediscovery of mathematics.
4. The aim of teaching mathematics is not to pass on recipes for computation for a user without reflection; it is to provide an introduction to the process of mathematical creation, and consequently educate thinking people.
5. Excessive formalism is discouraged; a formal language should be introduced only after proper preparations.
6. The process of mathematical creation is closely connected with play. In playfulness, the artistic nature of mathematics manifests itself

Tamás Varga is one of the most important figures in the history of Hungarian mathematics education. He was the central figure of the reform movement called Complex Mathematics Education from 1963 and 1978. His and his group's ideas were incorporated in the national curriculum of 1978. Although, national curriculum went through several changes since then, his legacy is still influencing teachers today. Varga and his group not only wanted to reform the content of mathematics education but its underlying approach. He wanted to shift mathematics education in a way that students' discovery, creativity, and the playfulness of mathematical activities would be the centre of it. His approach can be described as "teaching mathematics by guided discovery". (Gosztonyi, 2020)

Influence of Russian mathematics teaching tradition

Notable Hungarian mathematics teacher-figures were influenced by Russian traditions of teaching mathematics. Problems from Russian problem books (such as the problem book of Shklyarsky, Chentsov and Yaglom 1979) appear in the Hungarian talent-care teaching traditions. For example, some problems well-known in the Russian tradition also appear in the curriculum programme of the Pósa-camps. In KöMaL several problems and articles were inspired by Kvant, the Russian equivalent of the journal. The Russian “experience” also inspired the creation of specialized mathematics classes for the talented students. The first class was organized in 1962, promising talents, competition winners were invited into the class. László Lovász, Miklós Laczkovich, József Pelikán, and Lajos Pósa were members of the first class. (Fried, 2010) The friendship forged between the classmates probably influenced their view on mathematics and mathematics education.

Post-communist era (from 1990)

The traditions of gifted mathematics education have continued past the political changes. The system of specialized classes for gifted students is still working, KöMaL, the mathematics and physics journal for secondary students still plays an important role in talent care. In the new era, several competitions have been founded for various levels, ages (even adults and teachers can participate in some competitions) in different styles: there are competitions with multiple choice tests, relay style competitions, or competitions where one should write a detailed solution. There are competitions for individuals and for groups. There is even a competition that takes place in public parks around the country, mobilizing thousands of participants (“Bear math”, Medve matek in Hungarian). The role of competitions is twofold, first, competitions can offer an enjoyable and fun experience for the participants, in that way competitions popularize math in the general population. Second, competitions can help discover talents and by preparing for competition, talented and motivated students encounter delightful problems that can enrich them. Apart from competitions there are many extracurricular activities (free and paid): math circles, camps, experience days that student from all backgrounds can benefit from. Although talent care in Hungary is rather developed it has its share of challenges. Reaching out for students with disadvantages are a major issue, most notably reaching out to Gypsy/Roma communities. There are diverse programs, not necessarily mathematical, aiming for helping the situation, however there is still much work to do. Another issue is that girls and women are less represented in mathematical competitions, programmes, fields. They seem to be not able to reach their potential. To address the issue, several projects have been developed in recent years (Győri et. al., 2020). For example, in recent years there is a project called, Girls’ Day, a career orientation day for girls, introducing them to diverse mathematical and engineering fields, and related opportunities, to pick girls’ interest in STEM. Participation and preparation in the European Girls’ Mathematical Olympiad (EGMO) are also a great opportunity for girls to develop their interest and talent.

About the Pósa camps

To understand better what the Pósa camps are, first we write about Lajos Pósa. After, we write about different aspect of the camps: the structure of the camps, the structure of the problems, how students work together, and the types of discussion taking place with students.

About Lajos Pósa

Born in 1947 in Budapest, his father was a chemist, and his mother was a mathematics teacher. Pósa was considered a child-prodigy. He worked on problems with Pál Erdős and co-authored papers with him, from the age of 13. He attended the first specialized mathematics class from 1962 at Fazekas High School in Budapest, where he made life-long friendships with his classmates. His interest in teaching probably started

when he started university, he enjoyed giving his fellow students interesting problems and organizing extracurricular math program for students in high school. After graduating, he was associated with the new mathematics education in Hungary, the internationally renowned complex teaching of mathematics lead by Tamás Varga and his peers. Pósa taught in high school in an experimental setting as part of this group. Having experienced summer camps and other extracurricular activities, he wanted to launch his own camp, to create a place where talented students could engage problem-solving in a meaningful way. He envisioned an environment where students could ponder about problems in depth without the time-constraint typically presents in schools. See a more detailed account of his biography in relation to developing his teaching ideas in Győri & Juhász (2017).

About the structure of the camps

Since 2008, the system of the camps has been more solidified (Győri & Juhász, 2017). For each group there are two or three camps during the school year, each a weekend long. In each group there are 25-35 students mostly from the same grade. The programme spans through 5-6 years. It is clearly different from a typical math circle. The meetings are less frequent, but more intense. Between camps there are homework problems, and students are encouraged to revise materials from the previous camps, to solidify ideas and concepts better.

How are students selected? Most students are former participant of a talent-nurturing week-long summer math camp (invitation to these camps is based on achievements), here are also students who are recommended by parents and teachers, participation in competitions is also taken into consideration.

Problem-threads

The structure of the problems is the following. It can be seen as a network of problems, with different types of connections between the problems. We call a series of problems which are connected by the same idea/approach/topic a problem-thread in this context. Different problem-threads can be interconnected with each other; thus problem-threads create a problem-carpet with rich connections. This terminology is not only a way to talk about the structure of camps, but also important because connections between different problems are frequently discussed with the students as well. The connection within a problem-thread can be varied: it can be the core idea of a problem-solving strategy, it can be a common topic, it can be an underlying idea.

The theoretical tools to analyse the structure of the network of the problems have been developing in recent years. (Gosztonyi, 2014, 2017) Since, a few problem-threads are discussed in different papers, such as a cubic geometry problem-thread (Katona & Szűcs, 2017), briefly mentioned problem from a problem-thread concerning arithmetic progression (Juhász & Katona, 2019), problem-thread about a “rotating-door” and its connection to computational, algorithmic thinking (Bóra, 2020), and some problems from the proof of impossibility problem-thread (Artigue et al., 2020).

Group-work

In the camps there are different sessions consisting of frontal discussion with the whole group, group work with 2-4 students and individual work. Most notable is how groups work in the camps. Students work in groups with 2-4 members in each, during group-work time individual work is also permitted. Groups are determined by the preference of the students. Interestingly, most groups stay the same throughout the years, only minor changes are occurring between groups. Group work is the core of the camps, this is when students have prolonged time to think about problems and to develop their ideas. They span a considerable amount of time, 1,5-2,5 hours for 4-5 main problems from diverse topics, and there are also additional problems for those who finish with the main problems early. These extra problem-threads are usually connected to the main topics, but they are typically harder to solve, and they are not part of the main

problems of the camps, they can be described best as a side-path. Groups work in different rooms inside the building, so they can focus on their work, and they are not preoccupied with the progress of other groups. Teachers and helpers regularly visit the groups, check on their progress, discuss solutions, and pose follow-up questions and further problems. Since the groups are being separated it is possible to hold personalized discussions about the problems with them.

Group-work in the camps entails special rules. First, everybody works individually, that way some eventually reach a solution, others are still struggling. After a period of individual thinking, it is time to work together on the problems, the ones who still did not reach a solution are allowed to work together now, sharing ideas and progress. Individuals who have already solved the problem, cannot participate in the discussion, they should let the others discover the solutions themselves.

For students these rules are not easy to follow at first. Sometimes they are tempted to see group-work as a competition between groups, or they want to reach the solution earlier. Eventually, more and more students start to appreciate this type of group work. The intention of the organizers is to create a balance between individual and group work to harness the advantages of both.

Discussion with the groups

During group-works important discussions take place with the students. Throughout the years the experiences are accumulated regarding these discussions: typical fallacies, logical traps, difficult points, where students are struggling. With these experiences a system of possible hints and follow-up questions started to emerge. Teachers and helpers are learning these hints from two sources, from their individual experiences their time being a student in the camps (most helpers and teachers were a former student of Pósa), or from their “apprenticeship”-like experiences: when they join as helpers, they usually join a more experienced member of the helpers.

Meta-discussions with students

It is also important to note that the structure of the problems is also widely discussed with students. Together with the teacher, after each solution, they talk about possible connections with different, already solved problems from the camps. Apart from that, students also regularly reflect on why a problem situation proved difficult. For example, if there were a previous problem that made their thinking prone to overlook something. For concrete examples see (Bóra, 2020) and in this paper see the problem about dividing the plane with n lines followed by the problem about dividing 64 points on the chessboard with lines.

The motion problem-thread

Introduction

This motion problem-thread is about motion as a problem-solving idea. The Pósa method emphasizes more on problem-solving ideas, rather than teaching students advanced theorems or techniques. Learning about advanced material is seen more like a by-product of the method, rather than its goal. In the following section the starting problems from this problem-thread and other connected problems are presented. The thread starts around the fifth camp, when students are typically at the end of their grade 8 studies. This problem-thread is quite different from problems, where motion is studied as a subject. At the end of the section some examples of problems where motion is seen as a subject are presented.

This problem-thread demonstrates well that between problems there can be not only conventional connections. Here the underlying idea is a special point of view, which helps solve the problem.

Initial problem - introducing the idea of motion as a tool for solving problems

Initial problem

Solve for real x the following equation: $x^2 + \sqrt{x} = 18$.

This problem is for grade 8 students in the camps. In general, outside the context of the camps, this problem is great for students with not too much experience with functions or algebraic expressions.

By the time teachers visit the groups most likely the students have found out one solution, since it is easy to see that $x=4$ is a good solution. The question arises, are there any other solutions? The students are convinced there is no other solution. Some of them even start to sketch out a graph of the function. However, the question remains, how can one prove there are no other solutions? Students are left to ponder about that for a while. Eventually, each group reached a similar argument.

One side of the equation is a constant, it doesn't change with x , it is always 18, irrelevant to the value of x . The other side of the equation is changing. If x increases, then x^2 and $\sqrt{x}$ increase, thus $x^2 + \sqrt{x}$ increases, and vice versa, if decreases, then x^2 and $\sqrt{x}$ decrease, thus $x^2 + \sqrt{x}$ decreases. At $x = 4$ there is equilibrium, but changing x , the left side of the equation moves away from 18, while the right side always stays the same.

Overall, it is not a difficult problem. Some more advanced students take note that the function on the left side is strictly monotonic increasing and therefore there is only one solution, which is of course an acceptable solution, but what we would like to show here precedes this idea. So, we would like to show that if a function is strictly monotonic it follows that every value can be taken at most by the function. (So here the intention of the problem is not that we use a more advanced knowledge and apply it to the situation, instead this is a preparatory step towards this more advanced knowledge.)

One of the main goals of this problem is to introduce students to the idea of motion as a problem-solving method. Although, at the time of the discussion this idea is not revealed, it will be revealed at a later point, when they discuss the solution of the "Where to build the school?" problem (see later). Novelty in the solution for the students, that we solved a problem not related to motion with the idea of motion. This solution will be a reference point for students, to help them better grasp this idea.

It could be also interesting to note the surrounding problems, problems that are posed at the same time or right before this problem. Let us investigate some examples. First, there are other type of equations to solve for students, some Diophantine equations, starting with the following: solve for positive integers $x^2 = 2y^2$.

Building on the results from this problem, students soon need to prove the irrationality of $\sqrt{2}$ and $\sqrt{3}$ and similar problems, problems dealing with infinity, namely Hilbert's paradox of the Grand Hotel.

It is worth noting that students by that age mostly encounter problems from discrete topics (such as combinatorics, set theory, etc.). That way, this and similar problems are their introduction to problems dealing with continuity.

Problem group "the halving of 100 points"

100 points 50/50 - starting problem

100 points are given on the plane. Is there always a line that "halves" the points? "Halving" means here that one of the half-planes (defined by the line) contains 50 points and the other half-plane as well, and no point lies on the line.

Solution. It is an unusual problem for students, probably this is their first time to encounter something similar. Teacher gives struggling students the following hint: pick such a line at random, that all points are on the same side of the line and start moving the line towards the points. What happens? What could be a possible obstacle to solve the problem? After giving out the hint most groups can solve the problem.

As we shift (translate) this line towards the points, the points cross the line ideally one by one. Problem occurs when two or more points cross the line at the same time. It is only possible to occur, when there are two (or more) points defining a line which is parallel to the original line. One only needs to avoid such a situation. Pick the slope of the line such that it will not be parallel to any of the lines defined by any two points. There exists such a slope, while there are only finitely many points, defining finitely (is it a routine lighting round question for students: at most how many problematic slope are there) many slopes. If we shift this line, the points will cross individually, and eventually, there will be a time when there are 50 points on one side of the line and 50 points on the other side.

In a way, this solution has a similar flavour to proof of existence, which has its own problem-thread. The proof of existence problem-thread is introduced at a similar time to this problem. Although this specific problem is not a part of that problem-thread strictly speaking (better examples are chosen for represent the thread), this problem is still in the background, and helps pave the way towards it.

To find the solution is not easy for students, they struggle a lot. One direction they try out is to order the points by their x-coordinate. While this direction is not perfect it still can help students realize what the main obstacle is to find such a line.

Another typical but less common approach is to rotate the line. The idea can be made into a proper and precise solution. This solution differs from the first one, although both employ the idea of motion.

Again, it is interesting to see what kind of surrounding problems are around these problems. It is worth noting some problems related to combinatorial geometry, which requires a different approach, but the flavour of the problems is similar.

Combinatorial Geometry Problem 1

What is the maximum number of regions defined by n lines in the plane?

After this problem, at some point, later in the same camp, typically follows a connected problem. This connection is not announced for students, while this is also something they should discover themselves, imitating a bit, how real mathematical research works.

Combinatorial Geometry Problem 2

There are 64 points on the chessboard, each point is in the middle of a square. How many lines do you need, at least, that each point is in a different region?

The connection of the previous problem is obvious. From the previous problem, we get a theoretical minimum, namely 11. However, this knowledge is misleading at first, as it turns out later that it is impossible to solve it with only 11 lines. We can only solve it with 14 lines. There is an obvious solution with 14 lines (the lines between the squares). It is not easy to see why 14 lines is the minimum. One can prove it by the relaxation of the requirements. How many lines do you need at least, when you only need to put the 28 points on the perimeter of the chessboard into different regions?

The idea of relaxing the requirements is a useful tool in problem-solving, in the camps, and it has its own problem-thread. So, this problem is part of that “relaxation” problem-thread as well.

The circle versions

The 100 points 50/50 problem is a great starting point to ask further questions. The following questions are usually posed by the students, and they are part of the main problems in the camps.

100 points 50/50 with a circle

We can ask the same question, only this time instead of a line we need to “halve” the given 100 points with a circle. 50 points should be inside the circle, 50 points should be outside the circle, and no points should be on the circle.

100 points with a circle – specific ratio variant

There is also a variant of this question. Instead of asking for halving 100 points, we ask for to divide 1000 points by the circle to get a specific ratio.

The purpose of this variant to make students think more deeply about the previous problem and develop its underlying idea a step further. Instead of halving they need to modify the original idea to accommodate the new condition.

Solution. In the case of the circle. The problem arises when two or more points are on the circle. This is only possible, when the centre of the circle is on the perpendicular bisector of any two points. To find a proper centre of the circle we need to avoid all perpendicular bisectors generated by the points. As we have finitely many points, and so finitely many perpendicular bisectors, there should be points left for the centre of the circle. Let's choose such a point and start to increase the radius of the circle from 0. As we increase the radius, the points cross the circle one by one, and eventually there will be a time when inside are 50 points and outside are 50 points.

One of the circle variants is usually a follow-up homework problem. It helps students to reflect deeper on the solution of the initial problem.

During the solution of the circle version, there is a short-cut solution, which recalls the solution of the line version. As we have proved there is a line already dividing the points, we only need a circle with sufficiently big radius, in this way around the concerning areas the circle behaves like a line. To avoid this short-cut, students with this solution get the following problem.

$2n+3$ points version with the circle

There are $2n+3$ points on the plane in general position (there are no 4 points on the same circle or 3 points on the same line). Prove that there are 3 points from these $2n+3$ points, whose circumscribed circle divides the plane in such a way, that inside there are n points and outside there are n points as well.

Solution. Pick two points at random, start with a circle going through these two points. Change the radius of the circle. Since the points are in the general position, the circle goes through the other points one by one. In its extreme positions the disk of the circle becomes one of the half planes. Let us examine these extreme positions. From the other $2n + 1$ points, let x be in one of the half planes and the other $2n + 1 - x$ in the other half plane. The value of $n + \frac{1}{2}$ are between the two, so there will be a position of the circle that as we change the radius, the number of points inside the disk is changes from n to $n + 1$ (or vice versa).

Infinite version

During frontal-discussion, the question of infinity arises quite naturally.

Infinite version.

Infinitely many points are given on the plane. Is there always a line that “halves” the points? There should be infinitely many points on both open half-planes.

Given the solution of the original problems many students are convinced that there is always a line with the required property. They even sketch out a proof, like the finite version. Since there are counterexamples, these proofs, based on the finite version, are not right. Finding counterexamples of different kinds becomes the main interest for the students. The following questions arise through discussions with the students. Typically, students give an unbounded counterexample, where points are lying on the same line so the follow-up questions would be: Can you give a bounded counterexample of the previous problem? Most counterexamples are not bounded, only they usually lie on the same line. Can you give a counterexample,

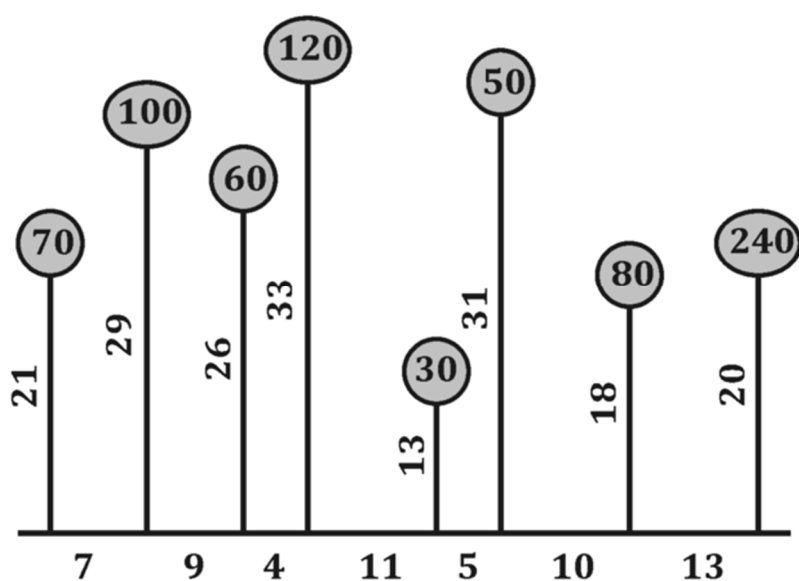
where points are in general position? (No three points are on the same line.) Many counterexamples are typically unbounded. So: Can you give a bounded counterexample, where points are in general position? Frontal discussion of this problem group can be tricky, teachers must pay extra attention which students are asked to present their solution. It is very easy to run into a counterexample which is also a counterexample to a later question.

Where should be the school built?

One or two camps later comes the following problem.

Where to construct the school?

On the following map we can see some villages. Numbers in the grey areas indicate the numbers of schoolchildren at a given village. On the different sections of the roads their lengths are shown. The villages want to build a school together alongside one of the roads or in a village. Where should be the school built, if the sum of the distance travelled by the students should be the smallest possible?



As a first hint, the teacher points out the following. ‘Let us observe the map carefully. Isn’t there unnecessary data?’ For the second hint, suggested not immediately (for example a half day later), the teacher suggests imagining the position of the school somewhere and let us start to move it and observe how the sum of the distances changes. Here comes the idea that motion helps to solve the problem. During the discussion with the whole class, the teacher draws the map on the table, starts to move a school made from paper around it and asks questions as he moves it. ‘Is it a good idea to construct the school in one of the villages?’ The answer is no. Reasoning: let us suppose we construct the school in one of the villages, let us try to move it towards the main road by x miles. Students living in the village must commute x meters more, other students from other villages must commute x meters less. There is no such village where more students live than in the rest of the villages together, so it is always beneficial to move the school out of a village towards the main road. From this observation we can also conclude that the data on the distances has no impact on where to construct the school. So, now we know that the school must be along the main road. Let us start from the left side of the main road and move the school alongside the road. If we construct the school to the right of the base of the first village, only for 70 students it is getting worse from the change, and $750 - 70 = 680$ students are benefitting from it. So, we can conclude we should move the school to the base of the next village. Is it better to move it to the right again? Yes, now for 170 students it gets worse, but for $750 - 170 = 580$ students it gets better. Continuing this process, we can conclude that the best possible location is at the base of the village with 30 students.

When presenting the problem, teachers do not give students the criteria of a good school, they ask the students instead, what do they think, what would be the good criteria. For example, we could try to minimise the maximum distance to the school, instead.

The connection with the other problems within this problem-thread is not obvious. After the solution teachers ask the students. "There were two antecedents of this problem. Can you find out which two I am referring to?" Students usually find out that 100 points 50/50 is a connecting problem first with some difficulties. The teacher needs to suggest that the nature of the connection is not conventional. To realize the connection with the first problem (the one with the equation), proves to be even more difficult. Even when the teacher let them know they usually seem confused. The teacher continues: "... So, we have the three connected problems now. Then, can you tell me what the connection is exactly?" In that way students learn about the underlying idea of motion as a problem-solving tool.

12 couples at a round table 4/4

12 old-fashioned couples are sitting at a round table. (An old-fashioned couple has one female and one male member.) The members of a couple are not necessarily sitting next to each other. Is it true that there always exist 8 consecutive places where 4 women and 4 men are sitting?

Solution. A possible hint for struggling students would be: let us pick 8 consecutive places. If there are 4 women and 4 men within this section we are done, otherwise try to move the selected section with 1 place to one direction. What could happen?

If we move the 8 places to one direction the number of women will change by one or stay the same. So, if there are 8 places with more than 4 women and another 8 with less than 4 women it follows that it must be an 8 with exactly 4 women. So let us suppose there is no 8 with more and another 8 with less than 5 women. In all 8s there must be at least 5 (or at most 3). Since there are 24 places and we can divide it into three disjunct 8s, then there are at least 15 (or at most 9) women in the group. We reached a contradiction; hence we proved the existence of an 8 with 4 women.

This solution also uses the idea of motion: we take one 8s and move it around the circle. This demonstrates it well how the idea of motion can be applied in different problems. This solution is a proof of existence; thus, the problem is part of the proof of existence problem-thread.

We can ask the same question only that this time we would like to find 10 consecutive places where there are 5 women and 5 men.

12 couples at a round table 5/5

12 old-fashioned couples are sitting at a round table. Is it true that there always exist 10 consecutive places where 5 women and 5 men are sitting?

We can refine the last solution to find a solution here. We can prove similarly that each 10 must contain at least 6 (at most 4) women. From this point we can't do the same. 8 was a divisor of 24, so we had three disjunct eights. So, what can we do? Instead, we consider all possible tens, let us count the number of women in all possible tens: we count at least $24 \cdot 6 = 144$ (at most $24 \cdot 4 = 96$) women. We counted each woman exactly 10 times (they are a part of exactly 10 tens). So, there are at least 14.4 (at most 9.6) women in the group which is a contradiction.

The next thing to consider is how to generalize the results. The teacher asks this from the students, to give them the opportunity to properly formulate the generalized statement.

N couples at a round table k/k

N old-fashioned couples are sitting at a round table. Is it true that there always exist $2k$ consecutive places where k women and k men are sitting? ($N \geq k$)

The same proof works. Students usually leave out the condition about N and k . So, it can be an opportunity to practise forming precise statements.

Further selected problems without discussion.

To demonstrate how far the idea of motion can take us in the context of high school mathematics here are two problems from the IMO (International Mathematical Olympiad). In both cases the idea of motion can be applied effectively to solve the problems.

Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line l going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S . This point, Q , takes over as the new pivot, and the line now rotates clockwise about Q , until it next meets a point of S . This process continues indefinitely. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times. (IMO, 2011, Problem 2)

A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines. (IMO, 2013, Problem 2)

Motion as a subject, a different problem-thread

Naturally, in the camps there are also problems when motion is studied as a subject as well, instead of being used as a tool. In this section we give examples dealing with motion in contrast problems introduced so far.

Sliding rod

There is a rod placed vertically against a wall. The rod slides until it lies flat on the floor. What is the trajectory of the midpoint of the rod?

This is a rather well-known problem. In the camps it is used to encourage students to experiment. This is one of the most important problems in the experiment-conjecture-prove problem thread. The experiment-conjecture-prove problem thread is about the importance of experimentations in problem solving. For this age group it is rather surprising this method. In relation to this problem the class often discuss about the differences and similarities of different scientific fields. More specifically, they talk about the role of experimentations in physics and mathematics. This problem is proposed typically in the first camp.

There is a related problem a few camps later.

Sliding right triangle

There is a right triangle its hypotenuse is placed against a wall. The triangle slides down until its hypotenuse lying on to the floor. What is the trajectory of the right-angled point of the triangle? How exactly does the point of the triangle move through the trajectory?

Another problem from the problem-thread to demonstrate the diversity of problems in this thread.

The wonderful lizard

In a distant island, scientists are observing a unique lizard. During the observation the followings are true:

- the total time of the observation is 6 minutes
- every scientist observes the lizard exactly for 1 minute (continuously, without breaking up the time)
- during any time of the observation at least one scientist is present
- the lizard can only move along a line and only in one direction
- every scientist observes that the lizard moves exactly 1 meter during his/her observation period

What is the maximum distance the lizard can move during this observation?

(Hint: it is more than 6 meters)

Discussions

In the next section, we write about current ongoing projects concerning the method and about the most important characteristics of the method.

Future direction of the Pósa method

The Pósa method has been developed in special circumstances (weekend camps, talented and motivated students), so adaptation of the method to a wider audience is not obvious. The organizers of the camps started a school experiment, where they taught three groups of high school students for four years in the spirit of the Pósa method. The analysis of the results is under work, the programme has just ended. Analysis of how problem-threads work in this programme can be found in the article of Matzsal et al. (2020).

Another notable project is “Flying school” from the past few years, where the organizers visit schools around Hungary and hold a 3–4-hour activity for grade 9 students (first year of high school in Hungary). The goal is twofold. First, the organizers wish to spread the joy of problem-solving for all the participants. Second, the organizers are looking for motivated and talented students who would like to join a 2-year long programme. During the programme each month there is a mathematical day, where they do mathematics in the spirit of Pósa method in Budapest together. The background of the participating schools is diverse. The aim is to reach students from under-developed regions as well, or students who have less access to talent nurturing opportunities.

General remarks about the method

To summarize, in the following section general characteristics of the method are discussed.

Problem-threads

Problem-threads are powerful tools for developing problem-solving skills. In the camps connections between problems are analysed, the relationships between solutions and ideas are openly discussed with students.

Problem posing

Problem posing also plays a crucial role in this method. In Hungary, usually there is not really a culture to practice how to pose meaningful, but reachable and interesting questions in a school setting. One of the benefits of problem posing for students is that they can better understand how research works in general, and especially how modern mathematical research works. Several interesting problems posed by students are build in into the curriculum.

Open questions

Students often think at the beginning of the programme, mathematics is a “closed” discipline, all questions are answered. They are rather surprised to learn about open research questions in mathematics, they can also understand and think about. In this problem-thread there was no such example, to see a problem-thread with open questions read the paper of first author from a different problem-thread (Bóra, 2020).

Design-thinking

The structure of the programme is changing overtime. With more and more experience, problems are restructured, and new problems are introduced. The system of hints given to students and important talking-points of the problems also benefit from the accumulating experience throughout the years. This natural process of reflection on the programme is very similar to a design-thinking approach.

Dialogues

Verbal discussions with each group, individuals or the whole class are crucial parts of the camps. Many important phases of problem solving are happening during these discussions, such as the refinement of ideas. There is a mostly unwritten tradition of how to guide students through problems.

Personality and atmosphere

weekend math camps focus on students, thus we can state this is very much a student-centred methodology. Not only satisfying their intellectual curiosity, but their overall wellbeing is also important for the organizers. They strive for forming an open, welcoming community, where students can be themselves, and pursue their love for mathematics without a worry.

References

- Artigue, M., Bosch, M., Doorman, M., Juhász, P., Kvasz, L., & Maass, K. (2020). Inquiry based mathematics education and the development of learning trajectories. *Teaching Mathematics and Computer Science*, 18(3), 63-89.
- Bóra, E. (2020). A computational thinking problem-thread for grade 7 students and above from the Pósa method. *Teaching Mathematics and Computer Science*, 18(3), 101-110.
- Fried, K. (2010). Case of Hungary. In *Russian mathematics education: History and world significance* (pp. 338-346).
- Győri, J. G., & Juhász, P. (2017). An extra-curricular gifted support programme in Hungary for exceptional students in mathematics. In *Teaching gifted learners in stem subjects* (pp. 89-106). Routledge.
- Győri, G. J., Fried, K., Köves, G., Oláh, V., & Pálfalvi, J. (2020). The traditions and contemporary characteristics of mathematics education in Hungary in the post-Socialist era. In *Eastern European mathematics education in the decades of change* (pp. 75-129). Springer, Cham.
- Gosztonyi, K. (2014). Séries de problèmes. L'exemple des Jeux avec l'infini de Rózsa Péter. *Bulletin de l'APMEP*, 507, 25-31. <https://www.apmep.fr/IMG/pdf/AAA14003.pdf>

Gosztonyi K. (2016) Mathematical Culture and Mathematics Education in Hungary in the XXth Century. In: Larvor B. (eds) Mathematical Cultures. Trends in the History of Science. Birkhäuser, Cham. https://doi.org/10.1007/978-3-319-28582-5_5

Gosztonyi, K. (2019). Conceiving teaching trajectories in the form of series of problems: a step for the theoretical reconstruction of the Hungarian Guided Discovery approach. In Eleventh Congress of the European Society for Research in Mathematics Education (No. 17). Freudenthal Group; Freudenthal Institute; ERME.

Gosztonyi, K. (2020). Tamás Varga's reform movement and the Hungarian Guided Discovery approach. *Teaching Mathematics and Computer Science*, 18(3), 11-28.

Juhász, Péter and Katona, Dániel (2019) Pósa method: Talent Nurturing in Weekend Math Camps. In: Including the Highly Gifted and Creative Students – Current Ideas and Future Directions: Proceedings of the 11th International Conference on Mathematical Creativity and Giftedness (MCG 11). Conference Proceedings in Mathematics Education (5). WTM – Verlag für wissenschaftliche Texte und Medien, Münster, pp. 270-276

Katona, D., & Szűcs, G. (2017). Pósa-method & cubic geometry: A sample of a problem thread for discovery learning of mathematics. *Differences in pedagogical theory and practice*, 17-34.

Koichu, B., & Andžāns, A. (2009). Mathematical creativity and giftedness in out-of-school activities. In *Creativity in mathematics and the education of gifted students* (pp. 285-307). Brill Sense.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Matzal, C., Manahan, K., Galaty, B., Wang, H., & Barbarics, M. (2020). Guided Discovery in Hungarian Education Using Problem Threads: The Pósa Method in Secondary Mathematics Classrooms. *Teaching Mathematics and Computer Science*, 18(1), 51-67.

Pólya, G. (1990). *How to solve it*. London, Penguin Books. (First published in 1945 by Princeton University Press.)

Pólya, G. (1954). *Induction and analogy in mathematics*. Princeton: Princeton University Press.

Rényi, A. (1967). *Dialogues on Mathematics*, San Francisco: Holden Day Publishers. (Originally published in 1965 as *Dialógusok a matematikáról*. Budapest: Akadémiai Kiadó.)

Rényi, A. (1972). *Letters on Probability* (trad. by László Vekerdí). Budapest: Akadémiai Kiadó. (Originally published in 1967 as *Levelek a valószínűségről*. Budapest: Akadémiai Kiadó.)

Shklyarsky, D. O., Chentsov, N. N., & Yaglom, I. M. (1979). *Selected Problems and Theorems in Elementary Mathematics: Arithmetic and Algebra*. M