

Maximum cliques in a graph without disjoint given subgraph

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Abstract

The generalized Turán number $\text{ex}(n, K_s, F)$ denotes the maximum number of copies of K_s in an n -vertex F -free graph. Let kF denote k disjoint copies of F . Gerbner, Methuku and Vizer [DM, 2019, 3130-3141] gave a lower bound for $\text{ex}(n, K_3, 2C_5)$ and obtained the magnitude of $\text{ex}(n, K_s, kK_r)$. In this paper, we determine the exact value of $\text{ex}(n, K_3, 2C_5)$ and described the unique extremal graph for large n . Moreover, we also determine the exact value of $\text{ex}(n, K_r, (k + 1)K_r)$ which generalizes some known results.

Keywords: Generalized Turán number, disjoint union, extremal graph.

1 Introduction

Let G be a graph with the set of vertices $V(G)$. For two graphs G and H , let $G \cup H$ denote the disjoint union of G and H , and kG denote k disjoint copies of G . We write $G + H$ for the join of G and H , the graph obtained from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$. We use K_n, C_n, P_n to denote the complete graph, cycle, and path on n vertices, respectively. Let $K_s(G)$ denote the number of copies of K_s in G .

For a graph F , the Turán number of F , denote by $\text{ex}(n, F)$, is the maximum number of edges in an F -free graph G on n vertex. In 1941, Turán [19] proved that the balanced complete r -partite graph on n vertices, called Turán graph $T_r(n)$, is the unique extremal graph of $\text{ex}(n, K_{r+1})$. Starting from this, the Turán problem has attracted a lot of attention. The study of disjoint copies of a given graph in the context of Turán numbers is very rich. The first result is due to Erdős and Gallai [5] who determined the Turán number of $\text{ex}(n, kK_2)$ for all n . Later Simonovits [18] and independently Moon [17]

determined the Turán number of disjoint copies of cliques. In [10] Gorgol initiated the systematic investigation of Turán numbers of disjoint copies of graphs and proved the following.

Theorem 1 (Gorgol [10]) *For every graph F and $k \geq 1$,*

$$\text{ex}(n, kF) = \text{ex}(n, F) + O(n).$$

In this paper we study the generalized Turán number of disjoint copies of graphs. The generalized Turán number $\text{ex}(n, T, F)$ is the maximum number of copies of T in any F -free graph on n vertices. Obviously, $\text{ex}(n, K_2, F) = \text{ex}(n, F)$. The earliest result in this topic is due to Zykov [23] who proved that $\text{ex}(n, K_s, K_r) = K_s(T_{r-1}(n))$.

Theorem 2 (Zykov [23]) *For all n ,*

$$\text{ex}(n, K_s, K_r) = K_s(T_{r-1}(n)),$$

and $T_{r-1}(n)$ is the unique extremal graph.

In recent years, the problem of estimating generalized Turán number has received a lot of attention. Many classical results have been extended to generalized Turán problem, see [1, 4, 11, 12, 15, 16, 20, 22].

Theorem 1 implies that the classical Turán number $\text{ex}(n, kF)$ and $\text{ex}(n, F)$ always have the same order of magnitude. However, this is not true for generalized Turán number. The function $\text{ex}(n, K_3, C_5)$ has attracted a lot of attentions, see [2, 6, 7], the best known upper bound is given by Lv and Lu,

Theorem 3 (Lv and Lu [14]) $\text{ex}(n, K_3, C_5) \leq \frac{1}{2\sqrt{6}}n^{\frac{3}{2}} + o(n^{\frac{3}{2}})$.

And Gerbner, Methuku and Vizer [8] proved $\text{ex}(n, K_3, 2C_5) = \Theta(n^2)$ [8]. This implies that the order of magnitudes of $\text{ex}(n, H, F)$ and $\text{ex}(n, H, kF)$ may differ. They also obtained a lower bound for $\text{ex}(n, K_3, 2C_5)$ which is obtained by joining a vertex to a copy of $T_2(n-1)$. In this paper, we show the graph $K_1 + T_2(n-1)$ is indeed the unique extremal graph for $\text{ex}(n, K_3, 2C_5)$.

Theorem 4 *For sufficiently large n ,*

$$\text{ex}(n, K_3, 2C_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor,$$

and $K_1 + T_2(n-1)$ is the unique extremal graph.

We also focus on the generalized the Turán number of disjoint copies of cliques. Since $\text{ex}(n, K_s, K_r)$ is known [23], it is natural to study the function $\text{ex}(n, K_s, kK_r)$. Gerbner, Methuku and Vizer [8] obtained the asymptotic value of $\text{ex}(n, K_s, kK_r)$.

Theorem 5 (Gerbner, Methuku and Vizer [8]) *If $s < r$, then*

$$\text{ex}(n, K_s, kK_r) = (1 + o(1)) \binom{r-1}{s} \left(\frac{n}{r-1}\right)^s.$$

If $s \geq r \geq 2$ and $k \geq 2$, then

$$\text{ex}(n, K_s, kK_r) = \Theta(n^x),$$

where $x = \left\lceil \frac{kr-s}{k-1} \right\rceil - 1$.

Liu and Wang [13] determined the exact value of $\text{ex}(n, K_r, 2K_r)$ for $r \geq 3$ and n sufficiently large. A new proof of $\text{ex}(n, K_r, 2K_r)$ can be found in [21] by Yuan and Yang. Gerbner and Patkós [9] determined $\text{ex}(n, K_s, 2K_r)$ for all $s \geq r \geq 3$ and n sufficiently large. In this paper, we determine the value of $\text{ex}(n, K_r, (k+1)K_r)$ for all $r \geq 2$, $k \geq 1$ and n sufficiently large.

Theorem 6 *There exists a constant $n_0(k, r)$ depending on k and $r \geq 2$ such that when $n \geq n_0(k, r)$,*

$$\text{ex}(n, K_r, (k+1)K_r) = K_r(K_k + T_{r-1}(n-k)),$$

and $K_k + T_{r-1}(n-k)$ is the unique extremal graph.

The detailed proofs of Theorems 4 and 6 will be presented in Sections 3 and 4, respectively.

2 Proof of Theorem 4

Suppose n is large enough and let G be an n -vertex $2C_5$ -free graph with $\text{ex}(n, K_3, 2C_5)$ copies of triangles. Since $K_1 + T_2(n-1)$ contains no $2C_5$, thus $K_3(G) \geq \lfloor (n-1)^2/4 \rfloor$. Next we will show that $G = K_1 + T_2(n-1)$. Since n is sufficiently large and by Theorem 3, G must contain a copy of C_5 , say $C = v_1v_2v_3v_4v_5v_1$. Then $G \setminus C$ contains no C_5 . By Theorem 3 again, we have

$$K_3(G \setminus C) \leq \frac{1}{2\sqrt{2}}(n-5)^{\frac{3}{2}} + o((n-5)^{\frac{3}{2}}).$$

We claim that there is at least one vertex in $V(C)$ whose neighborhood contains a copy of $6P_4$. To prove this, we need a theorem obtained by Bushaw and Kettle [3].

Theorem 7 (Bushaw and Kettle[3]) *For $k \geq 2$, $\ell \geq 4$ and $n \geq 2\ell + 2k\ell(\lfloor \ell/2 \rfloor + 1) \binom{\ell}{\lfloor \ell/2 \rfloor}$,*

$$\text{ex}(n, kP_\ell) = \binom{k\lfloor \ell/2 \rfloor - 1}{2} + (k\lfloor \ell/2 \rfloor - 1)(n - k\lfloor \ell/2 \rfloor + 1) + \lambda,$$

where $\lambda = 1$ if ℓ is odd, and $\lambda = 0$ if ℓ is even.

By Theorem 7, we know $\text{ex}(n, 6P_4) \leq \max\left\{\binom{872}{2}, 11(n-6)\right\}$. Now suppose no vertex in $V(C)$ contains $6P_4$ in its neighborhood. Then the number of triangles containing v_i is at most

$$e(G[N(v_i)]) \leq \text{ex}(n, 6P_4) = 11n + o(n).$$

Therefore, the total number of triangles satisfies

$$\begin{aligned} K_3(G) &\leq \frac{1}{2\sqrt{2}}n^{\frac{3}{2}} + o(n^{\frac{3}{2}}) + 55n + o(n) \\ &= \frac{1}{2\sqrt{2}}n^{\frac{3}{2}} + o(n^{\frac{3}{2}}) \\ &< \frac{(n-1)^2}{4}. \end{aligned}$$

The last inequality holds when n is large. A contradiction.

Therefore, we may assume that v_1 is the vertex in $V(C)$ such that $G[N(v_1)]$ contains a copy of $6P_4$. If $G \setminus v_1$ contains a copy of C_5 , then at least one copy of P_4 in $G[N(v_1)]$ does not intersect with this C_5 and hence we find two disjoint C_5 , a contradiction. Thus $G \setminus v_1$ is C_5 -free. So we have

$$K_3(G) \leq e(G \setminus v_1) + K_3(G \setminus v_1). \quad (2.1)$$

So if we have $e(G \setminus v_1) + K_3(G \setminus v_1) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$, then the proof is completed. To prove this, we need the following lemma.

Lemma 1 *Let $n \geq 2\binom{68}{3}$. If G is a C_5 -free graph on n vertices, then*

$$e(G) + K_3(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,$$

and equality holds if and only if $G = T_2(n)$.

Proof. For each integer n , let G_n be a C_5 -free graph of n vertices such that $e(G_n) + K_3(G_n)$ is maximum. For every n , if G_n is also triangle-free, then by Turán Theorem [19], $e(G_n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Hence, $e(G_n) + K_3(G_n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ and equality holds if and only if $G_n = T_2(n)$, we are done.

Next we shall prove that from $n \geq 2\binom{68}{2}$, each G_n is triangle-free. To do this, let us define a function

$$\phi(n) := e(G_n) + K_3(G_n) - \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Since $T_2(n)$ is C_5 -free and $e(T_2(n)) + K_3(T_2(n)) = \left\lfloor \frac{n^2}{4} \right\rfloor$, we have $\phi(n) \geq 0$. We claim

that from $n \geq 68$, if G_n contains a triangle, then

$$\phi(n) < \phi(n-1) - 1. \quad (2.2)$$

First suppose that $\delta(G_n) \geq \frac{n}{4} - 1$. Let xy be the edge of G_n which is contained in the most number of triangles. Set $W = N(x) \cap N(y) = \{z_1, \dots, z_w\}$. Since G_n is C_5 -free, $G_n[W]$ contains no edge unless $w \leq 2$. Let $D_0 = N(x) \setminus (W \cup \{y\})$, $D_i = N(z_i) \setminus (W \cup \{x, y\})$ for $1 \leq i \leq w$ and $D_{w+1} = N(y) \setminus (W \cup \{x\})$. We next show that D_i satisfy the following properties for $0 \leq i \leq w+1$.

- (P1) $|D_i| \geq \frac{n}{4} - w - 2$ for $i = 0, w+1$ and $|D_j| \geq \frac{n}{4} - 4$ for $1 \leq j \leq w$;
- (P2) $D_i \cap D_j = \emptyset$ for $0 \leq i \neq j \leq w+1$;
- (P3) There are no edges between D_i, D_j .

Since $\delta(G_n) \geq \frac{n}{4} - 1$, (P1) is clearly true. Since G_n is C_5 -free, it is easy to see that $D_i \cap D_j = \emptyset$ for $1 \leq i \neq j \leq w$. Suppose $D_0 \cap D_i \neq \emptyset$ or $D_{w+1} \cap D_i \neq \emptyset$ for some $1 \leq i \leq w$, by symmetry, let $v \in D_0 \cap D_i$. Then by the choice of xy , we have $w \geq 2$. For $1 \leq j \leq w$ and $j \neq i$, $vz_i y z_j x v$ is a copy of C_5 , a contradiction. Thus (P2) holds. Suppose uv is an edge with $u \in D_i, v \in D_j$, then $uz_i y z_j v u$ is a copy of C_5 if $i, j \in [1, w]$, $uz_i y x v u$ or $uz_i x y v u$ is a copy of C_5 if $i \in [1, w]$ and $j \in \{0, w+1\}$, $ux z_1 y v u$ is a copy of C_5 if $i = 0, j = w+1$, a contradiction. This implies (P3) holds.

Let $N = V(G_n) - W \cup \{x, y\} - \cup_{i=0}^{w+1} D_i$. By (P1) and (P2), we have

$$n = |N| + \sum_{i=0}^{w+1} |D_i| + w + 2 \geq |N| + 2\left(\frac{n}{4} - w - 2\right) + w\left(\frac{n}{4} - 4\right) + w + 2,$$

which implies $w \leq 2$, $|N| \leq \frac{n}{4} + 7$ and $D_i \neq \emptyset$ when $n \geq 61$. By the choice of xy , each vertex of D_i has at most two neighbors in $G_n[D_i]$ for $0 \leq i \leq w+1$ since there is no edge in 3 triangles. By (P3) and $\delta(G_n) \geq \frac{n}{4} - 1$, each vertex in D_i has at least $\frac{n}{4} - 4$ neighbors in N . Let $v_0 \in D_0$ and $v_1 \in D_{w+1}$. Because $n \geq 68$, we can deduce that $2(\frac{n}{4} - 4) > \frac{n}{4} + 7 \geq |N|$ and hence $N(v_0) \cap N(v_1) \cap N \neq \emptyset$. Then $uv_0 x y v_1 u$ is a copy of C_5 , where $u \in N(v_0) \cap N(v_1) \cap N$, a contradiction. We are done if the minimum degree is at least $\frac{n}{4} - 1$.

Therefore, there is one vertex v in G_n such that $d(v) < \frac{n}{4} - 1$ when $n \geq 68$. Because G_n is C_5 -free, $G_n[N(v)]$ is the disjoint union of stars and triangles which implies $e(G_n[N(v)]) \leq d(v)$. If we delete v from G_n , it will destroy at most $d(v)$

triangles and delete $d(v)$ edges. Hence,

$$\begin{aligned}
& \phi(n-1) - \phi(n) \\
&= \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - \{(e(G_n) + K_3(G_n)) - (e(G_{n-1}) + K_3(G_{n-1}))\} \\
&\geq \frac{2n-2}{4} - \{(e(G_n) + K_3(G_n)) - (e(G_n - v) + K_3(G_n - v))\} \\
&\geq \frac{2n-2}{4} - 2d(v) > \frac{2n-2}{4} - 2\left(\frac{n}{4} - 1\right) > 1.
\end{aligned}$$

Hence our claim(inequality 2.2) holds for $n \geq 68$.

Note that for $n_0 \geq 68$, if G_{n_0} contains no triangle, then $\phi(n_0) = 0$. Moreover, for every $n \geq n_0$, we have that G_n contains no triangles, either. Otherwise, we can find an integer n such that G_n contains a triangle but G_{n-1} is triangle-free. But then $\phi(n) \leq \phi(n-1) - 1 < 0$ by inequality 2.2, which is contrary to $\phi(n) \geq 0$. Now let n_0 be the first integer after 68 such that G_{n_0} is triangle-free. Then

$$0 \leq \phi(n_0) \leq \phi(n_0 - 1) - 1 < \phi(68) - (n_0 - 68) \leq \binom{68}{2} + \binom{68}{3} + 68 - n_0.$$

This implies $n_0 \leq 2\binom{68}{3}$. Thus G_n must be triangle-free for $n \geq 2\binom{68}{3} \geq n_0$. So $e(G_n) + K_3(G_n) = e(G_n) = \lfloor n^2/4 \rfloor$ and $G_n = T_2(n)$ by Turán Theorem [19]. The proof of Lemma 1 is completed. \square

Combining equation (2.1) and Lemma 1, we can see that when n is large, $K_3(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ and equality holds if and only if $G = K_1 + T_2(n-1)$. The proof of Theorem 4 is completed. \blacksquare

3 Proof of Theorem 6

We prove it by induction on r and in each case, we always assume $n \geq n_0(k, r) =$. The base case $r = 2$ is the celebrated Erdős-Gallai Theorem [5], which says that

$$\text{ex}(n, K_2, (k+1)K_2) = \max \left\{ \binom{2k+1}{2}, (n-k)k + \binom{k}{2} \right\}.$$

As $n \geq n_0(k, 2)$, we know $\text{ex}(n, K_2, (k+1)K_2) = K_2(K_k + T_1(n-k))$.

Let $r \geq 3$ and suppose that the result holds for all $r' < r$. Next we consider the case $\text{ex}(n, K_r, (k+1)K_r)$. Let G be a $(k+1)K_r$ -free graph on n vertices with $\text{ex}(n, K_r, (k+1)K_r)$ copies of K_r . We may assume that G contains k disjoint copies of K_r . Otherwise we can add some edges into G until the resulting graph contains k disjoint K_r . But at least one K_r in these k disjoint K_r is new which implies that the

number of K_r is increased, a contradiction. Let

$$I = \{X_1, \dots, X_k\}$$

be a set of k disjoint r -cliques in G , where X_i is a copy of K_r . Let $V(I) = \cup_{i=1}^k V(X_i)$ and $N = G \setminus V(I)$. Clearly, N contains no K_r . We say a vertex v in I is joined to an $(r-1)$ -clique in N if v is adjacent to all vertices of this $(r-1)$ -clique. For each X_i , $i \in [k]$, we have the following property.

Claim 1 *Each X_i contains at most one vertex which is joined to at least $kr+1$ disjoint $(r-1)$ -cliques in N .*

Proof. If not, suppose $u_1, u'_1 \in V(X_1)$ are both joined to $kr+1$ disjoint $(r-1)$ -cliques. First we can find an $(r-1)$ -clique joined to u_1 in N . Since u'_1 is also joined to at least $kr+1$ disjoint $(r-1)$ -cliques in N , we can find another $(r-1)$ -clique joined to u'_1 which does not intersect with the $(r-1)$ -clique joined to u . Together with $\{X_2, \dots, X_k\}$, we find a copy of $(k+1)K_r$, a contradiction. \square

By Claim 1, let $A = \{X_1, \dots, X_a\}$ be a subset of I such that there exists a vertex in X_i , say u_i , that is joined to at least $kr+1$ disjoint $(r-1)$ -cliques in N for each $i \in [a]$. Let $U = \{u_1, \dots, u_a\}$.

Since N is K_r -free, each K_r in G must intersect with some vertices in $V(I)$. Then all r -cliques can be divided into two classes: the set of cliques in which all vertices are contained in $V(N) \cup U$, and the set of cliques containing at least one vertex in $V(I) \setminus U$. We simply use $K_r(U)$ and $K_r(\overline{U})$ to denote the number of copies of K_r in these two classes, respectively.

Suppose a K_r in the first class contains s vertices in U and $r-s$ vertices in N , the number of K_r 's of this type is at most $\binom{a}{s} K_{r-s}(N)$. Since N is K_r -free and by Theorem 2, which says $\text{ex}(n, K_s, K_r) = K_s(T_{r-1}(n))$, we have $K_{r-s}(N) \leq K_{r-s}(T_{r-1}(n-kr)) \leq \binom{r-1}{r-s} \left(\frac{n-kr}{r-1}\right)^{r-s}$. Then

$$\begin{aligned} K_r(U) &\leq \sum_{s=1}^r \binom{a}{s} K_{r-s}(N) \\ &\leq a \left(\frac{n-kr}{r-1}\right)^{r-1} + \binom{a}{2} \binom{r-1}{r-2} \left(\frac{n-kr}{r-1}\right)^{r-2} + O(n^{r-3}). \end{aligned} \quad (3.1)$$

Next we calculate the size of $K_r(\overline{U})$. Each vertex $v \in V(I) \setminus U$ is joined to at most kr independent $(r-1)$ -cliques in N . Hence the number of K_r containing v and $r-1$

vertices of N is at most

$$\begin{aligned} K_{r-1}(G[N(v) \cap V(N)]) &\leq \text{ex}(n - kr, K_{r-1}, (kr + 1) \cdot K_{r-1}) \\ &= K_{r-1}(K_{kr} + T_{r-2}(n - 2kr)) \\ &\leq (kr) \left(\frac{n - 2kr}{r - 2} \right)^{r-2}, \end{aligned}$$

the second equality comes from the induction hypothesis. Any other copies of K_r in $K_r(\bar{U})$ contains at most $r - 2$ vertices in N and at least one vertex in $V(I) \setminus U$. So the number of such r -cliques is at most

$$\sum_{s=2}^r \left(\binom{kr}{s} - \binom{a}{s} \right) K_{r-s}(N) \leq \left(\binom{kr}{2} - \binom{a}{2} \right) \binom{r-1}{r-2} \left(\frac{n - kr}{r-1} \right)^{r-2} + O(n^{r-3}).$$

Hence,

$$K_r(\bar{U}) \leq \left(kr + \left(\binom{kr}{2} - \binom{a}{2} \right) \binom{r-1}{r-2} \right) \left(\frac{n - kr}{r-1} \right)^{r-2} + O(n^{r-3}). \quad (3.2)$$

Therefore, by inequality (3.1) and (3.2), we have

$$K_r(G) \leq a \left(\frac{n - kr}{r-1} \right)^{r-1} + \left(kr + \binom{kr}{2} \binom{r-1}{r-2} \right) \left(\frac{n - kr}{r-1} \right)^{r-2} + O(n^{r-3}). \quad (3.3)$$

On the other hand, since $K_k + T_{r-1}(n - k)$ is $(k + 1)K_r$ -free, we know that

$$K_r(G) \geq k \left(\frac{n - k}{r-1} \right)^{r-1} + O(n^{r-2}). \quad (3.4)$$

When n is greater than some constant $n_0(k, r)$, inequalities (3.3) and (3.4) hold mean $a = k$ and then $U = \{u_1, \dots, u_k\}$.

Let $G' = G \setminus U$. We claim that G' is also K_r -free. Suppose not, G' contains a r -clique, denote by X'_0 . Since each u_i is joined to at least $kr + 1$ independent copies of K_{r-1} 's in N , at least $(k - 1)r + 1$ of whom are disjoint with X'_0 for each $i \in [k]$. Then we can find a r -clique X'_1 such that $u_1 \in X'_1$ and $V(X'_1) \cap V(X'_0) = \emptyset$. Next, we claim that we may find another k independent r -cliques such that each is disjoint with X'_0 . Suppose we have found pairwise disjoint r -cliques X'_1, \dots, X'_{i-1} such that $u_j \in X'_j$ for $j \in [i - 1]$ and $i \leq k$. Then, in $G'[N(u_i)]$, there are at least $(k - 1)r + 1 - (i - 1)(r - 1) \geq 1$ independent $(r - 1)$ -cliques which disjoint with $\{X'_0, X'_1, \dots, X'_{i-1}\}$. That is we can choose a $(r - 1)$ -clique and thus a r -clique X'_i such $u_i \in X'_i$ and X'_0, X'_1, \dots, X'_i are pairwise disjoint. The procedure can keep going until we find k independent r -cliques X'_1, \dots, X'_k . Then X'_0, X'_1, \dots, X'_k forms a $(k + 1)K_r$, a contradiction.

Since G' is K_r -free, by Zykov's Theorem, $K_{r-i}(G') \leq K_{r-i}(T_{r-1}(n - k))$ and the

equality holds if and only if $G' = T_{r-1}(n - k)$. Thus

$$K_r(K_k + T_{r-1}(n - k)) \leq K_r(G) \leq \sum_{i=0}^r \binom{k}{i} K_{r-i}(G') = K_r(K_k + T_{r-1}(n - k)).$$

The condition of the equality holds means $G = K_k + T_{r-1}(n - k)$. The proof of Theorem 6 is completed. \blacksquare

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