



Generalized Turán problems for double stars

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ABSTRACT

We study the generalized Turán function $\text{ex}(n, H, F)$, when H or F is a double star $S_{a,b}$, which is a tree with a central edge uv , a leaves connected to u and b leaves connected to v . We determine $\text{ex}(n, K_k, S_{a,b})$ and $\text{ex}(n, S_{a,b}, F)$ for sufficiently large n , where F is either a 3-chromatic graph with an edge whose deletion results in a bipartite graph, or the 2-fan, i.e. two triangles sharing a vertex. We also give bounds on $\text{ex}(n, S_{a,b}, S_{c,d})$.

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1. Introduction

Ordinary Turán problems deal with the quantity $\text{ex}(n, F)$, which is the largest number of edges in n -vertex graphs that do not contain F as a subgraph. Turán's theorem [19] states that $\text{ex}(n, K_{k+1})$ is equal to the number of edges in the k -partite Turán graph, which is the complete k -partite graph with each part having order either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

Generalized Turán problems deal with $\text{ex}(n, H, F)$, which is the largest number of copies of H in n -vertex graphs that do not contain F as a subgraph. More formally, let $\mathcal{N}(H, G)$ denote the number of copies of H in G , and then we let $\text{ex}(n, H, F) = \max\{\mathcal{N}(H, G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\}$. After several sporadic results, starting with [20], the systematic study of generalized Turán problems was initiated by Alon and Shikhelman [2].

Győri, Wang and Woolfson [15] initiated the study of generalized Turán problems involving double stars. The double star $S_{a,b}$ consists of a central edge uv , a leaves connected to u and b leaves connected to v . We say that u and v are centers of the double star. Győri, Wang and Woolfson [15] showed the following.

Theorem 1.1 (Győri, Wang and Woolfson [15]). *If n is sufficiently large, then $\text{ex}(n, S_{a,b}, K_3) = \mathcal{N}(S_{a,b}, K_{m,n-m})$ for some m .*

The above theorem states that among triangle-free graphs, the number of copies of a double star is always maximized by a complete bipartite graph. Given a and b , it is straightforward to determine which complete bipartite graph contains the most copies of $S_{a,b}$ (i.e. to determine m), but we cannot handle this problem in such generality.

In generalized Turán problems, there exist other examples of this situation where the extremal number is almost explicitly determined, only missing some optimization of polynomials. Gerbner and Palmer [12] introduced the F -Turán-good graphs, which are graphs H such that $\text{ex}(n, H, F)$ is equal to the number of copies of H in the $(\chi(F) - 1)$ -partite Turán graph for n sufficiently large. We say that H is weakly F -Turán-good if for n sufficiently large we have $\text{ex}(n, H, F) = \mathcal{N}(H, G)$ for some complete $(\chi(F) - 1)$ -partite n -vertex graph G . We can reformulate Theorem 1.1 the following way: $S_{a,b}$ is weakly K_3 -Turán-good.

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Györi, Pach and Simonovits [14] showed that complete multipartite graphs are weakly K_k -Turán-good for any k . Gerbner [10] showed that if F is a 3-chromatic graph with a color-critical edge (which is an edge whose deletion decreases the chromatic number), then complete multipartite graphs are weakly F -Turán-good.

It is easy to see that counting copies of $S_{a,b}$ or $K_{a+1,b+1}$ in complete bipartite graphs is the same optimization problem. Indeed, every copy of $K_{a+1,b+1}$ contains exactly $(a+1)(b+1)$ copies of $S_{a,b}$, and every copy of $S_{a,b}$ is counted exactly once. In fact, Theorem 1.1 implies the above mentioned result of Györi, Pach and Simonovits [14] for $k=3$, as $\mathcal{N}(K_{a+1,b+1}, G) \leq (a+1)(b+1)\mathcal{N}(S_{a,b}, G) \leq (a+1)(b+1)\mathcal{N}(S_{a,b}, K_{m,n-m}) = \mathcal{N}(K_{a+1,b+1}, K_{m,n-m})$. More generally, we have the following.

Proposition 1.2. *Assume that H has a unique proper r -coloring, F has chromatic number $r+1$ and H is weakly F -Turán-good. If H' is obtained by adding edges to H and has chromatic number r , then H' is also weakly F -Turán-good. Moreover, the same complete r -partite graph contains the maximum number of copies of H and H' .*

Proof. Let us assume that there are p ways to obtain H' from H by adding edges, and H' contains q copies of H . Observe that if a copy of H is in a complete r -partite graph G_0 , then all the p ways to obtain H' create a subgraph of G_0 . Let n be large enough, and let G_0 be a complete r -partite n -vertex graph with $\mathcal{N}(H, G_0) = \text{ex}(n, H, F)$. Then, for any F -free n -vertex graph G we have $\mathcal{N}(H', G) \leq p\mathcal{N}(H, G)/q \leq p\text{ex}(n, H, F)/q = p\mathcal{N}(H, G_0)/q = \mathcal{N}(H', G_0)$, completing the proof. \square

In this paper, we examine further generalized Turán problems where the forbidden graph or the graph we count or both are double stars. Our first result extends Theorem 1.1 to 3-chromatic graphs with a color-critical edge in place of the triangle.

Theorem 1.3. *If F has chromatic number three and a color-critical edge, then $S_{a,b}$ is weakly F -Turán-good.*

We remark that the asymptotics have been known, as $\text{ex}(n, S_{a,b}, F) = (1+o(1))\text{ex}(n, S_{a,b}, K_3)$ for any 3-chromatic F by a theorem of Gerbner and Palmer [11]. Also note that using Proposition 1.2, Theorem 1.3 implies the above mentioned result of Gerbner [10], i.e. that complete bipartite graphs are weakly F -Turán-good.

The smallest example of a 3-chromatic graph with no color-critical edge is the 2-fan F_2 , which consists of two triangles sharing a vertex. It is easy to see that to $K_{a,b}$ with $a \geq 2$ and $b \geq 3$, we can add exactly one edge without creating a copy of F_2 . Erdős, Füredi, Gould and Gunderson [7] showed that if $n \geq 5$, then an F_2 -free graph has at most $\lfloor n^2/4 \rfloor + 1$ edges, one more than the Turán graph. Gerbner and Palmer [11] showed that C_4 is F_2 -Turán-good (the single additional edge cannot be in any copy of C_4). Gerbner [9] extended this to any $K_{a,a}$ with $a > 2$ in place of C_4 . Let $K_{a,b}^+$ denote the graph we obtain from $K_{a,b}$ by adding an edge to the part of size a . Gerbner and Patkós [13] showed that $\text{ex}(n, K_{a,b}, F_2) = \mathcal{N}(K_{a,b}, K_{m,n-m}^+)$ for some m . We prove a similar result when counting double stars instead of $K_{a,b}$.

Theorem 1.4. *For any positive integers a, b and sufficiently large n , we have $\text{ex}(n, S_{a,b}, F_2) = \mathcal{N}(S_{a,b}, K_{m,n-m}^+)$ for some m .*

Perhaps the most natural questions concerning generalized Turán problems for double stars are counting cliques or counting double stars when a double star is forbidden. We resolve both questions by the following results.

Proposition 1.5. *Let $n = p(a+b+1) + q$ with $q \leq a+b$. Then, for $k \geq 3$ we have $\text{ex}(n, K_k, S_{a,b}) = p\binom{a+b+1}{k} + \binom{q}{k}$.*

Concerning $\text{ex}(n, S_{a,b}, S_{c,d})$, we assume without loss of generality that $a \leq b$ and $c \leq d$, and separate two cases. If $c \leq a$, then we can assume $b < d$, as otherwise $\text{ex}(n, S_{a,b}, S_{c,d}) = 0$.

Let $r = \max\{(c+d)\binom{c+d-1}{a}\binom{c+d-1-a}{b}, d\binom{d-1}{a}\binom{d-1}{b}\}$ if $a \neq b$, and $r = \max\{\frac{c+d}{2}\binom{c+d-1}{a}\binom{c+d-1-a}{b}, \frac{d}{2}\binom{d-1}{a}\binom{d-1}{b}\}$ if $a = b$. We say that r is nice if $r = d\binom{d-1}{a}\binom{d-1}{b}$ for $a \neq b$ or $r = \frac{d}{2}\binom{d-1}{a}\binom{d-1}{b}$ for $a = b$.

Proposition 1.6. *Let $c \leq a \leq b < d$. Then, there is a constant $k = k(a, b, c, d)$ such that for any n , we have $rn - k \leq \text{ex}(n, S_{a,b}, S_{c,d}) \leq rn$. Moreover, if r is nice, dn is even and n is large enough, then $\text{ex}(n, S_{a,b}, S_{c,d}) = rn$. If r is not nice and $c+d+1$ divides n , then $\text{ex}(n, S_{a,b}, S_{c,d}) = rn$.*

Theorem 1.7. *Let $a \leq b$ and $a < c \leq d$. If n is sufficiently large, then $\text{ex}(n, S_{a,b}, S_{c,d}) = \mathcal{N}(S_{a,b}, K_{c,n-c})$.*

The rest of the paper is structured as follows. In Section 2, we state the well-known results we use, and afterwards we state and prove the lemmas needed to prove Theorems 1.3 and 1.4. Section 3 contains the proofs of the theorems and propositions stated above.

2. Preliminaries

We will use a theorem of Abbott, Hanson and Sauer [1]. Let $f(m)$ be equal to $m(m-1)$ if m is odd and to $(m-1)^2 + (m-2)/2$ if m is even.

Theorem 2.1 (Abbott, Hanson and Sauer [1]). *If a graph has $f(m)$ edges, then it contains a star or a matching with m edges.*

Note that $f(m)$ is the best possible bound, but we will only use the weaker result that any graph G contains a star or matching with at least $\sqrt{|E(G)|}$ edges.

We will also use some other well-known results.

Theorem 2.2 (Kővári, Sós and Turán [16]). *For any integers $s \leq t$, we have $\text{ex}(n, K_{s,t}) = O(n^{2-1/s})$.*

Theorem 2.3 (Erdős, Simonovits [4,5,18]). *For any graph F , if G is an F -free graph on n vertices with $\text{ex}(n, F) - o(n^2)$ edges, then we can obtain the $(\chi(F) - 1)$ -partite Turán graph from G by adding and deleting $o(n^2)$ edges.*

Theorem 2.4 (Simonovits [17]). *If F has a color-critical edge and n is sufficiently large, then $\text{ex}(n, F) = \text{ex}(n, K_{\chi(F)})$*

Let B_t denote the book graph with t pages, i.e. t triangles sharing an edge. Alon and Shikhelman [2] showed $\text{ex}(n, K_3, B_t) = o(n^2)$ for any fixed t . Another result from [2] is that $\text{ex}(n, H, F) = o(n^{|V(H)|})$ if and only if F is a subgraph of a blow-up of H . Combined with the removal lemma [6], this implies that if F has chromatic number three, then any F -free n -vertex graph G contains a set of $o(n^2)$ edges such that every triangle in G contains at least one of those edges.

We continue with the lemmas needed for the proof of Theorem 1.3.

Lemma 2.5. *Let t be a positive integer, $\beta < 1$ and n large enough. Let G be an n -vertex B_t -free graph with maximum degree $n/2 < \Delta < \beta n$. Then, $|E(G)| \leq \Delta(n - \Delta)$. Moreover, if G contains a triangle, then $|E(G)| \leq \Delta(n - \Delta) - cn$ for some c depending on β and t .*

Proof. Assume first that G is triangle-free. Let v be a vertex of G with degree Δ , and U be the set of neighbors of v . Then, U is an independent set, and the $n - \Delta$ vertices in $V(G) \setminus U$ each have degree at most Δ , completing the proof in this case.

Assume now that G contains a triangle uvw . Consider first the case that $\Delta < 5n/9$. Then, any two of the vertices u, v, w have less than t common neighbors, thus $d(u) + d(v) + d(w) < n + 3t$. This implies that there is a vertex of degree less than $(n + 3t)/3 < 11n/30$ (here we use the fact that n is large enough). We delete such a vertex and denote the remaining graph by G_1 . We repeat this procedure to obtain G_2 and so on, until we either reach a triangle-free graph G_i , or we reach a graph G_i with maximum degree Δ_i such that $\Delta_i \leq (n - i)/2$. Let $x = \Delta - n/2$, then $\Delta_{2x-1} \geq \Delta - (2x - 1) > (n - (2x - 1))/2$, thus in the second situation we have $i > 2x - 1$.

In the first case, G_i has maximum degree $\Delta_i \geq \Delta - i$, thus, similarly to the first paragraph of the proof, G_i contains an independent set of order $\Delta_i \geq \lfloor (n - i)/2 \rfloor$, thus G_i contains at most $\Delta_i(n - i - \Delta_i) \geq (\Delta - i)(n - i - (\Delta - i))$ edges. Therefore, G contains at most $(\Delta - i)(n - \Delta) + 11in/30$ edges, which completes the proof. In the second case, G_i has at most $(n - i)^2/4$ edges by Theorem 2.4, thus $|E(G)| \leq (n - i)^2/4 + 11in/30 = n^2/4 - 2ni/15 + i^2/4 \leq n^2/4 - 4xn/15 + x^2 = n^2/4 - 2x(n/8 - x/2) - xn/20 \leq n^2/4 - 7x^2/2 - xn/20 \leq n^2/4 - x^2 - cn$. Here the first inequality follows from the definition of i , the second inequality follows from $i \geq 2x$ using that $2n/15 > i/4$, the equation is simple reorganization, and the penultimate inequality uses that $n > 18x$. This completes the proof in this case.

Consider now the case $\Delta > 5n/9$. Let v be a vertex of degree Δ and U be the set of its neighbors. Let W be the set of vertices with degree at least $n/2 + t$, then W is an independent set by the B_t -free property. This also implies that U and W are disjoint, since $v \in W$. Let U' be the subset of U consisting of vertices that have at least $(n - \Delta)/2 + t$ neighbors in $V(G) \setminus U$, then U' is an independent set. We have at most $t|U \setminus U'|$ edges inside U . On the other hand, for every vertex $u \in U \setminus U'$ we have at least $(n - \Delta)/2 - t$ vertices outside U not connected to u .

Let us assume that there are k vertices of degree less than $n/2 + t$ outside U . Then, the number of edges in G is at most $k(n/2 + t) + (n - \Delta - k)\Delta + t|U \setminus U'|$. Indeed, the first term stands for the edges incident to $V(G) \setminus (U \cup W)$, the second term stands for the edges incident to W and the third term stands for the edges inside U . The above quantity can be rewritten as $\Delta(n - \Delta) - k(\Delta - n/2 - t) + t|U \setminus U'| \leq \Delta(n - \Delta) - k(n/18 - t) + t|U \setminus U'|$. It is easy to see that we are done if $k \geq 1$ and $|U \setminus U'| \leq n/20t$, or if $k \geq 19t$.

We also have $|E(G)| \leq t|U \setminus U'| + k(n/2 + t) + \Delta(n - \Delta) - |U \setminus U'|((n - \Delta)/2 - t)$, as the first term stands for the edges inside U , the second term stands for the edges inside $V(G) \setminus U$, the third term stands for the number of pairs (u, v) with $u \in U, v \in V(G) \setminus U$, and the fourth term stands for those such pairs that are not edges of $V(G)$. If $|U \setminus U'| > n/20t$ and $k < 19t$, then this bound is $\Delta(n - \Delta) - cn^2$ and we are done. If $k = 0$ and $|U \setminus U'| > 0$, then this bound is $\Delta(n - \Delta) - cn$, and if $k = 0$ and $|U \setminus U'| = 0$, then G is bipartite, thus triangle-free. This completes the proof. \square

We need a version for books of size increasing with n , when we also forbid another graph.

Lemma 2.6. Let $1/2 < \beta$, $\alpha < 1/9$, $2\alpha < 1 - \beta$ and F be a 3-chromatic graph with a color-critical edge. Let $n > n_0(\alpha, \beta, F)$ be an integer large enough, $t \leq \alpha n$ and $(1/2 + \alpha)n < \Delta < \beta n$ be positive integers. If G is an n -vertex F -free graph with maximum degree Δ such that the largest book in G is B_t , then $|E(G)| \leq \Delta(n - \Delta) - ctn$ for a constant $c = c(\alpha, \beta, F)$.

Proof. We start by picking an $\varepsilon > 0$ sufficiently small with respect to α , β and F . We will use that ε is small in multiple instances; we will not list the exact assumptions. We choose n large enough with respect to α , β , F and ε ; again we do not list the exact assumptions here.

Let v be a vertex of degree Δ and U be the set of its neighbors. Clearly, the property that H is F -free implies that a bipartite graph is forbidden inside U , thus there are at most $\varepsilon^4 n^2/3$ edges inside U by the Kővári-Sós-Turán theorem. Let W be the set of vertices outside U with degree at least $\Delta - \varepsilon n$. Assume that $|W| \leq n - \Delta - \varepsilon^3 n/2$. Then, the total number of edges in G is at most $\Delta|W| + (\Delta - \varepsilon n)(n - \Delta - |W|) + \varepsilon^4 n^2/3 = \Delta(n - \Delta) - \varepsilon n(n - \Delta - |W|) + \varepsilon^4 n^2/3 \leq \Delta(n - \Delta) - \varepsilon^4 n^2/6$, completing the proof. Thus, we will assume that $|W| > n - \Delta - \varepsilon^3 n/2$. Any two vertices of W have at least αn common neighbors (since ε is small enough), thus there are no edges inside W . This implies that all but at most $(\varepsilon^3/2 + \varepsilon^4/2)n^2$ edges of G go between U and W .

Let $u, u' \in U$ with $uu' \in E(G)$. Then, u and u' have at most $t \leq \alpha n$ common neighbors, in particular the number of neighbors of u in W plus the number of neighbors of u' in W is at most $n - \Delta + \alpha n$. Let U' denote the set of vertices in U with more than $|W| - \varepsilon n \geq n - \Delta - 2\varepsilon n > (n - \Delta + \alpha n)/2$ neighbors in W , then U' is an independent set. Every vertex of $U \setminus U'$ is connected to at most $|W| - \varepsilon n$ vertices of W , thus at least $\varepsilon n|U \setminus U'|$ edges are missing between U and W .

If $|U \setminus U'| \geq \varepsilon^2 n$, then the total number of edges is at most $|U||W| - \varepsilon^3 n^2 + (\varepsilon^3/2 + \varepsilon^4/2)n^2 = |U||W| - (\varepsilon^3 - \varepsilon^4)n^2/2 \leq \Delta(n - \Delta) - (\varepsilon^3 - \varepsilon^4)n^2/2$, completing the proof. Thus, we will assume that $|U \setminus U'| \leq \varepsilon^2 n$.

We found a bipartite subgraph G' of G with parts U' of size at least $n - \Delta - \varepsilon^2 n$ and W of size at least $n - \Delta - \varepsilon^3 n/2 \geq n - \Delta - \varepsilon^2 n$. Moreover, every vertex of G' is connected to all but at most $2\varepsilon n$ vertices of the other part of G' .

Let x be a vertex of G not in G' . If x has a set A of $|V(F)|$ neighbors in W , then the vertices in A have at least $|U'| - 2\varepsilon n|V(F)|$ common neighbors in U' . Because of the F -free property, x is connected to less than $|V(F)|$ of those neighbors, thus to at most $|V(F)|(2\varepsilon n + 1)$ vertices of U' . Similarly, if x has at least $|V(F)|$ neighbors in U' , then x has at most $|V(F)|(2\varepsilon n + 1)$ neighbors in W . Thus, either x is connected to at most $|V(F)|(2\varepsilon n + 1)$ vertices on both sides of G' , or x is connected to less than $|V(F)|$ vertices on one side.

Let us add now vertices outside G' to U' with at least $t + 1$ neighbors in W , one by one. First we add a vertex x_1 that has less than $|V(F)|$ neighbors in U' , if exists. Then, we add x_2 if it has less than $|V(F)|$ neighbors in $U' \cup \{x_1\}$. In general, we add a vertex x_i with less than $|V(F)|$ neighbors in $U' \cup \{x_1, \dots, x_{i-1}\}$, if exists. If there is no such vertex with at least $t + 1$ neighbors in W , then we stop and let $U'' = U' \cup \{x_1, \dots, x_{i-1}\}$. We execute the same procedure with the role of U' and W reversed, i.e., we take vertices outside G' with at least $t + 1$ neighbors in U' , and add them to W one by one if they have less than $|V(F)|$ neighbors among the vertices inside W or added to W this way. We denote by W' the set of vertices obtained this way.

Then, the number of edges inside U'' is at most $2|V(F)|\varepsilon^2 n$, as the at most $2\varepsilon^2 n$ vertices added to the independent set U' each contribute less than $|V(F)|$ such edges. Similarly, there are at most $2|V(F)|\varepsilon^2 n$ edges inside W' .

Claim 2.7. There is a p_0 such that if $p \geq p_0$ and we have a B_p in G with vertices x, y, z_1, \dots, z_p , edges xy, xz_i, yz_i for $i \leq p$ and $z_1, \dots, z_p \in U'' \cup W'$, then at least $c_0 p n$ edges are missing between U'' and W' for some constant c_0 depending only on F and β .

Proof. There is a $p_0(F)$ such that if $p \geq p_0(F)$, then $\text{ex}(2p, K_{|V(F)|, |V(F)|}) \leq \varepsilon p^2$ by the Kővári-Sós-Turán theorem. If we take a $K_{|V(F)|, |V(F)|}$ with one part in $\{z_1, \dots, z_p\}$ and other part avoiding x and y and add x and y , the resulting graph contains F . Therefore, at most εp^2 edges go from z_1, \dots, z_p to any set of p vertices not containing x, y and any z_i . Also there are at most εp^2 edges inside $\{z_1, \dots, z_p\}$, hence at most $\lceil n/p \rceil \varepsilon p^2 + 2p \leq 2p\varepsilon n$ edges are incident to z_1, \dots, z_p . Thus, at least $p(n - \Delta - \varepsilon^2 n - 2\varepsilon n)/2 \geq pn(1 - \beta - \varepsilon^2 - 2\varepsilon)/2 \geq pn(1 - \beta)/3$ edges are missing between U'' and W' if ε is small enough. \square

Assume first that each vertex of $V(G) \setminus V(G')$ is in $U'' \cup W'$. Observe that $U \subset U''$. Indeed, if a vertex $u \in U$ is connected to $t + 1$ vertices of U' , then $uv \in E(G)$ and u and v have $t + 1$ common neighbors, forming B_{t+1} , a contradiction. Therefore, $|U''| \geq \Delta$.

If $t < p_0(F)$, the statement we want to prove simply follows from Lemma 2.5. If $t \geq p_0(F)$, then Claim 2.7 with $p = t$ completes the proof, since the number of edges that are not between U'' and W' is at most $4|V(F)|\varepsilon^2 n \leq c_0 t n/2$.

Assume now that there are $1 \leq k \leq 2\varepsilon^2 n$ vertices not in $U'' \cup W'$. We claim that they each have at most $\max\{t, |V(F)|(2\varepsilon n + 1)\}$ neighbors in both U' and W . Indeed, if such a vertex v has more than t neighbors in W , and less than $|V(F)|$ neighbors in U' , then we add it to U'' . If v has at least $|V(F)|$ neighbors in U' , then v has at most $|V(F)|(2\varepsilon n + 1)$ neighbors in W . Similarly we obtain that if v has more than t neighbors in U' then v has at most $|V(F)|(2\varepsilon n + 1)$ neighbors in U' .

The k vertices not in $U'' \cup W'$ are also connected to at most $2\varepsilon^2 n$ vertices outside $U' \cup W$. Observe that $\max\{|V(F)|(2\varepsilon n + 1), t\} + 2\varepsilon^2 n^2 \leq (n - \Delta - \varepsilon n)/2$ for ε small enough, thus these k vertices are incident to at most $k(n - \Delta - \varepsilon n)$ edges. Then, $|U''| \geq \Delta - k$, thus $|U''||W'| \leq (\Delta - k)(n - \Delta)$. The number of edges in G is at most $(\Delta - k)(n - \Delta) + 4|V(F)|\varepsilon^2 n + k(n -$

$\Delta - \varepsilon n) \leq \Delta(n - \Delta) - k\varepsilon n/2$. This completes the proof if $t < 2p_0(F)$ or if $k > t/2$. In the remaining case, at least $t/2$ vertices of z_1, \dots, z_t are in $U'' \cup W'$, thus we can apply Claim 2.7 with $p = \lceil t/2 \rceil$. Then, at least $c_0 \lceil t/2 \rceil n$ edges are missing between U'' and W' , and the number of edges not between U'' and W' is at most $4|V(F)|\varepsilon^2 n + k|V(F)|(\varepsilon n + 1) \leq c_0 t n/3$ if ε is small enough. \square

Let us continue with the lemmas needed for the proof of Theorem 1.4. A *vertex cover* is a set of vertices incident to each of the edges.

Lemma 2.8. *If G is an n -vertex F_2 -free graph and A is a vertex cover with $|A| < n/4$, then $|E(G)| < |A|(n - |A|) + 1$.*

Proof. Assume that A is a vertex cover. Let us delete each vertex of A that is connected to at most $n/2$ vertices of $V(G) \setminus A$, and denote the resulting set by A' . Assume that there are $\ell > 1$ edges inside A' . Then, there is a matching or star in A' with $\lceil \sqrt{\ell} \rceil$ edges by Theorem 2.1.

Observe that a vertex outside A can be connected to at most $m + 1$ vertices of a matching with m edges inside A' , because of the F_2 -free property. For a star inside A' with m edges, at most one vertex outside A can be connected to its center and two other vertices. Therefore, $n - |A| - 1$ vertices outside A are each either connected to at most 2 vertices of the star, or are not connected to the center. The latter possibility occurs for less than $n/2$ vertices outside A , as the center of the star is in A' . This means that in both cases, at least $(m - 1)(n/2 - |A| - 1)$ edges are missing between A' and $V(G) \setminus |A|$.

The number of edges inside A is at most $\ell + |A||A \setminus A'|$, while the number of edges missing between A and $V(G) \setminus |A|$ is at least $(\lceil \sqrt{\ell} \rceil - 1)(n/2 - |A| - 1) + |A \setminus A'|(n/2 - |A|)$. Observe that $\ell \leq \binom{|A'|}{2} < |A'|^2/2$, thus $\sqrt{2\ell} < |A'| \leq n/2 - |A| - 1$, which implies that $\ell = \sqrt{\ell/2} \sqrt{2\ell} < (\lceil \sqrt{\ell} \rceil - 1)(n/2 - |A| - 1)$ if $\ell \geq 4$. For $\ell \leq 3$, it is obvious that $\ell < (\lceil \sqrt{\ell} \rceil - 1)(n/2 - |A| - 1)$ using that $|A| \geq 3$ to have at least two edges inside A' . We also have $|A||A \setminus A'| < |A \setminus A'|(n/2 - |A|)$, and the two inequalities combined mean that more edges are missing between A and $V(G) \setminus |A|$ than the number of edges inside A , which completes the proof.

Finally, if $\ell \leq 1$, then we have at most $\ell + |A||A \setminus A'|$ edges inside A and at least $|A \setminus A'|(n/2 - |A| - 1) \geq |A||A \setminus A'|$ edges are missing between A and $V(G) \setminus A$, completing the proof. \square

Lemma 2.9. *Let a and b be positive integers. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if G is an n -vertex F_2 -free graph with maximum degree $\Delta > n/2$ and at least $\Delta(n - \Delta) - \delta n^2$ edges, then G can be turned to a bipartite graph G' by deleting at most εn^2 edges.*

We remark that if $\Delta = n/2 + o(n)$, then this is the Erdős-Simonovits stability theorem.

Proof. By the Erdős-Simonovits stability theorem, there is an $\alpha > 0$ such that if G has at least $n^2/4 - \alpha n^2$ edges, then G can be made bipartite by deleting at most $\varepsilon n^2/2$ edges. We can assume that α is small enough. Assume first that $\Delta \leq n/2 + \alpha n$, and let $\delta = \alpha/2$. Then, G has at least $\Delta(n - \Delta) - \delta n^2 \geq n^2/4 - (\alpha^2 + \delta)n^2 \geq n^2/4 - \alpha n^2$ edges, thus the conclusion holds.

Let us assume now that $\Delta > n/2 + \alpha n$ and let v be a vertex of degree Δ and U be the set of its neighbors. Then, there are no independent edges inside U , hence there are at most n edges inside U . Observe that if $n - \Delta = o(n)$, then there are $o(n^2)$ edges in G , thus the conclusion holds in this case. Therefore, we can assume that $n - \Delta > \beta n$ for some $\beta > 0$.

Consider the bipartition of G into U and $V(G) \setminus U$. Let G' be the subgraph of G induced on $V(G) \setminus U$. We are done unless $|E(G')| \geq \varepsilon n^2 - n$. Let $u \in U$ and let W be the set of vertices in $V(G) \setminus U$ that are not adjacent to u . Then, W covers almost all the edges of G' . More precisely, there is a vertex w such that $W \cup \{w\}$ is a vertex cover of G' . Indeed, there can be only a star outside W in G' , as otherwise there are two independent edges in the neighborhood of u .

We pick $u \in U$ in such a way that W is the smallest, then at least $\Delta|W|$ edges are missing between U and $V(G) \setminus U$. By Lemma 2.8, if $|W| + 1 < (n - \Delta)/4$, then $|E(G')| < (|W| + 1)(n - \Delta - |W| - 1) + 1$. If $|W| < \varepsilon n$, this contradicts our assumption that $|E(G')| \geq \varepsilon n^2 - n$. Therefore, we can assume that $|W| \geq \varepsilon n$. The number of edges in G is at most $\Delta(n - \Delta) - \Delta|W| + (|W| + 1)(n - \Delta - |W| - 1) + 1 + n < \Delta(n - \Delta) - \delta n^2$ if δ is small enough, a contradiction.

If $|W| + 1 \geq (n - \Delta)/4$, then we use the simpler bound $|E(G')| \leq \text{ex}(|V(G')|, F_2) = |V(G')|^2/4 + 1$ due to Erdős, Füredi, Gould and Gunderson [7]. Thus, the number of edges in G is at most $\Delta(n - \Delta) - \Delta(n - \Delta)/4 + (n - \Delta)^2/4 + 1 + n = \Delta(n - \Delta) + \frac{n - \Delta}{4}(n - 2\Delta) + 1 + n \leq \Delta(n - \Delta) - \alpha\beta n/2 + n < \Delta(n - \Delta) - \delta n^2$ if δ is small enough, a contradiction completing the proof. \square

3. Proofs

Recall that Theorem 1.3 states that if F has chromatic number three and a color-critical edge, then $S_{a,b}$ is weakly F -Turán-good. Before the proof, we describe the main component of the proof of Theorem 1.1 in [15]. We let $f(x, y) = \binom{x-1}{a} \binom{y-1}{b} + \binom{y-1}{a} \binom{x-1}{b}$ if $a \neq b$ and $f(x, y) = \binom{x-1}{a} \binom{y-1}{b}$ if $a = b$. Then, $f(d(u), d(v))$ is the number of copies of $S_{a,b}$ with central edge uv . Györi, Wang and Woolfson observed that if G is a triangle-free graph with maximum degree $\Delta \geq n/2$, then $f(d(u), d(v)) \leq \max_{n/2 \leq \Delta' \leq \Delta} f(\Delta', n - \Delta')$. It is easy to see that $K_{\Delta', n - \Delta'}$ has at least as many edges as G , and each edge is the central edge of as least as many copies of $S_{a,b}$ as any edge of G .

Proof of Theorem 1.3. Let us assume indirectly that there is an infinite sequence of counterexamples, i.e. graphs G_1, G_2, \dots such that $\text{ex}(n_i, S_{a,b}, F) = \mathcal{N}(S_{a,b}, G_i) > \mathcal{N}(S_{a,b}, K_{m, n_i - m})$ for every m , and $n_i > n_{i-1}$. Let us consider first the case that there exists a constant $\beta < 1$ such that for each G_i , the maximum degree $\Delta_i < \beta n_i$.

First we show that $\Delta_i > n_i/2$. Indeed, otherwise $K_{\lfloor n_i/2 \rfloor, \lceil n_i/2 \rceil}$ has at least $|E(G_i)|$ edges, and each edge of $K_{\lfloor n_i/2 \rfloor, \lceil n_i/2 \rceil}$ is the central edge of $f(\lfloor n_i/2 \rfloor, \lceil n_i/2 \rceil)$ copies of $S_{a,b}$. On the other hand, in G_i each edge is the central edge of at most $f(\Delta_i, \Delta_i)$ copies of $S_{a,b}$. This, using the obvious monotonicity properties of $f(x, y)$ implies that $\mathcal{N}(S_{a,b}, G_i) \leq \mathcal{N}(S_{a,b}, K_{\lfloor n_i/2 \rfloor, \lceil n_i/2 \rceil})$, a contradiction.

Thus, we can assume that $\Delta_i > n_i/2$. Let us pick $\varepsilon > 0$ small enough. We choose an i such that n_i is large enough. Let $G = G_i$, $n = n_i$ and $\Delta = \Delta_i$. We have mentioned in Section 2 that a result of Alon and Shikhelman [2] combined with the removal lemma implies the following: G contains a set of at most εn^2 edges such that every triangle in G contains at least one of those edges.

Let B_t be the largest book in G with vertices u, v connected to each other and to z_1, \dots, z_t . Then, there are at most $\varepsilon t n^2$ triangles in G .

We pick α to satisfy the conditions of Lemma 2.6 and consider first the case when $t \geq \alpha n$. Then, there is no copy of $K_{|V(F)|, |V(F)|}$ with one part in $Z = \{z_1, \dots, z_t\}$. Indeed, such a copy together with u and v would contain F . This implies with the Kővári-Sós-Turán theorem that there are $o(n^2)$ edges incident to Z in G . Let us consider the copies of $S_{a,b}$ that contain a vertex from Z . We can count them by picking an edge incident to Z $o(n^2)$ ways, then $a+b$ other vertices $O(n^{a+b})$ ways, and then a copy of $S_{a,b}$ on these $a+b+2$ vertices, constant number of ways. This shows that there are $o(n^{a+b+2})$ copies of $S_{a,b}$ that contain a vertex from Z . Let us delete all the edges incident to Z , and add a copy of $K_{\lfloor t/2 \rfloor, \lceil t/2 \rceil}$ on Z . The resulting graph is clearly F -free and contains $\mathcal{N}(S_{a,b}, G) - o(n^{a+b+2}) + \Omega(n^{a+b+2}) > \mathcal{N}(S_{a,b}, G)$ copies of $S_{a,b}$, a contradiction.

Assume now that $t < \alpha n$. If $t = 0$, then Theorem 1.1 gives the proof. If $t > 0$, then we can apply Lemma 2.6 to show that G has at most $\Delta(n - \Delta) - ctn$ edges. Observe that for every edge uv we have that $d(u) + d(v) \leq n + t$.

Consider the maximum of the function $f(x, y)$ where $x + y \leq z$ and $x, y \leq \Delta$. Clearly this maximum is obtained by $f(\Delta', z - \Delta')$ for some $z/2 \leq \Delta' \leq \Delta$. We apply it for $z = n$, thus for $x + y \leq n$ we have $f(x, y) \leq f(m, n - m)$ for some $n/2 \leq m \leq \Delta$. If $x' + y' \leq n + g$ for some g , then we have that $f(x', y') \leq f(m', n + g - m') \leq f(m', n - m') + c'gn^{a+b-1} \leq f(m, n - m) + c'gn^{a+b-1}$ for some $n/2 \leq m' \leq \Delta$ and a constant c' depending only on a and b . Observe that each edge of $K_{m, n-m}$ is the central edge of $f(m, n - m)$ copies of $S_{a,b}$, and $K_{m, n-m}$ has $m(n - m) \geq \Delta(n - \Delta)$ edges.

For an edge uv , let $g(uv)$ denote the number of common neighbors of u and v . Clearly $\sum_{uv \in E(G)} g(uv) \leq 3\mathcal{N}(K_3, G) \leq 3\varepsilon n^2$. We have $d(u) + d(v) \leq n + g(uv)$, thus $f(d(u), d(v)) \leq f(\Delta'', n + g(uv) - \Delta'')$ for some $\Delta'' \leq \Delta$. We have $f(\Delta'', n + g(uv) - \Delta'') \leq f(\Delta'', n - \Delta'') + c'g(uv)n^{a+b-1} \leq f(m, n - m) + c'g(uv)n^{a+b-1}$. Therefore, $\mathcal{N}(S_{a,b}, G) = \sum_{uv \in E(G)} f(d(u), d(v)) \leq |E(G)|f(m, n - m) + \sum_{uv \in E(G)} c'g(uv)n^{a+b-1} \leq |E(G)|f(m, n - m) + 3c'\varepsilon n^{a+b+1} \leq (\Delta(n - \Delta) - ctn)f(m, n - m) + 3c'\varepsilon n^{a+b+1} \leq m(n - m)f(m, n - m) - ctnf(m, n - m) + 3c'\varepsilon n^{a+b+1} \leq m(n - m)f(m, n - m) - c''tn^{a+b+1} + 3c'\varepsilon n^{a+b+1}$ for some constant c'' depending on α, β, F . This completes the proof if ε is small enough.

Finally, if β does not exist, then we pick a $\gamma > 0$ small enough and we can pick an n -vertex F -free graph G with maximum degree $\Delta > (1 - \gamma)n$ and $\mathcal{N}(S_{a,b}, G) = \text{ex}(n, S_{a,b}, F)$ such that n is large enough. Let v be a vertex of maximum degree Δ and U be its neighborhood. Observe that if we delete the endvertex of a color-critical edge of F , we obtain a bipartite graph F'' . Then, there is no F'' on U , thus by the Kővári-Sós-Turán theorem, there are at most γn^2 edges inside U if n is large enough. As there are at most γn vertices outside U , there are at most γn^2 edges of G incident to a vertex outside U , thus $|E(G)| \leq 2\gamma n^2$. This implies that $\mathcal{N}(S_{a,b}, G) \leq 2\gamma n^{a+b+2}$, as we can pick the central edge first, and then we can pick each other vertex at most n ways. Clearly $\mathcal{N}(S_{a,b}, K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) > 2\gamma n^{a+b+2}$ if γ is small enough and n is large enough, giving us the contradiction that completes the proof. \square

Recall that Theorem 1.4 states that for sufficiently large n we have $\text{ex}(n, S_{a,b}, F_2) = \mathcal{N}(S_{a,b}, K_{m, n-m}^+)$ for some m . The proof is similar to that of Theorem 1.3. In particular, we use $f(x, y)$ and $g(uv)$.

Proof of Theorem 1.4. Similarly to Theorem 1.3, we assume indirectly that there is an infinite sequence of counterexamples, i.e. F_2 -free graphs G_1, G_2, \dots such that $|V(G_i)| = n_i$, $\text{ex}(n_i, S_{a,b}, F_2) = \mathcal{N}(S_{a,b}, G_i) > \mathcal{N}(S_{a,b}, K_{m, n_i - m})$ for every m , and $n_i > n_{i-1}$. Let us consider first the case that there exists a constant $\beta < 1$ such that for each G_i , the maximum degree $\Delta_i < \beta n_i$.

First observe that $\Delta_i > n_i/2$ by the same reasoning as in Theorem 1.3. Let $\varepsilon > 0$ be small enough and $n = n_i$ be large enough. Let $\Delta = \Delta_i$. We will also prove that $m, n - m \leq \Delta$. It is easy to see and follows from a more general theorem of Alon and Shikhelman [2] that $\text{ex}(n, K_3, F_2) \leq c_1 n$ for an absolute constant c_1 .

Let $G = G_i$ and observe that G is neither a bipartite graph, nor a 3-chromatic graph with a color-critical edge, as in that case G would be a subgraph of $K_{m, n-m}^+$ for some m . Assume first that each vertex has degree at least $3\varepsilon^{1/3}n$.

Claim 3.1. G has at most $(\Delta - 1)(n - \Delta + 1) - c_2 n$ edges for some c_2 that depends of ε but does not depend on n .

Proof of Claim. Assume that the statement does not hold, then by Lemma 2.9 there is a bipartite subgraph G' of G with parts A and B such that there are at most εn^2 edges inside A and B in G . We pick among such bipartite graphs the one

with the most edges. It implies that each vertex is connected to at least as many vertices in the other part as in its part, in particular to at least $3\varepsilon^{1/3}n/2$ vertices. This also implies that $|A|, |B| \geq 3\varepsilon^{1/3}n/2$.

Assume first that there is exactly one edge uv inside A and exactly one edge xy inside B . If no vertex of A is connected to both x and y , or no vertex of B is connected to both u and v , then $|E(G)| \leq \Delta(n - \Delta) - \min\{|A|, |B|\} + 2 \leq \Delta(n - \Delta) - 3\varepsilon^{1/3}n/2$, completing the proof.

Otherwise, it is easy to see that two independent edges on these four vertices, say ux and vy are missing from G . We can assume without loss of generality that uy and vx are in G . Let G_0 denote $K_{|B|, |A|}^+$ with parts A and B and the edge xy . Let us compare the number of copies of $S_{a,b}$ in G and in G_0 . These graphs share the edges with an endvertex not in $\{u, v, x, y\}$, and the endvertices either have the same degree or the degree is larger in G_0 . Moreover, the intersection of the neighborhoods of the endvertices is as small as possible, thus each such edge is the central edge of the same number of copies of $S_{a,b}$ in both graphs, or more copies of $S_{a,b}$ in G_0 . The same holds for the edges ux and vy . The edge uv in G is the central edge of $\binom{|B|}{a} \binom{|B|-a}{b}$ copies of $S_{a,b}$ in G , and the edge ux is the central edge of more than $\binom{|B|-1}{a} \binom{n-|B|-1}{b} + \binom{|B|-1}{b} \binom{n-|B|-1}{a}$ copies of $S_{a,b}$ in G_0 (in the case $a = b$, both quantities are divided by 2). Therefore, G_0 contains more copies of $S_{a,b}$ than G , a contradiction.

Assume now that there are $q \geq 2$ edges inside A and $q' \leq q$ edges inside B in G . First we will show that $\max\{|A|, |B|\} \geq \Delta - 1$. Let u be a vertex of degree Δ , and assume that u has s neighbors in the other part. We will deal with the case $u \in B$, the other case can be proved the same way, we omit the details. Then, there are at most $\Delta - 1 \leq n$ edges between the neighbors of u , thus at least $s(\Delta - s) - 2n$ edges are missing between A and B . Therefore, the number of edges in G is at most $|A|(n - |A|) + \varepsilon n^2 - s(\Delta - s) + 2n$. Furthermore, if $|A| - s \geq \varepsilon^{1/3}n$, then (using that $s \geq \Delta/2$), we have $s(\Delta - s) \geq |A|(\Delta - |A|) + (2|A| - \Delta)\varepsilon^{1/3}n - \varepsilon^{2/3}n^2 \geq |A|(\Delta - |A|) + 2\varepsilon^{1/3}n\varepsilon^{1/3}n - \varepsilon^{2/3}n^2 = |A|(\Delta - |A|) + \varepsilon^{2/3}n^2$. If ε is small enough, this is larger than $|A|(\Delta - |A|) + \varepsilon n^2 + 3n$, thus the number of edges in G is at most $|A|(n - |A|) + \varepsilon n^2 - s(\Delta - s) + 2n \leq |A|(n - |A|) - |A|(\Delta - |A|) - n = |A|(n - \Delta) - n$. Therefore, in the case $|A| - s \geq \varepsilon^{1/3}n$, we are done unless $|A| > \Delta$.

Assume that $|A| - s < \varepsilon^{1/3}n$. If B contains at least 2 neighbors of u , at least one of them has at most one common neighbor with u in part A , in particular has less than $\varepsilon^{1/3}n$ neighbors in A , a contradiction. Therefore, u has at most one neighbor in B , thus $s \geq \Delta - 1$, in particular $|A| \geq \Delta - 1$.

By Theorem 2.1, there is a star or matching of size at least \sqrt{q} inside A . If there is a matching M of size $m \geq \sqrt{q}$ inside A , then every vertex of B is connected to at most one endpoint of all but one of the edges of M , thus to at most $m + 1$ vertices of M . Therefore, at least $|B|(m - 1) \geq 3\varepsilon^{1/3}n/2(m - 1)$ edges are missing between A and B .

Assume that there is a star S of size $m \geq \sqrt{q}$ inside A with center u . Recall that at least $\varepsilon^{1/3}n$ vertices in B are connected to u . If $v \in B$ is connected to u and a leaf w of S , then no other vertex can be connected to both u and another leaf of S . This means that at least $\varepsilon^{1/3}n - 1$ vertices of B are connected only to u and w in S , thus there are at least $(\varepsilon^{1/3}n - 1)(m - 1)$ missing edges between A and B .

In both cases, at least $(\varepsilon^{1/3}n/2)\sqrt{q}$ edges are missing between A and B . On the other hand, there are at most $2q = \sqrt{q}2\sqrt{q} \leq \sqrt{q}2\sqrt{\varepsilon n} \leq (\varepsilon^{1/3}n/4)\sqrt{q}$ edges inside A and B . Therefore, the number of edges in G is at most $|A||B| - (\varepsilon^{1/3}n/4)\sqrt{q} \leq (\Delta - 1)(n - \Delta + 1) - (\varepsilon^{1/3}n/4)\sqrt{q}$. Here we used that A has order at least $\Delta - 1$. \square

Let us return to the proof of the theorem. We have $\sum_{uv \in E(G)} g(uv) \leq 3\mathcal{N}(K_3, G) \leq 3c_1n$. Then, for some $m \leq \Delta - 1$, we have $\mathcal{N}(S_{a,b}, G) = \sum_{uv \in E(G)} f(d(u), d(v)) \leq |E(G)|f(m, n - m) + \sum_{uv \in E(G)} c'g(uv)n^{a+b-1} \leq |E(G)|f(m, n - m) + 3c_1n^{a+b} \leq (\Delta - 1)(n - \Delta + 1)f(m, n - m) - c_2nf(m, n - m) + 3c_1n^{a+b} \leq m(n - m)f(m, n - m) - c_2nf(m, n - m) + 3c_1n^{a+b} \leq m(n - m)f(m, n - m) - c'n^{a+b+1} + 3c_1n^{a+b}$ for some constant c_1 depending on $\alpha, \beta, F, \varepsilon$. This completes the proof.

Let us assume now that there are vertices of degree less than $3\varepsilon^{1/3}n$ in G . We remove vertices of degree less than $3\varepsilon^{1/3}n$ one by one, as long as there is such a vertex. Let k be the number of vertices removed this way and G_1 be the resulting graph. Each time we remove a vertex, we remove at most $3\varepsilon^{1/3}n^{a+b+1}$ copies of $S_{a,b}$, thus we remove at most $3k\varepsilon^{1/3}n^{a+b+1}$ copies of $S_{a,b}$. If $k \geq \varepsilon^{1/4(a+b+1)}n$, then we place $K_{\lfloor k/2 \rfloor, \lfloor k/2 \rfloor}$ on those vertices, that has at least $c'k^{a+b+2} \geq c'k\varepsilon^{1/4}n^{a+b+1}$ copies of $S_{a,b}$ for some constant c' that depends only on a and b . Since ε is small enough, the number of copies of $S_{a,b}$ increases without creating F_2 , a contradiction. If $k < \varepsilon^{1/4(a+b+1)}n$, then $n - k$ is large enough, thus we know that G_1 contains at most $\mathcal{N}(S_{a,b}, K_{m, n-k-m}^+)$ copies of $S_{a,b}$ for some m with $m, n - k - m \leq \Delta$. Then, $K_{m, n-k-m}^+$ contains more copies of $S_{a,b}$ than G . Indeed, we pick k vertices in the part of order $n - m$, there are $\mathcal{N}(S_{a,b}, K_{m, n-k-m}^+)$ copies of $S_{a,b}$ that avoid them, and there are at least $k\binom{m}{a+1}\binom{n-m-k}{b} \geq c''k(1 - \beta - \varepsilon^{1/4(a+b+1)})^{a+b+1}n^{a+b+1}$ copies of $S_{a,b}$ that contain exactly one of them, for some constant c'' that depends only on a and b . If ε is small enough, then $c''k(1 - \beta - \varepsilon^{1/4(a+b+1)})^{a+b+1}n^{a+b+1} > 3k\varepsilon^{1/3}n^{a+b+1}$, giving a contradiction.

Finally, consider the case that there is no $\beta < 1$ such that for each i , $\Delta_i < \beta n_i$. Then, we can proceed exactly as in the proof of Theorem 1.3, thus we only give a sketch. We can pick a γ small enough and an F_2 -free graph n -vertex graph G with $\mathcal{N}(S_{a,b}, G) = \text{ex}(n, S_{a,b}, F_2)$, maximum degree at least $(1 - \gamma)n$ such that n is large enough. Let U be the neighborhood of a vertex of maximum degree, then there are at most n edges inside U and at most γn^2 other edges, thus at most $2\gamma n^{a+b+2}$ copies of $S_{a,b}$, which is less than $\mathcal{N}(S_{a,b}, K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor})$, a contradiction completing the proof. \square

Let us continue with Proposition 1.5. Let S_r denote the star with r leaves, i.e. $S_r = S_{r,0}$, and let $n = p(a+b+1) + q$ with $q \leq a+b$. Chase [3], proving a conjecture of Gan, Loh and Sudakov [8] showed that for $k \geq 3$ we have $\text{ex}(n, K_k, S_{a+b+1}) = p\binom{a+b+1}{k} + \binom{q}{k}$. Recall that Proposition 1.5 states that $\text{ex}(n, K_k, S_{a,b}) = p\binom{a+b+1}{k} + \binom{q}{k}$.

Proof of Proposition 1.5. Let $G = pK_{a+b+1} + K_q$ denote the vertex-disjoint union of p copies of K_{a+b+1} and a copy of K_q . Then, G is clearly $S_{a,b}$ -free (and S_{a+b+1} -free) and contains $p\binom{a+b+1}{k} + \binom{q}{k}$ copies of K_k . This gives the lower bound.

Let us assume that $a \leq b$ without loss of generality. We use induction on n to prove the upper bound. Let G be an $S_{a,b}$ -free n -vertex graph with the most copies of K_k . If every vertex of G has degree at most $a+b$, then we are done by the result of Chase. If a vertex has degree $d > a+b$, then all its neighbors have degree less than a . This way we found $a+b+1$ vertices with degree less than a , let G' be the graph obtained by deleting them. We deleted at most $(a+b+1)\binom{a-1}{k-1}$ copies of K_k . Let G'' be the n -vertex $S_{a,b}$ -free graph obtained by adding a K_{a+b+1} to G' . Then, G'' contains at least $\mathcal{N}(K_k, G) - (a+b+1)\binom{a-1}{k-1} + \binom{a+b+1}{k}$ copies of K_k . We have $(a+b+1)\binom{a-1}{k-1} = k(a+b+1)(a-1) \dots (a-k+1)/k! < k(a+b+1)\frac{a+b}{2} \dots \frac{a+b-k+2}{2}/k! = k\binom{a+b+1}{k}/2^{k-1} < \binom{a+b+1}{k}$, thus G'' contains more copies of K_k than G , a contradiction. \square

Let us turn to $\text{ex}(n, S_{a,b}, S_{c,d})$. Recall that Proposition 1.6 deals with the case $c \leq a \leq b < d$.

Proof of Proposition 1.6. In the calculations in this proof, we assume that $a \neq b$. If $a = b$, then our bounds on the number of copies of $S_{a,b}$ are always divided by 2 because of symmetry, thus the proof also works for that case.

Let G be an n -vertex $S_{c,d}$ -free graph. Let us assume there is a vertex u of degree more than $c+d$ in G . Then, each neighbor v of u has degree at most c , hence cannot be a center of a copy of $S_{a,b}$. As each vertex of $S_{a,b}$ is incident to a center, u cannot be in any copy of $S_{a,b}$, thus we can delete u . Therefore, we can assume that each vertex of G has degree at most $c+d$.

Assume now that a vertex u has degree more than d . Then, for any neighbor v of u , there are at most $c+d+1$ vertices (including u and v) that are connected to either u or v . Therefore, there are at most $s_1 = 2\binom{c+d-1}{a}\binom{c+d-1-a}{b}$ copies of $S_{a,b}$ with central edge uv . Moreover, if v has degree at most d , then there are at most $s_2 = \binom{d-1}{a}\binom{c+d-1-a}{b} + \binom{d-1}{b}\binom{c+d-1-b}{a}$ copies of $S_{a,b}$ with central edge uv . If u and v both have degree at most d , then clearly there are at most $s_3 = 2\binom{d}{a}\binom{d}{b}$ copies of $S_{a,b}$ with central edge uv . Observe that $s_2 < s_3$ and $s_2 < s_1$.

Let us assume that G contains m vertices of degree at most d , and there are x edges between vertices of degree at most d and more than d . Then, there are at most $(dm-x)/2$ edges between vertices of degree at most d and at most $((c+d)(n-m)-x)/2$ edges between vertices of degree more than d . This implies that the total number of copies of $S_{a,b}$ is at most $((c+d)(n-m)-x)s_1/2 + s_2x + (dm-x)s_3/2 \leq (c+d)(n-m)s_1/2 + dms_3/2$. This upper bound is linear in m , thus the maximum is taken at either $m=0$ or $m=|E(G)|$, and we have $|E(G)| \leq (c+d)n/2$. This completes the proof of the upper bound.

If r is nice, then the lower bounds are given by any d -regular triangle-free graph on n or $n-1$ vertices. If r is not nice, then the lower bounds are given by $\lfloor n/(c+d+1) \rfloor$ vertex disjoint copies of K_{c+d+1} . \square

Recall that Theorem 1.7 states that if $a \leq b$, $a < c \leq d$ and n is sufficiently large, then $\text{ex}(n, S_{a,b}, S_{c,d}) = \mathcal{N}(S_{a,b}, K_{c,n-c})$.

Proof of Theorem 1.7. Let G be an $S_{c,d}$ -free n -vertex graph. Let A denote the set of vertices with degree at most c and B denote the set of vertices of degree at least $c+d+1$. Then, every edge from B goes to A . Let $q = \min\{c, |B|\}$ and $x = \binom{c-1}{a}\binom{|A|-1}{b} + \binom{c-1}{b}\binom{|A|-1}{a}$ for $a \neq b$ and $x = \binom{c-1}{a}\binom{|A|-1}{b} + \binom{c-1}{b}\binom{|A|-1}{a})/2$ for $a = b$.

We consider two types of copies of $S_{a,b}$. There are $O(n)$ copies where the central edge is not incident to B , since there are $O(n)$ edges in an $S_{c,d}$ -free graph and each edge not between A and B has endvertices of degree at most $c+d$. The number of copies of $S_{a,b}$ with central edge between A and B is at most $|A|qx$. Indeed, there are at most $|A|q$ such edges, we pick a neighbors of the endvertex in A , and the other endvertex has at most $|A|-1$ other neighbors, we pick b of those vertices, and then repeat this with the role of a and b reversed, if $a \neq b$. Note that $|A|-1$ can be replaced here by the degree of the endvertex in B to obtain a better bound.

We are done if $q < c$. Therefore, B has at least c vertices, let v_1, \dots, v_c be the c vertices of largest degree (we choose arbitrarily in the case of ties). If v_c has degree $O(1)$, then we are also done, as only $|A|(c-1)$ edges between A and B would be central edges of x copies of $S_{a,b}$, the other edges between A and B are central edges of $O(1)$ copies.

If a vertex u has degree less than c , we can add an edge uv_i without creating $S_{c,d}$. We repeat this until every vertex has degree at least c . Assume next that there is an edge uv between two vertices of degree c . Then, we delete this edge and we can replace it with uv_i and vv_j for some i and j . Clearly the number of copies of $S_{a,b}$ increases. We repeat this until we eliminate every such edge.

Assume now that a vertex u of degree c is connected to a vertex w not among the vertices v_i . Then, we replace the edge uw with an edge uv_i for some i . We claim that the number of copies of $S_{a,b}$ does not decrease.

The number of copies of $S_{a,b}$ where uw is a leaf edge depends on the degree of w and of the other endvertex w' of the central edge, but that is always c , and the same holds for uv_i after the replacement. Here we use that there is no edge

between vertices of degree c , thus after picking $a - 1$ or $b - 1$ other neighbors of w or v_i , the neighbors of w' are not among the vertices picked earlier.

Consider now the copies of $S_{a,b}$ where uw is the central edge. If w has degree at most $c + d$, then we delete $O(1)$ such copies of $S_{a,b}$ and add more. If w has degree more than $c + d$, then every neighbor of w has degree c . The number of copies of $S_{a,b}$ where uw is the central edge depends only on the degrees of u and w , thus there are more copies with uv_i as the central edge.

Finally, assume that there is a vertex u of degree more than c but not in $\{v_1, \dots, v_c\}$. Then, the neighbors of u have degree more than c and at most $c + d$, thus u has degree at most $c + d$, and the neighbors of neighbors of u also have degree at most $c + d$ and more than c . This implies that u is in $O(1)$ copies of $S_{a,b}$. Let us delete the edges incident to u and add the edges uv_i for every $i \leq c$, then the number of copies of $S_{a,b}$ increases. We repeat this until every vertex but v_1, \dots, v_c has degree c . The resulting graph is $K_{c,n-c}$, completing the proof. \square

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Declaration of competing interest

There is no conflict of interest.

Data availability

No data was used for the research described in the article.

References

- [1] H.L. Abbott, F. Hanson, N. Sauer, Intersection theorems for systems of sets, *J. Comb. Theory, Ser. A* 12 (3) (1972) 381–389.
- [2] N. Alon, C. Shikhelman, Many T copies in H -free graphs, *J. Comb. Theory, Ser. B* 121 (2016) 146–172.
- [3] Z. Chase, A proof of the Gan-Loh-Sudakov conjecture, *Adv. Combin.* 10 (2020).
- [4] P. Erdős, Some recent results on extremal problems in graph theory, in: *Theory of Graphs, Internl. Symp. Rome, 1966*, pp. 118–123.
- [5] P. Erdős, On some new inequalities concerning extremal properties of graphs, in: P. Erdős, G. Katona (Eds.), *Theory of Graphs*, Academic Press, New York, 1968, pp. 77–81.
- [6] P. Erdős, P. Frankl, V. Rödl, The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, *Graphs Comb.* 2 (2) (1986) 113–121.
- [7] P. Erdős, Z. Füredi, R.J. Gould, D.S. Gunderson, Extremal graphs for intersecting triangles, *J. Comb. Theory, Ser. B* 64 (1995) 89–100.
- [8] W. Gan, P. Loh, B. Sudakov, Maximizing the number of independent sets of a fixed size, *Comb. Probab. Comput.* 24 (2015) 521–527.
- [9] D. Gerbner, On Turán-good graphs, *Discrete Math.* 344 (8) (2021) 112445.
- [10] D. Gerbner, A non-aligning variant of generalized Turán problems, *arXiv preprint*, arXiv:2109.02181, 2021.
- [11] D. Gerbner, C. Palmer, Counting copies of a fixed subgraph in F -free graphs, *Eur. J. Comb.* 82 (2019) 103001.
- [12] D. Gerbner, C. Palmer, Some exact results for generalized Turán problems, *Eur. J. Comb.* 103 (2022) 103519.
- [13] D. Gerbner, B. Patkós, Generalized Turán problems for intersecting cliques, *arXiv preprint*, arXiv:2105.07297, 2021.
- [14] E. Győri, J. Pach, M. Simonovits, On the maximal number of certain subgraphs in K_r -free graphs, *Graphs Comb.* 7 (1) (1991) 31–37.
- [15] E. Győri, R. Wang, S. Woolfson, Extremal problems of double stars, *arXiv preprint*, arXiv:2109.01536, 2021.
- [16] P. Kővári, V.T. Sós, P. Turán, On a problem of Zarankiewicz, *Colloq. Math.* 33 (1954) 50–57.
- [17] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: *Theory of Graphs, Proc. Colloq., Tihany, 1966*, Academic Press, New York, 1966, pp. 279–319.
- [18] M. Simonovits, Extremal graph problems with symmetrical extremal graphs. Additional chromatic conditions, *Discrete Math.* 7 (1974) 349–376.
- [19] P. Turán, Egy gráfelméleti szélsőértékfeladatról, *Mat. Fiz. Lapok* 48 (1941) 436–452.
- [20] A.A. Zykov, On some properties of linear complexes, *Mat. Sb.* 66 (2) (1949) 163–188.