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Book free 3-uniform hypergraphs

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ABSTRACT

A k-book in a hypergraph consists of k Berge triangles sharing a common edge. In this paper we prove that the number of the hyperedges in a k-book-free 3-uniform hypergraph on n vertices is at most $\frac{n^2}{8}(1+o(1))$.

1. Introduction

Let G be a graph. The vertex and the edge set of G are denoted by $V(G)$ and $E(G)$. If there are two triangles sitting on an edge in a graph, we call this a *diamond*. Whereas k triangles sitting on an edge is called a *k-book*, denoted by a B_k. Similarly, let H be a hypergraph and the vertex and the edge set of H be denoted by $V(H)$ and $E(H)$. A hypergraph is called *r-uniform* if each member of E has size r. A hypergraph $H=(V,E)$ is called *linear* if every two hyperedges have at most one vertex in common. A Berge cycle of length k, denoted by Berge-C_k, is an alternating sequence of distinct vertices and distinct hyperedges of the form $v_1, h_1, v_2, h_2, \ldots, v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_k v_1 \in h_k$. The hypergraph equivalent of k-books is defined similarly with k Berge triangles sharing a common edge. We say that this common edge is the base of the k-book.

The maximum number of edges in a triangle-free graph is one of the classical results in extremal graph theory and proved by Mantel in 1907 [13]. The extremal problem for diamond-free graphs follows from this. Given a graph G on n vertices and having $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges. Mantel showed that G contains a triangle. Rademacher (unpublished, and simplified later by Erdős in [6]) proved in the 1940s that the number of triangles in G is at least $\left\lfloor \frac{n}{2} \right\rfloor$. Erdős conjectured in 1962 [7] that the size of the largest book in G is at least $\frac{n}{6}$ and this was proved soon after by Edwards (unpublished, see also Khadźiivanov and Nikiforov [11] for an independent proof).

Theorem 1 *(Edwards [4], Khadźiivanov and Nikiforov [11]). Every n-vertex graph with more than $\frac{n^2}{4}$ edges contains an edge that is in at least $\frac{n}{6}$ triangles.*

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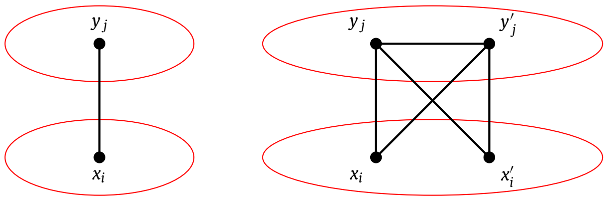

Fig. 1. Replacing every edge x_iy_i in the bipartite graph with two hyperedges $x_iy_jy'_j$ and $y_jy'_jx'_i$.

Both Rademacher's and Edwards' results are sharp. In the former, the addition of an edge to one part in the complete balanced bipartite graph (note that in G there is an edge contained in $\lfloor \frac{n}{2} \rfloor$ triangles) achieves the maximum. In the latter, every known extremal construction of G has $\Omega(n^3)$ triangles. For more details on book-free graphs we refer the reader to the following articles [2], [14] and [16]. We look into the equivalent problem in the case of hypergraphs.

Given a family of hypergraphs $\mathcal{F}$, we say that a hypergraph H is $\mathcal{F}$-free if for every $F \in \mathcal{F}$, the hypergraph H does not contain a F as a sub-hypergraph.

The systematic study of the Turán number of Berge cycles started with Lazebnik and Verstraëte [12], who studied the maximum number of hyperedges in an r-uniform hypergraph containing no Berge cycle of length less than five. Another result was the study of Berge triangles by Győri [8]. He proved that:

Theorem 2 *(Győri [8]). The maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraphs on n vertices is at most* $\frac{n^2}{8}$.

It continued with the study of Berge five cycles by Bollobás and Győri [3]. In [9], Győri, Katona, and Lemons proved the following analog of the Erdős-Gallai Theorem [5] for Berge paths. For other results see [1,10]. The particular case of determining the maximum number of the hyperedges of a triangle-free linear hypergraph on n vertices is equivalent to the famous $(6,3)$-problem, which is a special case of a general problem of Brown, Erdős, and Sós. The following theorem of Ruzsa and Szemerédi plays important role in our paper:

Theorem 3 *(Ruzsa and Szemerédi [15]). For any $\epsilon > 0$ there exists $n_0(\epsilon)$ such that if $n > n_0(\epsilon)$ then a Berge-triangle-free 3-uniform linear hypergraph on n vertices has at most ϵn^2 hyperedges.*

We continue the work on that and determine the maximum number of hyperedges for a k-book-free 3-uniform hypergraph. The main result is as follows:

Theorem 4. *For a given $k \geq 2$ and $\epsilon > 0$ there exists $n_1(k,\epsilon)$ such that if $n > n_1(k,\epsilon)$ then a 3-uniform B_k-free hypergraph H on n vertices can have at most* $\dfrac{n^2}{8} + \epsilon n^2$ *edges.*

The following example shows that this result is asymptotically sharp. Take a complete bipartite graph with color classes of size $\lceil \frac{n}{4} \rceil$ and $\lfloor \frac{n}{4} \rfloor$ respectively. Denote the vertices in each class with x_i and y_i respectively. Construct a graph by doubling each vertex and replacing each edge with two hyperedges as shown below (Fig. 1). So essentially, we have replaced every graph edge with two hyperedges. The construction does not contain a B_k, as it does not contain a Berge triangle. With this, the number of hyperedges is $2 \times \frac{n^2}{16} = \frac{n^2}{8}$.

2. Proof of Theorem 4

Fix $k \geq 2$, $\epsilon > 0$ and set

$$n_1(k,\epsilon) = \max\left\{18k+12, n_0\left(\frac{\epsilon}{6k^2-8k}\right)\right\}$$

where $n_0(.)$ is from Theorem 3. Suppose that $n > n_1(k,\epsilon)$. Let H be a B_k-free 3-uniform hypergraph on n vertices. We are interested in the 2-shadow, i.e., let G be a graph with vertex set $V(H)$ and

$$E(G) = \{ab \mid \{a,b\} \subset e \in E(H)\}.$$

If an edge in G lies in more than one hyperedge in H, we color it blue. Otherwise, we color it red. We define hypergraphs H_r and H_b in the following way. $V(H_r) = V(H_b) = V(H)$,

$$E(H_r) = \{e \in E(H) \mid e \text{ contains two or three red edges of } G\}$$

and $E(H_b) = E(H) \setminus E(H_r)$. Note that each hyperedge in H_b contains two or three blue edges of G.

Claim 5. *The number of hyperedges in H_r is at most $\frac{n^2}{8}$.*

Proof. Denote the subgraph of G formed by the red colored edges by G_r. Suppose that $|E(G_r)| \geq \frac{n^2}{4}+1$. By Theorem 1 we have a book of size $\frac{n}{6}$ in G_r. Denote the vertices of the $\frac{n}{6}$-book in G_r with u, v and x_i, $1 \leq i \leq \frac{n}{6}$ respectively where uv is the base of the book. Denote the third vertex of the hyperedge containing edge uv by w, set $X = \{x_i \mid 1 \leq i \leq \frac{n}{6}\}$ and for each $x_i \in X$ denote the hyperedge containing ux_i and vx_i by ux_iy_i and vx_iz_i respectively.

Set $E' := \emptyset$ and $X' := \emptyset$. Go through the vertices of X and perform the following procedure for each of them. At the beginning of the process no vertex is marked.

If the current vertex $x_i = w$ then mark it.

If x_i is unmarked then

- add x_i to X' and hyperedges ux_iy_i and vx_iz_i to E',
- if there exists $j > i$ such that $y_i = x_j$ then mark x_j,
- if there exists $\ell > i$ such that $z_i = x_\ell$ then mark x_ℓ.

By definition of red edges and the procedure (i.e. it adds two new hyperedges to E' forming a Berge triangle with uvw at each step handling an unmarked vertex but at most one: when $x_i = w$) the set of hyperedges $E' \cup \{uvw\}$ with vertex set $X' \cup \{u, v\}$ form a k'-book with base uvw, where $k' = |X'|$. Moreover at each step of the procedure whenever an unmarked vertex was added to X' then at most two more vertices became marked. Each unmarked vertex are in X' at the end of the procedure, therefore

$$k' \geq \frac{|X \setminus \{w\}|}{3} \geq \frac{n/6-1}{3}$$

at the end of the procedure and it is at least k by the definition of $n_1(k, \epsilon)$, but this is a contradiction.

Hence $|E(G_r)| \leq \frac{n^2}{4}$ and

$$|E(H_r)| \leq \frac{|E(G_r)|}{2} \leq \frac{n^2}{8}$$

by the definition of red colored edges. □

Now let us work on the sub-hypergraph H_b.

Claim 6. *Each pair of vertices is contained in at most $2k-2$ hyperedges of H_b.*

Proof. Suppose that $\{u, v\}$ is a pair of vertices which is contained in $2k-1$ hyperedges of H_b. Note that edge uv is colored blue. Denote the third vertices of hyperedges containing u and v by $x_1, \dots, x_{2k-1}$ and set $X = \{x_i \mid 1 \leq i \leq 2k-1\}$. Observe that for each i at least one of ux_i and vx_i is colored blue.

Set $E' := \emptyset$ and $X' := \emptyset$. Go through the vertices of X and perform the following procedure for each of them. At the beginning of the process no vertex is marked.

If the current vertex $x_i = x_{2k-1}$ and there is no marked vertex in X then do nothing.

Otherwise if x_i is unmarked then

- add x_i to X' and add ux_iv to E',
- if ux_i is colored blue denote a hyperedge containing it by ux_iy_i where $y_i \neq v$ and add ux_iy_i to E',
- otherwise vx_i is colored blue, so denote a hyperedge containing it by vx_iy_i where $y_i \neq u$ and add vx_iy_i to E',
- if there exists $j > i$ such that $y_i = x_j$ then mark x_j.

If at the end of the procedure there is no marked vertex in X then set $w = x_{2k-1}$ otherwise let w be an arbitrary marked vertex.

By definition of blue edges and the procedure (i.e. it adds two new hyperedges to E' forming a Berge triangle with uvw at each step handling an unmarked vertex but at most the last one) the set of hyperedges $E' \cup \{uvw\}$ with vertex set $X' \cup \{u, v\}$ form a k'-book with base uvw where $k' = |X'|$. Moreover if there is no marked vertex in X at the end of the process then $X' = X \setminus \{x_{2k-1}\}$, otherwise at each step of the procedure whenever an unmarked vertex was added to X' than at most one more vertex became marked and each unmarked vertex are in X' at the end of the procedure. Therefore $k' \geq k$, but it is a contradiction. □

We now give an upper bound on the number of hyperedges in H_b.

Claim 7. *The number of hyperedges in H_b is at most ϵn^2.*

Proof. Take a hyperedge xyz in the sub-hypergraph H_b. By Claim 6 there are at most $2k-2$ hyperedges of H_b containing each of the pairs of vertices xy, yz, and xz. If we deleted all such hyperedges barring xyz we would delete at most $6k-9$ hyperedges. Therefore there is a linear 3-uniform subhypergraph H'_b of H_b with $V(H'_b)=V(H_b)=V(H)$ and

$$|E(H'_b)| \geq \frac{|E(H_b)|}{6k-8}$$

(i.e. a greedy algorithm can find an appropriate H'_b).

Consider a hyperedge e in H'_b. Observe that H'_b is a B_k-free hypergraph, since it is a subhypergraph of H, therefore the number of Berge triangles sitting on the edge e is at most $k-1$. Apply the following greedy procedure until all the hyperedges are marked. In a step pick an unmarked hyperedge, mark it and delete an unmarked hyperedge of each Berge triangle containing the current hyperedge. Observe that this marked edge is not an edge of a triangle anymore. Define H''_b the following way. Let $V(H''_b)=V(H'_b)=V(H)$ and $E(H''_b)$ contains the remaining hyperedges of H'_b. Observe that at most $k-1$ edges were deleted in each step and marked edges were never deleted. Therefore

$$|E(H''_b)| \geq \frac{|E(H'_b)|}{k}.$$

Moreover H''_b is a Berge-triangle-free 3-uniform linear hypergraph therefore Theorem 3 can be applied with $\epsilon' = \frac{\epsilon}{6k^2-8k}$. We get that

$$\frac{|E(H_b)|}{6k^2-8k} \leq |E(H''_b)| \leq \frac{\epsilon n^2}{6k^2-8k}. \quad \square$$

Proof of Theorem 4. By definition $E(H)$ is a disjoint union of $E(H_r)$ and $E(H_b)$. By Claim 5 and Claim 7,

$$|E(H)| \leq |E(H_r)| + |E(H_b)| \leq \frac{n^2}{8} + \epsilon n^2. \quad \square$$

3. Conclusions

Recall that both Turán numbers of triangle-free graph and k-book-free graphs on n vertices are $\frac{n^2}{4}$, moreover Győri [8] proved that the maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraphs on n vertices is at most $\frac{n^2}{8}$. Given the similarities, we conjecture the following:

Conjecture 1. *For a given $k \geq 2$ every 3-uniform B_k-free hypergraph H on n vertices (n is large) has at most $\frac{n^2}{8}$ hyperedges.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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