On Helly numbers of exponential lattices*

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Abstract

Given a set $S \subseteq \mathbb{R}^2$, define the *Helly number of* S, denoted by $H(S)$, as the smallest positive integer N, if it exists, for which the following statement is true: for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^2$ such that the intersection of any N or fewer members of $\mathcal{F}$ contains at least one point of S, there is a point of S common to all members of $\mathcal{F}$.

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We prove that the Helly numbers of *exponential lattices* $\{\alpha^n \colon n \in \mathbb{N}_0\}^2$ are finite for every $\alpha > 1$ and we determine their exact values in some instances. In particular, we obtain $H(\{2^n \colon n \in \mathbb{N}_0\}^2) = 5$, solving a problem posed by Dillon (2021).

For real numbers $\alpha, \beta > 1$, we also fully characterize exponential lattices $L(\alpha, \beta) = \{\alpha^n \colon n \in \mathbb{N}_0\} \times \{\beta^n \colon n \in \mathbb{N}_0\}$ with finite Helly numbers by showing that $H(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is rational.

1 Introduction

Helly's theorem [11] is one of the most classical results in combinatorial geometry. It states that, for each $d \in \mathbb{N}$, if the intersection of any $d + 1$ or fewer members of a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is nonempty, then the entire family $\mathcal{F}$ has nonempty intersection. There have been numerous variants and generalizations of this famous result; see [1, 13] for example. One active direction of this research with rich connections to the theory of optimization, in particular to integer programming and LP-type problems [1, 4], is the study of variants of Helly's theorem with coordinate restrictions, which is captured by the following definition.

Let d be a positive integer. The *Helly number* of a set $S \subseteq \mathbb{R}^d$, denoted by $H(S)$, is the smallest positive integer N, if it exists, such that the following statement is true for every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$: if the intersection of any N or fewer members of $\mathcal{F}$ contains at least one point of S, then $\bigcap \mathcal{F}$ contains at least one point of S. If no such number N exists, then we write $H(S) = \infty$. Helly's theorem in this language can be restated as $H(\mathbb{R}^d) = d + 1$.

A classical result of this sort is *Doignon's theorem* [8] where the set S is the integer lattice $\mathbb{Z}^d$. This result, which was also independently discovered by Bell [3] and by Scarf [15], states that $H(\mathbb{Z}^d) \leq 2^d$. This is tight as for $Q = \{0, 1\}^d$ the intersection of any $2^d - 1$ sets in the family $\{\operatorname{conv}(Q \setminus \{x\}) \colon x \in Q\}$ contains a lattice point, but the intersection of all 2^d sets does not.

The theory of Helly numbers of general sets is developing quickly and there are many result of this kind [1, 13]. For example, De Loera, La Haye, Oliveros, and Roldán-Pensado [5] and De Loera, La Haye, Rolnick, and Soberón [6] studied the Helly numbers of differences of lattices and Garber [9] considered Hely numbers of crystals or cut-and-project sets.

The Helly number of a set S is closely related to the maximum size of a set that is empty in S. A subset $X \subseteq S$ is *intersect-empty* if $\left(\bigcap_{x \in X} \operatorname{conv}(X \setminus \{x\})\right) \cap S = \emptyset$. A convex polytope P with vertices in S is *empty in S* if P does not contain any points of S other than its vertices. In particular, an empty polytope does not contain points of S in the interior of its edges. For a discrete set S, we use $h(S)$ to denote the maximum number of vertices of an empty polytope in S. If there are empty polytopes in S with arbitrarily large number of vertices, then we write $h(S) = \infty$.

The following result by Hoffman [12] (which was essentially already proved by Doignon [8]) shows the close connection between intersect-empty sets and empty polygons in S and the Helly numbers of S; see also [2].

Proposition 1 ([12])**.** *If $S \subseteq \mathbb{R}^d$, then $H(S)$ is equal to the maximum cardinality of an intersect-empty set in S. If S is discrete, then $H(S) = h(S)$.*

Since all the sets S studied in this paper are discrete, we state all of our results using $h(\alpha)$ but, due to Proposition 1, our results apply to $H(\alpha)$ as well.

Very recently, Dillon [7] proved that the Helly number of a set S is infinite if S belongs to a certain collection of *product sets*, which are sets of the form $S = A^d$ with a certain kind of discrete set $A \subseteq \mathbb{R}$. His result shows, for example, that whenever p is a polynomial of degree at least 2 and $d \geq 2$, then $h(\{p(n) \colon n \in \mathbb{N}_0\}^d) = \infty$. However, there are sets for which Dillon's method gives no information, for example $\{2^n \colon n \in \mathbb{N}_0\}^2$. Thus, Dillon [7] posed the following question, which motivated our research.

Problem 1 (Dillon, [7])**.** *What is $h(\{2^n \colon n \in \mathbb{N}_0\}^2)$?*

In this paper, we study the Helly numbers of *exponential lattices* $L(\alpha)$ and $L(\alpha, \beta)$ in the plane where $L(\alpha) = \{\alpha^n \colon n \in \mathbb{N}_0\}^2$ and $L(\alpha, \beta) = \{\alpha^n \colon n \in \mathbb{N}_0\} \times \{\beta^n \colon n \in \mathbb{N}_0\}$ for real numbers $\alpha, \beta > 1$. In particular, we prove that Helly numbers of exponential lattices $L(\alpha)$ are finite and we provide several estimates that give exact values for α sufficiently large, solving Problem 1. We also show that Helly numbers of exponential lattices $L(\alpha, \beta)$ are finite if and only if $\log_\alpha(\beta)$ is rational. Finally, we introduce some new open problems, for example, it is not even known whether the Helly numbers of the sets $\{\alpha^n \colon n \in \mathbb{N}_0\}^d$ with $d > 2$ are finite.

2 Our results

For a real number $\alpha > 1$ and the exponential lattice $L(\alpha) = \{\alpha^n \colon n \in \mathbb{N}_0\}^2$, we abbreviate $h(L(\alpha))$ by $h(\alpha)$.

As our first result, we provide finite bounds on the numbers $h(\alpha)$ for any $\alpha > 1$. The upper bounds are getting smaller as α increases and reach their minimum at $\alpha = 2$.

Theorem 2. *For every real $\alpha > 1$, the maximum number of vertices of an empty polygon in $L(\alpha)$ is finite. More precisely, we have $h(\alpha) \leq 5$ for every $\alpha \geq 2$,*

$h(\alpha) \le 7$ *for every* $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$, *and*

$$h(\alpha) \le 3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$$

for every $\alpha \in (1, \frac{1+\sqrt{5}}{2})$.

We note that if $\alpha = 1 + \frac{1}{x}$ for $x \in (0, \infty)$, then the bound from Theorem 2 becomes $h(1 + \frac{1}{x}) \le O(x \log_2(x))$. Moreover, we show that the breaking points of α for our upper bounds are determined by certain polynomial equations; see Section 3.

We also consider the lower bounds on $h(\alpha)$ and provide the following estimate.

Theorem 3. *We have* $h(\alpha) \ge 5$ *for every* $\alpha \ge 2$ *and* $h(\alpha) \ge 7$ *for every* $\alpha \in \left[\frac{1+\sqrt{5}}{2}, 2\right)$. *For every* $\alpha \in \left(1, \frac{1+\sqrt{5}}{2}\right)$, *we have*

$$h(\alpha) \ge \left\lfloor \sqrt{\frac{1}{\alpha - 1}} \right\rfloor .$$

If $\alpha = 1 + \frac{1}{x}$ where $x \in (0, \infty)$, then the lower bound from Theorem 3 becomes $h(1 + \frac{1}{x}) \ge \lfloor \sqrt{x} \rfloor$. So with decreasing α, the parameter $h(\alpha)$ indeed grows to infinity.

By combining Theorems 2 and 3, we get the precise value of the Helly numbers of $L(\alpha)$ with $\alpha \ge (1 + \sqrt{5})/2$. In particular, for $\alpha = 2$, we obtain a solution to Problem 1.

Corollary 4. *We have* $h(\alpha) = 5$ *for every* $\alpha \ge 2$ *and* $h(\alpha) = 7$ *for every* $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$.

We prove the following result which shows that even a slight perturbation of S can affect the value $h(S)$ drastically. We use the *Fibonacci numbers* $(F_n)_{n \in \mathbb{N}_0}$, which are defined as $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every integer $n \ge 2$.

Proposition 5. *We have* $h(\{F_n \colon n \in \mathbb{N}_0\}^2) = \infty$.

We recall that $F_n = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}$ for every $n \in \mathbb{N}_0$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio* and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi$ is its conjugate. Since $\psi < 1$, this formula shows that the points of $\{F_n \colon n \in \mathbb{N}_0\}^2$ are approaching the points of the scaled exponential lattice $\frac{\varphi}{\sqrt{5}} \cdot L(\varphi) = \{\frac{\varphi}{\sqrt{5}} \cdot \varphi^n \colon n \in \mathbb{N}_0\}^2$. Thus, Proposition 5 is in sharp contrast with the fact that $h(\frac{\varphi}{\sqrt{5}} \cdot L(\varphi)) = h(\varphi) \le 7$, which follows from Theorem 2 and from the fact that affine transformations of any set $S \subseteq \mathbb{R}^d$ do not change $h(S)$. We also note Dillon's method [7] does not imply $h(\{F_n \colon n \in \mathbb{N}_0\}^2) = \infty$.

We also consider the more general case of exponential lattices where the rows and the columns might use different bases. For real numbers $\alpha > 1$ and $\beta > 1$, let $L(\alpha, \beta)$ be the set $\{\alpha^n : n \in \mathbb{N}_0\} \times \{\beta^n : n \in \mathbb{N}_0\}$. Note that $L(\alpha) = L(\alpha, \alpha)$ for every $\alpha > 1$.

As our last main result, we fully characterize exponential lattices $L(\alpha, \beta)$ with finite Helly numbers $h(L(\alpha, \beta))$, settling the question of finiteness of Helly numbers of planar exponential lattices completely.

Theorem 6. *Let $\alpha > 1$ and $\beta > 1$ be real numbers. Then, $h(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is a rational number.*

Moreover, if $\log_\alpha(\beta) \in \mathbb{Q}$, that is, $\beta = \alpha^{p/q}$ for some $p, q \in \mathbb{N}$, then

$$\left\lfloor \frac{1}{pq} \left\lfloor \sqrt{\frac{1}{\alpha^{1/q} - 1}} \right\rfloor \right\rfloor \leq h(L(\alpha, \beta)) \leq pq \cdot h(\alpha^p).$$

The proof of Theorem 6 is based on Theorem 2 and on the theory of continued fractions and Diophantine approximations.

Open problems

First, it is natural to try to close the gap between the upper bound from Theorem 2 and the lower bound from Theorem 3 and potentially obtain new precise values of $h(\alpha)$.

Second, we considered only the exponential lattice in the plane, but it would be interesting to obtain some estimates on the Helly numbers of exponential lattices $\{\alpha^n : n \in \mathbb{N}_0\}^d$ in dimension $d > 2$. In particular, are these numbers finite?

We also mention the following conjecture of De Loera, La Haye, Oliveros, and Roldán-Pensado [5], which inspired the research of Dillon [7].

Conjecture 1 ([5])**.** *If $\mathcal{P}$ is the set of prime numbers, then $h(\mathcal{P}^2) = \infty$.*

Using computer search, Summers [16] showed that $h(\mathcal{P}^2) \geq 14$.

3 Proof of Theorem 2

Here, we prove Theorem 2 by showing that the number $h(\alpha)$ is finite for every $\alpha > 1$. This follows from the upper bounds $h(\alpha) \leq 5$ for $\alpha \geq 2$, $h(\alpha) \leq 7$ for every $\alpha \geq [\frac{1+\sqrt{5}}{2}, 2)$, and

$$h(\alpha) \leq 3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$$

for any $\alpha \in (1, \frac{1+\sqrt{5}}{2})$.

We start by introducing some auxiliary definitions and notation. Let $\alpha > 1$ be a real number and consider the exponential lattice $L(\alpha)$. For $i \in \mathbb{N}_0$, the *ith column* of $L(\alpha)$ is the set $\{(\alpha^i, \alpha^n) \colon n \in \mathbb{N}_0\}$. Analogously, the *ith row* of $L(\alpha)$ is the set $\{(\alpha^n, \alpha^i) \colon n \in \mathbb{N}_0\}$.

For a point p in the plane, we write $x(p)$ and $y(p)$ for the x- and y-coordinates of p, respectively. Let P be an empty convex polygon in $L(\alpha)$. Let e be an edge of P connecting vertices u and v where $x(u) < x(v)$ or $y(u) < y(v)$ if $x(u) = x(v)$. We use $\overline{e}$ to denote the line determined by e and oriented from u to v. The *slope of* e with $x(u) < x(v)$ is the slope of $\overline{e}$, that is, $\frac{y(v)-y(u)}{x(v)-x(u)}$.

We distinguish four types of edges of P; see part (a) of Figure 1. Roughly speaking, the type of an edge is exactly the quadrant where the normal vector to this edge points to (up to the boundaries of the quadrants). First, assume $x(u) \neq x(v)$ and $y(u) \neq y(v)$. We say that e is of *type I* if the slope of e is negative and P lies to the right of $\overline{e}$. Similarly, e is of *type II* if the slope of e is positive and P lies to the right of $\overline{e}$. An edge e has *type III* if the slope of e is negative and P lies to the left of $\overline{e}$. Finally, *type IV* is for e with positive slope and with P lying to the left of $\overline{e}$. It remains to deal with horizontal and vertical edges of P. A horizontal edge e is of type II if P lies below $\overline{e}$ and is of type III otherwise. Similarly, a vertical edge e is of type IV if P lies to the left of $\overline{e}$ and is of type III otherwise.

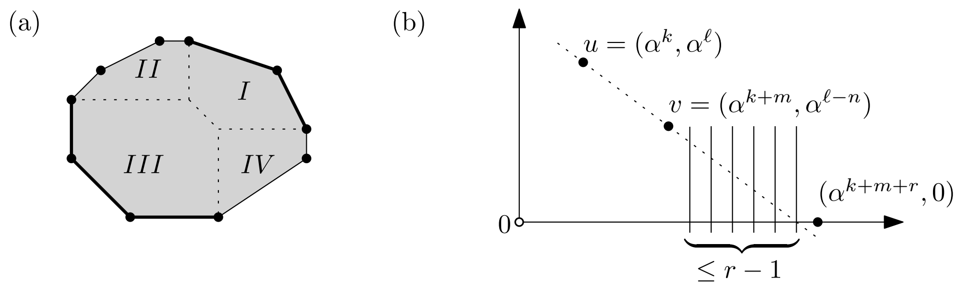

Figure 1: (a) The four types of edges of a convex polygon. (b) An illustration of the proof of Lemma 7.

Note that each edge of P has exactly one type and that the types partition the edges of P into four convex chains. We first provide an upper bound on the number of edges of those chains of P and then derive the bound on the total number of edges of P by summing the four bounds. We start by estimating the number of edges of P of type I.

Lemma 7. *The polygon P has at most $\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$ edges of type I.*

Proof. First, let $r = \lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$ and note that $r \geq 1$ as $\alpha > 1$. Let e be the left-most edge of P of type I and let u and v be the vertices of e. Since e is of type

I, we have $u = (\alpha^k, \alpha^\ell)$ and $v = (\alpha^{k+m}, \alpha^{\ell-n})$ for some positive integers k, ℓ, m, and n.

We will show that the point $(\alpha^{k+m+r}, 0)$ lies above the line $\overline{e}$. Since there are at most $r-1$ columns of $L(\alpha)$ between the vertical line containing v and the vertical line containing $(\alpha^{k+m+r}, 0)$ and the point $(\alpha^{k+m+r}, 0)$ is below the lowest row of $L(\alpha)$, it then follows that there are at most r edges of P of type I; see part (b) of Figure 1.

Since the line $\overline{e}$ contains u and v, we see that

$$\overline{e} = \{(x, y) \in \mathbb{R}^2 \colon (\alpha^\ell - \alpha^{\ell-n})x + (\alpha^{k+m} - \alpha^k)y = \alpha^{k+\ell+m} - \alpha^{k+\ell-n}\}.$$

It suffices to check that by substituting the coordinates of the point $(\alpha^{k+m+r}, 0)$ into the equation of the line $\overline{e}$ gives a left side that is at least $\alpha^{k+\ell+m} - \alpha^{k+\ell-n}$. The left side equals $\alpha^{k+\ell+m+r} - \alpha^{k+\ell+m-n+r}$ and thus we want

$$\alpha^{k+\ell+m+r} - \alpha^{k+\ell+m-n+r} \geq \alpha^{k+\ell+m} - \alpha^{k+\ell-n}.$$

By dividing both sides by $\alpha^{k+\ell}$ and by rearranging the terms, we can rewrite this expression as

$$\alpha^{-n}(1 - \alpha^{m+r}) \geq \alpha^m - \alpha^{m+r}.$$

Since $m, r > 0$ and $\alpha > 1$, we get $(1 - \alpha^{m+r}) < 0$ and thus the left side is increasing as n increases, so we can assume $n = 1$, leading to

$$\alpha^{-1} - \alpha^{m+r-1} \geq \alpha^m - \alpha^{m+r}.$$

We can again rearrange the inequality as

$$\alpha^r - \alpha^{r-1} - 1 \geq -\alpha^{-1-m},$$

where the right side is negative and approaches 0 as m tends to infinity, so we can replace it by 0, obtaining

$$\alpha^r - \alpha^{r-1} \geq 1.$$

This inequality is satisfied by our choice of r. □

We now estimate the number of edges of P that are of type III.

Lemma 8. *The polygon P has at most $2\lceil \log_\alpha \left(\frac{\alpha+1}{\alpha}\right) \rceil + 1$ edges of type III for $1 < \alpha < 2$ and at most 2 such edges for $\alpha \geq 2$.*

Proof. Let $t = \lceil \log_\alpha \left(\frac{\alpha+1}{\alpha}\right) \rceil$ and $s = t + 1$ for $\alpha \in (1, 2)$ and $t = 1 = s$ for $\alpha \geq 2$. Suppose for contradiction that there are $s + t + 1$ edges of P of type III. Let $v_1, \ldots, v_{s+t+2}$ be the vertices of the convex chain that is formed by edges of P of type III. We use Q to denote the convex polygon with vertices $v_1, \ldots, v_{s+t+2}$. Note that Q is empty in $L(\alpha)$ as P is empty and $Q \subseteq P$.

Let v' be the point $(x(v_{s+2}), \alpha \cdot y(v_{s+2}))$, that is, v' is the point of $L(\alpha)$ that lies just above v_{s+2}; see part (a) of Figure 2. We will show that the point v' lies below the line $\overline{v_1 v_{s+t+2}}$. Since v' lies in the same column of $L(\alpha)$ as v_{s+2}, this then implies that v' lies in the interior of Q, contradicting the fact that Q is empty in $L(\alpha)$.

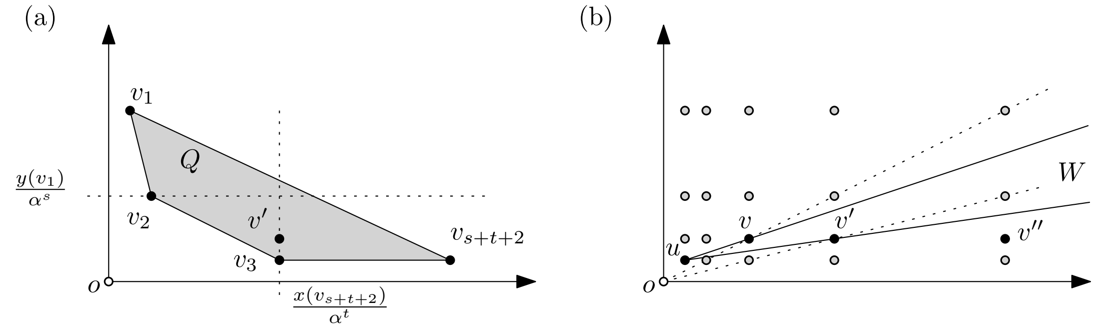

Figure 2: (a) An illustration of the proof of Lemma 8 for $s = 1 = t$. (b) An illustration of Lemma 9.

Note that $x(v') \leq \frac{x(v_{s+t+2})}{\alpha^t}$ and $y(v') \leq \frac{y(v_1)}{\alpha^s}$ as all edges $v_i v_{i+1}$ are of type III and thus the x- and y-coordinates decrease by a multiplicative factor at least α for each such edge. Since the only vertical edge might be $v_1 v_2$ and the only horizontal edge might be $v_{s+t+1} v_{s+t+2}$, the x- or y-coordinates indeed decrease by the factor at least α at each step.

Let $v_1 = (\alpha^k, \alpha^\ell)$ and $v_{s+t+2} = (\alpha^{k+m}, \alpha^{\ell-n})$ for some positive integers k, ℓ, m, n. Note that $m, n \geq s + t$. The line determined by v_1 and v_{s+t+2} is then

$$\{(x, y) \in \mathbb{R}^2 \colon (\alpha^\ell - \alpha^{\ell-n})x + (\alpha^{k+m} - \alpha^k)y = \alpha^{k+\ell+m} - \alpha^{k+\ell-n}\}.$$

Since $x(v') \leq \frac{x(v_{s+t+2})}{\alpha^t}$ and $y(v') \leq \frac{y(v_1)}{\alpha^s}$, it suffices to check

$$(\alpha^\ell - \alpha^{\ell-n})\frac{\alpha^{k+m}}{\alpha^t} + (\alpha^{k+m} - \alpha^k)\frac{\alpha^\ell}{\alpha^s} < \alpha^{k+\ell+m} - \alpha^{k+\ell-n}.$$

After dividing by $\alpha^{k+\ell+m}$, this can be rewritten as

$$\alpha^{-t} + \alpha^{-s} < 1 - \alpha^{-m-n} + \alpha^{-t-n} + \alpha^{-s-m}.$$

Since $m, n \geq s + t$, the right hand side is decreasing with increasing m and n and thus we only need to prove

$$\alpha^{-s} + \alpha^{-t} \leq 1.$$

If $\alpha \geq 2$, then $s = 1 = t$ and this inequality becomes $2/\alpha \leq 1$, which is true. If $\alpha \in (1, 2)$, then $s = t + 1$ and the inequality becomes $1 + 1/\alpha \leq \alpha^t$, which is also true by our choice of t. □

It remains to bound the number of edges of P that are of types II and IV. Observe that if we switch the x- and y- coordinates of P, then edges of type II become edges of type IV and vice versa. Since the exponential lattice $L(\alpha)$ is symmetric with respect to the line $x = y$, we see that it suffices to estimate the number of edges of type II. To do so, we use the following auxiliary result.

Lemma 9. *Let u be a point of $L(\alpha)$ and let v and v' be two points of $L(\alpha)$ that are consecutive in a row R of $L(\alpha)$ that lies above the row containing u; see part (b) of Figure 2. If v and v' lie to the right of u, then all points of $L(\alpha)$ that lie above R in the interior of the angle W spanned by the rays $\overline{uv}$ and $\overline{uv'}$ lie on at most $\lceil \log_\alpha(\frac{\alpha}{\alpha-1}) \rceil$ lines containing the origin.*

Proof. Similarly as in Lemma 7, we set $r = \lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$ and note that $r \geq 1$. We can assume without loss of generality that $u = (1, 1)$ as otherwise it suffices to scale the points of $L(\alpha)$ with an affine transformation. Since v and v' are consecutive on R, they both lie in the same closed halfplane determined by the line $x = y$. We first assume that the point v lies below or on the line $x = y$.

Let o be the origin and consider the lines $\overline{ov}$ and $\overline{ov'}$. Then, the part of the line $\overline{ov}$ above the row R is (not necessarily strictly) above $\overline{uv}$; see part (b) of Figure 2. Similarly, the part of the line $\overline{ov'}$ above R is above $\overline{uv'}$. It follows that only points of $L(\alpha)$ that lie on a line $\overline{ow}$, where w is a point of $L(\alpha)$ to the right of v on R, can lie in the interior of W.

Let v'' be the point $(\alpha^r \cdot x(v'), y(v'))$, that is, v'' is the point of $L(\alpha)$ that lies r columns to the right of v' on R, We will show that the part of the line $\overline{ov''}$ above R lies below $\overline{uv'}$. This will conclude the proof as all points of $L(\alpha)$ that lie in the interior of W above R have to then lie on one of the r lines $\overline{ow}$ with w lying between v and v'' on R.

It suffices to compare the slopes of the lines $\overline{ov''}$ and $\overline{uv'}$. Let $v' = (\alpha^m, \alpha^n)$ for some positive integers m and n. Then, the slope of $\overline{ov''}$ is

$$\frac{y(v'') - y(o)}{x(v'') - x(o)} = \frac{y(v')}{\alpha^r \cdot x(v')} = \frac{\alpha^n}{\alpha^{m+r}}$$

and the slope of $\overline{uv'}$ equals

$$\frac{y(v') - y(u)}{x(v') - x(u)} = \frac{y(v') - 1}{x(v') - 1} = \frac{\alpha^n - 1}{\alpha^m - 1}.$$

Thus, we want

$$\frac{\alpha^n - 1}{\alpha^m - 1} \geq \frac{\alpha^n}{\alpha^{m+r}}.$$

We can rewrite this inequality as

$$\alpha^{m+n+r} - \alpha^{m+r} \geq \alpha^{n+m} - \alpha^n,$$

which can be further rewritten by dividing both sides with α^n as

$$\alpha^{m+r}(1 - \alpha^{-n}) \geq \alpha^m - 1.$$

The left side is increasing with increasing n, so we can assume $n = 1$ and, by dividing both sides with α^m, we obtain

$$\alpha^r(1 - \alpha^{-1}) \geq 1 - \alpha^{-m}.$$

Now, the term α^{-m} on the right side approaches 0 from above with increasing m, so we can replace it by 0 obtaining

$$\alpha^r - \alpha^{r-1} \geq 1.$$

This inequality is satisfied by our choice of r.

Now, assume that the point v lies above the line $x = y$. Then, the proof proceeds analogously as in the previous case. The part of the line $\overline{ov}$ above the row R is (not necessarily strictly) below $\overline{uv}$. Similarly, the part of the line $\overline{ov'}$ above R is below $\overline{uv'}$. Then, only points of $L(\alpha)$ that lie on a line $\overline{ow}$, where w is a point of $L(\alpha)$ to the left of v on R, can lie in the interior of W above R. Considering the point $(\alpha^{-r} \cdot x(v'), y(v'))$ of $L(\alpha)$ that lies r columns to the left of v' on R, we can show with analogous computations as before that the part of the line $\overline{ov''}$ above R lies above $\overline{uv'}$. This concludes the proof. □

Now, we can apply Lemma 9 to obtain an upper bound on the number of edges of P of type II.

Lemma 10. *The polygon P has at most $\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil + 1$ edges of type II.*

Proof. Again, let $r = \lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$. Let u be the leftmost vertex of the convex chain C determined by the edges of P of type II. Similarly, let v be the second leftmost vertex of C. Note that since the edge uv is of type II, the vertex v lies in a row R of $L(\alpha)$ above the row containing u and v is also to the right of u. Let v' be the point $(\alpha \cdot x(v), y(v))$, that is, the point of $L(\alpha)$ that is to the right of v on R. Then, by Lemma 9, all points of $L(\alpha)$ that lie above R and in the interior of the angle W spanned by the rays $\overline{uv}$ and $\overline{uv'}$ lie on at most r lines containing the origin.

Since P is empty in $L(\alpha)$, all vertices of C besides u, v, and possibly v' lie in W above R. Since all edges of C are of type II, every line determined by the origin and by a point of $L(\alpha)$ from the interior of W contains at most one vertex of C. Note that if v' is a vertex of C, then the only vertices of C are u, v, v'. Thus, in total C has at most $r + 2$ vertices and therefore at most $r + 1$ edges. □

We recall that, by symmetry, the same bound applies for edges of type IV and thus we get the following result.

Corollary 11. *The polygon P has at most $\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil + 1$ edges of type IV.* $\square$

Since each edge of P is of one of the types I–IV, it immediately follows from Lemmas 7, 8, 10, and from Corollary 11 that the number of edges of P is at most

$$3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1} \right) \right\rceil + 2 + 2 \left\lceil \log_\alpha \left(\frac{\alpha+1}{\alpha} \right) \right\rceil + 1 \leq 5 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1} \right) \right\rceil + 3,$$

as $\log_x \left(\frac{x}{x-1}\right) \geq \log_x \left(\frac{x+1}{x}\right)$ for every $x > 1$. In particular, this gives $h(2) \leq 8$ and $h\left(\frac{1+\sqrt{5}}{2}\right) \leq 13$. To obtain better bounds that are tight for $\alpha \geq \frac{1+\sqrt{5}}{2}$, we observe that not all types can appear simultaneously. To show this, we will use one last auxiliary result.

Let p and q be points lying on the same row R of $L(\alpha)$, each contained in an edge of P. We note that p and q do not need to be distinct and that they can also be interior points of an edge of P. Let L and L' be two lines containing p and q, respectively. If the slopes of L and L' are negative, then we call the part of the plane between L and L' below R a *slice of negative slope*; see part (a) of Figure 3 Analogously, *a slice of positive slope* is the part of the plane between L and L' above R if L and L' have positive slope.

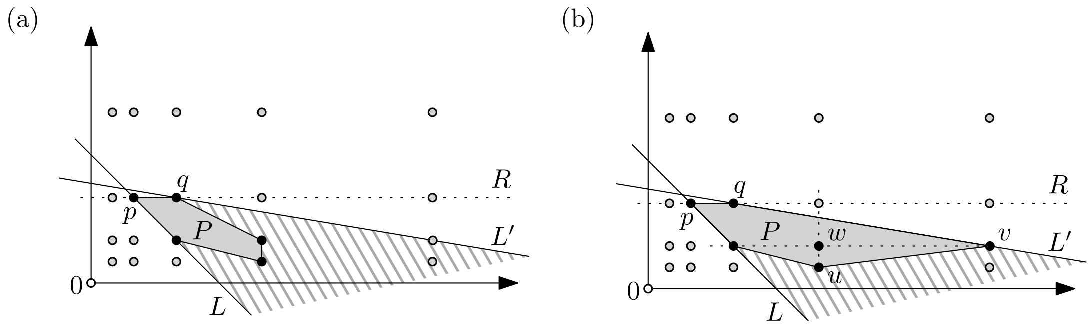

Figure 3: (a) An example of a slice of negative slope. The slice is denoted by dark gray stripes. (b) An illustration of the proof of Lemma 12 for negative slopes.

Lemma 12. *If the empty polygon P is contained in a slice of negative slope, then there is no non-vertical edge of P of type IV. Similarly, if P is contained in a slice of positive slope, then there is no edge of type I.*

Proof. It suffices to prove the statement for slices of negative slope as the proof of the statement for the positive slope is analogous. Suppose for contradiction that there is a non-vertical edge uv of type IV in a slice of negative slope determined by lines L and L' and points p and q as in the definition of a slice. Without loss of generality, we assume $x(u) < x(v)$.

Consider the point $w = (x(u), y(v))$ of $L(\alpha)$. Since uv is non-vertical, we have $w \notin \{u, v\}$. We claim that w is in the interior of P, contradicting the assumption that P is empty in $L(\alpha)$. Since uv is of type IV, the point u lies below the row containing w. However, since p is contained in an edge of P and P is in the slice, the boundary of P intersects this row to the left of w. Analogously, v is to the right of the column containing w and thus the boundary of P intersects this column above w. Then, however, w lies in the interior of P. □

Finally, we can now finish the proof of Theorem 2.

Proof of Theorem 2. First, we observe that if all vertices of P lie on two columns of $L(\alpha)$, then P can have at most four vertices. So we assume that this is not the case. Let u be the leftmost vertex of P with the highest y-coordinate among all leftmost vertices of P. Let e_1 and e_2 be the edges of P incident to u. We denote the other edge of P adjacent to e_1 as e and the other edge of P adjacent to e_2 as e'. We also use t_I, t_{II}, t_{III}, and t_{IV} to denote the number of edges of P of type I, II, III, and IV, respectively.

First, assume that e_1 is vertical. If e_2 is horizontal, then, since u is the top vertex of e_1 and P is not contained in two columns of $L(\alpha)$, the point $(\alpha \cdot x(u), y(u)/\alpha)$ of $L(\alpha)$ lies in the interior of P, which is impossible as P is empty in $L(\alpha)$.

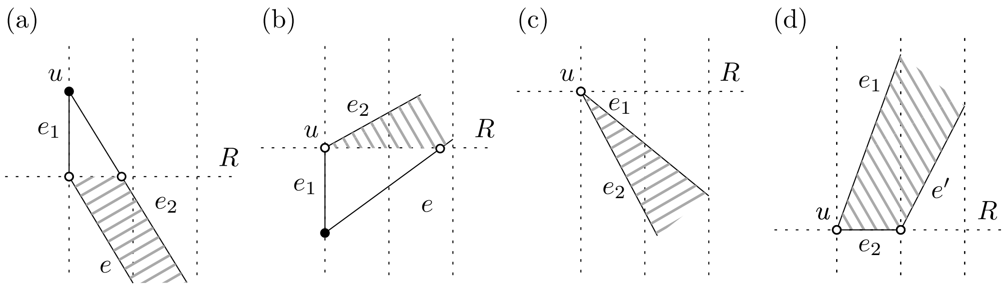

Figure 4: An illustration of the proof of Theorem 2. A slice is denoted by grey strips and its points p and q by empty circles. For example, in part (a), the slice is formed by the region between lines containing the edges e and e_2.

If e_1 is vertical and the slope of e_2 is negative, then there is no edge of type II. Thus, the edge e intersects the row R of $L(\alpha)$ containing the other vertex of e_1 and $\overline{e}$ has negative slope. Then, the part of P below R is contained in the slice of negative slope determined by $\overline{e_2}$ and $\overline{e}$; see part (a) of Figure 4. By Lemma 12, there is no non-vertical edge of type IV in P. By Lemmas 7 and 8, the total number of edges of P is thus at most

$$t_I + t_{III} + 1 \leq \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 2 \left\lceil \log_\alpha \left(\frac{\alpha+1}{\alpha} \right) \right\rceil + 2$$

for $\alpha \in (1, 2)$. Moreover, for $\alpha \geq 2$ this bound may be reduced by one.

If e_1 is vertical and the slope of e_2 is positive, then, since P is empty, there is no edge of type III besides e_1 as otherwise the point $(\alpha \cdot x(u), y(u))$ of $L(\alpha)$ is in the interior of P. The edge e intersects the row R of $L(\alpha)$ containing u and $\overline{e}$ has positive slope. Thus, the part of P above R is contained in the slice of positive slope determined by $\overline{e_2}$ and $\overline{e}$; see part (b) of Figure 4. By Lemma 12, there is no edge of type I in P. By Lemma 10 and Corollary 11, the total number of edges of P is then at most

$$t_{II} + 1 + t_{IV} \leq 2 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3.$$

In the rest of the proof, we can now assume that none of the edges e_1 and e_2 is vertical. We can label them so that the slope of e_1 is larger than the slope of e_2.

First, assume that the slope of e_1 is positive and the slope of e_2 is negative. Then, since the vertices of P do not lie on two columns of $L(\alpha)$, the point $(\alpha \cdot x(u), y(u))$ is contained in the interior of P, which is impossible as P is empty in $L(\alpha)$.

If the slopes of e_1 and e_2 are both non-positive, then there is no edge of type II besides the possibly horizontal edge e_1 as u is the leftmost vertex of P. By Lemma 12, there is also no non-vertical edge of type IV as P is contained in the slice of negative slopes determined by $\overline{e_1}$ and $\overline{e_2}$ or by $\overline{e}$ and $\overline{e_2}$ if e_1 is horizontal; see part (c) of Figure 4. Thus, by Lemmas 7 and 8, the number of edges of P is at most

$$t_I + 1 + t_{III} + 1 \leq \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 2 \left\lceil \log_\alpha \left(\frac{\alpha+1}{\alpha} \right) \right\rceil + 3$$

for $\alpha \in (1, 2)$. Moreover, for $\alpha \geq 2$ this bound may be reduced by one.

If the slopes of e_1 and e_2 are both non-negative, then there is no edge of type III besides the possibly horizontal edge e_2 (note that a vertical edge of type III would have u as its bottom vertex, which is impossible by the choice of u). Then, P is contained in the slice of positive slope determined by $\overline{e_1}$ and $\overline{e_2}$ or, if e_2 is horizontal, by $\overline{e_1}$ and $\overline{e'}$; see part (d) of Figure 4. Lemma 12 then implies that there is also no edge of type I. We thus have at most

$$t_{II} + 1 + t_{IV} \leq 2 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$$

edges of P by Lemma 10 and Corollary 11.

Altogether, the upper bound on the number of edges of P is

$$\max\left\{\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 2\left\lceil \log_\alpha\left(\frac{\alpha+1}{\alpha}\right)\right\rceil + 3,\ \ 2\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 3\right\}$$

for $\alpha \in (1,2)$. Moreover, the first term may be reduced by one for $\alpha \geq 2$. This becomes 5 for $\alpha \geq 2$, $h(\alpha) \leq 7$ for $\alpha \geq [\frac{1+\sqrt{5}}{2}, 2)$, and at most $3\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 3$ otherwise, since $\left\lceil \log_\alpha\left(\frac{\alpha+1}{\alpha}\right)\right\rceil \leq \left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil$ for every $\alpha \in (1, \frac{1+\sqrt{5}}{2})$. $\square$

4 Proof of Theorem 3

We prove the lower bounds on $h(\alpha)$ through the following three propositions.

Proposition 13. *For every $\alpha \geq 2$, we have $h(\alpha) \geq 5$.*

Proof. It is easy to check that $\mathrm{conv}\{(1,\alpha^2),(\alpha,\alpha),(\alpha^2,1),(\alpha^2,\alpha),(\alpha,\alpha^2)\}$ is an empty polygon in $L(\alpha)$ with 5 vertices for any α; see Figure 5. $\square$

Proposition 14. *For every $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$, we have $h(\alpha) \geq 7$.*

Proof. Let $k = k(\alpha)$ be a sufficiently large integer, and let

$$Q(\alpha) = \{(1,\alpha^k),(\alpha^{k-2},\alpha^{k-1}),(\alpha^{k-1},\alpha^{k-2}),(\alpha^k,1),(\alpha^k,\alpha),(\alpha^{k-1},\alpha^{k-1}),(\alpha,\alpha^k)\};$$

see Figure 5. We will show that $\mathrm{conv}(Q(\alpha))$ is an empty polygon in $L(\alpha)$ with 7 vertices.

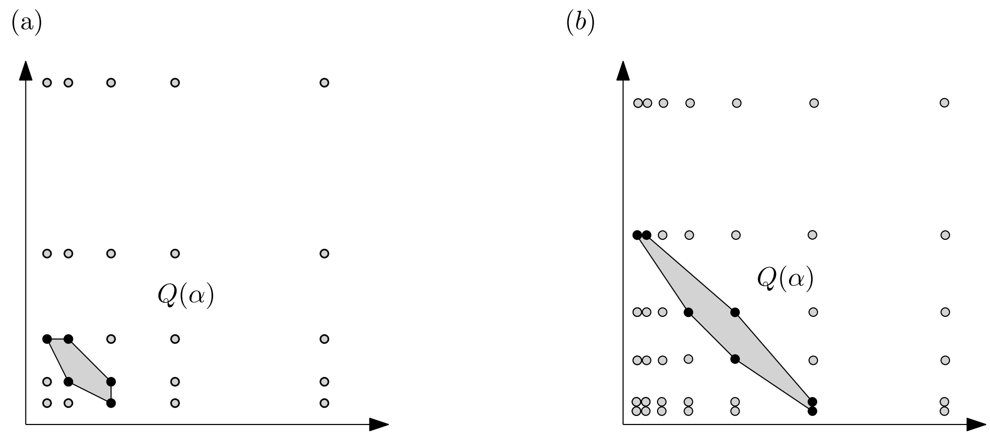

Figure 5: (a) An illustration of the proof of Proposition 13. (b) An illustration of the proof of Proposition 14.

First, we show that $Q(\alpha) \setminus \{(\alpha^{k-1}, \alpha^{k-1})\}$ is in convex position. For this, by symmetry, it is enough to check that $\{(\alpha^{k-1}, \alpha^{k-2}), (\alpha^k, 1), (1, \alpha^k)\}$ is oriented counterclockwise. This is the case exactly if $\alpha^{k-1} - \alpha^k + \alpha^{k-2} - 1 < 0$. By rearranging we get $\alpha^{k-2}(\alpha+1-\alpha^2) < 1$, which holds for any k, since $\alpha+1-\alpha^2 \leq 0$ as $\alpha \geq (1+\sqrt{5})/2$.

Now, to show that the set $Q(\alpha)$ is in convex position, it is sufficient to check that $\{(1, \alpha^k), (\alpha^k, \alpha), (\alpha^{k-1}, \alpha^{k-1})\}$ is oriented counterclockwise. This holds exactly if $\alpha^{k-1} - \alpha^k + \alpha^{k-1} - \alpha \geq 0$. By rearranging we get $2\alpha^{k-2}(2-\alpha) \geq 1$. Since $1 < \alpha < 2$, this holds if k is sufficiently large.

Thus, $\operatorname{conv}(Q(\alpha))$ has 7 vertices. To show that $\operatorname{conv}(Q(\alpha))$ is empty in $L(\alpha)$, we remark that points of the exponential lattice $L(\alpha)$ with both coordinates smaller than α^{k-1} are below the line through $(\alpha^{k-1}, \alpha^{k-2})$ and $(\alpha^{k-2}, \alpha^{k-1})$. Further, points with at least one coordinate larger than α^{k-1} are either above the line through $(1, \alpha^k)$ and (α, α^k) or to the right of the line through $(\alpha^k, 1)$ and (α^k, α). □

Proposition 15. *For every $\alpha > 1$, we have $h(\alpha) \geq \left\lfloor \sqrt{\frac{1}{\alpha-1}} \right\rfloor$.*

Proof. For a positive integer k, let $P(k) = \{(\alpha^i, \alpha^{k-i}) : 1 \leq i \leq k\}$. Since $P(k)$ is contained in the hyperbola $h = \{(x,y) \in \mathbb{R}^2 \colon x, y > 0, xy = \alpha^k\}$, the points of $P(k)$ are in convex position, and $\operatorname{conv}(P(k))$ has k vertices. We will show that if $k \leq \sqrt{\frac{1}{\alpha-1}}$, then $\operatorname{conv}(P(k))$ is empty.

For points (x,y) of $L(\alpha)$ above h, we have $xy \geq \alpha^{k+1}$. Further, points (x,y) of $L(\alpha)$ with $xy \geq \alpha^{k+2}$ are separated from h by the hyperbola $h' = \{(x,y) \in \mathbb{R}^2 \colon x, y > 0, xy = \alpha^{k+1}\}$. Thus, it is sufficient to check that h' is above the line ℓ connecting $(1, \alpha^k)$ with $(\alpha^k, 1)$. The closest point of h' to ℓ is $(\alpha^{(k+1)/2}, \alpha^{(k+1)/2})$, thus it is sufficient to check that this point is above ℓ. This holds if $2\alpha^{(k+1)/2} - \alpha^k - 1 \geq 0$ and we show that this inequality is satisfied for $k \leq \sqrt{\frac{1}{\alpha-1}}$.

Let $\alpha = 1 + s^2$ with some $s \in (0,1)$. In this notation, $k \leq 1/s$ and we need to prove that $2(1+s^2)^{(k+1)/2} \geq (1+s^2)^k + 1$. Since $(1+s^2)^{(k+1)/2} \geq 1 + s^2\frac{k+1}{2}$ by the Bernoulli inequality, and $(1+s^2)^k \leq e^{s^2 k}$, it is sufficient to prove the stronger inequality $2(1 + s^2\frac{k+1}{2}) \geq e^{s^2 k} + 1$. By taking the derivative of both sides with respect to k we have $s^2 \leq s^2 e^{s^2 k}$, thus it is enough to check the inequality when k is maximal. If $k = 1/s$, it is equivalent to $1 + s + s^2 \geq e^s$, which holds for $s \in (0,1)$ as $e^s = 1 + s + \sum_{n=2}^{\infty} \frac{s^n}{n!} \leq 1 + s + s^2 \sum_{n=2}^{\infty} \frac{1}{n!} = 1 + s + s^2(e-2) < 1 + s + s^2$. □

5 Proof of Proposition 5

Let us denote $\mathcal{F} = \{F_n \colon n \in \mathbb{N}_0\}^2$. For every positive integer k, we show that $h(\mathcal{F}) \geq k+1$. We will show that the points (F_{i+2}, F_i) with odd $i \in \{1, \ldots, 2k+1\}$ are vertices of an empty convex polygon P in $\mathcal{F}$.

First, we show that the points (F_{i+2}, F_i) with odd $i \in \{1, \dots, 2k+1\}$ are in convex position. To show that, it suffices to show that the slopes of lines determined by three consecutive such points are decreasing. That is, we want to prove

$$\frac{F_i - F_{i-2}}{F_{i+2} - F_i} > \frac{F_{i+2} - F_i}{F_{i+4} - F_{i+2}}$$

for every odd $i \in \{1, \dots, 2k-3\}$. Since $F_k = F_{k-1} + F_{k-2}$ for every $k \geq 2$, this inequality can be rewritten as

$$\frac{F_{i-1}}{F_{i+1}} > \frac{F_{i+1}}{F_{i+3}}.$$

Thus, we want to show that $F_{i-1} \cdot F_{i+3} > F_{i+1}^2$ for odd i. This is indeed true, as $F_{i-1} \cdot F_{i+3} - F_{i+1}^2 = (-1)^{i+1-2} F_2^2 > 0$ by the Catalan identity of the Fibonacci numbers.

To show that the polygon P is empty in $\mathcal{F}$, consider the line $L = \{(x, y) \in \mathbb{R}^2 \colon y = x/\varphi^2\}$. Any point (F_{i+2}, F_i) with odd i lies below L because

$$\frac{F_{i+2}}{\varphi^2} = \frac{1}{\varphi^2} \cdot \frac{\varphi^{i+3} - \psi^{i+3}}{\sqrt{5}} > \frac{\varphi^{i+1} - \psi^{i+1}}{\sqrt{5}} = F_i$$

since $\varphi^2 > \psi^2$ and $i+3$, $i+1$ are both even implying $\psi^{i+3}, \psi^{i+1} > 0$. Analogously, all points (F_{i+2}, F_i) with even i lie above L. For any i, every point (F_j, F_i) with $j \leq i+1$ lies above L, because $F_i \geq F_{j-1} > F_j/\varphi^2$. Each point (F_{i+2}, F_i) with odd i lies at vertical distance less than $1/2$ from L as

$$\frac{F_{i+2}}{\varphi^2} = \frac{1}{\varphi^2} \cdot \frac{\varphi^{i+3} - \psi^{i+3}}{\sqrt{5}} = \frac{\varphi^{i+1} - \psi^{i+1}}{\sqrt{5}} + \frac{\varphi^2 \psi^{i+1} - \psi^{i+3}}{\varphi^2 \sqrt{5}} \leq F_i + \frac{\varphi^2 \psi^2 - \psi^4}{\sqrt{5}}$$
$$< F_i + \frac{1}{2}.$$

Any point (F_{i+2}, F_j) with $j \leq i-1$ lies below L at vertical distance at least $1/2$ since the distance is either at least $F_i - F_j \geq 1$ if i is odd or it is at least $F_i - F_j - \frac{1}{2} \geq \frac{1}{2}$ if i is even. Thus the only points of $\mathcal{F}$ lying between the parallel lines $y = x/\varphi^2 - 1/2$ and L are the points (F_{i+2}, F_i) with i odd. It follows that P is an empty convex polygon in $\mathcal{F}$ and $h(\mathcal{F}) \geq k+1$.

6 Proof of Theorem 6

Let $\alpha, \beta > 1$ be two real numbers. We prove that $h(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is a rational number.

6.1 Finite upper bound

First, assume that $\log_\alpha(\beta) \in \mathbb{Q}$. We will use Theorem 2 to show that the number $h(L(\alpha, \beta))$ is finite. Since $\log_\alpha(\beta) \in \mathbb{Q}$ and $\alpha, \beta > 1$, there are positive integers p and q such that $\beta = \alpha^{p/q}$. Suppose for contradiction that there is an empty polygon P in $L(\alpha, \beta)$ with at least $pq \cdot h(\alpha^p) + 1$ vertices. Note that this number of vertices is finite by Theorem 2. For $k \in \{0, \ldots, q-1\}$, we call a row of $L(\alpha, \beta)$ *congruent to* k if it is of the form $\{\alpha^n : n \in \mathbb{N}_0\} \times \beta^m$ for some integer m congruent to k modulo q. Analogously, a column of $L(\alpha, \beta)$ is *congruent to* $\ell \in \{0, \ldots, p-1\}$ if it is of the form $\alpha^m \times \{\beta^n : n \in \mathbb{N}_0\}$ for some m congruent to ℓ modulo p.

Now, since P contains at least $pq \cdot h(\alpha^p) + 1$ vertices, the pigeonhole principle implies that there are integers $k \in \{0, \ldots, q-1\}$ and $\ell \in \{0, \ldots, p-1\}$ such that at least $h(\alpha^p) + 1$ vertices of P that all lie in rows congruent to k and in columns congruent to ℓ. Let P' be the convex polygon that is spanned by these vertices. We claim that the polygon P' is not empty in $L(\alpha, \beta)$. Since $P' \subseteq P$, we get that P is also not empty in $L(\alpha, \beta)$, which contradicts our assumption about P.

To show that P' is not empty in $L(\alpha, \beta)$, consider the subset L of $L(\alpha, \beta)$ that contains only points of $L(\alpha, \beta)$ that lie in rows congruent to k and in columns congruent to ℓ. Clearly, vertices of P' lie in L and L is an affine image of $L(\alpha^p)$, which is scaled by the factors α^ℓ and $\beta^k = \alpha^{kp/q}$ in the x- and y-direction, respectively. Since affine mappings preserve incidences and P' has at least $h(\alpha^p) + 1$ vertices, it follows that P' is not empty in L. Since $L \subseteq L(\alpha, \beta)$, P' is not empty in $L(\alpha, \beta)$ either.

6.2 Finite lower bound

Let $\log_\alpha(\beta) \in \mathbb{Q}$ and $\beta = \alpha^{p/q}$ for some relative prime positive integers p and q. Observe that in this case $L(\alpha, \beta) \subset L(\alpha^{1/q})$. Thus, if an empty polygon in $L(\alpha^{1/q})$ is a subset of $L(\alpha, \beta)$, then it is an empty polygon in $L(\alpha, \beta)$.

Let $k = \left\lfloor \sqrt{1/(\alpha^{1/q} - 1)} \right\rfloor$ and consider the set $P = \{(\alpha^{i/q}, \alpha^{(k-i)/q}) : 1 \leq i \leq k\}$. It is an empty polygon in $L(\alpha^{1/q})$, as it is shown in the proof of Proposition 15. Since its subset $P' = \{(\alpha^{i/q}, \alpha^{(k-i)/q}) : 1 \leq i \leq k \text{ with } q|i \text{ and } p|k-i\}$ is a subset of $L(\alpha, \beta)$ and an empty polygon in $L(\alpha^{1/q})$, it is an empty polygon in $L(\alpha, \beta)$ with $\lfloor k/pq \rfloor$ vertices.

6.3 Infinite lower bound

Now, assume that $\log_\alpha(\beta) \notin \mathbb{Q}$. We will find a subset of $L(\alpha, \beta)$ forming empty convex polygon in $L(\alpha, \beta)$ with arbitrarily many vertices. To do so, we use the theory of continued fractions, so we first introduce some definitions and notation.

6.3.1 Continued fractions

Here, we recall mostly basic facts about continued fractions, which we use in the proof. Most of the results that we state can be found, for example, in the book by Khinchin [14].

For a positive real number r, the *(simple) continued fraction of* r is an expression of the form

$$r = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where $a_0 \in \mathbb{N}_0$ and $a_1, a_2, \ldots$ are positive integers. The simple continued fraction of r can be written in a compact notation as

$$[a_0; a_1, a_2, a_3, \ldots].$$

For every $n \in \mathbb{N}_0$, if we denote $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$ and set $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$, then the numbers p_n and q_n satisfy the recurrence

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \tag{1}$$

for each $n \in \mathbb{N}$. Observe that if r is irrational, then its continued fraction has infinitely many coefficients. Also, it follows from (1) that $\frac{p_n}{q_n} < r$ for n even and $\frac{p_n}{q_n} > r$ for n odd.

For example, if $r = \log_2(3)$, we get the continued fraction $[1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]$ and the sequence $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}, \frac{485}{306}, \ldots\right)$. For $r = \frac{1+\sqrt{5}}{2}$, we have $[1; 1, 1, 1, \ldots]$ and $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots\right)$.

We will call the fractions $\frac{p_n}{q_n}$ the *convergents* of r. A *semi-convergent* of r is a number $\frac{p_{n-1}+ip_n}{q_{n-1}+iq_n}$ where $i \in \{0, 1, \ldots, a_{n+1}\}$. Note that each convergent of r is also a semi-convergent of r. The names are motivated by the use of convergents and semi-convergents as rational approximations of an irrational number r.

A rational number $\frac{p}{q}$ is a *best approximation* of an irrational number r, if any fraction $\frac{p'}{q'} \neq \frac{p}{q}$ with $q' < q$ satisfies

$$\left| q' \left(r - \frac{p'}{q'} \right) \right| > \left| q \left(r - \frac{p}{q} \right) \right|.$$

A rational number $\frac{p}{q}$ is a *best lower approximation of* r if

$$q' \left(r - \frac{p'}{q'} \right) > q \left(r - \frac{p}{q} \right) \geq 0$$

for all rational numbers $\frac{p'}{q'}$ with $\frac{p'}{q'} \leq r$, $\frac{p}{q} \neq \frac{p'}{q'}$, and $0 < q' \leq q$. Similarly, $\frac{p}{q}$ is a *best upper approximation of* r if

$$q'\left(r - \frac{p'}{q'}\right) < q\left(r - \frac{p}{q}\right) \leq 0$$

for all rational numbers $\frac{p'}{q'}$ with $\frac{p'}{q'} \geq r$, $\frac{p}{q} \neq \frac{p'}{q'}$, and $0 < q' \leq q$.

It is a well known fact that convergents are best approximations of r [14]. The following lemma about best lower and best upper approximations is a recent result of Hančl and Turek [10]. Our definitions of best lower or upper approximations correspond to their definitions of best lower or upper approximations *of the second kind*. The lemma follows from Theorem 4.5 of [10].

Lemma 16 ([10])**.** *Let r be a real number with $r = [a_0; a_1, a_2, \dots]$ and let $\frac{p_n}{q_n}$ be the nth convergent of r for each $n \in \mathbb{N}_0$. Then, the following two statements hold.*

1. *The set of best lower approximations of r consists of semi-convergents $\frac{p_{n-1}+ip_n}{q_{n-1}+iq_n}$ of r with n odd and $0 \leq i < a_{n+1}$.*

2. *The set of best upper approximations of r consists of semi-convergents $\frac{p_{n-1}+ip_n}{q_{n-1}+iq_n}$ of r with n even and $0 \leq i < a_{n+1}$, except for the pair $(n, i) = (0, 0)$.*

Finally, a real number r is *restricted* if there is a positive integer M such that all the partial denominators a_i from the continued fraction of r are at most M. The restricted numbers are exactly those numbers r that are badly approximable by rationals [14], that is, there is a constant $c > 0$ such that for every $\frac{p}{q} \in \mathbb{Q}$ we have $\left|r - \frac{p}{q}\right| > \frac{c}{q^2}$.

We divide the rest of the proof of Theorem 6 into two cases, depending on whether $\log_\alpha(\beta)$ is restricted or not.

6.3.2 Unrestricted case

First, we assume that $\log_\alpha(\beta)$ is not restricted. Let $[a_0; a_1, a_2, a_3, \dots]$ be the continued fraction of $\log_\alpha(\beta)$ with $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ for every $n \in \mathbb{N}_0$. Then, for every positive integer m, there is a positive integer $n(m)$ such that $a_{n(m)+1} \geq m$. We use this assumption to construct, for every positive integer m, a convex polygon with at least m vertices from $L(\alpha, \beta)$ that is empty in $L(\alpha, \beta)$.

For a given m, consider the integer $n(m)$ and let W be the set of points

$$w_i = \left(\alpha^{p_{n(m)-1}+ip_{n(m)}}, \beta^{q_{n(m)-1}+iq_{n(m)}}\right)$$

where $i \in \{0, 1, \dots, a_{n(m)+1}\}$. That is, we consider points where the exponents form semi-convergents $\frac{p_{n(m)-1}+ip_{n(m)}}{q_{n(m)-1}+iq_{n(m)}}$ to $\log_\alpha(\beta)$. We abbreviate $p_{n,i} = p_{n(m)-1} + ip_{n(m)}$

and $q_{n,i} = q_{n(m)-1} + iq_{n(m)}$. Observe that $|W| \geq m$. We will show that W is the vertex set of an empty convex polygon in $L(\alpha, \beta)$. To do so, we assume without loss of generality that $n(m)$ is even so that $\frac{\beta^{q_{n(m)}}}{\alpha^{p_{n(m)}}} > 1$. The other case when $n(m)$ is odd is analogous.

First, we show that W is in convex position. In fact, we prove that all triples $(w_{i_1}, w_{i_2}, w_{i_3})$ with $i_1 < i_2 < i_3$ are oriented counterclockwise. It suffices to show this for every triple (w_i, w_{i+1}, w_{i+2}). To do so, we need to prove the inequality

$$\frac{y(w_{i+2}) - y(w_{i+1})}{x(w_{i+2}) - x(w_{i+1})} = \frac{\beta^{q_{n,i+2}} - \beta^{q_{n,i+1}}}{\alpha^{p_{n,i+2}} - \alpha^{p_{n,i+1}}} > \frac{\beta^{q_{n,i+1}} - \beta^{q_{n,i}}}{\alpha^{p_{n,i+1}} - \alpha^{p_{n,i}}} = \frac{y(w_{i+1}) - y(w_i)}{x(w_{i+1}) - x(w_i)}.$$

After dividing by $\frac{\beta^{q_{n(m)-1}}}{\alpha^{p_{n(m)-1}}}$, this can be written as

$$\frac{\beta^{(i+2)q_{n(m)}} - \beta^{(i+1)q_{n(m)}}}{\alpha^{(i+2)p_{n(m)}} - \alpha^{(i+1)p_{n(m)}}} > \frac{\beta^{(i+1)q_{n(m)}} - \beta^{iq^{n(m)}}}{\alpha^{(i+1)p_{n(m)}} - \alpha^{ip_{n(m)}}}.$$

If divide both sides by $\frac{\beta^{(i+1)q_{n(m)}} - \beta^{iq_{n(m)}}}{\alpha^{(i+1)p_{n(m)}} - \alpha^{ip_{n(m)}}}$, then the above inequality becomes

$$\frac{\beta^{q_{n(m)}}}{\alpha^{p_{n(m)}}} > 1.$$

This is true as $n(m)$ is even.

It remains to prove that the polygon Q with the vertex set W is empty in $L(\alpha, \beta)$. Suppose for contradiction that there is a point (α^p, β^q) of $L(\alpha, \beta)$ lying in the interior of Q. Let i be the minimum positive integer from $\{1, \ldots, a_{n(m)+1}\}$ such that $q < q_{n,i}$. Such an i exists as (α^p, β^q) is in the interior of Q. We then have $q_{n,i-1} < q < q_{n,i}$. Since (α^p, β^q) is in the interior of Q and W lies below the line $x = y$, we have $\frac{p}{q} > \log_\alpha(\beta)$. So it is enough to prove that (α^p, β^q) does not lie above the line $\overline{w_{i-1}w_i}$.

We have $p_{n,i} - \log_\alpha(\beta)q_{n,i} < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1}$ as $\frac{p_{n,i}}{q_{n,i}}$ is a best upper approximation of $\log_\alpha(\beta)$ and $q_{n,i-1} < q_{n,i}$. This implies $\frac{\beta^{q_{n,i-1}}}{\alpha^{p_{n,i-1}}} < \frac{\beta^{q_{n,i}}}{\alpha^{p_{n,i}}}$, or equivalently that w_i lies above the line determined by w_{i-1} and the origin.

Now if (α^p, β^q) lies above the line $\overline{w_{i-1}w_i}$, then it also lies above the line determined by w_{i-1} and the origin. Thus, $\frac{\beta^{q_{n,i-1}}}{\alpha^{p_{n,i-1}}} < \frac{\beta^q}{\alpha^p}$, implying

$$p - \log_\alpha(\beta)q < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1},$$

which means that $\frac{p}{q}$ is a better upper approximation of $\log_\alpha(\beta)$ than $\frac{p_{n,i-1}}{q_{n,i-1}}$. Thus, there exists a best upper approximation $\frac{p^*}{q^*}$ of $\log_\alpha(\beta)$ with $q_{n,i-1} < q^* < q_{n,i}$. This contradicts part 2 of Lemma 16 as $\frac{p^*}{q^*}$ is not a semi-convergent of $\log_\alpha(\beta)$.

6.3.3 Restricted case

Now, assume that the number $\log_\alpha(\beta)$ is restricted. Let $[a_0; a_1, a_2, a_3, \dots]$ be the continued fraction of $\log_\alpha(\beta)$ with $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ for every $n \in \mathbb{N}_0$. Let $M = M(\alpha, \beta)$ be a number satisfying

$$a_n \leq M \tag{2}$$

for every $n \in \mathbb{N}_0$ and let $c = c(\alpha, \beta) > 0$ be a constant such that

$$\left|\log_\alpha(\beta) - \frac{p}{q}\right| > \frac{c}{q^2} \tag{3}$$

holds for every $\frac{p}{q} \in \mathbb{Q}$. Recall that $\frac{\alpha^{p_n}}{\beta^{q_n}} < 1$ for even n and $\frac{\alpha^{p_n}}{\beta^{q_n}} > 1$ for odd n. Note also that the sequence $\left(\frac{\alpha^{p_n}}{\beta^{q_n}}\right)_{n \in \mathbb{N}_0}$ converges to 1 as $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0}$ converges to $\log_\alpha(\beta)$. Moreover, the terms of $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0}$ with odd indices form a decreasing subsequence and the terms with even indices determine an increasing subsequence.

Let $n_0 = n_0(\alpha, \beta)$ be a sufficiently large positive integer and let V be the set of points $v_n = (\alpha^{p_n}, \beta^{q_n})$ for every odd $n \geq n_0$. Note that V is a subset of $L(\alpha, \beta)$.

We first show that V is in convex position. In fact, we prove a stronger claim by showing that the orientation of every triple $(v_{n_1}, v_{n_2}, v_{n_3})$ with $n_1 < n_2 < n_3$ is counterclockwise. It suffices to show this for every triple (v_{n-4}, v_{n-2}, v_n). To do so, we prove that the slopes of the lines determined by consecutive points of V are increasing, that is,

$$\frac{y(v_n) - y(v_{n-2})}{x(v_n) - x(v_{n-2})} = \frac{\beta^{q_n} - \beta^{q_{n-2}}}{\alpha^{p_n} - \alpha^{p_{n-2}}} > \frac{\beta^{q_{n-2}} - \beta^{q_{n-4}}}{\alpha^{p_{n-2}} - \alpha^{p_{n-4}}} = \frac{y(v_{n-2}) - y(v_{n-4})}{x(v_{n-2}) - x(v_{n-4})}$$

for every even $n \geq n_0$. By dividing both sides of the inequality with $\frac{\beta^{q_{n-2}}}{\alpha^{p_{n-2}}}$, we rewrite this expression as

$$\frac{\beta^{q_n - q_{n-2}} - 1}{\alpha^{p_n - p_{n-2}} - 1} > \frac{1 - \beta^{q_{n-4} - q_{n-2}}}{1 - \alpha^{p_{n-4} - p_{n-2}}}.$$

Using (1), this is the same as

$$\frac{\beta^{a_n q_{n-1}} - 1}{\alpha^{a_n p_{n-1}} - 1} > \frac{1 - \beta^{-a_{n-2} q_{n-3}}}{1 - \alpha^{-a_{n-2} p_{n-3}}}.$$

The above inequality can be rewritten as

$$(\beta^{a_n q_{n-1}} - 1)(1 - \alpha^{-a_{n-2} p_{n-3}}) > (\alpha^{a_n p_{n-1}} - 1)(1 - \beta^{-a_{n-2} q_{n-3}}),$$

where $\beta^{q_{n-1}} > \alpha^{p_{n-1}} > 1$ as $n-1$ is even. Therefore, if the above inequality holds for $a_n = 1$, then it holds for any a_n as this number is always at least 1. Thus, it suffices to show

$$(\beta^{q_{n-1}} - 1)(1 - \alpha^{-a_{n-2}p_{n-3}}) > (\alpha^{p_{n-1}} - 1)(1 - \beta^{-a_{n-2}q_{n-3}}). \tag{4}$$

We prove this using the following simple auxiliary lemma.

Lemma 17. *Consider the function $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by $f(x,y) = (x-1)(1-1/y)$. Let $x, y, x', y' > 1$ be real numbers such that $1 - \frac{1}{y} - \frac{x}{x'} > 0$. Then, $f(x', y) > f(x, y')$.*

Proof. We have

$$\begin{aligned} f(x',y) - f(x,y') &= (x'-1)\left(1 - \frac{1}{y}\right) - (x-1)\left(1 - \frac{1}{y'}\right) \\ &= x' - \frac{x'-1}{y} - x + \frac{x-1}{y'} > x' - \frac{x'}{y} - x = x'\left(1 - \frac{1}{y} - \frac{x}{x'}\right) > 0, \end{aligned}$$

where the last inequality follows from $1 - \frac{1}{y} - \frac{x}{x'} > 0$. $\square$

Now, by choosing $x = \alpha^{p_{n-1}}$, $x' = \beta^{q_{n-1}}$, $y = \alpha^{a_{n-2}p_{n-3}}$, and $y' = \beta^{a_{n-2}q_{n-3}}$, the inequality (4) becomes $f(x', y) > f(x, y')$. In order to prove it, we just need to verify the assumptions of Lemma 17. We clearly have $x, x', y, y' > 1$. It now suffices to show $1 - \frac{1}{y} - \frac{x}{x'} > 0$. By (3), we obtain that $q_{n-1}\log_\alpha(\beta) - p_{n-1} \geq c/q_{n-1}$, thus

$$\frac{x}{x'} = \frac{\alpha^{p_{n-1}}}{\beta^{q_{n-1}}} \leq \alpha^{-c/q_{n-1}}.$$

Now, to bound q_{n-1} in terms of p_{n-3}, equation (1) gives

$$\begin{aligned} q_{n-1} &= a_{n-1}q_{n-2} + q_{n-3} \leq (M+1)q_{n-2} = (M+1)(a_{n-2}q_{n-3} + q_{n-4}) \\ &\leq (M+1)^2 q_{n-3} \leq 2\log_\beta(\alpha)(M+1)^2 p_{n-3}, \end{aligned}$$

where we used (2) and $q_{n-4} \leq q_{n-3} \leq q_{n-2}$, $q_{n-3} \leq 2\log_\beta(\alpha)p_{n-3}$ for n large enough. It follows that $q_{n-1} \leq M'p_{n-3}$ for a suitable constant $M' = M'(\alpha, \beta) > 0$. Thus,

$$1 - \frac{1}{y} - \frac{x}{x'} \geq 1 - \alpha^{-a_{n-2}p_{n-3}} - \alpha^{-c/q_{n-1}} \geq 1 - \alpha^{-a_{n-2}p_{n-3}} - \alpha^{-c/(M'p_{n-3})},$$

which is at least

$$\frac{c\ln\alpha}{2M'p_{n-3}} - \frac{1}{\alpha^{a_{n-2}p_{n-3}}}$$

as $1-c\ln\alpha/(2M'p_{n-3}) \geq e^{-2c\ln\alpha/(2M'p_{n-3})} = \alpha^{-c/(M'p_{n-3})}$ if $0 < c\ln\alpha/(2M'p_{n-3}) < 1/2$. The last expression is positive if $n \geq n_0$ and n_0 is sufficiently large so that p_{n-3} is large enough.

It remains to show that the convex polygon P with the vertex set V is empty in $L(\alpha, \beta)$. We proceed analogously as in the unrestricted case. Suppose for contradiction that there is a point (α^p, β^q) of $L(\alpha, \beta)$ lying in the interior of P. Then, let $v_n = (\alpha^{p_n}, \beta^{q_n})$ be the lowest vertex of P that has (α^p, β^q) below. Such a vertex v_n exists, as V contains points with arbitrarily large y-coordinate. By the choice of v_n, we obtain $q_{n-2} < q < q_n$. Since (α^p, β^q) is in the interior of P and V lies below the line $x = y$, we have $\frac{p}{q} > \log_\alpha(\beta) > \frac{p_{n-1}}{q_{n-1}}$. Moreover, since all triples from V are oriented counterclockwise, the point (α^p, β^q) lies above the line $\overline{v_{n-2}v_n}$.

Let

$$w_i = (\alpha^{p_{n-2}+ip_{n-1}}, \beta^{q_{n-2}+iq_{n-1}})$$

where $i \in \{0, 1, \ldots, a_n\}$ similarly as in the proof of the unrestricted case. There, it was shown that all the triples w_{i-1}, w_i, w_{i+1} are oriented counterclockwise, thus all the points w_i with $i \in \{1, \ldots, a_n - 1\}$ lie below the line $\overline{v_{n-2}v_n}$. Thus, if (α^p, β^q) lies above the segment connecting v_{n-2} and v_n, then there is an i such that (α^p, β^q) lies above the segment connecting w_{i-1} and w_i. As in the last two paragraphs of the proof of the unrestricted case, the position of (α^p, β^q) implies the inequality $p - \log_\alpha(\beta)q < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1}$, and the contradiction follows from part 2 of Lemma 16, as there can be no best upper approximation of $\log_\alpha(\beta)$ which is not a semi-convergent of $\log_\alpha(\beta)$.

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