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Improved upper bound on the Frank number of 3-edge-connected graphs

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ABSTRACT

In an orientation O of the graph G, an arc e is deletable if and only if $O - e$ is strongly connected. For a 3-edge-connected graph G, the Frank number is the minimum k for which G admits k strongly connected orientations such that for every edge e of G the corresponding arc is deletable in at least one of the k orientations. Hörsch and Szigeti conjectured the Frank number is at most 3 for every 3-edge-connected graph G. We prove an upper bound of 5, which improves the previous bound of 7.

1. Introduction

The graphs in this paper are finite and without loops or multiple edges. We recommend the excellent book by Bondy and Murty [2] for the concepts and notations used here.

A graph G is defined by its vertex set V and edge set E. An *orientation* of G is a directed graph $D = (V, A)$ such that each edge $uv \in E$ is replaced by exactly one of the arcs (u, v) or (v, u). A *circuit*

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is a directed cycle. A graph is *cubic* if every vertex has degree 3. A graph is *k-edge-connected* if and only if the removal of any $k-1$ edges leaves a connected graph. The *edge-connectivity* of a graph G, denoted by $\kappa'(G)$, is the smallest k for which the graph is k-edge-connected. A 2-edge-connected graph is often called *bridgeless*.

A directed graph is *strongly connected* if and only if selecting two arbitrary vertices x and y, there is a directed (x, y)-path. An orientation of G is *k-arc-connected* if and only if the removal of any $k-1$ arcs leaves a strongly connected directed graph.

Theorem 1.1 (*Robbins*). *A graph has a strongly connected orientation if and only if it is* 2*-edge-connected.*

The following theorem is a fundamental result in the theory of directed graphs [6].

Theorem 1.2 (*Nash-Williams*). *A graph has a k-arc-connected orientation if and only if it is* 2*k-edge-connected.*

This theorem has the following consequence: If we fix a 2-arc connected orientation of a 4-edge-connected graph, then any arc can be removed and the remaining digraph is still strongly connected. This situation changes for 3-edge-connected graphs and their orientations. This motivated András Frank to raise some questions on 3-edge-connected graphs and their orientations. These concepts and Frank's question appeared first in the paper by Hörsch and Szigeti [5].

In an orientation O of G, the arc e is *deletable* if and only if $O-e$ is strongly connected. For a 3-edge-connected graph G, the *Frank number*, denoted by $F(G)$, is the minimum k for which G admits k strongly connected orientations such that for every edge e of G the corresponding arc is deletable in at least one of the k orientations. Why 3-edge-connected graphs? Suppose G has a cut of size at most 2. In any orientation of G, the removal of any of these edges results in either a directed cut or a directed graph that is not even connected. Hence no set of orientations can satisfy the conditions. On the other hand, if G is 4-edge-connected, then G admits a 2-arc-connected orientation by Theorem 1.2. This orientation yields that $F(G) = 1$ by definition. Consequently, $F(G) = 1$ if and only if G is 4-edge-connected. Thus the problem is interesting only if the edge-connectivity of G is 3, i.e. $\kappa'(G) = 3$. In the sequel, we consider only graphs with edge-connectivity 3.

Hörsch and Szigeti [5] showed that any 3-edge-connected graph G satisfies $F(G) \leq 7$. Prior to that, DeVos et al. proved a more general result with a weaker bound [3]: For every 3-edge-connected graph G, there exists a partition of $E(G)$ into at most nine sets $\{X_1, X_2, \ldots, X_m\}$ so that $G \setminus X_i$ is 2-edge-connected for every $1 \leq i \leq m$. Our main result improves the best known upper bound on the Frank number.

Theorem 1.3. *For every* 3*-edge-connected graph the Frank number is at most* 5.

Independently, Goedgebeur et al. [4] proved for every 3-edge-connected graph the Frank number is at most 4.

The paper is organized as follows. In the second section, we introduce the main tools and results, which we use in our proof. The third section is dedicated to the proof of our main result. We conclude by discussing the limits of our proof technique.

2. Preliminaries

For some integer k, a *k-flow* (o, v) on a graph G consists of an orientation o of the edges of G and a valuation $v : E(G) \mapsto \{0, \pm 1, \pm 2, \ldots, \pm(k-1)\}$ such that at every vertex the sum of the values on incoming edges equals the sum on the outgoing edges. A k-flow (o, v) is *nowhere-zero* if the value of v is not 0 for any edge of G. A nowhere-zero k-flow on G is *all-positive* if the value $v(e)$ is positive for every edge e of G. Every nowhere-zero k-flow can be transformed to an all-positive nowhere-zero k-flow by changing the orientation of the edges with negative $v(e)$ and changing negative values of $v(e)$ to $-v(e)$. Inspired by their ideas and approach, we use the following result by Goedgebeur et al. [4].

Lemma 2.1. *Let G be a 3-edge-connected graph, and let (o, v) be an all-positive nowhere-zero k-flow on G. Any edge of G, which receives value 1 in (o, v) is deletable in o.*

Indeed, an orientation arising from a flow is always strongly connected, and the removal of any arc of value 1 cannot create a directed cut since the flow is nowhere-zero. Using a slightly stronger Lemma, Goedgebeur et al. [4] proved the following result, which we state without proof.

Theorem 2.2. *If a graph G admits a nowhere-zero 4-flow, then $F(G) \leq 2$.*

It is well-known that every 3-edge-colorable cubic graph admits a nowhere-zero 4-flow. Tutte posed the following stronger claim, which inspired a vast amount of research.

Conjecture 2.3 (*Tutte's 4-Flow Conjecture*)**.** *Every bridgeless graph without a Petersen-minor has a nowhere-zero 4-flow.*

Everyone believes the validity of this conjecture. This partly explains why the only known examples of graphs with Frank number 3 are created from the Petersen graph using certain operations [1]. Each of the constructed graphs contains the Petersen graph as a minor.

Since we consider 3-edge-connected graphs only, we can use the following result by Jaeger [7], which was later improved by Seymour [9].

Theorem 2.4 (*Jaeger*)**.** *Every bridgeless graph has a nowhere-zero 8-flow.*

Theorem 2.5 (*Seymour*)**.** *Every bridgeless graph has a nowhere-zero 6-flow.*

Let H denote an Abelian group. An *H-flow* on an oriented graph D is an assignment of values of H to the arcs of D such that for each vertex v, the sum of the values on the incoming arcs is the same as the sum of the values on the outgoing arcs. For a graph G, an H-flow is defined using any orientation D of G since H is Abelian. A *nowhere-zero H*-flow on G is an H-flow, where $0 \in H$ is not assigned to any edge. The following is a useful corollary of a theorem by Tutte [8,10]:

Fact 2.6. *If H and H' are two finite Abelian groups of the same order, then the graph G has an H-flow if and only if G has an H'-flow.*

In what follows, we particularly use a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a 3-edge-connected graph G. We use 0/1 vectors with three coordinates to denote elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The flow condition at a vertex v implies the following property: in each coordinate, the sum of values on edges incident to v is 0. Hence for a fixed vertex v, in each coordinate, there exist an even number of edges incident to v that have value 1.

3. Improvement of the upper bound

Surprisingly, the weaker flow result of the two, Theorem 2.4 of Jaeger is the one, that is useful for our purposes. The main idea of the proof is the following. We fix a nowhere-zero 8-flow of a 3-edge-connected graph G, which exists by Theorem 2.4. We create five other nowhere-zero k-flows of G in such a way that we control the set of edges with value 1, and we apply Lemma 2.1. Since every edge of G receives value 1 in at least one of the flows, we are done. Let us recall our main theorem.

Theorem 1.3 For every 3-edge-connected graph the Frank number is at most 5.

Proof. Combining Theorem 2.4 and Fact 2.6, we consider a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow on G. For $i \in \{1, 2, 3\}$, let G_i denote the subgraph of G induced by those edges of E, which have value 1 at the ith coordinate. Note that these subgraphs might have some edges in common, more precisely the number of nonzero coordinates of each edge is the same as the number of subgraphs to which it belongs. By the nowhere-zero property, every edge is contained in at least one of these subgraphs.

By the flow condition, the subgraphs G_1, G_2, G_3 are Eulerian. We can think of an Eulerian trail as a directed graph. Thus we can partition each edge set $E(G_i)$ into edge-disjoint circuits. We may use

Table 1
The possible orientations and values of e in (O_1, f_1).

Value of e in the original $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow		Final orientation and value of e in (O_1, f_1)	
$(1, 0, 0)$	row 1	same as in o_1, 1	
$(0, 1, 0)$	row 2	same as in o_2, 2	
$(0, 0, 1)$	row 3	same as in o_3, 4	
$(1, 1, 0)$	row 4	same as in o_2, 3	if e has the same orientation in o_1 and o_2,
	row 5	same as in o_2, 1	if e has different orientations in o_1 and o_2.
$(1, 0, 1)$	row 6	same as in o_3, 5	if e has the same orientation in o_1 and o_3,
	row 7	same as in o_3, 3	if e has different orientations in o_1 and o_3.
$(0, 1, 1)$	row 8	same as in o_3, 6	if e has the same orientation in o_2 and o_3,
	row 9	same as in o_3, 2	if e has different orientations in o_2 and o_3.
$(1, 1, 1)$	row 10	same as in o_3, 7	if e has the same orientation in all three of o_1, o_2, o_3,
	row 11	same as in o_3, 5	if the orientation of e in o_1 is different from that in o_2 and o_3,
	row 12	same as in o_3, 3	if the orientation of e in o_2 is different from that in o_1 and o_3,
	row 13	same as in o_3, 1	if the orientation of e in o_3 is different from that in o_1 and o_2.

a vertex in different Eulerian trails. For $i = 1, 2, 3$, let us fix an orientation o_i defined by the edge-disjoint circuits of $E(G_i)$. At this point, it might occur that an edge of G has different orientations in different subgraphs G_i. The next aim is to select, for each $i \in \{1, 2, 3\}$, an appropriate positive value v_i to send along o_i.

Let us emphasize that after fixing the pair (o_i, v_i) for each G_i, we define another flow (O_1, f_1) in the next step. The orientation and value of an arc in (O_1, f_1) is determined by the superposition of the chosen (o_i, v_i) pairs or triples. We always assign a positive value. If the orientations go opposite, then we let the largest value determine the direction and subtract the values going in opposite direction.

Let us construct an all-positive nowhere-zero 8-flow (O_1, f_1) by sending values $v_1 = 1$, $v_2 = 2$ and $v_3 = 4$ along the fixed orientations o_i of G_i, respectively. Indeed, this is a flow, and there can be no arcs of value 0. The maximum value of an arc is 7, if this edge was of type $(1, 1, 1)$ in the original $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow, and the edge received the same orientation in all three orientations o_i. Since in every edge-cut, the sum of the flow values in the two directions is the same, there are no directed cuts if we take the superposition of the flows o_1, o_2, o_3. Thus O_1 is strongly connected.

In order to prove $F(G) \leq 5$, we create a set of strongly-connected orientations $\mathcal{O}$ of G such that $|\mathcal{O}| = 5$, and any edge of G is deletable in at least one of the orientations of $\mathcal{O}$. Let O_1 be the first member of $\mathcal{O}$. By Lemma 2.1, the arcs of value 1 in (O_1, f_1) are deletable with respect to O_1. What are these arcs? We summarize the possible final orientations and values of the arcs in Table 1.

There are three types of arcs of value 1 in (O_1, f_1). We create four other flows such that the larger valued arcs of O_1 receive value 1 in at least one orientation. We refer to any edge of G as one which belongs to the corresponding row of Table 1 with respect to (O_1, f_1). The next goal is to create four other strongly connected orientations such that every type of edge defined by Table 1 gets flow value 1 in at least one of those orientations, and hence is deletable in those orientations by Lemma 2.1. We achieve this by combining the following two things: we reverse the orientations o_i in G_i, denoted by $-o_i$, for each $i \in \{1, 2, 3\}$ if necessary, and we change the values v_i appropriately. By reversing the orientations, we can switch the role of a fixed edge with respect to its role described in Table 1, and by changing the values we can change the final values of the edges.

Table 2
Cross-table for clearing all cases referring to Table 1.

Original value	(O_1, f_1)	(O_2, f_2)	(O_3, f_3)	(O_4, f_4)	(O_5, f_5)
(1, 0, 0) row 1	value 1				
(0, 1, 0) row 2				value 1	
(0, 0, 1) row 3					value 1
(1, 1, 0) row 4				value 1	
(1, 1, 0) row 5	value 1				
(1, 0, 1) row 6			value 1		
(1, 0, 1) row 7		value 1			value 1
(0, 1, 1) row 8			value 1		
(0, 1, 1) row 9		value 1			
(1, 1, 1) row 10					value 1
(1, 1, 1) row 11		value 1		value 1	
(1, 1, 1) row 12			value 1		
(1, 1, 1) row 13	value 1				

We define the following two 10-flows (O_2, O_3) and two 8-flows (O_4, O_5) based on the orientations o_1, o_2, o_3. Again, if an edge is present in several circuits, we superimpose the values as before[3] to make the flow (O_i, f_i) all-positive for all $i \in \{1, 2, 3, 4, 5\}$. We define O_2 by keeping the same orientations as in O_1, but sending value 4 along o_1, value 2 along o_2, and value 3 along o_3. The only difference between O_3 and O_2 is that the orientations of the circuits in o_3 are reversed, i.e. the orientations are defined by the superposition of o_1, o_2 and $-o_3$. The orientation O_4 is defined by the superposition of o_1, $-o_2$ and o_3 and we send value 2 along o_1, value 1 along $-o_2$ and value 4 along o_3. The only difference between O_5 and O_4 is that the values along $-o_2$ and o_3 are swapped. We indicate in Table 2 which edges receive value 1 in the corresponding orientations.

By Lemma 2.1, it is clear that $\mathcal{O} = \{O_1, O_2, O_3, O_4, O_5\}$ yields $F(G) \leq 5$. □

Let us continue with some comments and observations on the proof.

Remark 3.1. It does not really matter what kind of flow we use to construct an orientation of $\mathcal{O}$. However, there are two key observations: since the orientation arises from flows, it cannot have a directed cut. Therefore, it is a strongly connected orientation, and by the nowhere-zero property and the 3-edge-connectivity of G an arc of value 1 cannot be the only arc going in the opposite direction in any cut.

Remark 3.2. Note that we do not claim that these are the only deletable edges of $O_i \in \mathcal{O}$. It may happen that some arcs with higher values are also deletable. Hence our result is slightly stronger than just proving $F(G) \leq 5$ since we only relied on the arcs of value 1.

Discussion

We do not see how to use directly Theorem 2.5 to improve the Frank number of 3-edge-connected graphs. A natural question arises, can we use a similar proof technique to achieve an even better upper bound? Goedgebeur et al. [4] proved if G admits a nowhere-zero all-positive 4-flow, then $F(G) \leq 2$. On the other hand, if we start with an arbitrary all-positive k-flow for some $k \geq 5$, then we either need to have some control over the arcs with value 1 or need to know some underlying structure, which we can exploit. In this proof technique, it seems vital to have those Eulerian subgraphs. To find them, we had to use an equivalent H-flow for an appropriate group H. However, the even degrees within those subgraphs was guaranteed by the fact that the corresponding coordinate was 0/1 valued. Hence it is unclear to us, what kind of H can be used here other than $\mathbb{Z}_2^m$.

[3] The larger value decides the direction except in O_2 and O_3, where 2+3 decides the direction instead of 4.

Suppose that $H = \mathbb{Z}_2^m$ for some $m \geq 2$. Let us introduce the notation A_k for those arcs of the original H-flow, which have exactly k non-zero coordinates ($k \in \{1, 2, \ldots, m\}$). Similarly to our proof, an all-positive H-flow gives rise to m Eulerian subgraphs G_i (for $i \in \{1, 2, \ldots, m\}$) and we can fix an orientation o_i on each of them. How many different types of arcs can get value 1? We carefully choose values v_i and consider orientations (O_j, f_j) as the superposition, while no arc gets value 0 in the process.

The existence of the arcs of A_2 ensures that the values v_i must be pairwise different, otherwise there is a possibility that an arc gets value 0 in the superposition. Thus at most one of the families in A_1 can get value 1 (the ones within G_i if and only if $v_i = 1$) in each orientation. Hence with this technique, we definitely need at least m orientations, consequently $m \leq F(G)$.

There are $2\binom{m}{2}$ possible types of arcs corresponding to the arcs of A_2. At most two of them can have value 1 in a fixed orientation (O, f). Indeed, if there were at least three types of such arcs, then we would choose two such that the first one is non-zero at coordinates i_1, i_2, the second one at coordinates j_1, j_2 such that $v_{i_1} - v_{i_2} = 1 = v_{j_1} - v_{j_2}$, and i_1, i_2, j_1, j_2 are pairwise different by the fact that the values v_i must be distinct. This could possibly lead to some arcs with final value 0, whenever an arc xy of A_4 has 1's at coordinates i_1, i_2, j_1, j_2. In particular, this happens when the corresponding four orientations are chosen in such a way that xy is oriented in the same direction in G_{i_1} and G_{j_2}, but in the opposite direction in the other two Eulerian subgraphs. That yields $\binom{m}{2} \leq F(G)$. Thus if one would like to improve our result, then m should be at most 3. Since $m = 2$ corresponds to the 4-flows, that circles back to Conjecture 2.3, the only possibility is $m = 3$.

From the previous observations, one can deduce that we need at least three orientations to deal with the arcs of A_1. In each of these orientations, there is an Eulerian subgraph G_i for $i \in \{1, 2, 3\}$ such that $v_i = 1$. These orientations cannot have two types of arcs with value 1 from A_2, since that would be possible only if the three values were 1, 2 and 3. Again this contradicts the non-zero property of our flow $(1+2-3 = 0)$. Consequently, at least three types of arcs are missing after these three orientations from the arcs of A_2. Hence at least two other orientations are needed. Therefore our upper bound of 5 is best if $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

We naturally asked ourselves whether using a different group H can help improving the upper bound of 5. In the revised version of their paper, Goedgebeur, Máčajová and Renders [4] achieved this feat.

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