Extremal Results for Graphs Avoiding a Rainbow Subgraph

Zhen He[a,b] Peter Frankl[b] Ervin Győri[b] Zequn Lv[b,c] Nika Salia[b,d]
Casey Tompkins[b] Kitti Varga[b,e,f] Xiutao Zhu[g]

Submitted: Nov 14, 2022; Accepted: Oct 29, 2023; Published: Jan 26, 2024

Abstract

We say that k graphs $G_1, G_2, \ldots, G_k$ on a common vertex set of size n contain a rainbow copy of a graph H if their union contains a copy of H with each edge belonging to a distinct G_i. We provide a counterexample to a conjecture of Frankl on the maximum product of the sizes of the edge sets of three graphs avoiding a rainbow triangle. We propose an alternative conjecture, which we prove under the additional assumption that the union of the three graphs is complete. Furthermore, we determine the maximum product of the sizes of the edge sets of three graphs or four graphs avoiding a rainbow path of length three.

Mathematics Subject Classifications: 05C35

1 Introduction

The classical theorem of Mantel [8] asserts that the maximum number of edges in an n-vertex graph containing no triangle is $\lfloor n^2/4 \rfloor$. This result was generalized by Turán [9], who showed that the maximum number of edges in an n-vertex graph with no complete graph K_r as a subgraph is obtained by taking a complete $(r-1)$-partite graph with parts of size $\lfloor n/(r-1) \rfloor$ or $\lceil n/(r-1) \rceil$.

Many natural generalizations of these theorems have been considered. For a graph G, let $E(G)$ denote the edge set of G and let $e(G) = |E(G)|$. Of particular importance to the present work is an extremal problem due to Keevash, Saks, Sudakov, and Verstraëte [7].

[a] School of Mathematics and Statistics, Beijing Jiaotong University.
[b] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences.
[c] Department of Mathematical Sciences, Tsinghua University.
[d] King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.
[e] Department of Computer Science and Information Theory, Budapest University of Technology and Economics.
[f] ELKH-ELTE Egerváry Research Group.
[g] School of mathematics, Nanjing University of Aeronautics and Astronautics.

They considered (among other problems) the maximum of $e(G_1) + e(G_2) + \cdots + e(G_k)$ across k graphs on a common vertex set of size n with the property that there is no K_r with each of its edges coming from a distinct $E(G_i)$. Such a K_r is referred to as a *rainbow* K_r. For this problem, there are two natural constructions. On the one hand, if $k \geqslant \frac{r^2-1}{2}$, then we take k identical copies of the Turán graph. On the other hand, when $\binom{r}{2} \leqslant k < \frac{r^2-1}{2}$, it is better to take $\binom{r}{2} - 1$ copies of the complete graph and let the remaining graphs have empty edge sets, and of course for $k < \binom{r}{2}$, it is optimal to take all k graphs to be complete. Keevash, Saks, Sudakov, and Verstraëte [7] also proved that for 3-chromatic color-critical graphs and sufficiently large n, an analogous result holds in which either a construction consisting of k copies of a Turán graph, or a construction consisting of complete graphs and graphs with no edges is optimal. Recently Chakraborti, Kim, Lee, Liu, and Seo [3] showed that the same holds for 4-chromatic color-critical graphs and almost all color-critical graphs of chromatic number at least 5, partially verifying a conjecture from [7]. In the context of extremal set theory, rainbow extremal problems have also been considered earlier, for example by Hilton [6].

After one has bounds on $e(G_1) + e(G_2) + \cdots + e(G_k)$, it is natural to consider maximizing other objective functions over $e(G_1)$, $e(G_2)$,..., $e(G_k)$. The problem of maximizing $\min\big(e(G_1), e(G_2), e(G_3)\big)$ while avoiding a rainbow triangle was considered by Aharoni, DeVos, de la Maza, Montejano, and Šámalin [1], answering a question of Diwan and Mubayi [4]. Let P_r denote the path with r vertices. The problem of maximizing $\min\big(e(G_1), e(G_2), \ldots e(G_k)\big)$ while avoiding a rainbow P_4 was considered by Babiński and Grzesik [2]. For the problem of maximizing the product $e(G_1)e(G_2)e(G_3)$ while avoiding a rainbow triangle, Frankl [5] gave the following conjecture.

Conjecture 1 (Frankl)**.** Let G_1, G_2, G_3 be graphs on a common vertex set of size n with no rainbow triangle. Then

$$e(G_1)e(G_2)e(G_3) \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor^3.$$

Taking G_1, G_2, G_3 to be 3 copies of the complete bipartite graph with almost equal parts attains this bound. Frankl proved that under the additional assumption $E(G_1) \subseteq E(G_2)$ and $E(G_1) \subseteq E(G_3)$, Conjecture 1 holds. We show that Frankl's conjecture does not hold in the general case.

Let γ be the maximum of

$$\frac{x^2}{2}\left(\frac{x^2}{2} + \frac{(1-x)^2}{2}\right)\left(x(1-x) + \frac{(1-x)^2}{2}\right) \tag{1}$$

on $[0, 1]$, and assume γ is attained at $x = x_0$. Note that $\frac{1}{52} < \gamma < \frac{1}{51}$ (and $x_0 \approx 0.793$). We have the following.

Theorem 2. *There exist graphs G_1, G_2, G_3 on a common vertex set of size n and with no rainbow triangle such that*

$$e(G_1)e(G_2)e(G_3) \geqslant \gamma n^6\big(1 - o(1)\big).$$

Proof. Let $[n] = X \cup Y$ be a partition of $[n]$ with X having size approximately $x_0 n$ and Y having size approximately $(1 - x_0)n$. Let G_1 consist of a complete graph on X, and G_2 consist of the union of a complete graph on X and a complete graph on Y and let G_3 consist of a complete graph on Y as well as all edges between X and Y. Observe that the product $e(G_1)e(G_2)e(G_3)$ is asymptotically γn^6. □

Moreover, we believe that the expression in Theorem 2 is asymptotically best possible.

Conjecture 3. For three graphs G_1, G_2, G_3 on a common vertex set of size n with no rainbow triangle, we have

$$e(G_1)e(G_2)e(G_3) \leqslant \gamma n^6 \big(1 + o(1)\big).$$

We prove Conjecture 3 under the additional assumption that every pair of vertices forms an edge in at least one of the graphs G_i. This result is presented in Section 2.

In Section 3 we consider graphs which avoid a rainbow path of length 3. For three or four graphs on a common vertex set of size n avoiding a rainbow P_4, we asymptotically determine the maximum value of the product of the sizes of their edge sets.

Finally, in Section 4 we present several natural questions concerning graphs avoiding certain rainbow subgraphs.

2 Proof of Conjecture 3 with an additional assumption

For convenience, we say that a pair of vertices (or an edge) in the n-vertex ground set is colored if it belongs to at least one of the sets $E(G_1)$, $E(G_2)$, $E(G_3)$. An edge is t-colored if it belongs to exactly t of the sets $E(G_1)$, $E(G_2)$, $E(G_3)$.

Theorem 4. *Let G_1, G_2, G_3 be graphs on a common vertex set of size n with no rainbow triangle. If n is sufficiently large and every pair of vertices on the ground set is colored, then the construction described in Theorem 2 maximizes $e(G_1)e(G_2)e(G_3)$.*

Proof. Let n be sufficiently large, and let G_1, G_2, G_3 be graphs on a common vertex set of size n with no rainbow triangle and assume that every edge is colored and $e(G_1)e(G_2)e(G_3)$ is maximal. For a vertex v and $i \in [3]$, let $N_i(v)$ be the set of neighbors of v in G_i which are not neighbors of v in G_l for all $l \neq i$. For any $\{i, j, l\} = \{1, 2, 3\}$, we denote the set of neighbors of the vertex v in G_i and G_j but not in G_l by $N_{ij}(v)$.

Assume $e(G_1)e(G_2)e(G_3) \geqslant \gamma n^6 \big(1 - o(1)\big)$. Then we have

$$e(G_1) + e(G_2) + e(G_3) \geqslant 3\sqrt[3]{e(G_1)e(G_2)e(G_3)} > 0.8n^2 + \frac{n}{2}$$

for any sufficiently large n. Since three-colored edges cannot be adjacent to any edge with at least two colors, the number of three-colored edges is at most $n/2$. Hence the number of two-colored edges is at least $0.3n^2$.

Claim 5. *The graph containing all edges with at least two colors is the union of vertex disjoint cliques such that every edge of each clique has the same coloring.*

Proof. Let $e = uv$ be a two-colored edge, and assume that e has colors 1 and 2. If $w \in N_3(v)$, then uw is one-colored with color 3. That is $N_3(v) = N_3(u)$ and

$$N_{13}(u) = N_{23}(u) = \emptyset. \tag{2}$$

Let $w \in N_{12}(v)$. Then uw is not colored with color 3. Suppose by the way of contradiction that uw is a one-colored edge. Without loss of generality, we may assume that uw is of color 1. Then by the maximality of the coloring, there is a vertex w' such that $w' \notin \{v, u, w\}$ and edges uw' and ww' are colored with colors 1 and 3 in any order. This is a contradiction since for any coloring of the edge $w'v$, the triangle $w'vu$ or the triangle $w'vw$ is a rainbow triangle. Hence $w \in N_{12}(u)$, thus

$$N_{12}(u) = N_{12}(v). \tag{3}$$

Then the claim follows from (2) and (3). $\square$

Let A be a clique of maximum size in the graph consisting of edges of at least two colors. Let a be the size of A. Then $\frac{(a-1)n}{2} \geqslant 0.3n^2$ since the maximum degree in the graph of the two-colored edges is $a-1$ and we observed earlier that there are at least $0.3n^2$ two-colored edges. Hence we have $a \geqslant 0.6n$. Since $a > n/2$ and there are at least $0.3n^2$ two-colored edges, it follows $\binom{a}{2} + \binom{n-a}{2} \geqslant 0.3n^2$. Thus we have $a \geqslant 0.723n$.

Without loss of generality, we may assume that the edges of A are colored with colors 1 and 2. Then we have $e(A) \geqslant 0.26n^2$ and it follows $e(G_3) \leqslant 0.24n^2$. From the maximality of the product $e(G_1)e(G_2)e(G_3)$ and Claim 5, we have that all one-colored edges are of color 3. Indeed, otherwise we could change the color of all the one-colored edges to color 3 and since $e(G_3) < e(A)$, this would increase the product. Moreover, for any maximal clique B in the graph different from A and consisting of edges of at least two colors, one of those colors must be color 3 (for otherwise, changing one of the colors in B to color 3 would increase the product).

By maximality, it is easy to observe that there is at most one clique colored with colors i and j for all $1 \leqslant i < j \leqslant 3$. Let a be the size of clique colored with colors 1 and 2, let b be the size of clique colored with colors 1 and 3, and let c be the size of clique colored with colors 2 and 3. Let d be the number of three-colored edges. Then G_1 consists of a complete graph of size a, a complete graph of size b and a matching of size d, the graph G_2 consists of complete graphs of size a and c and a matching of size d, and G_3 consists of all edges not in A. Without loss of generality we may assume $c \geqslant b$. We may also assume that we do not have $b = c = 0$, for otherwise the product is only $O(n^5)$. Then the following holds:

$$\left(\binom{a}{2} + \binom{b}{2} + d\right)\left(\binom{a}{2} + \binom{c}{2} + d\right) \leqslant \left(\binom{a}{2} + \binom{b}{2}\right)\left(\binom{a}{2} + \binom{c+2d}{2}\right).$$

Therefore $d = 0$, otherwise by changing the coloring, we could increase the product. Thus $a + b + c = n$ and since $a \geqslant 0.72n$, we have

$$\left(\binom{a}{2} + \binom{b}{2}\right)\left(\binom{a}{2} + \binom{c}{2}\right) \leqslant \left(\binom{a}{2} + \binom{b+c}{2}\right)\binom{a}{2}.$$

Therefore without loss of generality, we may assume $b = 0$, otherwise by changing the coloring, we could increase the product. Thus, we have obtained that G_1, G_2 and G_3 have the form of the construction in the proof of Theorem 2. That is, we have a partition of the ground set into two parts X and Y, and each edge inside X is colored with colors 1 and 2, each edge inside Y is colored with colors 2 and 3, and all edges between X and Y are colored with 3. The maximum product of the number of edges among such constructions is obtained by a construction of the form described in the proof of Theorem 2 since it maximizes the expression (1) (and thus as a consequence $e(G_1)e(G_2)e(G_3) = \gamma n^6\big(1 + o(1)\big)$). □

3 Results about 4-vertex paths

Next we obtain some results about graphs avoiding a rainbow P_4 (where, recall, P_4 denotes the path with 4 vertices).

Theorem 6. *There exist graphs G_1, G_2, G_3 on a common vertex set of size n and with no rainbow P_4 such that*

$$e(G_1)e(G_2)e(G_3) \geqslant n^6 \left(\frac{1}{256} - o(1)\right).$$

Proof. Let $[n] = X \cup Y$ be a partition of $[n]$ with X and Y of sizes approximately $0.5n$. Let G_1 consist of a complete graph on X, let G_2 consist of a complete graph on Y, and let G_3 consist of a complete graph on X and a complete graph on Y. Observe that $e(G_1)e(G_2)e(G_3) = n^6\big(\frac{1}{256} - o(1)\big)$. □

Theorem 7. *For three graphs G_1, G_2, G_3 on a common vertex set of size n with no rainbow P_4, we have*

$$e(G_1)e(G_2)e(G_3) \leqslant n^6 \left(\frac{1}{256} + o(1)\right).$$

Proof. Let V be the vertex set of size n. Let G_1, G_2, G_3 be graphs on V with no rainbow P_4 and let $G = \big(V, E(G_1) \cup E(G_2) \cup E(G_3)\big)$. Let H be the graph induced by all edges with three colors. Then H is P_4-free, so $e(H) \leqslant \mathrm{ex}(n, P_4) \leqslant n$. Hence we can assume that there is no edge with three colors.

For any vertex $v \in V$ and for any $i \in [3]$, let $d_i(v)$ denote the degree of v in G_i. For any $v \in V$, if $d_i(v) = o(n)$ for some $i \in [3]$, then we assume $d_i(v) = 0$, since the edges of G_i incident to v would not affect the asymptotic value of $e(G_1)e(G_2)e(G_3)$. Let $c(v)$ be the subset of $\{1, 2, 3\}$ such that $d_i(v) \geqslant 1$ for every $i \in c(v)$ and $d_i(v) = 0$ for every $i \notin c(v)$. Let

$$V = \left(\bigcup_{i=1}^{3} A_i\right) \cup \left(\bigcup_{1 \leqslant i < j \leqslant 3} A_{ij}\right) \cup A_{123}$$

be a partition of V, where A_i is the set of vertices v with $c(v) = \{i\}$, A_{ij} is the set of vertices v with $c(v) = \{i, j\}$ and A_{123} is the set of vertices v with $c(v) = \{1, 2, 3\}$. We call the sets A_i, A_{ij} and A_{123} the parts of G. Let $a_i = |A_i|$ for any $i \in [3]$, $a_{ij} = |A_{ij}|$ for any $1 \leqslant i < j \leqslant 3$ and $a_{123} = |A_{123}|$. If one part of this partition has size $o(n)$, then we assume that the size of this part is 0, since the edges incident to this partition class would not contribute to the asymptotic value of $e(G_1)e(G_2)e(G_3)$. Note that there are no edges in $G[A_{123}]$, there are no edges between any two sets from $\{A_{12}, A_{13}, A_{23}, A_{123}\}$, and there are no edges between A_{i_1} and $A_{i_2} \cup A_{i_3} \cup A_{i_2 i_3}$ for any $\{i_1, i_2, i_3\} = \{1, 2, 3\}$.

Assume G satisfies the properties above and maximizes $e(G_1)e(G_2)e(G_3)$. Then we may assume that $G[A_i]$ is complete in G_i for any $i \in \{1, 2, 3\}$ and $G[A_{ij}]$ is complete in G_i and G_j for any $1 \leqslant i < j \leqslant 3$. We may also assume that every vertex in A_i is connected to every vertex in A_{123} with an edge of color i for any $i \in [3]$ and every vertex in A_{ij} is connected to every vertex in A_i and in A_j with an edge of color i and of color j, respectively, for any $1 \leqslant i < j \leqslant 3$.

Let $e_i = e(G_i)$ for any $i \in [3]$ and let

$$k(v) := \frac{d_1(v)}{e_1} + \frac{d_2(v)}{e_2} + \frac{d_3(v)}{e_3}$$

for each $v \in V$. Then

$$\sum_{v \in V} k(v) = \sum_{v \in V} \sum_{i=1}^{3} \frac{d_i(v)}{e_i} = \sum_{i=1}^{3} \sum_{v \in V} \frac{d_i(v)}{e_i} = 6.$$

Lemma 8. *For any $v \in V$, we have*

$$k(v) = \frac{6}{n} + o\left(\frac{1}{n}\right).$$

Proof. Let u, v be two vertices in V. By the maximality of $e_1e_2e_3$, putting u from its part to the part of v, we obtain

$$e_1e_2e_3 \geqslant \big(e_1 - d_1(u) + d_1(v) - 1\big)\big(e_2 - d_2(u) + d_2(v) - 1\big)\big(e_3 - d_3(u) + d_3(v) - 1\big).$$

Hence $k(u) + o\big(\frac{1}{n}\big) \geqslant k(v)$. Similarly, by putting v from its part to the part of u, we obtain $k(v) + o\big(\frac{1}{n}\big) \geqslant k(u)$. Hence $k(v) + o\big(\frac{1}{n}\big) = k(u)$. Thus $k(v) = \frac{6}{n} + o\big(\frac{1}{n}\big)$ holds for any $v \in V$. ☐

By Theorem 6, we may assume that $e_1e_2e_3 \geqslant n^6/256$. Then $e_1 + e_2 + e_3 \geqslant 3\sqrt[3]{n^6/256} > 0.47n^2$. Observe that A_{123} cannot contain any edges. Moreover, if any of the sets A_1, A_2 or A_3 is empty, then the set A_{123} is empty as well.

Case 1: A_i is not empty for any $i \in [3]$.

Let v_i be a vertex in A_i for any $i \in [3]$, and let $\{i, j, k\} = \{1, 2, 3\}$. Applying Lemma 8 to v_i for any $i \in [3]$, we obtain

$$k(v_i) = \frac{a_i + a_{ij} + a_{ik} + a_{ijk}}{e_i} = \frac{6}{n} + o\left(\frac{1}{n}\right).$$

Hence

$$e_1 + e_2 + e_3 = \frac{n}{6}\left(\sum_{i=1}^{3} a_i + 2\sum_{1\leqslant i<j\leqslant 3} a_{ij} + 3a_{123}\right) + o(n^2) > 0.47n^2.$$

Then $a_{123} > 0.8n$. Since there are no edges in $G[A_{123}]$, and so each edge is colored at most twice, it follows that $e_1 + e_2 + e_3 \leqslant 2\left(\binom{n}{2} - \binom{0.8n}{2}\right) < 0.47n^2$, a contradiction.

Case 2: One of the parts A_i, say A_1, is empty and the other two are not.

In this case A_{123} is empty. Applying Lemma 8 to some vertex in A_2 and in A_3, we obtain

$$\frac{a_2 + a_{12} + a_{23}}{e_2} + o\left(\frac{1}{n}\right) = \frac{a_3 + a_{13} + a_{23}}{e_3} + o\left(\frac{1}{n}\right) = \frac{6}{n} + o\left(\frac{1}{n}\right).$$

Hence $e_2 + e_3 = \frac{n}{6}(n + a_{23}) + o(n^2)$. Then $e_2e_3 \leqslant \left(\frac{n}{12}\right)^2(n + a_{23})^2 + o(n^4)$. By $e_1 \leqslant (n - a_{23})^2/2$, we know

$$\begin{aligned} e_1e_2e_3 \leqslant \left(\frac{n}{12}\right)^2 \frac{(n + a_{23})^2(n - a_{23})^2}{2} + o(n^6) &\leqslant n^6\left(\frac{1}{288} + o(1)\right) \\ &< n^6\left(\frac{1}{256} + o(1)\right). \end{aligned}$$

Case 3: Two of the parts A_i, say A_1 and A_2, are empty and the third one is not.

In this case A_{123} is empty. If A_{13} is empty, then moving all vertices in A_3 to A_{23} would increase the product. Hence we may assume that A_{13} and A_{23} are not empty. Applying Lemma 8 to some vertex in A_{13}, in A_3 and in A_{23}, we obtain

$$\begin{aligned} \frac{a_3 + a_{13}}{e_3} + \frac{a_{13}}{e_1} + o\left(\frac{1}{n}\right) &= \frac{a_3 + a_{13} + a_{23}}{e_3} + o\left(\frac{1}{n}\right) \\ &= \frac{a_3 + a_{23}}{e_3} + \frac{a_{23}}{e_2} + o\left(\frac{1}{n}\right) = \frac{6}{n} + o\left(\frac{1}{n}\right). \end{aligned}$$

Hence $\frac{a_{13}}{e_1} = \frac{a_{23}}{e_3} + o\left(\frac{1}{n}\right)$ and $\frac{a_{13}}{e_3} = \frac{a_{23}}{e_2} + o\left(\frac{1}{n}\right)$. So $e_3^2 = e_1e_2 + o(n^4)$ and $\frac{a_{13}e_2}{a_{23}} = \frac{a_{23}e_1}{a_{13}} + o(n^2)$. By $e_1 = (a_{12}^2 + a_{13}^2)/2 + o(n^2)$ and $e_2 = (a_{12}^2 + a_{23}^2)/2 + o(n^2)$, we have

$$a_{13}^2\left(a_{12}^2 + a_{23}^2\right) = a_{23}^2\left(a_{12}^2 + a_{13}^2\right) + o(n^4).$$

Hence $a_{12} = 0$ or $a_{13} = a_{23} + o(n)$. If $a_{12} = 0$, then $e_3 - \frac{a_3^2}{2} \geqslant \max\{e_1, e_2\}$. Then since $|A_3| = \Omega(n)$, we know $e_3^2 = e_1e_2 + \Omega(n^4)$, a contradiction with $e_3^2 = e_1e_2 + o(n^4)$. So we have $a_{13} = a_{23} + o(n)$. Thus by

$$\frac{a_{13}}{e_1} = \frac{a_{23}}{e_3} + o\left(\frac{1}{n}\right),$$

we obtain $e_1 = e_3 + o(n^2)$. Hence,

$$\frac{(a_{12}^2 + a_{13}^2)}{2} = \frac{(n - a_{12})^2}{2} - a_{13}a_{23} + o(n^2) = \frac{(n - a_{12})^2}{2} - a_{13}^2 + o(n^2).$$

Then $(n^2 - 2na_{12})/2 = 3a_{13}^2/2 + o(n^2) \leqslant n^2\big(3/8 + o(1)\big)$, so $n\big(1/2 + o(1)\big) \geqslant a_{12} \geqslant n\big(1/8 - o(1)\big)$. It follows that

$$\begin{aligned}
e_1 + e_2 + e_3 &= \frac{a_{12}^2 + a_{13}^2}{2} + \frac{a_{12}^2 + a_{23}^2}{2} + \left(\frac{(n - a_{12})^2}{2} - a_{13}a_{23}\right) + o(n^2) \\
&= a_{12}^2 + \frac{(n - a_{12})^2}{2} + \frac{(a_{13} - a_{23})^2}{2} + o(n^2) \\
&= a_{12}^2 + \frac{(n - a_{12})^2}{2} + o(n^2) \\
&= \frac{3}{2}\left(a_{12} - \frac{n}{3}\right)^2 + \frac{n^2}{3} + o(n^2) \\
&\leqslant \frac{51n^2}{128} + o(n^2) < 0.47n^2.
\end{aligned}$$

Case 4: A_i is empty for all $i \in [3]$.

First, observe that A_{123} is empty. Without loss of generality, we can assume $a_{12} \leqslant a_{13} \leqslant a_{23}$. Then the following holds:

$$\begin{aligned}
&\left(a_{12}^2 + a_{13}^2\right)\left(a_{13}^2 + a_{23}^2\right)\left(a_{12}^2 + a_{23}^2\right) \\
&\qquad \leqslant \left(\left(a_{13} + \frac{a_{12}}{2}\right)^2 + \left(a_{23} + \frac{a_{12}}{2}\right)^2\right)\left(a_{13} + \frac{a_{12}}{2}\right)^2\left(a_{23} + \frac{a_{12}}{2}\right)^2.
\end{aligned}$$

The preceding inequality is easy to verify by taking the difference of the right- and left-hand sides, multiplying out and pairing off the negative terms with the positive ones and by using that $a_{12} \leqslant a_{13} \leqslant a_{23}$. It follows that $a_{12} = 0$, since otherwise we could increase the product $e_1e_2e_3$ by moving half of the vertices from A_{12} to A_{13} and the other half to A_{23}. Then

$$\begin{aligned}
e_1e_2e_3 = \frac{\left(a_{13}^2 + a_{23}^2\right)a_{13}^2a_{23}^2}{8} + o(n^6) &= \frac{\left(n^2 - 2a_{13}a_{23}\right)(a_{13}a_{23})^2}{8} + o(n^6) \\
&\leqslant n^6\left(\frac{1}{256} + o(1)\right).
\end{aligned}$$

□

Theorem 9. *There exist graphs G_1, G_2, G_3, G_4 on a common vertex set of size n and with no rainbow P_4 such that*

$$e(G_1)e(G_2)e(G_3)e(G_4) \geqslant n^8\left(\frac{1}{4096} - o(1)\right).$$

Proof. Let $[n] = X \cup Y$ be a partition of $[n]$ with X and Y of sizes approximately $0.5n$. Let G_1, G_2 consist of a complete graph on X and G_3, G_4 consist of a complete graph on Y. It is easy to see that $e(G_1)e(G_2)e(G_3)e(G_4) = n^8\big(\frac{1}{8^4} - o(1)\big)$. □

Theorem 10. *For four graphs G_1, G_2, G_3, G_4 on a common vertex set of size n and with no rainbow P_4, we have*

$$e(G_1)e(G_2)e(G_3)e(G_4) \leqslant n^8\left(\frac{1}{8^4} + o(1)\right).$$

Proof. Let V be the vertex set of size n. Let G_1, G_2, G_3, G_4 be graphs on V, and let $G = \big(V, E(G_1) \cup E(G_2) \cup E(G_3) \cup E(G_4)\big)$. Let H be the graph induced by all edges with at least three colors. Then H is P_4-free. Hence $e(H) \leqslant \mathrm{ex}(n, P_4) \leqslant n$. So we can assume that there is no edge with at least three colors.

For any vertex $v \in V$ and for any $i \in [4]$, let $d_i(v)$ denote the degree of v in G_i. For any $v \in V$, if $d_i(v) = o(n)$ for some $i \in [4]$, then we assume $d_i(v) = 0$, since edges from G_i incident to v would not contribute asymptotically to the product $e(G_1)e(G_2)e(G_3)e(G_4)$. Let $c(v)$ be the subset of $\{1, 2, 3, 4\}$ such that $d_i(v) \geqslant 1$ for every $i \in c(v)$ and $d_i(v) = 0$ for every $i \notin c(v)$. Let

$$V = \left(\bigcup_{i=1}^{4} A_i\right) \cup \left(\bigcup_{1\leqslant i<j\leqslant 4} A_{ij}\right) \cup A$$

be a partition of V, where A_i is the set of vertices v with $c(v) = \{i\}$, and A_{ij} is the set of vertices v with $c(v) = \{i, j\}$ and A is the set of vertices v with $|c(v)| \geqslant 3$. Let $a_i = |A_i|$ for any $i \in [4]$, $a_{ij} = |A_{ij}|$ for any $1 \leqslant i < j \leqslant 4$ and $a = |A|$. If one part of this partition has size $o(n)$, then we assume the size of this part is 0, since edges incident to this partition class would not contribute to the asymptotic value of the product $e(G_1)e(G_2)e(G_3)e(G_4)$. Note that there are no edges in $G[A]$, there are no edges between A and $\bigcup_{1\leqslant i<j\leqslant 4} A_{ij}$, no edges between any two sets in $\{A_{ij} : 1 \leqslant i < j \leqslant 4\}$ and no edges between A_i and $\big(\bigcup_{j\in[4]\setminus\{i\}} A_j\big) \cup \big(\bigcup_{k,l\in[4]\setminus\{i\}:\, k<l} A_{kl}\big)$.

Assume G satisfies the properties above and maximizes the product of the size of the edge sets. By Theorem 9, we may assume $e(G_1)e(G_2)e(G_3)e(G_4) \geqslant n^8\big(1/4096 + o(1)\big)$. Therefore $e(G_1) + e(G_2) + e(G_3) + e(G_4) \geqslant n^2/2 + o(n^2)$. Since there are at most $n^2/2$ edges in G and no edges with at least three colors, we know that the number of edges with two colors is at least as great as the number of missing edges in G. Then

$$\sum_{1\leqslant i<j\leqslant 4} \frac{a_{ij}^2}{2} > \sum_{1\leqslant i<j\leqslant 4} \frac{a_{ij}}{2}(n - a_i - a_j - a_{ij}).$$

Hence there are $i, j \in [4]$, say 1 and 2, such that $\frac{a_{12}^2}{2} > \frac{a_{12}}{2}(n - a_1 - a_2 - a_{12})$, so $2a_{12} + a_1 + a_2 > n$. Let $B = A_1 \cup A_2 \cup A_{12}$ and $C = A_3 \cup A_4 \cup A_{34}$, let $b = |B|$ and let $c = |C|$. Then $b > n/2$ and $b + c \leqslant n$. Clearly, there are no edges of color 3 or 4 adjacent to any vertex in B and all edges of colors both 3 and 4 are in $G[A_{34}]$. It follows that

$$e(G_3) + e(G_4) \leqslant \frac{(n - a_{12} - a_1 - a_2)^2}{2} + \frac{a_{34}^2}{2} \leqslant \frac{(n-b)^2 + c^2}{2}.$$

Clearly, there are no edges of color 1 or 2 adjacent to any vertex in C, all edges of colors both 1 and 2 are in $G[A_{12}]$, and there are no edges between A_{12} and $V \setminus (B \cup C)$. Thus,

$$e(G_1) + e(G_2) \leqslant \frac{(n - a_{34} - a_3 - a_4)^2}{2} + \frac{a_{12}^2}{2} - a_{12}(n - b - c).$$

By $2a_{12} + a_1 + a_2 > n$, we have

$$e(G_1) + e(G_2) \leqslant \frac{(n-c)^2}{2} + \frac{b^2}{2} - b(n - b - c) = \frac{(n-b-c)^2 + 2b^2}{2}.$$

Then

$$e(G_1)e(G_2)e(G_3)e(G_4) \leqslant \left(\frac{(n-b-c)^2+2b^2}{4}\right)^2 \left(\frac{(n-b)^2+c^2}{4}\right)^2.$$

Since $(n-b)^2+c^2+(n-b-c)^2+2b^2 \leqslant 2(n-b)^2+2b^2$ and $(n-b-c)^2+2b^2 \geqslant 2b^2 \geqslant (n-b)^2+b^2$, we obtain

$$((n-b)^2+c^2)((n-b-c)^2+2b^2) \leqslant 4(n-b)^2b^2 \leqslant \frac{n^4}{4}.$$

Thus $e(G_1)e(G_2)e(G_3)e(G_4) \leqslant \frac{n^8}{8^4}$. □

4 Concluding Remarks

We conclude by mentioning some open problems. First, it seems plausible that for 6 graphs $G_1, G_2, \ldots, G_6$ avoiding a rainbow K_4, the asymptotically optimal configuration for maximizing the product of the edge-densities is simply 6 copies of the Turán graph with 3 parts.

Second, for $2k$ graphs avoiding a rainbow P_4 and maximizing the product of the edge-densities, one can take an independent set of size $n(2k-1)/(4k-1)$ and $2k$ cliques, each of size $n/(4k-1)$, each containing edges of one of the colors, as well as the edges between each of the cliques and the independent set in their respective colors. It appears this construction may be asymptotically optimal.

Finally, for k graphs avoiding a rainbow path P_{k+1} and maximizing the product of the edge-densities, one could take two disjoint cliques of size $n/2$, each consisting of k-colored edges from two different sets of k colors.

Acknowledgments

We would like to thank the referees for their detailed remarks, which greatly improved the presentation of the paper. The research of Frankl, Győri and Salia was partially supported by the National Research, Development and Innovation Office NKFIH, grants K132696, SNN-135643 and K126853. The research of Tompkins was supported by NKFIH grant K135800.

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