On 3-uniform hypergraphs avoiding a cycle of length four

Beka Ergemlidze[a] Ervin Győri[b] Abhishek Methuku[c]
Nika Salia[d] Casey Tompkins[b]

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Abstract

We show that the maximum number of edges in a 3-uniform n-vertex hypergraph without a Berge cycle of length four is at most $(1+o(1))\frac{n^{3/2}}{\sqrt{10}}$. This improves earlier estimates by Győri and Lemons, and by Füredi and Özkahya.

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1 Introduction

Given a hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ and $E(\mathcal{H})$ denote the set of vertices and edges of $\mathcal{H}$. A hypergraph is called r-uniform if all of its edges have size r. Berge [1] introduced the following definitions of a path and a cycle in a hypergraph.

Definition 1. A *Berge cycle* of length $\ell \geqslant 2$ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \dots, v_\ell\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ. A *Berge path* of length ℓ is a set of $\ell+1$ distinct vertices $\{v_1, \dots, v_{\ell+1}\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that for $1 \leqslant i \leqslant \ell$ we have $\{v_i, v_{i+1}\} \subseteq e_i$.

Let $\mathrm{ex}_r(n, BC_\ell)$ denote the maximum number of edges in a r-uniform n-vertex hypergraph without a Berge cycle of length ℓ. In the case $r = 2$ we write simply $\mathrm{ex}(n, C_\ell)$.

A well-known result of Bondy and Simonovits [3] asserts that for all $\ell \geqslant 2$ we have $\mathrm{ex}(n, C_{2\ell}) = O(n^{1+1/\ell})$, however, the order of magnitude is only known to be sharp in the cases $\ell = 2, 3, 5$. Erdős, Rényi and Sós [5] proved the asymptotic result $\mathrm{ex}(n, C_4) =$

[a] Department of Mathematics and Statistics, University of South Florida, Tampa, Florida, USA (beka.ergemlidze@gmail.com).
[b] Alfréd Rényi Institute of Mathematics, Budapest, Hungary (gyori.ervin@renyi.hu, CTompkins496@gmail.com).
[c] ETH Zürich, Zürich, Switzerland (abhishekmethuku@gmail.com).
[d] Department of Mathematics, King Fahd University of Petroleum and Minerals, 31261, Dhahran, Saudi Arabia (salianika@gmail.com).

$\frac{n^{3/2}}{2} + o(n^{3/2})$, see also [4, 8]. Győri and Lemons [11] extended this result (and the Bondy-Simonovits theorem) and showed in particular that $\mathrm{ex}_r(n, BC_4) = O(n^{3/2})$ for all $r \geqslant 3$. It follows from the results of Füredi and Özkahya [9] that $\mathrm{ex}_3(n, BC_4) \leqslant (1 + o(1))\frac{2}{3}n^{3/2}$ (see Theorem 2 in [9]). In this note, we significantly improve this bound as follows.

Theorem 2.

$$\mathrm{ex}_3(n, BC_4) \leqslant (1 + o(1))\frac{n^{3/2}}{\sqrt{10}}.$$

Note that, the best known lower bound $\mathrm{ex}_3(n, BC_4) \geqslant (1 - o(1))\frac{n^{3/2}}{3\sqrt{3}}$ comes from a construction of Bollobás and Győri [2] with a more general version stated in [10]. We take a C_4-free bipartite graph with color classes of size $n/3$ and $\frac{(2n/3)^{3/2}}{2\sqrt{2}} = \frac{n^{3/2}}{3\sqrt{3}}$ edges asymptotically. Fix one of the classes and for each vertex v in that class, we take an additional vertex v' and add it to every edge in the graph incident to v. This results in a 3-uniform hypergraph on n vertices with $\frac{n^{3/2}}{3\sqrt{3}}$ edges asymptotically, and it is easy to verify this hypergraph contains no Berge C_4.

Related results. Let us briefly mention some important related results where one or more short Berge cycles are forbidden. Recall that a hypergraph without a Berge cycle of length two is linear (i.e., any two hyperedges intersect in at most one vertex). The famous $(6, 3)$-problem is equivalent to determining $\mathrm{ex}_3(n, \{BC_2, BC_3\})$. This was considered by Ruzsa and Szemerédi in their classical paper [13], where they showed that $n^{2-\frac{c}{\sqrt{\log n}}} < \mathrm{ex}_3(n, \{BC_2, BC_3\}) = o(n^2)$ for some constant $c > 0$. Lazebnik and Verstraëte [12] studied hypergraphs containing no Berge cycle of length less than five (i.e., girth five) and showed that $\mathrm{ex}_3(n, \{BC_2, BC_3, BC_4\}) = \frac{1}{6}n^{3/2} + o(n^{3/2})$. Ergemlidze, Győri and Methuku [6] strengthened their result by showing that the same bound holds even if one does not forbid the Berge triangle i.e., they showed $\mathrm{ex}_3(n, \{BC_2, BC_3, BC_4\}) \sim \mathrm{ex}_3(n, \{BC_2, BC_4\})$. Bollobás and Győri [2] studied hypergraphs containing no Berge five cycle and showed that $(1+o(1))\frac{n^{3/2}}{3\sqrt{3}} \leqslant \mathrm{ex}_3(n, BC_5) \leqslant \sqrt{2}n^{3/2} + 4.5n$. Ergemlidze, Győri and Methuku [7] improved this result by showing that $\mathrm{ex}_3(n, BC_5) < (1+o(1))0.254n^{3/2}$. Moreover, in [6], the same authors also studied the analogous question for linear hypergraphs and determined the bound asymptotically by showing that $\mathrm{ex}_3(n, \{BC_2, BC_5\}) = n^{3/2}/3\sqrt{3} + o(n^{3/2})$.

2 Proof of the upper bound in Theorem 2

Now we prove Theorem 2. Let $\mathcal{H}$ be a 3-uniform hypergraph with no Berge C_4 and no isolated vertices. A block $\mathcal{B}$ of a hypergraph $\mathcal{H}$ is defined to be a maximal subhypergraph of $\mathcal{H}$ with the property that for any two edges $e, f \in E(\mathcal{B})$, there is a sequence of edges of $\mathcal{H}$, $e = e_1, e_2, \ldots, e_t = f$, such that $|e_i \cap e_{i+1}| = 2$ for all $1 \leqslant i \leqslant t - 1$ and $V(\mathcal{B}) = \cup_{h \in E(\mathcal{B})} h$. It is easy to see that the blocks of $\mathcal{H}$ define a unique partition of $E(\mathcal{H})$.

For a block $\mathcal{B}$ and an edge $h \in E(\mathcal{B})$, we say h is a *leaf* if there exists $x \in h$ such that the only edge of $\mathcal{B}$ incident to x is h. Let $\mathcal{B}'$ be the set of non-leaf edges of $\mathcal{B}$. By the definition, if $\mathcal{B}'$ contains at least two edges it contains two edges sharing two vertices of $\mathcal{H}$. Let two such edges be $\{v, u, w\}$ and $\{v, u, w'\}$. If there is an edge $\{w, w', v\}$ or $\{w, w', u\}$, note that at most one such edge may exist, then these three edges induce $K_4^{(3)-}$, the 3-uniform hypergraph on 4-vertices and 3 edges, and $\mathcal{B} = K_4^{(3)-}$ since $\mathcal{H}$ is Berge C_4-free hypergraph. If neither $\{w, w', v\}$ nor $\{w, w', u\}$ is an edge, then since $\{v, u, w\}$ is not a leaf edge there is an edge in $\mathcal{B}$ incident with vertices v and w or vertices u and w, without loss of generality we assume there is an edge $\{v, w, v'\}$, for some vertex v' distinct from v, u, w, w'. Similarly, we have an edge $\{v, w', v''\}$ or $\{u, w', v''\}$ for some vertex v'' distinct from v, u, w, w'. This is a contradiction since w, u, w', v, w induces a Berge C_4 in $\mathcal{H}$ in this order. Therefore we have that the set of non-leaf edges of a block $\mathcal{B}$ is either empty, a single edge, or $K_4^{(3)-}$. Even more, if the set of non-leaf edges of $\mathcal{B}$ is $E(K_4^{(3)-})$, then $\mathcal{B}$ does not contain a leaf edge. Thus, the following classification of the blocks into *type 1* and *type 2* blocks is indeed partitioning of the set of all blocks $B(\mathcal{H}) := \{\mathcal{B} \mid \mathcal{B}$ is a block in $\mathcal{H}\}$.

- We say $\mathcal{B} \in B(\mathcal{H})$ is *type 1* if there exists an edge $e \in E(\mathcal{B})$ such that for all distinct $f_1, f_2 \in E(\mathcal{B})$, $f_1, f_2 \neq e$, we have $|e \cap f_i| = 2$, for $i = 1, 2$ and $f_1 \cap f_2 \subseteq e$. (Note that if a block consists of a single edge it is a *type 1* block since it trivially satisfies the condition.)

- We say $\mathcal{B} \in B(\mathcal{H})$ is *type 2* if $\mathcal{B} = K_4^{(3)-}$.

Define the 2-shadow of $\mathcal{H}$ to be the graph on the same set of vertices as $\mathcal{H}$ whose edges are all pairs of vertices $\{x, y\}$ for which there exists an edge $e \in E(\mathcal{H})$ such that $\{x, y\} \subset e$. We denote the 2-shadow of $\mathcal{H}$ by $\partial\mathcal{H}$. The proof of Theorem 2 will proceed by estimating the number of 3-paths (3-vertex paths) in the 2-shadow of $\mathcal{H}$ in two different ways. To this end, we introduce several notions of the degree of a vertex. Given a vertex v in a hypergraph $\mathcal{H}$, $d(v)$ denotes the classical hypergraph degree of v, in particular $d(v) = |\{h \in E(\mathcal{H}) : v \in h\}|$. Let $d_s(v)$ be the (graph) degree of v in the 2-shadow of the hypergraph, in particular $d_s(v) = |\{e \in E(\partial\mathcal{H}) : v \in e\}|$. The *excess degree* of the vertex v to be $d_{ex}(v) = d_s(v) - d(v)$. Finally, we define the *block degree* $d_b(v)$ to be the total number of blocks containing an edge that contains v.

Notice that for every 4-cycle x_1, x_2, x_3, x_4, x_1 of $\partial\mathcal{H}$, there exists three distinct integers $1 \leqslant i < j < k \leqslant 4$ such that $\{x_i, x_j, x_k\} \in E(\mathcal{H})$, otherwise $\mathcal{H}$ contains a copy of Berge C_4. We call this edge a *representative edge* of this 4-cycle. Note that each 4-cycle of $\partial\mathcal{H}$ has either 1, 2 or 3 representative edges since $\mathcal{H}$ is Berge C_4-free hypergraph. Two edges of $\mathcal{H}$ sharing two vertices yield a C_4 in $\partial\mathcal{H}$. However, these are not the only types of C_4's in $\partial\mathcal{H}$. We call a 4-cycle x_1, x_2, x_3, x_4, x_1 in $\partial\mathcal{H}$ *rare* if the sub-hypergraph of $\mathcal{H}$ induced by the vertices $\{x_1, x_2, x_3, x_4\}$ does not contain two disjoint edges e and f with both containing $\{x_1, x_3\}$ or $\{x_2, x_4\}$. In the following claim, we show that the number of such cycles is small.

Claim 3. *For every $a, b \in V(\mathcal{H})$, there are at most two 3-paths not contained in a single edge of $\mathcal{H}$ with endpoints a and b.*

Proof. Suppose, by contradiction, that there are three distinct vertices v_1, v_2, v_3 different from a and b such that a, v_i, b forms a 3-path of $\partial\mathcal{H}$ for all integers $1 \leqslant i \leqslant 3$. It follows that there are three Berge paths a, e_i, v_i, f_i, b, for integers $1 \leqslant i \leqslant 3$ in $\mathcal{H}$. Note that those edges are not necessarily distinct. But we have $e_i \neq f_i$ for $i \neq j$, since $\{a, v_i\} \subset e_i$ and $\{b, v_j\} \subset f_j$ since $\mathcal{H}$ is 3-uniform. Note that if $e_2 = e_3$, then $e_2 = \{a, v_2, v_3\}$, hence $e_1 \neq e_2$. Similarly we have either $f_1 \neq f_2$ or $f_1 \neq f_3$. We may assume, without loss of generality, that $e_1 \neq e_2, e_3$. It follows that either $a, e_1, v_1, f_1, b, f_2, v_2, e_2, a$ or $a, e_1, v_1, f_1, b, f_3, v_3, e_3, a$ is a Berge C_4, a contradiction. □

We now define a particular type of 3-path in $\partial\mathcal{H}$. A 3-path, x_1, x_2, x_3, is called *good* if $\{x_1, x_2, x_3\} \notin E(\mathcal{H})$ and there is no $x \in V(\mathcal{H})$ such that x, x_1, x_2, x_3, x is a rare cycle of $\partial\mathcal{H}$. From Claim 3 it follows that for every $a, b \in V(\mathcal{H})$ there are at most two good 3-paths with endpoints a and b.

Claim 4. *There are at most* $6\,|E(\mathcal{H})|$ *rare* 4*-cycles in* $\partial\mathcal{H}$*.*

Proof. We fix an edge $\{a, b, c\} \in E(\mathcal{H})$. It suffices to show that the edge $\{a, b, c\}$ is representative of at most 6 rare 4-cycles (that is, $\{a, b, c\}$ is contained in the vertex set of at most 6 rare 4-cycles). Suppose by contradiction that this is not true. Observe that there are three possible positions for a fixed vertex v among the vertices of a rare 4-cycle in $\partial\mathcal{H}$ containing $\{a, b, c\}$. By the pigeonhole principle, there are 3 distinct vertices v_1, v_2, v_3 different from a, b, or c with the same position in the 4-cycle. Without loss of generality, we may assume they form a 4-cycle in the order v_i, a, c, b, v_i. Therefore from the definition of a rare 4-cycle, there are at least three 3-paths not contained in a single edge of $\mathcal{H}$ from a to b, a contradiction to Claim 3. □

Using Claim 4, it is easy to see that the number of 3-paths in $\partial\mathcal{H}$ which are not good is at most $3\,|E(\mathcal{H})| + 3 \cdot 6\,|E(\mathcal{H})| = 21\,|E(\mathcal{H})|$. Here we use the fact that each rare 4-cycle induces an edge of $\mathcal{H}$.

By conditioning on the middle vertex of the 3-path, we have the following estimate on the number of 3-paths in $\partial\mathcal{H}$:

$$\#(\text{3-paths in } \partial\mathcal{H}) = \sum_{v \in V(\mathcal{H})} \binom{d_s(v)}{2} = \sum_{v \in V(\mathcal{H})} \binom{d(v) + d_{ex}(v)}{2}.$$

The following claim provides an upper bound on the number of good 3-paths in $\partial\mathcal{H}$.

Claim 5.

$$\#(\text{good 3-paths in } \partial\mathcal{H}) \leqslant 2\binom{n}{2} - 4 \sum_{v \in V(\mathcal{H})} \binom{d_b(v)}{2}.$$

Proof. Fix a vertex v and consider two adjacent edges $\{v, x_1, x_2\}$ and $\{v, y_1, y_2\}$ such that they belong to the different blocks; clearly the vertices v, x_1, x_2, y_1, y_2 are all distinct. We claim that there is at most one good 3-path, namely x_i, v, y_j, between x_i and y_j, for each $i, j \in \{1, 2\}$. Suppose this is not the case, then without loss of generality, there exists $u \neq v$ such that x_1, u, y_1 is a good 3-path. By the definition of a good 3-path, there are two distinct edges $h_x, h_y \in \mathcal{H}$ such that $x_1, u \in h_x$ and $y_1, u \in h_y$. If $\{v, x_1, x_2\}$, $\{v, y_1, y_2\}$,

h_x and h_y are all different edges, then clearly there is a Berge 4-cycle. Therefore either $\{v, x_1, x_2\} = h_x$ or $\{v, y_1, y_2\} = h_y$. Hence we have $u \in \{x_2, y_2\}$, without loss of generality we may assume $u = x_2$. Observe that the 4-cycle x_1, x_2, y_1, v of $\partial\mathcal{H}$ contains a good 3-path and so by definition the 4-cycle x_1, x_2, y_1, v is not a rare 4-cycle. Hence we have a contradiction to the statement that edges $\{v, x_1, x_2\}$ and $\{v, y_1, y_2\}$ belong to different blocks. We conclude that there is at most one good path between x_i and y_j. So there are at least $4\sum_{v\in V(\mathcal{H})}\binom{d_b(v)}{2}$ pairs of vertices which have at most one good 3-path between them. From Claim 3, for each pair of vertices, there are at most two of good 3-paths in $\partial\mathcal{H}$. These observations complete the proof of Claim 5. $\square$

Thus, since the number of 3-paths which are not good is at most $21\,|E(\mathcal{H})|$, we have

$$\sum_{v\in V(\mathcal{H})}\binom{d(v)+d_{ex}(v)}{2} = \#(\text{3-paths in } \partial\mathcal{H}) \leqslant 2\binom{n}{2} - 4\sum_{v\in V(\mathcal{H})}\binom{d_b(v)}{2} + 21\,|E(\mathcal{H})|\,. \tag{1}$$

Claim 6. *We have* $\sum_{v\in V(\mathcal{H})} d_{ex}(v) \geqslant |E(\mathcal{H})|$ *and* $\sum_{v\in V(\mathcal{H})} d_b(v) \geqslant |E(\mathcal{H})|$.

Proof. First, we prove lower bounds on the sums $\sum_{v\in V(\mathcal{H})} d_{ex}(v)$ and $\sum_{v\in V(\mathcal{H})} d_b(v)$. For each block $\mathcal{B}$ and $v \in V(\mathcal{B})$, let $d^{\mathcal{B}}_{ex}(v)$ denote an excess degree of v inside the hypergraph $\mathcal{B}$. If $\mathcal{B}$ is type 1, then every vertex $v \in V(\mathcal{B})$ has $d^{\mathcal{B}}_{ex}(v) \geqslant 1$, so for type 1 blocks, $\sum_{v\in V(\mathcal{B})} d^{\mathcal{B}}_{ex}(v) \geqslant |V(\mathcal{B})|$. It is easy to see that for every block $\mathcal{B}$ we have $|V(\mathcal{B})| > |E(\mathcal{B})|$, so $\sum_{v\in V(\mathcal{B})} d^{\mathcal{B}}_{ex}(v) > |E(\mathcal{B})|$, for every type 1 block $\mathcal{B}$.

If $\mathcal{B}$ is a type 2 block, then $\sum_{v\in V(\mathcal{B})} d^{\mathcal{B}}_{ex}(v) = 3 = |E(\mathcal{B})|$. Therefore,

$$\sum_{v\in V(\mathcal{B})} d^{\mathcal{B}}_{ex}(v) \geqslant |E(\mathcal{B})|$$

for every block $\mathcal{B}$ in $B(\mathcal{H})$. This together with the fact that the blocks define a partition of the edges $E(\mathcal{H})$ implies

$$\sum_{v\in V(\mathcal{H})} d_{ex}(v) = \sum_{\mathcal{B}\in B(\mathcal{H})}\sum_{v\in V(\mathcal{B})} d^{\mathcal{B}}_{ex}(v) \geqslant \sum_{\mathcal{B}\in B(\mathcal{H})} |E(\mathcal{B})| = |E(\mathcal{H})|\,.$$

On the other hand, a simple double-counting argument yields

$$\sum_{v\in V(\mathcal{H})} d_b(v) = \sum_{\mathcal{B}\in B(\mathcal{H})} |V(\mathcal{B})| \geqslant \sum_{\mathcal{B}\in B(\mathcal{H})} |\mathcal{B}| = |E(\mathcal{H})|\,.$$ $\square$

Using Claim 6, we have the upper bound

$$4\,|E(\mathcal{H})| = 3\,|E(\mathcal{H})| + |E(\mathcal{H})| \leqslant \sum_{v\in V(\mathcal{H})}(d(v)+d_{ex}(v)).$$

Since $\binom{x}{2}$ is a convex function, by Jensen's inequality we have

$$\binom{\frac{1}{n}\sum_{v\in V(\mathcal{H})}(d(v)+d_{ex}(v))}{2} \leqslant \frac{1}{n}\sum_{v\in V(\mathcal{H})}\binom{d(v)+d_{ex}(v)}{2}.$$

Combining the above two inequalities we get

$$n\binom{\frac{4|E(\mathcal{H})|}{n}}{2} \leqslant \sum_{v\in V(\mathcal{H})}\binom{d(v)+d_{ex}(v)}{2}. \tag{2}$$

Similarly, by Claim 6 and Jensen's inequality, we have

$$n\binom{\frac{|E(\mathcal{H})|}{n}}{2} \leqslant \sum_{v\in V(\mathcal{H})}\binom{d_b(v)}{2}. \tag{3}$$

Combining (1), (2) and (3) we obtain

$$n\binom{\frac{4|E(\mathcal{H})|}{n}}{2} + 4n\binom{\frac{|E(\mathcal{H})|}{n}}{2} \leqslant 2\binom{n}{2} + 21\,|E(\mathcal{H})|\,. \tag{4}$$

Rearranging (4) yields the desired bound,

$$|E(\mathcal{H})| \leqslant (1+o(1))\frac{n^{3/2}}{\sqrt{10}}.$$

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