On Turán problems with bounded matching number

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Abstract

Very recently, Alon and Frankl initiated the study of the maximum number of edges in n-vertex F-free graphs with matching number at most s. For fixed F and s, we determine this number apart from a constant additive term. We also obtain several exact results.

1 Introduction

A basic problem in extremal graph theory is the following. Given a positive integer n and a graph F, how many edges can an n-vertex graph have if it does not contain F as a subgraph? More generally, given n and a family $\mathcal{F}$ of graphs, how many edges can an n-vertex graph have if it does not contain any member of $\mathcal{F}$ as a subgraph? We denote the largest number of edges by $\mathrm{ex}(n,\mathcal{F})$. In the case $\mathcal{F}$ contains only one graph, we write $\mathrm{ex}(n,F)$ instead of $\mathrm{ex}(n,\{F\})$.

One of the earliest results concerning these numbers is due to Turán [8], who showed that $\mathrm{ex}(n,K_{k+1}) = |E(T(n,k))|$, where the Turán graph $T(n,k)$ is the complete k-partite n-vertex graph with each part of order $\lfloor n/k\rfloor$ or $\lceil n/k\rceil$. Another fundamental result is due to Erdős and Gallai [5], who showed that $\mathrm{ex}(n,M_{s+1}) = \max\{|E(G(n,s))|, \binom{2s+1}{2}\}$, where the matching M_{s+1} consists of $s+1$ independent edges and $G(n,s)$ has s vertices of degree $n-1$ and $n-s$ vertices of degree s. Chvátal and Hanson [3] determined $\mathrm{ex}(n,K_{1,k+1},M_{s+1})$ (the case $s=k$ was solved earlier in [1]).

Very recently, Alon and Frankl [2] combined the above results and considered forbidding a graph F and M_{s+1} at the same time. Let $G(n,k,s)$ denote the complete k-partite n-vertex graph with one part of order $n-s$ and each other part of order $\lfloor s/k\rfloor$ or $\lceil s/k\rceil$. Alon and Frankl [2] showed that $\mathrm{ex}(n,\{K_{k+1},M_{s+1}\}) = \max\{|E(G(n,k,s))|, |E(T(2s+1,k))|\}$, in particular for n sufficiently large we have $\mathrm{ex}(n,\{K_{k+1},M_{s+1}\}) = |E(G(n,k,s))|$. Moreover, for any F with chromatic number $k+1$ and a color-critical edge (an edge whose deletion decreases the chromatic number), they showed that $\mathrm{ex}(n,\{F,M_{s+1}\}) = |E(G(n,k,s))|$, provided $s > s_0(F)$ and $n > n_0(F)$.

First we prove a generalization of this second result.

Theorem 1.1. *If $\chi(F) > 2$ and n is large enough, then $\mathrm{ex}(n, \{F, M_{s+1}\}) = \mathrm{ex}(s, \mathcal{F}) + s(n-s)$, where $\mathcal{F}$ is the family of graphs obtained by deleting a color class from F.*

We remark that isolated vertices of members of $\mathcal{F}$ are important here. For example, if F is an odd cycle $C_{2\ell+1}$ (or more generally, if F is 3-chromatic with a color-critical edge), then $\mathcal{F}$ contains the graph consisting of an edge and $2\ell-1$ isolated vertices. If $s \geq 2\ell+1$, then $\mathrm{ex}(s, \mathcal{F}) = 0$, while if $s \leq 2\ell+1$, then $\mathrm{ex}(s, \mathcal{F}) = \binom{s}{2}$.

Observe that if F has a color-critical edge, then $\mathcal{F}$ contains a graph F' with chromatic number $k := \chi(F-1)k$ and a color-critical edge. By a result of Simonovits [7], we have that $\mathrm{ex}(s, \mathcal{F}) = |E(T(s, k-1))|$ if s is large enough. Therefore, the above theorem indeed generalizes the second result of Alon and Frankl [2]. We also have the following.

Corollary 1.2. *If $\chi(F) > 2$, then $\mathrm{ex}(n, \{F, M_{s+1}\}) = s(n-s) + O(1)$.*

In the case F is bipartite, we can also determine $\mathrm{ex}(n, \{F, M_{s+1}\})$ apart an additive constant term. Let F be a bipartite graph and let $p = p(F)$ denote the smallest possible order of a color class in a proper two-coloring of F. If $p > s$, then $G(n, s)$ and K_{2s+1} are both F-free, thus the Erdős-Gallai theorem [5] gives the exact value of $\mathrm{ex}(n, \{F, M_{s+1}\})$.

Proposition 1.3. *If F is bipartite and $p = p(F) \leq s$, then $\mathrm{ex}(n, \{F, M_{s+1}\}) = (p-1)n + O(1)$. Moreover, there is a $K = K(F, s)$ such that for any n, there is an n-vertex $\{F, M_{s+1}\}$-free graph with $|E(G)| = \mathrm{ex}(n, \{F, M_{s+1}\})$ that has vertices $v_1, \dots, v_{p-1}$ and at least $n-K$ vertices u such that the neighborhood of u is $\{v_1, \dots, v_{p-1}\}$. Furthermore, the vertices with neighborhood different from $\{v_1, \dots, v_{p-1}\}$ each have degree at last p.*

The lower bound is given by $K_{p-1,n-p+1}$. It is clearly not the extremal graph though. Now we describe two candidates.

Construction 1. Let $\mathcal{F}_0$ denote the family of graphs obtained by deleting $p-1$ vertices from F and let $\mathcal{F}_1 = \mathcal{F}_0 \cup \{M_{s-p+2}\}$. Then we can add an $\mathcal{F}_1$-free graph to the larger class of $K_{p-1,n-p+1}$ and all edges to the smaller class. The resulting graph is clearly $\{F, M_{s+1}$-free and has $(p-1)(n-p+1) + \binom{p-1}{2} + \mathrm{ex}(n-p+1, \mathcal{F}_1)$ edges. Note that $\mathcal{F}_1$ contains $K_{1,|V(F)|-p}$, thus $ex(n-p+1, \mathcal{F}_1) = O(1)$.

Construction 2. Assume that F is connected. We take $K_{p-1,n+p-2s}$, and on the remaining $2s-2p+1$ vertices, we take an F-free graph with $\mathrm{ex}(2s-1, F)$ edges. Clearly, none of the components of this graph contains F, and the largest matchings have size at most $p-1+s-p$.

We remark that the second construction can easily be improved for some specific F. For example, if F is a path P_4 on 4 vertices, we can take $K_{p-1,n-3s+2p-1}$ and $s-p$ triangles. We claim that if F contains a cycle and s is large enough, then the second construction contains more edges. Indeed, compared to the first construction, we lose $O(s)$ edges and gain $\omega(s)$ edges.

Assume now that F is a forest and observe that $\mathcal{F}_1$ contains a matching of order at most $|V(F)|-p+1$. Indeed, if F has v non-isolated vertices, then there are at most $v-1$ edges between the two parts, thus at most $p-1$ vertices of the part of order $|V(F)|-p$ have degree more than 1. If we delete those vertices, we obtain a matching. This implies that $\mathrm{ex}(n-p+1,\mathcal{F}_1)$ does not depend on s.

Now assume that F is a tree with parts of different order, i.e., $|V(F)|> 2p$. Assume furthermore that s and n are sufficiently large, and for simplicity assume that $2s-1$ is divisible by $|V(F)|-1$. In this case $s/(|V(F)|-1)$ copies of $K_{|V(F)|-1}$ forms an F-free graph, thus $\mathrm{ex}(2s-1,F)\ge(|V(F)|-2)(2s-1)/2$. Now, compared to Construction 1, the second construction loses $(2s-1)(p-1)+c$ edges, where c does not depend on s. On the other hand, Construction 2 gains at least $(2s-1)(|V(F)|-2)/2>(2s-1)(p-1/2)$, thus Construction 2 is better. Note that essentially the same argument also works if $2s-1$ is not divisible by $|V(F)|-1$.

We believe that for other trees Construction 1 is better than Construction 2 for every s, moreover, Construction 1 is extremal. The Erdős-Sós conjecture [4] states that for any tree F, we have $\mathrm{ex}(n,F)\le(|V(F)|-2)n/2$. It is known for several classes of trees. In particular, it was shown for paths by Erdős and Gallai [5].

Proposition 1.4. *Let F be a balanced tree, i.e., $|V(F)|=2p(F)$ and let $p(F)\le s$. Assume that the Erdős-Sós conjecture holds for F. Then for sufficiently large n, we have* $\mathrm{ex}(n,\{F,M_{s+1}\})=(p-1)(n-p+1)+\binom{p-1}{2}$.

The above proposition determines $\mathrm{ex}(n,\{P_{2\ell},M_{s+1}\})$ for sufficiently large n. We can also deal with odd paths.

Proposition 1.5. *Let $2\le\ell\le s$. If ℓ divides $s-\ell+1$, then for sufficiently large n we have that* $\mathrm{ex}(n,\{P_{2\ell+1},M_{s+1}\})=(\ell-1)(n-2s+\ell-1)+\binom{p-1}{2}+(s-\ell+1)(2\ell-1)$. *If ℓ does not divide $s-\ell+1$, then let $t:=\lfloor(s-\ell+1)/\ell\rfloor$. For sufficiently large n, we have that* $\mathrm{ex}(n,\{P_{2\ell+1},M_{s+1}\})=(\ell-1)(n-\ell+1-2\ell t)+1+\binom{p-1}{2}+t\binom{2\ell}{2}$.

2 Proofs

Let us start with the proof of Theorem 1.1 that we restate here for convenience.

Theorem. *If $\chi(F)>2$ and n is large enough, then* $\mathrm{ex}(n,\{F,M_{s+1}\})=\mathrm{ex}(s,\mathcal{F})+s(n-s)$, *where $\mathcal{F}$ is the family of graphs obtained by deleting a color class from F.*

Proof. Let G_0 be an s-vertex $\mathcal{F}$-free graph with $\mathrm{ex}(s,\mathcal{F})$ edges. Let us add $n-s$ new vertices and connect each of them to each vertex of G_0. The resulting graph is clearly M_{s+1}-free, since s vertices are incident to all the edges, and F-free by the definition of $\mathcal{F}$. This gives the lower bound.

To show the upper bound, consider an $\{F,M_{s+1}\}$-free n-vertex graph G. Let $v_1,\dots,v_n$ be the vertices of G in decreasing order of their degrees. Observe that $d(v_{s+1})\le 2s$. Indeed, otherwise we can pick greedily a matching M_{s+1} the following way. In step i, we pick v_i and a neighbor of v_i we have not picked earlier. This way we have at most $2i-2$ forbidden neighbors, thus we can pick a new one even at step $s+1$, a contradiction.

Observe also that G has at most $\sum_{i=1}^{2s} d(v_i) \leq sum_{i=1}^{s} d(v_i) + 2s^2$ edges. Indeed, the at most $2s$ vertices of a largest matching are incident to every edge, and $2s$ vertices are incident to at most $\sum_{i=1}^{2s} d(v_i)$ edges. The upper bound on this quantity follows from $d(v_{s+1}, \dots, d(v_{2s}) \leq 2s$.

We claim that $d(v_s) \geq n - 3s^2$. Indeed, otherwise $sum_{i=1}^{s} d(v_i) + 2s^2 \leq (s-1)(n-1) + n - s^2 \leq s(n-s)$ and we are done. This implies that $v_1, \dots, v_s$ have at least $n - s - 3s^3$ common neighbors. Let $U = \{v_1, \dots, v_s\}$. Observe that $G[U]$ is $\mathcal{F}$-free, otherwise we would find an F by picking at most $|V(F)|$ of their common neighbors as the missing color class.

We claim that there is no edge outside U. Indeed, otherwise we could find M_{s+1} greedily as earlier: first we pick the edge outside U, and then in step $i+1$, we pick v_i and a neighbor of v_i we have not picked earlier. This is doable since v_i has at least $n - 3s^2 \geq 2i$ neighbors. The number of edges is at most $\mathrm{ex}(s, \mathcal{F}) + s(n-s)$, where the first term is an upper bound on the number of edges inside U, while the second term is an upper bound on the number of edges with one endpoint inside U and the other endpoint outside U. This completes the proof. ■

Let us continue with the proof of Proposition 1.3 that we restate here for convenience.

Proposition. *If F is bipartite and $p = p(F) \leq s$, then $\mathrm{ex}(n, \{F, M_{s+1}\}) = (p-1)n + O(1)$. Moreover, there is a $K = K(F, s)$ such that for any n, there is an n-vertex $\{F, M_{s+1}\}$-free graph with $|E(G)| = \mathrm{ex}(n, \{F, M_{s+1}\})$ that has vertices $v_1, \dots, v_{p-1}$ and at least $n - K$ vertices u such that the neighborhood of u is $\{v_1, \dots, v_{p-1}\}$. Furthermore, the vertices with neighborhood different from $\{v_1, \dots, v_{p-1}\}$ each have degree at last p.*

Proof. The lower bound is given by $K_{p-1,n-p+1}$, or by Construction 1 or Construction 2.

Let G be an n-vertex $\{F, M_{s+1}\}$-free graph. Let U denote the set of at most $2s$ vertices of a largest matching, then every edge of G is incident to at least one vertex of U. Every p-set in U has less than $q := |V(F)| - p$ common neighbors. As there are at most $\binom{2s}{p}$ p-sets in U, there are at most $\binom{2s}{p}$ vertices outside U that are adjacent to at least p sets.

Let W denote the set of the other at least $n - \binom{2s}{p}(|V(F)| - p) - 2s$ vertices outside U. Then vertices of W have degree at most $p-1$. Note that by choosing K sufficiently large, we can assume that n is sufficiently large. In particular, if at most $\binom{2s}{p-1} \max\{|V(F)|, 2s\}$ vertices in W with degree $p-1$, then the number of edges is at most $(p-2)n + O(1)$ and we are done. Otherwise, at least $\max\{|V(F)|, 2s\}$ vertices of W have the same $p-1$ neighbors $v_1, \dots, v_{p-1}$.

For any other vertex of W, we change its neighborhood to $v_1, \dots, v_{p-1}$ to obtain G'. If G' contained F or M_{s+1}, that would contain some of the vertices whose neighborhood was changed. But the could be replaced by vertices with the same neighborhood already in G, to obtain F or M_{s+1} in G. Therefore, G' is $\{F, M_{s+1}\}$-free. Clearly $|E(G')| \geq |E(G)|$, hence if G has $\mathrm{ex}(n, \{F, M_{s+1}\})$ edges, then so does G'. It is easy to see that G' has $(p-1)n + O(1)$ edges and the desired additional property. ■

Let us continue with the proof of Proposition 1.4 that we restate here for convenience.

Proposition. *Let F be a balanced tree, i.e., $|V(F)|= 2p(F)$ and let $p(F) \leq s$. Assume that the Erdős-Sós conjecture holds for F. Then for sufficiently large n, we have* $\mathrm{ex}(n, \{F, M_{s+1}\}) = (p-1)(n-p+1) + \binom{p-1}{2}$.

Proof. The lower bound is given by Construction 1, which is $G(n, p-1)$ in this case. Indeed, if we delete $p-1$ vertices in one of the parts of F and leave only a leaf, then the resulting graph is a single edge and some isolated vertices. As $\mathcal{F}_1$ contains this graph, $\mathrm{ex}(n-p+1, \mathcal{F}_1) = 0$.

For the upper bound, let G be a graph ensured by Proposition 1.3. Thus, G has n vertices, $\mathrm{ex}(n, \{F, M_{s+1}\})$ edges, G is $\{F, M_{s+1}\}$-free, and G contains a set $U = \{v_1, \dots, v_{p-1}\}$ such that all but K vertices have neighborhood U. Let W denote the set of vertices with neighborhood U and $U' := V(G) \setminus (U \cup W)$.

Observe that there is no edge inside W since $\mathrm{ex}(n-p+1, \mathcal{F}_1) = 0$.

Claim 2.1. *There is no edge between U and U'.*

Proof. First we show that if $F \neq K_2$, then F has a vertex x that is adjacent to at least one, but at most $p-1$ leaves and exactly one neighbor of degree greater than 1. Indeed, let F' be the graph we obtain by deleting the leaves of F, then F' has at least two leaves. Those vertices in F have one neighbor of degree greater than 1 and at least 1 leaf neighbor. As there are at most $2p-2$ leaves in F, at least one of these two vertices have at most $p-1$ leaf neighbors.

Assume that $v_i u$ is an edge between U and U' and let u' be a neighbor of u outside U (this exists otherwise $u \in W$). Now we map x to u its non-leaf neighbor to v_i, and we map the leaf neighbors of x to u' and $p-2$ other neighbors of u. We map the remaining vertices of the part of F containing these leaves to arbitrary vertices in U, and the remaining vertices of the other part of F to arbitrary vertices in W. This way we find a copy of F in G, a contradiction. ■

Let us return to the proof of the proposition. Since the Erdős-Sós conjecture holds for F, we have $\mathrm{ex}(|U'|, F) \leq (p-1)|U'|$, thus there are at most $(p-1)|U'|$ edges inside U'. Then $|E(G)| \leq \binom{p-1}{2} + (p-1)(n-p+1-|U'|) + \mathrm{ex}(|U'|, F) \leq \binom{p-1}{2} + (p-1)(n-p+1)$, completing the proof. ■

We finish the paper with the proof of Proposition 1.5 that we restate here for convenience.

Proposition. *Let $2 \leq \ell \leq s$. If ℓ divides $s-\ell+1$, then for sufficiently large n we have that* $\mathrm{ex}(n, \{P_{2\ell+1}, M_{s+1}\}) = (\ell-1)(n-2s+\ell-1) + \binom{p-1}{2} + (s-\ell+1)(2\ell-1)$. *If ℓ does not divide $s-\ell+1$, then let $t := \lfloor (s-\ell+1)/\ell \rfloor$. For sufficiently large n, we have that* $\mathrm{ex}(n, \{P_{2\ell+1}, M_{s+1}\}) = (\ell-1)(n-\ell+1-2\ell t) + 1 + \binom{p-1}{2} + t\binom{2\ell}{2}$.

Proof. The lower bounds are given by the following graphs. If ℓ divides $s-\ell+1$, then we take $G(n-2s+2\ell-2, \ell-1)$, and on the remaining $2s-2\ell+2$ vertices, we take $(s-\ell+1)/\ell$ copies of $K_{2\ell}$. Each component is $P_{2\ell+1}$-free, and the largest matching is of size $\ell-1$ in the large component, and of size $s-\ell+1$ in the clique components.

If ℓ does not divide $s-\ell+1$, then we similarly take copies of $K_{2\ell}$ on at most $s-\ell+1$ vertices, i.e., we take t copies. On the remaining $n-2\ell t$ vertices, we take $G(n-2\ell t, p-1)$ and add another edge. Again each component is $P_{2\ell+1}$-free, but this time the largest matching is of size ℓ in the large component. However, the remaining components have order $t2\ell < 2s-2\ell+2$, thus the largest matching in those components have size at most $s-\ell$.

Let us continue with the upper bounds. We apply Proposition 1.3 to obtain an extremal n-vertex graph G with vertices $U=\{v_1,\dots,v_{p-1}\}$, such that the set W of vertices with neighborhood U contains all but at most K vertices. Moreover, the vertices of $U'=V(G)\setminus(U\cup W)$ have degree at least ℓ. We will use multiple times the following simple observation: changing the neighborhood of a vertex u to U does not create F or M_{s+1}. Indeed, we could replace the vertex u in any forbidden configuration to any other common neighbor of the vertices of U to create another copy without containing any of the new edges,

It is easy to see that there is at most one edge inside W. We claim that there is an extremal graph G' satisfying the above properties without any edges between U and U'.

Consider a component C of U'. If C does not contain $P_{2\ell}$, then it contains at most $\mathrm{ex}(|V(C)|,P_{2\ell})=|V(C)|(\ell-1)$ edges. Then we can change the neighborhood of vertices in C to U. The resulting graph is also $\{P_{2\ell+1},M_{s+1}\}$-free and the number of edges does not decrease. We apply these to all the $P_{2\ell}$-free components. In the resulting graph G', every vertex of U' is in a component containing a $P_{2\ell}$, in particular is the endvertex of a $P_{\ell+1}$ inside U'. As every vertex of U is the endvertex of a $P_{2\ell-1}$ outside U', an edge between U and U' would give a $P_{2\ell+1}$ in G', a contradiction.

Consider now a component C of G' in U' with $v>2\ell$ vertices. A theorem of Kopylov [6] gives an upper bound on the number of edges inside P_k-free connected graphs. It shows that $|E(G'[C])|\le \max\{\binom{2\ell-1}{2}+v-2\ell+1, |E(G(v,\ell-1))|+1\}\le v(\ell-1)$. Therefore, again, we can change the neighborhood of vertices in C to U without decreasing the number of edges.

Consider now a component C of G' in U' with less than 2ℓ vertices. Then C has at most $|V(C)|(\ell-1)$ edges, thus again, we can change the neighborhood of vertices in C to U without decreasing the number of edges.

Consider now a component C of G' in U' with 2ℓ vertices that is M_ℓ-free. By the Erdős-Gallai theorem, we know that C contains at most $\binom{2\ell}{2}-\ell+1\le 2\ell(\ell-1)$ edges, thus again, we can change the neighborhood of vertices in C to U without decreasing the number of edges.

We obtained that each component in U' has 2ℓ vertices and contains a matching M_ℓ, thus adding the missing edges inside that component would not increase the largest matching in G', nor it would create $P_{2\ell+1}$. Therefore, U' consists of copies of $K_{2\ell}$. Clearly there are at most t copies. Clearly, 2ℓ vertices in a $K_{2\ell}$ add $\ell(2\ell-1)$ edges, while 2ℓ vertices in W add $2\ell(\ell-1)$ edges. Therefore, it is worth to pick the maximum number of 2ℓ-cliques.

If there is no edge inside W or ℓ does not divide $s-\ell+1$, then we are done. In the remaining case, we can only add $t-1$ copies of $K_{2\ell}$. Compared to this graph, we can delete 2ℓ vertices from W including the endvertices of the extra edge from G' and add one more copy of $K_{2\ell}$. This way we removed $2\ell(\ell-1)+1$ edges and added $\ell(2\ell-1)$ edges without creating F or M_{s+1}. The number of edges increases, a contradiction completing the proof. ■

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