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THE PROFILE POLYTOPE OF NONTRIVIAL INTERSECTING FAMILIES*

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Abstract. The profile vector of a family $\mathcal{F}$ of subsets of an n-element set is $(f_0, f_1, \ldots, f_n)$, where f_i denotes the number of the i-element members of $\mathcal{F}$. In this paper we determine the extreme points of the set of profile vectors for the class of nontrivial intersecting families.

Key words. profile polytope, intersecting, nontrivial intersecting

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1. Introduction and preliminaries. Let $[n] = \{1, \ldots, n\}$ be our underlying set. If $F \subseteq [n]$, then $\overline{F}$ denotes the complement of F. Let $\mathcal{F}$ be a family of subsets of $[n]$ (i.e., $\mathcal{F} \subseteq 2^{[n]}$). Let $\overline{\mathcal{F}} := \{F \subset [n] : \overline{F} \in \mathcal{F}\}$. A family is called *intersecting* if any two members have a nonempty intersection. Intersecting families of sets have attracted a lot of researchers; see, e.g., Chapter 2 of the book [9]. Let us start with a well-known and trivial statement.

PROPOSITION 1. *The maximum size of an intersecting family is* 2^{n-1}.

The maximum size is achieved, e.g., by the family of all subsets containing a given fixed element. A family is called *k-uniform* if all its members have cardinality k. Let $\mathcal{F}_k$ denote the subfamily of the k-element subsets in $\mathcal{F}$: $\mathcal{F}_k = \{F : F \in \mathcal{F}, |F| = k\}$.

THEOREM 2 (Erdős, Ko, and Rado [2]). *Let* $k \le n/2$. *Then the maximum size of a k-uniform intersecting family is* $\binom{n-1}{k-1}$.

Let us call an intersecting family *trivial* if all its members contain a given fixed element, and nontrivial otherwise. The maximum in the above theorem is again achieved by the largest trivial intersecting family.

THEOREM 3 (Hilton and Milner [10]). *Let* $k \le n/2$. *Then the maximum size of a nontrivial k-uniform intersecting family is* $1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$.

The maximum is given by the Hilton–Milner type family $HM(k)$, which we define next. $HM(k)$ contains $A = \{2, \ldots, k+1\}$ and every k-element set which contains 1 and intersects A. Moreover, Hilton and Milner [10] also showed that $HM(k)$ is the unique maximum if $3 < k < n/2$, up to isomorphism (i.e., in every maximum family there is a fixed point contained in all but one of the members). If $k = 3$, then there is another extremal family $\{F \in \binom{[n]}{3} : |F \cap [3]| \ge 2\}$.

We will also use the following generalized Hilton–Milner type families. Let $B = [m]$. Then $HM(k, m) = \{F \subset [n] : |F| = k, n \in F, |F \cap B| \ge 1\}$. This is a k-uniform, trivial intersecting family, but if we add B, it becomes a nontrivial intersecting family.

We will use the following simple observation.

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LEMMA 4. *Let m be the maximum cardinality of the members of a nontrivial intersecting family $\mathcal{F}$, and let $i<m$ and $i \le n/2$. Then $|\mathcal{F}_i| \le |HM(i,m)|$. Moreover, if $i < n/2$, equality holds if and only if $\mathcal{F}_i$ consists of all the sets containing a fixed element x and intersecting an m-element set B' with $x \notin B'$ (i.e., $\mathcal{F}_i$ is isomorphic to $HM(i,m)$) or $i=2$, $m=3$, and $\mathcal{F}_i = \{\{a,b\},\{a,c\},\{b,c\}\}$.*

Proof. If $\mathcal{F}_i$ is nontrivial, then $|\mathcal{F}_i| \le |HM(i,i)|+1 \le |HM(i,m)|$. Here we have equality only if $i=2$ and $m=3$, since $HM(i,m)$ contains $HM(i,i)$ and we obtain new members by picking n, any element of $[m]\setminus[i]$, and any further elements if $i>2$. If $\mathcal{F}_i$ is trivial, all its members contain a fixed element x. There is a set F in $\mathcal{F}$ which does not contain x because of the nontriviality, and the i-element sets of $\mathcal{F}$ also intersect F. One can easily see that $\mathcal{F}_i$ is a subset of a generalized Hilton–Milner type family then, and $HM(i,m)$ is the largest of those. ∎

We will use the Kruskal–Katona theorem [11, 12]. Given a k-uniform family $\mathcal{F} \subset 2^{[n]}$, its *shadow* is

$$\Delta\mathcal{F} := \{G \subset [n] : |G| = k-1, \text{ there exists } F \in \mathcal{F} \text{ with } G \subset F\}.$$

The *shade* of $\mathcal{F}$ is $\nabla\mathcal{F} := \{G \subset [n] : |G| = k+1, \text{ there exists } F \in \mathcal{F} \text{ with } G \supset F\}$. Given two sets $F, G \subset 2^{[n]}$, we say that F is before G in the *colexicographical order* or *colex order* if the largest element of the symmetric difference of F and G is in G. Let $\mathcal{C}_k^\ell$ denote the family of the first ℓ sets from $\binom{[n]}{k}$ in the colex order.

Given two positive integers ℓ and i, there is a unique way to write ℓ in the form $\ell = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}$ with $n_i > n_{i-1} > \ldots > n_j \ge j \ge 1$. This form is called the *cascade form* of ℓ. The cascade form can be found in a greedy way: we pick the largest n_i such that $\binom{n_i}{i} \le \ell$, then the largest n_{i-1} such that $\binom{n_i}{i} + \binom{n_{i-1}}{i-1} \le \ell$, and so on.

The Kruskal–Katona shadow theorem [11, 12] states that if $\mathcal{F}$ is a k-uniform family with $|\mathcal{F}| = \ell$, then $|\Delta\mathcal{F}| \ge |\Delta\mathcal{C}_k^\ell|$. It is not hard to calculate the cardinality of $|\Delta\mathcal{C}_k^\ell|$: if $\ell = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_j}{j}$, then $|\Delta\mathcal{C}_k^\ell| = \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \cdots + \binom{n_j}{j-1}$.

There is a simpler version of the shadow theorem due to Lovász [13]. It states that if $\mathcal{F}$ is a k-uniform family with $|\mathcal{F}| = \binom{x}{k}$, then $|\Delta\mathcal{F}| \ge \binom{x}{k-1}$. Here x is not necessarily an integer and $\binom{x}{k}$ is defined to be $\frac{x(x-1)\ldots(x-k+1)}{k!}$. This is a weaker bound, but easier to use. We will use both versions of the shadow theorem later.

1.1. Profile polytopes. The profile polytopes were introduced by Erdős, Frankl, and Katona in [3]. Recall that $\mathcal{F}_i$ denotes the subfamily of the i-element subsets in $\mathcal{F}$. Its size $|\mathcal{F}_i|$ is denoted by f_i. The vector $\mathbf{p}(\mathcal{F}) = (f_0, f_1, \ldots, f_n)$ in the $(n+1)$-dimensional Euclidian space $\mathbb{R}^{n+1}$ is called the *profile* or *profile vector* of $\mathcal{F}$.

If Λ is a finite set in $\mathbb{R}^d$, its *convex hull* $\operatorname{conv}(\Lambda)$ is the set of all convex combinations of the elements of Λ. A point of Λ is an *extreme point* if it is not a convex combination of other points of Λ. It is easy to see that the convex hull of a set is equal to the convex hull of the extreme points of the set.

Let $\mathbf{A}$ be a class of families of subsets of $[n]$. We denote by $\Lambda(\mathbf{A})$ the set of profiles of the families belonging to $\mathbf{A}$:

$$\Lambda(\mathbf{A}) = \{\mathbf{p}(\mathcal{F}) : \mathcal{F} \in \mathbf{A}\}.$$

The *profile polytope* of $\mathbf{A}$ is $\operatorname{conv}(\Lambda(\mathbf{A}))$. We are interested in the extreme points of $\Lambda(\mathbf{A})$. We simply call them the extreme points of $\mathbf{A}$.

Suppose we are given a weight function $w : \{0, \ldots, n\} \to \mathbb{R}$, and the weight of a family $\mathcal{F}$ is defined to be $\sum_{F \in \mathcal{F}} w(|F|)$, which is equal to $\sum_{i=0}^n w(i) f_i$. Usually we are interested in the maximum of the weight of the families in a class **A**. So we want to maximize this sum, i.e., find a family $\mathcal{F}_0 \in \mathbf{A}$ and an inequality $\sum_{i=0}^{n} w(i) f_i = w(\mathcal{F}) \leq w(\mathcal{F}_0) = c$. This is a linear inequality, and it is always maximized in an extreme point.

Given a class (or property) of families, the first natural question in extremal combinatorics is the maximum cardinality such a family can have. When it is answered, often some simple weight functions are considered and the maximum weight of such a family is studied. Determining the extreme points answers these questions for every (linear) weight function.

Erdős, Frankl, and Katona [3] determined the extreme points of the intersecting Sperner families. In their next paper [4], the extreme points of the profile polytope of the intersecting families were determined. Now we define these. Let coordinate i of $\mathbf{a}$ be 0 if $i < n/2$, $\binom{n-1}{i-1}$ if $i = n/2$ and $\binom{n}{i}$ if $i > n/2$. Let $k \leq n/2$. Coordinate i of $\mathbf{a}_k$ is 0 if $i < k$, $\binom{n-1}{i-1}$ if $k \leq i \leq n-k$, and $\binom{n}{i}$ if $i > n-k$. Let Γ_a be the set of vectors that we can get from any of the vectors $\mathbf{a}_k$ and $\mathbf{a}$, if we replace an arbitrary set of coordinates by 0. Note that if n is even, then $\mathbf{a} = \mathbf{a_{n/2}}$.

THEOREM 5 (Erdős, Frankl, and Katona [4]). *The set of extreme points of the intersecting families is* Γ_a.

The corresponding intersecting families are the following. $\mathcal{A}_k$ consists of the sets which have sizes at least k and contain the element n, and of every other set which has size greater than $n-k$. $\mathcal{A}$ consists of all the sets with size greater than $n/2$ and the sets which have sizes $n/2$ and contain n. These families are obviously intersecting and their profile vectors are $\mathbf{a}_k$ and $\mathbf{a}$. We can delete full levels and the families are still intersecting; in the corresponding vectors some coordinates are changed to 0.

Since then several other classes of families have been considered (see, e.g., [1, 5]), generalizations have been studied [7, 8], and profile polytopes were applied for counting subposets [6]. Note that most of the classes of families where the profile polytope has been studied are *hereditary,* i.e., if we remove some members of a family in the class, the resulting family still belongs to the class. It makes determining the extreme points easier, as we do not have to deal with negative weights, and all extreme points can be achieved by changing some coordinates of a few essential ones to 0. However, in this paper we determine the extreme points of the nontrivial intersecting families, which is not a hereditary property.

In the next section we define what is needed to state our main theorem. We prove an important special case in section 3, and finish the proof by a case analysis in section 4.

2. The main theorem. Let us start with some simple observations. A nontrivial intersecting family cannot contain the empty set or a singleton. It might contain the full set, but that does not change the intersecting property, nor the nontrivial property. It means that for a weight function w if $w(n) > 0$, the maximum family contains the full set, and if $w(n) < 0$, it does not. Moreover, changing only $w(n)$ does not change the other parts of the maximum family, and hence we can basically forget about n. More precisely, $(p_0, p_1, \ldots, p_{n-2}, p_{n-1}, 0)$ is an extreme point if and only if $(p_0, p_1, \ldots, p_{n-2}, p_{n-1}, 1)$ is an extreme point.

Now we define several vectors, which are going to be the extreme points of the nontrivial intersecting families. Then we state our main theorem, and after that we show that these vectors indeed correspond to nontrivial intersecting families and are extreme points (note that for most classes of families where profile polytopes have been studied, these statements are trivial, but not for the nontrivial intersecting families).

That part also makes it easier to understand where these definitions come from. All these vectors are in the $(n+1)$-dimensional Euclidean space, but coordinates 0,1, and n are always 0. Let $H \subset \{2,3,\dots,n-2,n-1\}$ be a nonempty set of indices, h be its smallest element, and h' be its largest element.

Let $\mathbf{b}_H = (b_0,\dots,b_n)$ with

$$b_i = \begin{cases} 0 & \text{if } i \notin H, \\ |HM(i,h')| & \text{if } i \in H \text{ and } i < h', \\ |HM(i,h')|+1 & \text{if } i = h'. \end{cases}$$

Let $\Gamma_b = \{\mathbf{b}_H : h+h' \le n\}$.

Let $\mathbf{c}_H = (c_0,\dots,c_n)$ with

$$c_i = \begin{cases} 0 & \text{if } i \notin H, \\ \binom{n-1}{i-1} & \text{if } i \in H \text{ and } i \le n-h, \\ \binom{n}{i} & \text{otherwise.} \end{cases}$$

Let $\Gamma_c = \{\mathbf{c}_H : h+h' > n\}$.

Let $\mathbf{d}_H = (d_0,\dots,d_n)$ with

$$d_i = \begin{cases} 0 & \text{if } i \notin H, \\ |HM(i,h')| & \text{if } i \in H \text{ and } i < h', \\ 1 & \text{if } i = h'. \end{cases}$$

Let $\mathbf{d} = (0,0,3,1,0,\dots,0)$ and $\Gamma_d = \{\mathbf{d}_H : |H| > 1, h+h'' \le n\} \setminus \{\mathbf{d}\}$, where h'' is the second largest element of H.

Let us consider the set P of vectors $(e_0,\dots,e_n)$ satisfying the following properties:

1. Every e_i is a nonnegative integer, $e_0 = e_1 = e_n = 0$.
2. $x := \sum_{i=2}^{n-1} e_i \ge 3$.
3. $\sum_{i=2}^{n-1} i e_i \le (x-1)n$.

Now we show the connection between P and nontrivial intersecting families. For two vectors $\mathbf{p} = (p_0,\dots,p_n)$ and $\mathbf{p}' = (p'_0,\dots,p'_n)$, we say that $\mathbf{p}' \le \mathbf{p}$ if $p'_i \le p_i$ for every $0 \le i \le n$.

LEMMA 6. *(i) If a nontrivial intersecting family does not contain* $[n]$*, its profile is in* P*.*

(ii) If $\mathbf{p} \in P$ *and there is no* $\mathbf{p}' \in P$ *different from* $\mathbf{p}$ *with* $\mathbf{p}' \le \mathbf{p}$*, then* $\mathbf{p}$ *is the profile of a nontrivial intersecting family.*

Proof. To show (i), observe that for the profile of a nontrivial intersecting family obviously $e_0 = e_1 = 0$ holds, and also we need at least three members in the family, as any two members trivially intersect. The third property is needed; otherwise an element of the underlying set would be covered x times, i.e., by every set, contradicting the nontriviality.

Let us prove now (ii). We are given a vector $\mathbf{p}$ and we are going to construct a nontrivial intersecting family $\mathcal{F}$ with profile $\mathbf{p}$. Observe that $\mathbf{p}$ shows how many k-element sets must be in the family for every k. Let us denote the sizes of the sets by $a_1,\dots,a_\ell$ in decreasing order. We choose the first (the largest) set F_1 of size a_1 arbitrarily. Let B_i be the set of vertices which are not covered by each of the first i sets $F_1,\dots,F_i$ (only by at most $i-1$ of them), and then $B_1 = \overline{F_1}$ and $B_i \supset B_{i-1}$ for every $i > 1$. We choose the second set F_2 of size a_2 in such a way that F_2 intersects F_1 and also F_2 contains B_1, if possible.

If it is not possible, then we claim that we have $x = 3$. Indeed, in that case we have $a_1 + a_2 \leq n$, and thus they together with the next set F_3 of size a_3 have their profile in P, which means no other set can be in the family because of our assumption on the minimality of $\mathbf{p}$. Then we pick F_2 of size a_2 such that it intersects F_1 in a single element, and then we pick F_3 of size a_3 such that it contains an element of $F_1 \setminus F_2$ and an element of $F_2 \setminus F_1$ and does not contain any element in $F_1 \cap F_2$. This is doable as $a_3 \geq 2$. The resulting family is clearly nontrivial intersecting.

If $x > 3$, we choose every F_i of size a_i in such a way that it contains B_{i-1}, if possible. Note that in this case it automatically intersects $F_1, \dots, F_{i-1}$. Indeed, F_i contains B_1, which is also contained in $F_2, \dots, F_{i-1}$. F_i also contains B_2, which intersects F_1 (we also use that B_1 and B_2 are not empty).

Now assume that when we add a set F_i, it is just enough or too small to cover every vertex in B_{i-1}, i.e., $a_i \leq |B_{i-1}|$. Then $i = \ell$, i.e., F_i is the last set (as the resulting profile vector is in P). We have to choose F_i in such a way that it intersects the other sets. As every vertex is covered at least $i-2$ times, all we have to do is to put an arbitrary vertex of B_{i-1} in F_i, and then the new set intersects all but one of the earlier sets, say, F_j. We have to choose a vertex in B_{i-1} contained in F_j, and then other vertices from B_{i-1} arbitrarily. As only vertices in B_{i-1} are used, no vertex is covered i times, and hence the family is nontrivial. ∎

Let Γ_e be the set of the extreme points of P. Now we can state our main theorem.

THEOREM 7. *The extreme points of the profile polytope of the nontrivial intersecting families are the elements of* $\Gamma_b \cup \Gamma_c \cup \Gamma_d \cup \Gamma_e$, *and additionally the vectors we get from these if we change the last coordinate from* 0 *to* 1.

To prove this statement, we have to show that the points listed are indeed extreme points and that there are no other extreme points. The first part is the easier task, and we will deal with it in the rest of this section. We give an example nontrivial intersecting family for each of the vectors $\mathbf{v} = \mathbf{v}_H \in \Gamma_b \cup \Gamma_c \cup \Gamma_d \cup \Gamma_e$ and also show that $\mathbf{v}$ is an extreme point, by showing a weight function such that $\mathbf{v}$ is the unique maximum.

Let us describe first the general approach to find such a weight function. We start by assuming that if $i \notin H$, then $w(i)$ is negative, and, moreover, $w(i)$ is so small compared to the other weights $w(j)$ that if a family contains even one i-element set, its total weight is negative. On the other hand, there is a $1 < j < n$ with $w(j) > 0$, and thus there is a family of positive weight. This shows that no i-element sets can be in the family of maximum weight. Similarly, we can say that for some $i \in H$ its weight is very large compared to the other weights. It implies that the family of maximum weight contains as many i-element sets as possible, i.e., $|HM(i,j)|$, where j is the largest nonzero coordinate of $\mathbf{v}_H$. We describe these ideas in more detail in the proof of the following lemma.

LEMMA 8. *The elements of* Γ_b *are extreme points of the nontrivial intersecting families.*

Proof. For $\mathbf{b}_H \in \Gamma_b$ we have to show a family $\mathcal{B}_H$ which has $\mathbf{b}_H$ as its profile, and a weight w which is maximized at $\mathbf{b}_H$. Let $\mathcal{B}_H = [h'] \cup \left(\bigcup_{i \in H} HM(i,h')\right)$, i.e., the union of $HM(i,h')$ for every $i \in H$, and additionally $[h']$. This family is obviously nontrivial intersecting, as each of its members except for $[h']$ contains n and intersects $[h']$.

Now we are going to show a weight function that is maximized only by families with profile $\mathbf{b}_H$. Let w be a weight such that if $i \notin H$, then $w(i) = -2^{2n}$. It is going to be so small compared to the other weights that no i-element sets can be in the maximum family $\mathcal{F}$. All other sets have weight at most 2^n, and there are less than 2^n sets in $\mathcal{F}$, and hence positive weight can only be achieved without these negative sets. Let $w(h) = 2^n$; it is very large compared to the other positive weights (but still very small compared to the absolute value of the negative weights), and all other weights are 1. Then a single h-element set has larger weight than all the other sets with positive weight, and thus the maximum family $\mathcal{F}$ contains as many h-element sets as possible. If $h < n/2$ and $h \neq h'$, then by Lemma 4 the maximum number of h-element sets is $|HM(h,h')|$, and the largest family of h-element sets is isomorphic to $HM(h,h')$, with one exceptional case where $h = 2$ and $h' = 3$ and the three h-element sets form a triangle.

It is easy to see that in this case, if $n = 5$, then we can add seven three-element sets to obtain a nontrivial intersecting family with profile $(0,0,3,7,0,0) = \mathbf{b}_H$. If $n \geq 6$, then the added three-element sets form a nontrivial intersecting family, thus there are at most $|HM(3,3)| + 1$ of them, and hence the profile of the family is at most $\mathbf{b}_H$ in every coordinate.

Let us return to the nonexceptional cases. Without loss of generality we can assume that $\mathcal{F}_h$ is equal to $HM(h,h')$. If $h = h'$, then by Theorem 3 the largest family of h-element sets has cardinality $|HM(h,h)| + 1$. In the other cases, observe that the only set of size at most h' that does not contain n and intersects every member of $HM(h,h')$ is $[h']$. Therefore, every other member of $\mathcal{F}$ should contain the fixed point n except for $[h']$. Also, every member of $\mathcal{F}$ should intersect $[h']$, and hence $\mathcal{F}$ is a subfamily of $\mathcal{B}_H$. Then $\mathcal{B}_H$ is the unique maximum.

Finally, if $h = n/2$, then the only nonzero coordinate is $\binom{n-1}{n/2-1}$ at coordinate $n/2$. This vector is an extreme point of the class of intersecting families, and thus it is an extreme point of this smaller family as well. ∎

Lemma 9. *The elements of* Γ_c *are extreme points of the nontrivial intersecting families.*

Proof. These are the elements of Γ_a which correspond to nontrivial intersecting families. They are extreme points of the larger set (of all intersecting families), and thus they are extreme points of the smaller set as well. ∎

Lemma 10. *The elements of* Γ_d *are extreme points of the nontrivial intersecting families.*

Proof. For $\mathbf{d}_H \in \Gamma_d$ we define the family

$$\mathcal{D}_H = \cup_{i \in H,\, i \neq h'} HM(i,h') \cup \{[h']\}.$$

It is the same as $\mathcal{B}_H$, except we removed most of the h'-element sets. Let w be the weight function described in the proof of Lemma 8. We use almost the same weight here, and we set $w'(h') = -1$ and $w'(i) = w(i)$ for every $i \neq h'$. Just as in Lemma 8, we need to have the largest number of h-element sets in the family $\mathcal{F}$ of maximum weight, that is, $|HM(h,h')|$. Then we need an h'-element set in $\mathcal{F}$, with the exception of one case as in Lemma 4. Without loss of generality, $\mathcal{F}_h = HM(h,h')$ and $\mathcal{F}_{h'} \supset \{[h']\}$. However, there is no point in having more h'-element sets in $\mathcal{F}$. For every other $i \in H$, there are at most $|HM(i,h')|$ sets of size i in $\mathcal{F}$, completing the proof. This proof does not work for $\mathbf{d}$ because of the exceptional case of Lemma 4. Indeed $\mathbf{d}$ is not an extreme point (unless $n = 3$, in which case $\mathbf{d} \in \Gamma_c$), as $(0, 0, 3, 0, \ldots, 0)$ and $(0, 0, 3, |HM(3, 3)| + 1, 0, \ldots, 0)$ are both profile vectors. ∎

LEMMA 11. *The elements of* Γ_e *are extreme points of the nontrivial intersecting families.*

Proof. Observe first that if $\mathbf{v} = (e_0, \ldots, e_n) \in P$ and we increase e_i, the resulting vector is in P, as we cannot violate any of the properties. It means that the extreme points of P are minimal. More precisely, if $\mathbf{e}$ is an extreme point and $\mathbf{e}' \geq \mathbf{e}$ (with $\mathbf{e}' \neq \mathbf{e}$), then $\mathbf{e}'$ cannot be an extreme point, as it is a convex combination of the following two elements of P: $\mathbf{e}$ and $2\mathbf{e}' - \mathbf{e}$. By Lemma 6, $\mathbf{e}$ is the profile vector of a nontrivial intersecting family.

We need to show that the extreme points of P are extreme points of the nontrivial intersecting families as well. Let P' be the set of the profile vectors of those nontrivial intersecting families which do not contain $[n]$. Then $P' \subset P$ by Lemma 6.

An extreme point $\mathbf{p}$ of the larger set P which also belongs to the smaller set P' must be an obviously extreme point of P' as well. Thus there exists a weight function w where it gives the maximum in P'. One can easily see that if we change $w(n)$ to a negative number, then $\mathbf{p}$ has the largest weight among every profile vector of nontrivial intersecting families. ∎

3. The main lemma. Our most important special case is when there are only two nonempty levels $1 < i < m < n$ with $m > n/2$ and $i + m \leq n$. For other values of i and m, it is going to be easy to see that Theorem 7 holds (we do it inside the proof of the main theorem in section 4). Thus, the lemma below contains the most complicated part of the proof.

LEMMA 12. *Let* $(f_0, f_1, f_2, \ldots, f_n)$ *be the profile vector of a nontrivial intersecting family* $\mathcal{F}$. *Let us assume that* m *is the maximum cardinality in* $\mathcal{F}$, $m > n/2$, *and* $i + m \leq n$. *Then there is a* $0 \leq \lambda \leq 1$ *such that* $(f_i, f_m) \leq \lambda(0, \binom{n}{m}) + (1 - \lambda)(|HM(i, m)|, |HM(m, m)| + 1)$.

We will use the following simple observations.

PROPOSITION 13. *(i) If* $x \leq y$, *then* $\binom{x}{k-1}/\binom{x}{k} \geq \binom{y}{k-1}/\binom{y}{k}$.
(ii) Let $0 \leq c'$, $0 < \alpha, a, b, c, b'$ *with* $bc' \leq cb'$, $b/c \leq \alpha$, *and* $c \geq c'$. *Then*

$$\frac{\alpha a + b}{a + c} \leq \frac{\alpha a + b'}{a + c'}.$$

Proof. The first statement easily follows from the definition of $\binom{x}{k}$.

By rearranging the desired inequality of (ii), we obtain the equivalent form $\alpha ac' + ab + bc' \leq \alpha ac + ab' + cb'$. Recall that we have $bc' \leq cb'$. The other terms can be rewritten as $\frac{b-b'}{c-c'} \leq \alpha$. We have $\frac{b-b'}{c-c'} \leq \frac{b-bc'/c}{c-c'} = \frac{b(c-c')/c}{c-c'} = b/c \leq \alpha$. ∎

Now we are ready to prove Lemma 12.

Proof of Lemma 12. We use induction on $n - m - i$. Recall that $HM(m) = HM(m, m) \cup \{[m]\}$; for simplicity we will use this notation in the proof. Observe that for the base case $i + m = n$ we have that $HM(i, m) \cup HM(m)$ consists of all the i-sets and m-sets containing n, except that it contains $[m]$ instead of its complement. Thus $HM(i, m) \cup HM(m)$ has $\binom{n}{i}$ members, just like any maximal nontrivially intersecting family on these two levels. Let us choose $\lambda = \frac{|HM(i,m)| - f_i}{|HM(i,m)|}$, then by definition $f_i \leq (1 - \lambda)|HM(i, m)|$, and we need

$$f_m \le \lambda \binom{n}{m} + (1-\lambda)|HM(m)| = \binom{n}{m} - \frac{f_i\binom{n}{m}}{|HM(i,m)|} + \frac{f_i|HM(m)|}{|HM(i,m)|} = \binom{n}{m} - f_i.$$

This holds for every intersecting family, even the trivial one. For nontrivial intersecting families, we have $f_i \le |HM(i,m)|$ by Lemma 4, and thus we have $\lambda \ge 0$, completing the proof of the base step.

Let us continue with the induction step. Let us consider $\nabla\mathcal{F}_i$, which is the shade of $\mathcal{F}_i$, and let $g_{i+1} = |\nabla\mathcal{F}_i|$. Then $\nabla\mathcal{F}_i \cup \mathcal{F}_m$ is obviously nontrivially intersecting, and thus by the induction hypothesis there is a $0 \le \lambda \le 1$ such that $(g_{i+1}, f_m) \le \lambda(0, \binom{n}{m}) + (1-\lambda)(|HM(i+1,m)|, |HM(m)|)$. We will show that the same λ works for f_i, i.e., $(f_i, f_m) \le \lambda(0, \binom{n}{m}) + (1-\lambda)(|HM(i,m)|, |HM(m)|)$. As the values in coordinate m do not change, all we need to prove is that $f_i \le (1-\lambda)|HM(i,m)|$ if $g_{i+1} \le (1-\lambda)|HM(i+1,m)|$. It is enough to show that $f_i/|HM(i,m)| \le g_{i+1}/|HM(i+1,m)|$, or equivalently $g_{i+1}/f_i \ge |HM(i+1,m)|/|HM(i,m)|$. As $HM(i+1,m) = \nabla HM(i,m)$, the last of the above inequalities means that the size of the shade of $\mathcal{F}_i$ is proportionally the smallest if $\mathcal{F}_i$ is $HM(i,m)$.

We will use the Kruskal–Katona theorem. To use it in the form we have stated it, we will consider the complement family, as the shade of a family is the shadow of its complement.

Observe that $\overline{HM(i,m)}$ is an initial segment of the colex ordering if we reorder the elements of $[n]$. Indeed, members of $\overline{HM(i,m)}$ completely avoid a given element z, and then we take all the $(n-i)$-sets but those that contain an m-element set B. By reordering, we can assume that $z = n$ and $B = \{n-m, \dots, n-1\}$. The sets containing n are the last in the colex order, and a superset F of B cannot be before a set $G \in \overline{HM(i,m)}$, as the largest element of $F \setminus G$ is in B, while every element of $G \setminus F$ is less than $n-m$.

The cascade form of $|\overline{HM(i,m)}|$ is $\binom{n-2}{n-i} + \binom{n-3}{n-i-1} + \binom{n-4}{n-i-2} + \cdots + \binom{n-m-1}{n-i-m+1} = \sum_{j=2}^{m+1}\binom{n-j}{n-i-j+2}$. Let $\mathcal{G}$ be a nonempty $(n-i)$-uniform family with $|\mathcal{G}| < |\overline{HM(i,m)}|$ and cascade form $|\mathcal{G}| = \sum_{j=2}^{m'}\binom{n_j}{n-i-j+2}$. Observe that $n_2 \le n-2$. This implies that for any h, $n_h \le n-h$.

We partition $\overline{HM(i,m)}$ into m parts: $\mathcal{H}_2$ consists of the first $\binom{n-2}{n-i}$ sets of $\overline{HM(i,m)}$ in the colex order, $\mathcal{H}_3$ consists of the next $\binom{n-3}{n-i-1}$ sets, and so on. $\mathcal{H}_j$ for $j \le m+1$ consists of $\binom{n-j}{n-i-j+2}$ sets that come after $\mathcal{H}_2, \dots, \mathcal{H}_{j-1}$, i.e., after the first $\binom{n-2}{n-i} + \binom{n-3}{n-i-1} + \binom{n-4}{n-i-2} + \cdots + \binom{n-j+1}{n-i-j+3}$ sets in the colex order. We also partition $\mathcal{G}$ into m parts: for $2 \le j < m+1$, $\mathcal{G}_j$ similarly consists of $\binom{n_j}{n-i-j+2}$ sets of $\mathcal{G}$ that come after $\mathcal{G}_2, \dots, \mathcal{G}_{j-1}$ in the colex order. Then $\mathcal{G}_{m+1}$ consists of all the remaining $\sum_{j=m+1}^{m'}\binom{n_j}{n-i-j+2}$ sets of $\mathcal{G}$. Let us note that $\mathcal{G}_j$ can be empty if $j > 2$.

Let us assume that $n_2 = n-2$, $n_3 = n-3, \dots, n_h = n-h$, and $n_{h+1} < n-h-1$. Let $\mathcal{H}^* = \cup_{j=1}^{h}\mathcal{H}_j$, $\mathcal{H}^{**} = \cup_{j=h+1}^{m+1}\mathcal{H}_j$, $\mathcal{G}^* = \cup_{j=1}^{h}\mathcal{G}_j$, $\mathcal{G}^{**} = \cup_{j=h+1}^{m+1}\mathcal{G}_j$. Observe that we have $|\mathcal{H}^*| = |\mathcal{G}^*|$ and $|\Delta\mathcal{H}^*| \le |\Delta\mathcal{G}^*|$ since $\mathcal{H}^*$ is an initial segment of the colex ordering. We also have $|\mathcal{H}^{**}| \ge \binom{n-h-1}{n-i-h+1}$ and $|\mathcal{G}^{**}| < \binom{n-h-1}{n-i-h+1}$.

Let $a := |\mathcal{H}^*|$, $c := |\mathcal{H}^{**}|$, $\alpha = |\Delta\mathcal{H}^*|/|\mathcal{H}^*|$, $b := |\Delta\mathcal{H}^{**} \setminus \Delta\mathcal{H}^*|$, $b' = |\Delta\mathcal{G}| - |\Delta\mathcal{G}^*|$, $c' := |\mathcal{G}^{**}|$, and $\alpha' = |\Delta\mathcal{G}^*|/|\mathcal{G}^*|$. Our goal is to apply (ii) of Proposition 13. By the above, we have $c > c'$. Now we will show that the other conditions are satisfied as well.

We let $p_\ell := \binom{n-\ell}{n-i-\ell+1} = |\Delta\mathcal{H}_\ell \setminus \Delta(\bigcup_{\ell'=2}^{\ell-1}\mathcal{H}_{\ell'})|$, i.e., the number of sets added to the shadow of $\bigcup_{\ell'=2}^{\ell}\mathcal{H}_{\ell'}$ by $\mathcal{H}_\ell$. Observe first that $p_\ell/|\mathcal{H}_\ell| = (n-i-\ell+2)/(i-1)$, and thus $p_\ell/|\mathcal{H}_\ell|$ decreases as ℓ increases. This implies that $p_\ell/|\mathcal{H}_\ell| \le p_{h+1}/|\mathcal{H}_{h+1}|$ for every $\ell > h+1$. Therefore, we have that

$$\frac{b}{c} = \frac{|\Delta\mathcal{H}^{**} \setminus \Delta\mathcal{H}^*|}{|\bigcup_{\ell=h+1}^{m+1} \mathcal{H}_\ell|} = \frac{\sum_{\ell=h+1}^{m+1} p_\ell}{|\bigcup_{\ell=h+1}^{m+1} \mathcal{H}_\ell|} \le \frac{\frac{p_{h+1}}{|\mathcal{H}_{h+1}|}|\bigcup_{\ell=h+1}^{m+1} \mathcal{H}_\ell|}{|\bigcup_{\ell=h+1}^{m+1} \mathcal{H}_\ell|} = \frac{p_{h+1}}{|\mathcal{H}_{h+1}|}. \tag{3.1}$$

Similarly, we have that

$$\alpha = \frac{|\Delta\mathcal{H}^*|}{|\mathcal{H}^*|} = \frac{|\Delta \cup_{j=1}^h \mathcal{H}_j|}{|\cup_{j=1}^h \mathcal{H}_j|} = \frac{\sum_{j=1}^h p_j}{|\bigcup_{j=1}^h \mathcal{H}_j|} \ge \frac{\frac{p_{h+1}}{|\mathcal{H}_{h+1}|}|\bigcup_{j=1}^h \mathcal{H}_j|}{|\bigcup_{j=1}^h \mathcal{H}_j|} = \frac{p_{h+1}}{|\mathcal{H}_{h+1}|} \ge \frac{b}{c},$$

where the last inequality uses (3.1).

Let $x < n-h-1$ be defined by $\binom{x}{n-i-h+1} := \binom{n_{h+1}}{n-i-h+1} + \binom{n_{h+2}}{n-i-h} + \cdots + \binom{n_{m'}}{n-i-m'+2} = |\bigcup_{\ell=h+1}^{m+1} \mathcal{G}_\ell|$. We have $|\Delta\mathcal{G}| \ge |\Delta\mathcal{H}^*| + \binom{n_{h+1}}{n-i-h} + \binom{n_{h+2}}{n-i-h-1} + \cdots + \binom{n_{m'}}{n-i-m'+1}$ by the Kruskal–Katona theorem. We claim that

$$\binom{n_{h+1}}{n-i-h} + \binom{n_{h+2}}{n-i-h-1} + \cdots + \binom{n_{m'}}{n-i-m'+1} \ge \binom{x}{n-i-h}. \tag{3.2}$$

Indeed, the left hand side is the sharp lower bound on the size of the shadow of an $(n-i-h+1)$-uniform family of size $\binom{x}{n-i-h+1}$ by the Kruskal–Katona theorem, while the right hand side is the not necessarily sharp lower bound on the size of the same family by Lovász's version of the shadow theorem.

Consider now the number of sets added to the shadow of $\cup_{\ell'=2}^{\ell}\mathcal{G}_{\ell'}$ by $\mathcal{G}_\ell$. Since $|\mathcal{G}_\ell| = \binom{n_\ell}{n-i-\ell+2}$, it gives at least an additional $\binom{n_\ell}{n-i-\ell+1}$ to the shadow even if all the sets that come before $\mathcal{G}_\ell$ in the colex order are in $\cup_{\ell'=2}^{\ell}\mathcal{G}_{\ell'}$. Therefore, we have $\frac{b'}{c'} \ge \frac{\binom{n_{h+1}}{n-i-h}+\binom{n_{h+2}}{n-i-h-1}+\cdots+\binom{n_{m'}}{n-i-m'+1}}{\binom{x}{n-i-h+1}} \ge \frac{\binom{x}{n-i-h}}{\binom{x}{n-i-h+1}} \ge \frac{\binom{n-h-1}{n-i-h}}{\binom{n-h-1}{n-i-h+1}} = p_{h+1}/|\mathcal{H}_{h+1}| \ge \frac{b}{c}$. In the inequalities here we used the observation at the top of the paragraph, then (3.2), then (i) of Proposition 13, and finally (3.1).

Now we can apply (ii) of Proposition 13 to show that $\frac{\alpha a+b}{a+c} \le \frac{\alpha a+b'}{a+c'} \le \frac{\alpha' a+b'}{a+c'}$. This means $|\Delta\overline{HM(i,m)}|/|\overline{HM(i,m)}| \le |\Delta\mathcal{G}|/|\mathcal{G}|$. By taking the complements, we obtain that $|\nabla HM(i,m)|/|HM(i,m)| \le |\nabla\mathcal{G}'|/|\mathcal{G}'|$ for any i-uniform family $\mathcal{G}'$ with $|\mathcal{G}'| \le |HM(i,m)|$. In particular, $|\nabla HM(i,m)|/|HM(i,m)| \le g_{i+1}/f_i$, completing the proof. ∎

4. Proof of the main theorem.

In this section we finish the proof of Theorem 7. It is easy to see that we can consider only families not containing $[n]$. It is enough to show that if a profile vector $\mathbf{p}$ of a nontrivial intersecting family $\mathcal{F}$ gives the unique maximum for a weight function w, then $\mathbf{p} \in \Gamma_b \cup \Gamma_c \cup \Gamma_d \cup \Gamma_e$.

An important observation is that if $F \in \mathcal{F}$, $F \subset G$, and G has positive weight, then G is in the maximum family (as adding it would not violate any of the properties). In the proof we often start with fixing the maximum size m of members; it implies that larger sets (except possibly $[n]$) do not have positive weight. Note that if $w(m) > 0$, then $\mathcal{F}_m$ is nontrivial intersecting. Indeed, if $\mathcal{F}_m$ is trivial, then all its members contain a given element x and there is a set $F \in \mathcal{F}$ of smaller size not containing x. But then all the m-element sets which contain F are in $\mathcal{F}$, even those which do not contain x, a contradiction.

We continue the proof with a case analysis.

Case 1. $w(i) \le 0$ *for every* $1 < i < n$.

Case 1a. $w(i) < 0$ *for every* $1 < i < n$. Obviously $\mathbf{p} \in P$, as in all the cases, by Lemma 6. But in this special case we will show that $\mathbf{p}$ is also an extreme point of P, and thus it is in Γ_e.

If $\mathbf{p}$ is not an extreme point of P, i.e., there is an element $\mathbf{p}'$ of P with larger weight, then either $\mathbf{p}'$ also corresponds to a nontrivial intersecting family (a contradiction), or there is another $\mathbf{p}'' \in P$ with $\mathbf{p}'' \leq \mathbf{p}'$ by Lemma 6. But then $\mathbf{p}''$ has even larger weight. As $\mathbf{p}''$ cannot correspond to a nontrivial intersecting family, there is an even smaller vector in P (with even larger weight). It cannot continue forever, as each coordinate is a nonnegative integer. We arrive at a vector which does correspond to a nontrivial intersecting family and hence has a larger weight than $\mathbf{p}$, a contradiction.

Case 1b. $w(i)=0$ *for some* $1<i<n$. Obviously $w(\mathbf{p}) \leq 0$, and $w(HM(i))=0$. The profile of $HM(i)$ is in Γ_b.

Case 2. $w(i)>0$ *for some* $1<i<n$. Let m be the maximum size in $\mathcal{F}$. We will use Lemma 4 several times.

Case 2a. $w(m)<0$. There is an m-element set F in $\mathcal{F}$. Obviously the only reason it is in the family is that without it the family would be trivial, and hence every other member of $\mathcal{F}$ contains a fixed point. Then for every level i the maximum weight is given either by the empty family (in case $w(i) \leq 0$) or $HM(i,m)$. Then we take the union of these uniform families (on every level i, the empty family of $HM(i,m)$), and we add $[m]$. The resulting family is nontrivial intersecting, and its profile vector is in Γ_d.

Case 2b. $w(m) \geq 0$ *and* $m \leq n/2$. For every level below m, the maximum of $w(\mathcal{F}_i)$ is either 0 or given by $HM(i,m)$. In particular, on level m clearly $HM(m) = HM(m,m) \cup \{[m]\}$ gives the maximum weight. This already makes sure the family is nontrivial intersecting, and hence for every other level j with $j<m$ we can choose $HM(j,m)$ or the empty family, depending on whether $w(j)$ is positive or not. The union of these uniform families is nontrivial intersecting and has the largest possible weight on every level up to m. Its profile is in Γ_b.

Case 2c. $w(m) \geq 0$ *and* $m>n/2$. Let m_0 be the size of the smallest member of the family $\mathcal{F}$.

Case 2c1. $w(m) \geq 0$, $m>n/2$, *and* $m+m_0>n$. Let us consider the following modified weight function. Let $w'(i)$ be the same as $w(i)$ if $m_0 \leq i \leq m$ and negative otherwise. Obviously the maximum nontrivial intersecting family for w' is also $\mathcal{F}$. Let us examine the intersecting family $\mathcal{F}'$ with maximum weight w' now. One can easily see using Theorem 5 that the profile of $\mathcal{F}'$ can be obtained from an $\mathbf{a}$ with $m_0 \leq j \leq n/2$ or from $\mathbf{a}$ by changing some coordinates to 0 (those with negative weight w'). If $w(m)=0$, then $\mathcal{F}'$ might contain no m-element sets, but even in this case we can add every m-element set to $\mathcal{F}'$ without decreasing the weight (and without ruining the intersecting property). The resulting family $\mathcal{F}''$ is nontrivial intersecting, and $w'(\mathcal{F}'')=w'(\mathcal{F}') \geq w'(\mathcal{F})=w(\mathcal{F})$, and thus $\mathcal{F}''$ must have the same profile as $\mathcal{F}$. The profile of $\mathcal{F}''$ is in Γ_c.

Case 2c2. $w(m) \geq 0$, $m>n/2$ *and* $m+m_0 \leq n$. Let H be the set of nonempty levels. Recall that coordinate i of $\mathbf{a}$ is 0 if $i<n/2$, $\binom{n-1}{i-1}$ if $i=n/2$, and $\binom{n}{i}$ if $i>n/2$. Let $\mathbf{a}'$ be the vector we get from $\mathbf{a}$ when we change the coordinates not in H to 0. We will show that $\mathbf{p}=\mathbf{b}_H$, by showing that there is a λ such that $\lambda \mathbf{b}_H + (1-\lambda)\mathbf{a}' \geq \mathbf{p}$. We have that $\mathbf{b}_H$ and $\mathbf{a}'$ are both 0 in the negative coordinates, and thus the weight of either $\mathbf{b}_H$ or $\mathbf{a}'$ is at least as large as the weight of $\mathbf{p}$. But that was the unique maximum, and thus $\mathbf{p}$ is equal to either $\mathbf{b}_H$ or $\mathbf{a}'$. As $\mathbf{p}$ has a nonzero coordinate below $n/2$, $\mathbf{p}$ cannot be equal to $\mathbf{a}'$.

Let $i \leq n/2$ be such that $f_i/|HM(i,m)| =: \lambda$ is maximal. Then $\lambda \mathbf{b}_H$ has at least f_j in coordinate j for every $j \leq n/2$. Let us consider now a coordinate $k>n/2$ with $w(k)>0$.

If the family $\mathcal{F}_i \cup \mathcal{F}_k$ is trivially intersecting, then $f_k \le |HM(k,m)|$, while $\mathbf{b}_H$ and $\mathbf{a}'$ both have at least $|HM(k,m)|$ in coordinate k, and thus so does $\lambda\mathbf{b}_H + (1-\lambda)\mathbf{a}'$, completing the proof.

If the family $\mathcal{F}_i \cup \mathcal{F}_k$ is nontrivially intersecting, we can apply Lemma 12. It implies that there is a λ' such that $(f_i, f_k) \le ((1-\lambda')|HM(i,k)|, \lambda'\binom{n}{k} + (1-\lambda')(|HM(k,k)| + 1)$. Coordinate i shows that $((1-\lambda')|HM(i,k)| \ge \lambda|HM(i,m)|$. Since $|HM(i,m)| \ge |HM(i,k)|$, this implies that $\lambda \le 1-\lambda'$. Consider now coordinate k. We have

$$(\star) \qquad f_k \le \lambda'\binom{n}{k} + (1-\lambda')(|HM(k,k)|+1) \le \lambda'\binom{n}{k} + (1-\lambda')|HM(k,m)|.$$

Since $|HM(k,m)| \le \binom{n}{k}$, increasing λ' increases the right hand side of $(\star)$. Since $\lambda' \le 1-\lambda$, the right hand side is at most $(1-\lambda)\binom{n}{k} + \lambda|HM(k,m)|$, which is coordinate k of $\lambda\mathbf{b}_H + (1-\lambda)\mathbf{a}'$, completing the proof.

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