



A Non-aligning Variant of Generalized Turán Problems

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Abstract. In the so-called generalized Turán problems we study the largest number of copies of H in an n -vertex F -free graph G . Here we introduce a variant, where F is not forbidden, but we restrict how copies of H and F can be placed in G . More precisely, given an integer n and graphs H and F , what is the largest number of copies of H in an n -vertex graph such that the vertex set of that copy does not contain and is not contained in the vertex set of a copy of F ? We solve this problem for some instances, give bounds in other instances, and we use our results to determine the generalized Turán number for some pairs of graphs.

1. Introduction

One of the most basic question of extremal graph theory is due to Turán [26]: given a graph F and an integer n , what is the largest number of edges an n -vertex graph can have if it does not contain a subgraph isomorphic to F ? This quantity is denoted by $\text{ex}(n, F)$ and is a subject of extensive research. In particular, Turán determined $\text{ex}(n, K_{k+1})$ for every n and k . It is the number of edges in the graph $T_k(n)$, which is the n -vertex complete k -partite graph with each part having order $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. $T_k(n)$ is now called the *Turán graph*.

A natural generalization is the following. Instead of the edges, we count the copies of another subgraph H , i.e., subgraphs that are isomorphic to H . We denote by $\text{ex}(n, H, F)$ the largest number of copies of H in n -vertex F -free graphs, i.e., graphs that do not contain F as a (not necessarily induced) subgraph. Problems concerning this quantity are often simply called *generalized Turán problems*. For graphs H and G , let $\mathcal{N}(H, G)$ denote the number of copies

of H in G . Then $\text{ex}(n, H, F) = \max\{\mathcal{N}(H, G) : G \text{ has } n \text{ vertices and is } F\text{-free}\}$. If for an n -vertex F -free graph G we have $\text{ex}(n, H, F) = \mathcal{N}(H, G)$, then we say that G is an *extremal graph* with respect to H and F (we simply say that G is extremal if H and F are clear from the context).

The first result concerning $\text{ex}(n, H, F)$ is due to Zykov [27], who showed that the Turán graph is extremal with respect to K_k and K_r . After several sporadic results, the systematic study of $\text{ex}(n, H, F)$ was initiated by Alon and Shikhelman [1].

Here we study a variant of $\text{ex}(n, H, F)$, where F is not forbidden, but we only count copies of H that are far from copies of F in the following sense.

Definition 1.1. We say that two subgraphs G' and G'' of a graph G *align* in G if $V(G') \subseteq V(G'')$ or $V(G'') \subseteq V(G')$.

Definition 1.2. Given graphs H , F and G , we denote by $\mathcal{N}_{\text{na}}(H, F, G)$ the number of copies of H in G that do not align with any copy of F in G . When F is clear from the context, we say that a copy of H is *nice* in G if it does not align with any copy of F in G . Given graphs H and F , we let $\text{ex}_{\text{na}}(n, H, F) := \max\{\mathcal{N}_{\text{na}}(H, F, G) : G \text{ is an } n\text{-vertex graph}\}$.

By definition, $\text{ex}_{\text{na}}(n, H, F) \geq \text{ex}(n, H, F)$. We remark that the situation is very different depending on whether H or F has more vertices. When looking at a copy of H in G , we either have to examine the subgraphs or the supergraphs of it to decide if it aligns with a copy of F .

The study of the case $|V(H)| > |V(F)|$ was suggested by the author in [12]. The reason is that counting copies of H that do not align with any F was used there to prove a generalized Turán result. Let us describe now the general approach. Observe that an n -vertex F -free graph contains at most $c \binom{n}{|V(H)| - |V(F')|} \text{ex}(n, F', F)$ copies of H that align with a copy of F' for some constant $c = c(H)$. Thus $\text{ex}(n, H, F) \leq \text{ex}_{\text{na}}(n, H, F') + c \binom{n}{|V(H)| - |V(F')|} \text{ex}(n, F', F)$.

Let us continue with a connection to another well-studied notion in extremal graph theory, namely inducibility. Let us denote by $i(H, n)$ the largest number of *induced* copies of a graph H in any n -vertex graph. If $|V(H)| \geq |V(F)|$ and H does not contain F , then induced copies of H do not align with any copy of F , thus $\text{ex}_{\text{na}}(n, H, F) \geq i(H, n)$. It is well known and easy to see that $i(H, n) = \Theta(n^{|V(H)|})$. The Θ notation here and later, and also O and o later mean that n goes to infinity and the other parameters are fixed. In particular, the constant hidden here depends on H (see [23] for a proof of a lower bound). The arguments above imply the following.

Observation 1.3. If $|V(H)| \geq |V(F)|$ and H does not contain F , then we have $\text{ex}_{\text{na}}(n, H, F) = \Theta(n^{|V(H)|})$.

We remark that this behaviour is very different from $\text{ex}(n, H, F)$.

Let us assume now that $|V(H)| < |V(F)|$. Let $\mathcal{F}$ denote the family of graphs obtained by adding edges (but no vertices) to F . If no member of $\mathcal{F}$ contains H as an induced subgraph, then induced copies of H do not align with

any copy of F , thus $\text{ex}_{\text{na}}(n, H, F) \geq i(H, n)$. In the remaining cases when H is an induced subgraph of some $F' \in \mathcal{F}$, the order of magnitude of $\text{ex}_{\text{na}}(n, H, F)$ can be different from both $n^{|V(H)|}$ and $\text{ex}(n, H, F)$, as shown by the following example. We denote by P_k the path with k vertices.

Proposition 1.4. *Let $3 < k < \ell$ and $\ell - k$ odd. Then $\text{ex}_{\text{na}}(n, P_k, P_\ell) = \Theta(n^{k-2})$.*

The case where $\ell - k$ even is very different, as we discuss in Sect. 3. We note that for any k and ℓ , $\text{ex}(n, P_k, P_\ell) = \Theta(n^{\lceil k/2 \rceil})$ by a result of Győri, Salia, Tompkins and Zamora [18] (who also determined the asymptotics, and in some cases they obtained exact results).

Let us return to the connection to inducibility. In some cases, every copy of H that does not align with any F must be induced. It is the case when H is a complete k -partite graph (but not K_k) and $F = K_{k+1}$, or more generally F can be obtained by adding an edge to a complete k -partite graph H' such that the largest partite set of H' has no more vertices than the smallest partite set of H .

Brown and Sidorenko [4] proved the following.

Theorem 1.5. (Brown, Sidorenko, [4]) *(i) If H is complete multipartite, then the n -vertex graph containing the most induced copies of H is also complete multipartite.*

(ii) If H is complete bipartite, then the n -vertex graph containing the most induced copies of H is also complete bipartite. Moreover, there is a $c = c_H > 0$ such that if G is a complete multipartite n -vertex graph but not bipartite, then G contains at most $i(H, n) - cn^{|V(H)|}$ induced copies of H .

We remark that they do not state the moreover part of (ii), but it follows from their proof and we will use it later. This result implies that $\text{ex}_{\text{na}}(n, H, K_{k+1})$ is given by some complete multipartite graph if H is complete k -partite but not K_k . In particular $\text{ex}_{\text{na}}(n, K_{a,b}, K_3) = \mathcal{N}(K_{a,b}, T)$ for some complete bipartite n -vertex graph T (this clearly holds also when $K_{a,b}$ is the complete graph K_2). The missing case $\text{ex}_{\text{na}}(n, K_k, K_{k+1})$ is also known: it is the largest number of maximal cliques of size k . Byskov [5] showed that the Turán graph $T_k(n)$ contains the most such cliques.

We can extend the above results to $\text{ex}_{\text{na}}(n, H, K_{k+1})$ if H is complete k' -partite for any k' .

Theorem 1.6. *Let K be a complete multipartite graph. Then $\text{ex}_{\text{na}}(n, K, K_{k+1}) = \mathcal{N}_{\text{na}}(K, K_{k+1}, T)$ for a complete r -partite graph T for some $r \geq k$.*

We remark that r does not have to be equal to k , similarly to Theorem 1.5. We will prove a more general version where we count multiple graphs, with weights, at the same time. We will use it later.

Theorem 1.7. *Let $H_1, \dots, H_\ell, H'_1, \dots, H'_m$ be complete multipartite graphs, $\alpha_i \geq 0$ and $\beta_j \geq 0$. Then among n -vertex graphs G , $x(G) := \sum_{i=1}^\ell \alpha_i \mathcal{N}_{\text{na}}(H_i, K_{k_i+1}, G) + \sum_{j=1}^m \beta_j \mathcal{N}(H'_j, G)$ is maximized by a complete multipartite graph T .*

Clearly, Theorem 1.6 is implied by Theorem 1.7, except that in the former theorem we state that T has at least k parts. But that is trivial, as if T had fewer than k parts, we could make it k -partite by adding edges without creating any copy of K_{k+1} .

Let us return to generalized Turán problems. Győri, Pach and Simonovits [17] showed that if K is complete multipartite, then $\text{ex}(n, K, K_{k+1}) = \mathcal{N}(K, T)$ for some complete k -partite graph T . Schelp and Thomasson [25] (extending a result of Bollobás [2]) showed that if we add up the numbers of different complete multipartite graphs with weights, where the weight is non-negative unless the corresponding graph is complete, then again a complete multipartite graph is extremal. See [11] for more generalized Turán results where we count multiple graphs.

The proofs of most of the results mentioned above use the same technique, Zykov symmetrization [27], that we will also use in the proof of Theorem 1.7. This makes the graph more symmetric (and eventually complete multipartite) without creating copies of K_k . Thus we cannot use it in the case of other forbidden graphs F .

We know that in generalized Turán problems, forbidding K_k or forbidding any k -chromatic graph is closely related by the following result.

Theorem 1.8. (Gerbner and Palmer [14]) *Let H be a graph and F be a graph with chromatic number k , then $\text{ex}(n, H, F) \leq \text{ex}(n, H, K_k) + o(n^{|V(H)|})$.*

If F has chromatic number k and K has chromatic number less than k , then Theorem 1.8 gives an asymptotic result for $\text{ex}(n, K, F)$. Theorem 1.6 gives an improvement in the error term in some cases: if $\text{ex}_{\text{na}}(n, K, K_{k+1}) = \text{ex}(n, K, K_{k+1})$ (which is the case if $k = 2$), then we have $\text{ex}(n, K, F) \leq \text{ex}(n, K, K_{k+1}) + O(n^{|V(K)|-k-1})\text{ex}(n, K_{k+1}, F)$. Indeed, in an F -free graph, we can count the copies of K that align with a K_{k+1} by picking a K_{k+1} and then picking the other $|V(K)| - k - 1$ vertices. For some F , this gives an improvement in the error term.

We can use Theorem 1.7 to give an exact bound in some cases. We say that an edge e of a graph is color-critical, if by deleting e we obtain a graph with smaller chromatic number. In extremal problems, k -chromatic graphs with a color-critical edge often behave very similarly to K_k .

Theorem 1.9. *Let F be a 3-chromatic graph with a color-critical edge and $a, b \geq 2$. Then for large enough n we have $\text{ex}(n, K_{a,b}, F) = \text{ex}(n, K_{a,b}, K_3) = \mathcal{N}(K_{a,b}, K_{t,n-t})$ for some t .*

Note that in the case both a and b are large enough, in F -free graphs every copy of $K_{a,b}$ is induced, thus Theorem 1.5 implies the above theorem in that case. We have mentioned that in [12], a result that fits in our setting was used to prove a sharp bound for a generalized Turán problem. In more details, F was a $(k+1)$ -chromatic graph with a color-critical edge such that $\text{ex}(n, K_{k+1}, F) = o(n^k)$. H was a member of a large class of graphs, that includes very balanced complete k -partite graphs, and the extremal graph was $T_k(n)$. For those graphs H in the case $k = 2$, and any 3-chromatic graph F with

a color-critical edge (without any assumption on $\text{ex}(n, K_{k+1}, F)$), Theorem 1.9 shows that $T_2(n)$ is the extremal graph.

Let us show a simpler example where $\text{ex}_{\text{na}}(n, H, F) = \text{ex}(n, H, F)$. Gerbner and Patkós [15] showed that if $2 < t < t'$, then $\text{ex}(n, K_{2,t}, K_{2,t'}) = (1 + o(1)) \binom{t'-1}{t} \binom{n}{2}$. The upper bound is proved by taking 2 vertices u and v , $\binom{n}{2}$ ways, and then counting the copies of $K_{2,t}$ where u and v form the smaller part. As they have at most $t' - 1$ common neighbors, there are at most $\binom{t'-1}{t}$ such copies of $K_{2,t}$. In our case, we can proceed the same way: if u and v have at most $t' - 1$ common neighbors, we get the same upper bound. If u and v have at least t' common neighbors, then every copy of $K_{2,t}$ where u and v form the smaller part aligns with a $K_{2,t'}$, thus we get the upper bound 0.

One of the most studied generalized Turán problems is $\text{ex}(n, K_3, C_\ell)$. Let $\mathcal{C}_\ell = \{C_3, C_4, \dots, C_\ell\}$. Győri and Li [16] showed that $\text{ex}(n, K_3, C_{2k+1}) = O(\text{ex}(n, C_{2k}))$ and $\text{ex}(n, K_3, C_{2k+1}) = \Omega(\text{ex}(n, C_{2k}))$ (their proof also shows the same bounds for $\text{ex}(n, C_3, C_{2k})$). A theorem of Bondy and Simonovits [3] states that $\text{ex}(n, C_{2k}) = O(n^{1+1/k})$. A conjecture of Erdős and Simonovits [7] states that this bound is sharp, moreover $\text{ex}(n, C_{2k}) = \Theta(n^{1+1/k})$. This is known only in the cases $k = 2, 3, 5$, see e.g. [22].

We can extend the result of Győri and Li to our setting.

Proposition 1.10. *For any $k \geq 2$ we have that $\text{ex}_{\text{na}}(n, K_3, C_{2k+1}) = O(n^{1+1/k})$.*

Surprisingly, the case of even cycles is more complicated. One can easily show (and we will show in the proofs) that the number of triangles in a C_ℓ -free graph G is at most $(\ell - 3)|E(G)|/3$, and similarly, the number of triangles not aligning with any C_ℓ in an arbitrary graph G' is at most $(\ell - 3)|E(G')|/3$. In the case ℓ is even, this immediately shows that $\text{ex}(n, K_3, C_{2k}) = O(\text{ex}(n, C_{2k}))$. However, G' can have more than $\text{ex}(n, C_{2k})$ edges, and indeed the same upper bound does not hold.

Theorem 1.11. *(i) $n^{2-o(1)} = \text{ex}_{\text{na}}(n, K_3, C_4) = o(n^2)$. (ii) For any $k \geq 3$ we have that $\text{ex}_{\text{na}}(n, K_3, C_{2k}) = O(n^{1+\frac{1}{k-1}})$ and $\text{ex}_{\text{na}}(n, K_3, C_{2k}) = \Omega(\text{ex}(n, C_{2k-2}))$.*

The rest of the paper is organized as follows. Section 2 contains the proofs of Propositions 1.4 and 1.10 and of Theorems 1.7, 1.9 and 1.11. We finish the paper with some concluding remarks in Sect. 3.

2. Proofs

Let us prove Proposition 1.4, that we restate here for convenience.

Proposition. *Let $3 < k < \ell$ and $\ell - k$ odd. Then $\text{ex}_{\text{na}}(n, P_k, P_\ell) = \Theta(n^{k-2})$.*

Proof. The lower bound is given by the following construction: we take a copy of P_k with vertices $v_1, \dots, v_k$ in this order, and blow up each vertex except for v_2 and v_{k-1} to linear size. More precisely, for $1 \leq i \leq k$, $2 \neq i \neq k - 1$ we add v_i^j for $j \leq n/k$ and connect each v_i^j to v_{i-1} , to each v_{i-1}^m , to v_{i+1} and to each v_{i+1}^m . Then we obtain $\Omega(n^{k-2})$ copies of P_k with vertices

$v_1^{m_1}, v_2, v_3^{m_3}, \dots, v_{k-2}^{m_{k-2}}, v_{k-1}, v_k^{m_k}$ in this order. As the first and last vertices of these paths have degree 1, these paths cannot be extended to a longer path.

The only way such a path P can align with a copy P' of P_ℓ is if the endpoints of P' are $v_1^{m_1}$ and $v_k^{m_k}$ and P' leaves P and returns to P at least once. Then P' visits the vertices of P in an order $u_1, u_2, \dots, u_{k-1}, u_k$, where $u_1 = v_1^{m_1}$

and $u_k = v_k^{m_k}$. Let $f(i)$ denote the number of vertices on P between u_i and u_{i+1} , and $f'(i)$ denote the number of vertices on P' between u_i and u_{i+1} (including u_i and u_{i+1}). We say that P' *jumps over forward* a vertex v_j whenever u_i is before and u_{i+1} is after v_j in P . We say that P' *jumps over backward* a vertex v_j whenever u_i is after and u_{i+1} is before v_j in P .

Observe that $f(i)$ and $f'(i)$ have the same parity, because the construction is a bipartite graph. Observe that $\sum_{i=1}^{k-1} f(i)$ is even, as we count every vertex but u_1 and u_k an even number of times: if P' jumps over a vertex in one direction, then P' also has to jump over that vertex in the other direction. This implies that P' has an even number of vertices not in P , a contradiction to our assumption that $\ell - k$ is odd.

For the upper bound, let G be an n -vertex graph, and let us pick a copy of P_k with vertices $v_1, \dots, v_k$ in this order such that this copy does not align with any P_ℓ , the following way. First we pick the vertices different from v_1, v_2, v_{k-1}, v_k , and then the edges v_1v_2 and $v_{k-1}v_k$. There are $O(n^{k-4})$ ways to pick the vertices, so if there are at most ℓn ways to pick both the edges v_1v_2 and $v_{k-1}v_k$, then we are done.

Assume without loss of generality that there are more than ℓn choices for v_1v_2 , let E_1 be the set of these edges and G_1 be the graph on $V(G)$ with edge set E_1 . Then we remove each vertex with degree fewer than ℓ in G_1 to obtain G_2 . Then we remove each vertex with degree less than ℓ in G_2 , and so on. At the end we obtain a graph G' with minimum degree at least ℓ . Then G' is non-empty, as we deleted fewer than ℓn edges from E_1 . Thus there are at least ℓ vertices in G' .

Let us now pick v_1v_2 arbitrarily from $E(G')$. Then there is an edge that can be picked as $v_{k-1}v_k$ so that the resulting P_k does not align with any P_ℓ , we pick such an edge. With $v_3, \dots, v_{k-2}$, they form a P_k that does not align with any P_ℓ . We will extend this P_k to a P_ℓ , obtaining a contradiction. Observe that v_1 has at least ℓ neighbors in G' , we pick one not on the path so far. We can extend the path to a P_ℓ the same way, by picking a neighbor of the vertex previously picked: the penultimate vertex still has at least one neighbor in G' that is not on the path so far, thus we can finish the path to obtain a P_ℓ , finishing the proof with a contradiction. a

□

Let us continue with the proof of Theorem 1.7, that we restate here for convenience.

Theorem. Let $H_1, \dots, H_\ell, H'_1, \dots, H'_m$ be complete multipartite graphs, $\alpha_i \geq 0$ and $\beta_j \geq 0$. Then among n -vertex graphs G , $x(G) := \sum_{i=1}^\ell \alpha_i \mathcal{N}_{\text{na}}(H_i, K_{k_i+1}, G) + \sum_{j=1}^m \beta_j \mathcal{N}(H'_j, G)$ is maximized by a complete multipartite graph T .

Proof. Let G be a graph on n vertices, and for a vertex v of G , let $d^*(v)$ denote the sum of the weights of copies of H_i and H'_j that are counted towards $x(G)$ and contain v . More precisely, let $d_i(v)$ denote the number of copies of H_i in G that do not align with any K_{k_i+1} and contain v , let $d'_j(v)$ denote the number of copies of H'_j in G that contain v , and let $d^*(v) = \sum_{i=1}^\ell \alpha_i d_i(v) + \sum_{j=1}^m \beta_j d'_j(v)$. Similarly, let $d^*(u, v)$ denote the sum of the weights of copies of H_i and H'_j that are counted towards $x(G)$ and contain both u and v .

We will apply Zykov's symmetrization. Consider two non-adjacent vertices u and v and assume without loss of generality that $d^*(u) \leq d^*(v)$. Then we "symmetrize" u to v : we remove all the edges incident to u , and for each edge vw , we add the edge uw to obtain another graph G' .

Claim 2.1. We have $x(G') \geq x(G)$. Moreover, if $d^*(u) < d^*(v)$, then $x(G') > x(G)$.

Proof. Let G_0 be the graph we obtain by deleting the edges incident to u . We have $x(G_0) \geq x(G) - d^*(u)$. Indeed, $d^*(u)$ accounts for removing the nice copies of H_i containing u and for removing the copies of H'_j containing u . It is possible that there are some other copies of H_i in G that align with some cliques K_{k_i+1} , but those cliques all contain u ; in that case they become nice after deleting the edges incident to u , thus $x(G)$ may decrease by less than $d^*(u)$.

We will show that $x(G') \geq x(G_0) + d^*(u)$. First we will show that $d^*(u, v)$ does not decrease when we symmetrize u to v .

Consider a nice copy H_i^* of H_i in G that contains both u and v . Then u and v belong to the same partite set of H_i^* . Thus H_i^* is in G' , as the common neighbors of u and v in G are also common neighbors of u and v in G' . If H_i^* aligns with a copy $K_{k_i+1}^*$ of K_{k_i+1} in G' , then $K_{k_i+1}^*$ cannot be in G , thus has to contain u (the only vertex incident to new edges), but then $K_{k_i+1}^*$ cannot contain v . Therefore, we can replace u with v in $K_{k_i+1}^*$ we obtain a copy of K_{k_i+1} in G that aligns with H_i^* , a contradiction. This shows that the number of copies of H_i aligning with no K_{k_i+1} and containing both u and v does not decrease.

Consider now a copy of H'_j in G that contains both u and v . Then this copy is also in G' , thus the number of copies of H'_j containing both u and v does not decrease.

This shows that when comparing $x(G')$ to $x(G_0)$, we have an increase by $d^*(u, v)$ when counting only those copies of H_i and H_j that contain both u and v . We will show that there is an increase of $d^*(v) - d^*(u, v)$ when counting the copies of H_i and H'_j that contain u but not v .

Let us consider a nice copy H_i^* in G that contains v but not u . Let H_i^{**} be obtained from H_i^* by replacing v with u , then H_i^{**} is in G' . If H_i^{**} aligns with a copy $K_{k_i+1}^{**}$ of K_{k_i+1} in G' , then $K_{k_i+1}^{**}$ does not contain v . Indeed,

if $|V(H_i)| \geq k_i$, then $K_{k_i+1}^{**}$ is a subgraph of H_i^{**} , and if $|V(H_i)| < k_i$, then H_i^{**} is a subgraph of $K_{k_i+1}^{**}$, thus $K_{k_i+1}^{**}$ contains u , hence it cannot contain v . Let $K_{k_i+1}^*$ be obtained from $K_{k_i+1}^{**}$ by replacing u with v . Then $K_{k_i+1}^*$ is in G and aligns with H_i^* , a contradiction. This means that for each copy of H_i contributing to $d^*(v) - d^*(u, v)$, we have a copy of H_i in G' containing u but not v and contributing to $x(G')$. Similarly, if H_j' contains v but not u , then replacing v with u creates a copy of H_j' in G' . This shows that the total weight of nice copies of H_i and copies H_j' that contain u and not v in G' is at least $d^*(v) - d^*(u, v)$.

We have shown that the total weight of nice copies of H_i and copies of H_j' containing u does not decrease, moreover, it increases if $d^*(v) > d^*(u)$. We also have to deal with copies of H_i not containing u . They obviously are also in G' , but they could stop being nice, which would decrease $x(G)$. However, if such a copy H_i^* aligns with a copy $K_{k_i+1}^*$ of K_{k_i+1} in G' , then $K_{k_i+1}^*$ cannot be in G , thus has to contain u . This implies that $K_{k_i+1}^*$ does not contain v . Let $K_{k_i+1}^{**}$ be obtained from $K_{k_i+1}^*$ by replacing u with v . then $K_{k_i+1}^{**}$ is in G and aligns with H_i^* in G , a contradiction. $\square$

We choose an arbitrary ordering $v_1, \dots, v_n$ of the vertices of G . We will apply such symmetrization steps repeatedly, as long as there are two non-adjacent vertices v_i and v_j such that their neighborhood is not exactly the same, i.e. the symmetrization changes the graph. If $d^*(v_i) < d^*(v_j)$, then we symmetrize v_i to v_j ; if $d^*(v_i) > d^*(v_j)$, then we symmetrize v_j to v_i . If $d^*(v_i) = d^*(v_j)$, then we check the smallest index i' of vertices with the exact same neighborhood as v_i , and the smallest index j' of vertices with the exact same neighborhood as v_j . We symmetrize v_i to v_j if $i' > j'$ and we symmetrize v_j to v_i if $i' < j'$.

Claim 2.2. *There are finitely many symmetrization steps.*

Proof. Steps where $d^*(v) \neq d^*(u)$ can happen at most $(\ell + m)n!2^n$ times, as the number of copies of some H_i or H_j' increases, but that number is bounded by $n!2^n$ for every i and j .

Between two such steps, v_i is symmetrized to some other vertex at most $n2^{i-2}$ times. This can be shown by induction on i ; for $i = 1$, observe that v_1 is not symmetrized. For $i = 2$, v_2 is only symmetrized to a vertex that has the same neighborhood as v_1 , thus from that point v_2 has the same neighborhood as v_1 . Indeed, if some other vertex v_i is symmetrized to v_j , then either v_j is connected to both v_1 and v_2 or none of them, thus this will hold for v_i after the symmetrization.

If the statement holds for i , observe that v_{i+1} can only be symmetrized only to some v_j with $j \leq i$ (or a vertex with the same neighborhood). However, v_{i+1} may not have the same neighborhood as v_j after a step where v_j is symmetrized to another vertex. In this case we may symmetrize v_{i+1} again to v_j (or a vertex with the exact same neighborhood as v_j at that point). This means that v_{i+1} is symmetrized to v_j (or a vertex with the exact same neighborhood as v_j at that point) again only after v_j was symmetrized to another vertex, thus at most $1 + n2^{j-2}$ times by induction. Therefore, altogether v_{i+1} is

symmetrized to other vertices at most $\sum_{j=1}^i 1 + n2^{j-2} \leq n2^{i-1}$ times, finishing the proof. $\square$

After finitely many steps, we arrive to a graph G^* that cannot be changed by symmetrization, thus non-adjacent vertices have the same neighborhood, which means being non-adjacent is an equivalence relation, i.e. G^* is complete multipartite. $\square$

Let us continue with the proof of Theorem 1.9, that we restate here for convenience.

Theorem. *Let F be a 3-chromatic graph with a color-critical edge and $a, b \geq 2$. Then for large enough n we have that $\text{ex}(n, K_{a,b}, F) = \text{ex}(n, K_{a,b}, K_3) = \mathcal{N}(K_{a,b}, K_{t,n-t})$ for some t .*

Proof. Let G be an n -vertex F -free graph. Let us denote by B_t the book graph, which consists of t triangles sharing an edge. Let t be the largest integer such that G contains a book B_t . Let us pick a suitably small constant c . We separate two cases.

Case 1. $t \leq cn$.

For any given triangle in G , there are at most $\alpha := 3\binom{t-1}{a-1}\binom{n}{b-2} + 3\binom{t-1}{b-1}\binom{n}{a-2} + \binom{t-1}{a}\binom{n}{b-3} + \binom{t-1}{b}\binom{n}{a-3} \leq c'n^{a+b-3}$ copies of $K_{a,b}$ that align with that triangle, for some constant c' . Indeed, at least two vertices of the triangle are in one of the partite sets U_1 of $K_{a,b}$, thus we actually found a $K_{a,b}$ with an additional edge. That edge with the vertices from the other partite set U_2 form a book, thus the vertices of U_2 are chosen from a set of order at most t .

First we pick two vertices of the triangle, and assume that they are in U_1 , while the third vertex is in U_2 , where $|U_2| = a$. Then we need to pick the other vertices of U_2 from a book containing the triangle, thus there are at most $\binom{t-1}{a-1}$ ways to pick those vertices, and at most $\binom{n}{b-2}$ ways to pick the other vertices not in the triangle. By symmetry, there are at most $3\binom{t-1}{b-1}\binom{n}{a-2}$ copies of $K_{a,b}$ that align with the triangle and contain two vertices from the partite set of order b and one vertex from the partite set of order a . Finally, if all the three vertices of the triangle is in the same partite set U_1 of $K_{a,b}$, then we need to pick the vertices of U_2 from a book containing that triangle, thus there are at most $\binom{t-1}{a}\binom{n}{b-3} + \binom{t-1}{b}\binom{n}{a-3}$ ways to do this.

Therefore, we have that $\mathcal{N}(K_{a,b}, G) \leq \mathcal{N}_{\text{na}}(K_{a,b}, K_3, G) + \alpha\mathcal{N}(K_3, G)$. By Theorem 1.7, we have $\mathcal{N}_{\text{na}}(K_{a,b}, K_3, G) + \alpha\mathcal{N}(K_3, G) \leq \mathcal{N}_{\text{na}}(K_{a,b}, K_3, T) + \alpha\mathcal{N}(K_3, T)$ for a complete multipartite graph T . If T has more than two partite sets, then Theorem 1.5 implies that $\mathcal{N}_{\text{na}}(K_{a,b}, K_3, T)$ is smaller by $\Omega(n^{a+b})$ than $\mathcal{N}_{\text{na}}(K_{a,b}, K_3, T')$ for some complete bipartite graph T' , using that $\mathcal{N}_{\text{na}}(K_{a,b}, K_3, G)$ is the number of induced copies of $K_{a,b}$ in G . As $\alpha\mathcal{N}(K_3, T) \leq cn^{a+b}$, this shows that $\mathcal{N}(K_{a,b}, G) \leq \mathcal{N}_{\text{na}}(K_{a,b}, K_3, G) + \alpha\mathcal{N}(K_3, G) \leq \mathcal{N}_{\text{na}}(K_{a,b}, K_3, T) + \alpha\mathcal{N}(K_3, T) \leq \mathcal{N}_{\text{na}}(K_{a,b}, K_3, T') - \Omega(n^{a+b}) + \alpha\mathcal{N}(K_3, T') < \mathcal{N}_{\text{na}}(K_{a,b}, K_3, T') = \mathcal{N}(K_{a,b}, T')$. If T has at most two partite sets, then $\mathcal{N}(K_3, T) = 0$, thus we have $\mathcal{N}(K_{a,b}, G) \leq \mathcal{N}(K_{a,b}, T)$.

Case 2. $t > cn$.

As F is 3-chromatic with a color-critical edge, for some k we have that F is contained in the graph we obtain from $K_{k,k+2}$ by adding an additional edge inside the larger partite set. Consider a copy of B_t in G , let x and y be the vertices of B_t that are connected to each other vertex of that B_t and let U be the set of the other t vertices in that book. Let U' denote the set of $n - t - 2$ vertices of G not in the book. It is easy that there is no copy of $K_{k,k}$ such that one partite set is in U and the other partite set is in $U \cup U'$, as that would form $K_{k,k+2}$ with x and y . Then by the Kővári-T.Sós-Turán theorem [20], there are $o(n^2)$ edges between U and U' and inside U .

This shows that there are $o(n^{a+b})$ copies of $K_{a,b}$ containing an edge between U and U' or inside U . If $n - t = o(n)$, then there are also $o(n^{a+b})$ copies of $K_{a,b}$ not inside U , thus we are done.

Assume now that $n - t = \Theta(n)$. The copies of $K_{a,b}$ are either inside U (at most $\text{ex}(t, K_{a,b}, F)$ copies) or inside U' ((at most $\text{ex}(n - t - 2, K_{a,b}, F)$ copies) or contain x or y or an edge between U and U' ($o(n^{a+b})$ copies). Recall that we have $\text{ex}(m, K_{a,b}, F) = \mathcal{N}(K_{a,b}, T) + o(m^{a+b})$ for some m -vertex complete bipartite graph T using Theorem 1.8 and that $\text{ex}(m, K_{a,b}, K_3) = \mathcal{N}(K_{a,b}, T)$ for some complete bipartite graph (a result of Győri, Pach and Simonovits, mentioned at the introduction).

Therefore, we have $\mathcal{N}(K_{a,b}, G) = \mathcal{N}(K_{a,b}, T_1) + \mathcal{N}(K_{a,b}, T_2) + o(n^{a+b})$, where T_1 is a complete bipartite graph on t vertices and T_2 is a complete bipartite graph on $n - t - 2$ vertices. On the other hand, let us take a copy of T_1 and a copy of T_2 and connect the vertices of the larger partite sets of them. The resulting graph is bipartite, thus F -free. Clearly we created $\Theta(n^{a+b})$ additional copies of $K_{a,b}$, thus this graph contains more copies of $K_{a,b}$ than G . $\square$

Let us continue with the proof of Proposition 1.10 that we restate here for convenience.

Proposition. $\text{ex}_{\text{na}}(n, K_3, C_{2k+1}) = O(n^{1+1/k})$.

Proof. For the upper bound, we follow a proof Füredi and Özkahya [10], who gave an upper bound on $\text{ex}(n, K_3, C_{2k+1})$. First we claim that the number of triangles not aligning with any C_ℓ in an arbitrary graph G is at most $(\ell - 3)|E(G)|/3$. Indeed, consider a vertex v . The number of triangles containing v is equal to the number of edges in its neighborhood. If there is a $P_{\ell-1}$ in the neighborhood of v , then the triangles formed by the edges of this path with v are not nice. Therefore, the number of nice triangles containing v is at most the number of edges in the graph G' , which we obtain by restricting G to the neighborhood of v and deleting all the edges of paths $P_{\ell-1}$. Then G' is a $P_{\ell-1}$ -free graph on $d(v)$ vertices, thus has at most $(\ell - 3)d(v)/2$ edges by a well-known theorem of Erdős and Gallai [6]. Adding up for each vertex, we obtain an upper bound $(\ell - 3)|E(G)|$, where each triangle is counted three times.

Let us consider now an arbitrary graph G , and take a random 3-coloring of its vertices, where every vertex gets color red, blue or green with probability $1/3$. Let G^* be obtained from G by deleting the monochromatic edges, and

afterwards deleting each edge that is not contained in any 3-chromatic nice triangle. Clearly, every triangle of G is in G' with probability at least $2/9$. Therefore, there is a 3-coloring such that at least $2/9$ of the nice triangles of G is in G^* . Obviously they are also nice triangles in G^* , thus it is enough to show that there are $O(n^{1+1/k})$ nice triangles in G^* .

Consider first the subgraph G_1 of G^* having the edges with blue and red end-vertices. We claim that G_1 does not contain C_{2k} . Indeed, let uv be an edge of such a C_{2k} . Then uv is in a 3-chromatic nice triangle uvw , thus w is green. This means w is not in the C_{2k} , thus adding uw and wv we obtain a C_{2k+1} in G that aligns with the triangle uvw , a contradiction. Using the Bondy-Simonovits theorem [3], this implies that G_1 has $O(n^{1+1/k})$ edges. By the same argument for the colors blue and green and the colors green and red we obtain that G^* has $O(n^{1+1/k})$ edges. By the first paragraph of the proof, we also have that G^* contains $O(n^{1+1/k})$ nice triangles, completing the proof. $\square$

Let us continue with the proof of Theorem 1.11, that we restate here for convenience.

Theorem. (i) $n^{2-o(1)} = \text{ex}_{\text{na}}(n, K_3, C_4) = o(n^2)$.

(ii) For any $k \geq 3$ we have that $\text{ex}_{\text{na}}(n, K_3, C_{2k}) = O(n^{1+\frac{1}{k-1}})$ and $\text{ex}_{\text{na}}(n, K_3, C_{2k}) = \Omega(\text{ex}(n, C_{2k-2}))$.

Proof. Let us start with the proof of (i) and let B_2 denote the book with 2 pages, i.e. two triangles sharing an edge (in other words, the 4-vertex graph with 5 edges). Alon and Shikhelman [1] showed that $n^{2-o(1)} = \text{ex}(n, K_3, B_2) = o(n^2)$. Observe that if an edge uv of a graph is contained in at least two triangles, then each triangle containing uv aligns with a B_2 . Thus we can delete every such edge without decreasing the number of triangles that do not align with any copy of B_2 . This way we obtain a graph where every edge is contained in at most one triangle, i.e. a B_2 -free graph. This shows that $\text{ex}_{\text{na}}(n, K_3, B_2) = \text{ex}(n, K_3, B_2)$. A triangle aligns with a C_4 if and only if it aligns with a B_2 , thus we have $\text{ex}_{\text{na}}(n, K_3, C_4) = \text{ex}_{\text{na}}(n, K_3, B_2) = \text{ex}(n, K_3, B_2)$.

Let us continue with the proof of (ii). For the upper bound, we proceed similarly to the proof of Proposition 1.10, thus from now on we assume familiarity with that proof and avoid repeating some details. We take an arbitrary graph G and a random 3-coloring of its vertices, where every vertex gets color red, blue or green with probability $1/3$. Let G^* be obtained from G by deleting the monochromatic edges, and afterwards deleting each edge that is not contained in any 3-chromatic nice triangle. Then, as in Proposition 1.10, it is enough to show that there are $O(n^{1+\frac{1}{k-1}})$ nice triangles in G_1 , which consists of the edges with blue and red end-vertices.

Consider the copies of C_{2k-2} in G_1 . For any edge $e = uv$ of it, there is a green vertex $w(e)$ connected to both u and v such that $uvw(e)$ is a nice triangle in G . If there are two edges e and e' in a copy of C_{2k-2} such that $w(e) \neq w(e')$, then we can use both $w(e)$ and $w'(e)$ to extend the cycle and find a C_{2k} containing $u, v, w(e)$, thus aligning with a triangle, a contradiction.

Thus we have that each vertex of that C_{2k-2} is connected to the same w , i.e. we found a wheel in G with center w , such that each triangle in the wheel is nice in G .

Claim 2.3. *Any two copies of C_{2k-2} in G_1 with different vertex sets share at most one vertex.*

Proof. Assume indirectly that copies C and C' of C_{2k-2} both contain u and v . Let w (resp. w') be the centers of the wheels corresponding to C and C' . Assume first that $w \neq w'$. Observe that we can go from u to v on a path using all the vertices of C and w . With w' , this extends to a C_{2k} in G that aligns with the triangles in C , a contradiction. Assume now that $w = w'$. Then there is a vertex contained by both C and C' , say u that is connected to a vertex u' of C' that is not in C . Then we can go through the vertices of C ending with u , then to u' and then to w to obtain a C_{2k} that aligns with the triangles of C , a contradiction. $\square$

The above claim implies that there are $O(n^2)$ copies of C_{2k-2} in G'' . Indeed, any pair of vertices can belong to the vertex set of only one such copy, thus there are at most $\binom{n}{2}$ such vertex sets, and there are at most $(2k-3)!/2$ copies of C_{2k-2} on a set of $2k-2$ vertices. We will use a supersaturation result of Simonovits, that appeared in [8] without proof (see [19] for a proof). It claims that there exist constants c and c' such that if we have $|E(H)| \geq cn^{1+1/k}$ for a graph H , then H contains at least $c' \left(\frac{|E(H)|}{|V(H)|} \right)^{2k}$ copies of C_{2k} . Applying it with G_1 and C_{2k-2} , we obtain that $|E(G_1)| \leq cn^{1+\frac{1}{k-1}}$ for a large enough c .

From this point we can again follow the proof of Proposition 1.10: $|E(G^*)| \leq 3cn^{1+\frac{1}{k-1}}$, thus the number of nice triangles in G^* is $O(n^{1+\frac{1}{k-1}})$, which implies that the number of nice triangles in G is $O(n^{1+\frac{1}{k-1}})$, completing the proof of the upper bound.

For the lower bound, consider an n -vertex bipartite C_{2k-2} -free graph G with $\Omega(\text{ex}(n, C_{2k-2}))$ edges, and let us double the vertices on one side. It means that we replace each vertex v with two vertices v_1, v_2 , and for each edge uv , we add the edges uv_1 and uv_2 . Let G' be the graph obtained this way, then clearly G' has $|E(G')|$ triangles. We claim that each of those is nice. Observe that a cycle in G' corresponds to a walk in G if we consider the visits to v_1 and v_2 as visiting v twice. A copy C of C_{2k} corresponds to a walk that has $2k$ vertices with repetitions. As there are no cycles of length less than $2k$ in G , there are only two possibilities. One is that the walk is on a single edge in G , but then the corresponding cycle has at most 4 vertices in G' , a contradiction. The other possibility is that the walk in G is a C_{2k} , i.e. C does not visit both vertices corresponding to a vertex of G . But each triangle T contains two vertices corresponding to a vertex of G , thus T does not align with C , completing the proof. $\square$

3. Concluding Remarks

- Let us return to $\text{ex}_{\text{na}}(n, P_k, P_\ell)$. If $0 < \ell - k$ is even, then the upper bound of Proposition 1.4 still holds, but we can improve it. We briefly give a sketch of the arguments. Let H be the graph consisting of k internally vertex disjoint paths of length $\ell - k$ connecting two vertices u and v . This graph is called a theta graph and it is well-known [9] that $\text{ex}(n, H) = O(n^{1+1/(\ell-k)})$. We pick copies $v_1v_2 \dots v_k$ of P_k not aligning with any P_ℓ by picking first the edge v_1v_2 , then v_3v_4 , and so on. One can show that each time, the possible edges to pick do not contain H , giving the upper bound $O(n^{k/2+k/(2(\ell-k))})$ if k is even, and $O(n^{(k+1)/2+(k-1)/(2(\ell-k))})$ if k is odd.

We also have the lower bound $\text{ex}_{\text{na}}(n, P_k, P_\ell) \geq \text{ex}(n, P_k, P_\ell) = \Theta(n^{\lceil k/2 \rceil})$. We can improve it in the case $k > 4$ is even. Let us again give a sketch. One of the constructions for $\text{ex}(n, P_k, P_\ell)$ goes by blowing up every second vertex of k -vertex path $v_1v_2 \dots v_k$ to linear size. In the case k is even, we have some freedom here, as we need to only avoid blowing up two adjacent vertices, thus we can blow up $v_1, v_4, v_6, \dots, v_k$. For our case, we modify it slightly: we blow up v_3 as well, but instead of placing a complete bipartite graph between the vertices replacing v_3 and v_4 , we place a bipartite graph G' with girth at least $\ell - k$. We consider the copies of P_k where the i th vertex is v_i or a replacement of v_i . Then a copy of a longer path containing every vertex of such a path has a circle inside G' , but this means more than $\ell - k$ additional vertices. It is easy to see that this way we get $\Omega(|E(G')|n^{k/2-1})$ copies of P_k not aligning with any P_ℓ . This gives a lower bound increased by a factor of roughly $n^{2/3(\ell-k)}$, using a theorem of Lazebnik, Ustimenko and Woldar [21].

- One could examine the following question. If we are given an integer n and graphs F , F' and H , at most how many copies of H can align with some F in an n -vertex F' -free graph? Clearly, this number plus $\text{ex}_{\text{na}}(n, H, F)$ is an upper bound on $\text{ex}(n, H, F')$. We have examined some problems where we combined a trivial upper bound on this number and a non-trivial bound on $\text{ex}_{\text{na}}(n, H, F)$.

Let us show an example where the opposite happens. More precisely, we have one more twist: we consider copies of H aligning with any copy of any graph from a family of graphs. Gerbner, Győri, Methuku and Vizer [13] showed that $\text{ex}(n, C_4, C_{2k}) = (1 + o(1))\binom{k-1}{2}\binom{n}{2}$ using the following idea. We say that a pair of vertices is *fat* if they have at least k common neighbors. We count separately the 4-cycles having two fat pairs of opposite vertices and the 4-cycles having at most one fat pair of opposite vertices. They showed that there are $O(n^{1+1/k})$ 4-cycles of the first type and at most $(1 + o(1))\binom{k-1}{2}\binom{n}{2}$ of the second type. This second statement is a simple observation: we pick two vertices $\binom{n}{2}$ ways to be a non-fat pair of opposite vertices and then we have to pick the other two vertices out of their at most $k - 1$ common neighbors. Observe that here we count the copies of C_4 that do not align with any graph obtained

from a C_4 by adding $k - 2$ new common neighbors to both opposite pairs (where the common neighbors of the two pairs may coincide).

- Let us mention an example where handling together the copies of H that align with F and those that do not align with F gives more than our approach. It is a simple proof of Qian, Xie and Ge [24]. Let H be a graph obtained by adding a vertex to K_ℓ and connecting it to an arbitrary number of vertices. They showed that $\text{ex}(n, H, K_k) = \mathcal{N}(H, T_{k-1}(n))$ for any $k > \ell$. The number of copies of H that align with $K_{\ell+1}$ is maximized by the Turán graph using Zykov's theorem. However, it is not clear whether the number of copies of H that do not align with $K_{\ell+1}$ is maximized by the Turán graph. What helps is that the surplus in the number of copies of H that align with H can also be used.
- Finally, let us discuss a variant. In the special case where F contains H , we say that a copy H^* of H *strongly aligns* with a copy F^* of F in G if F^* contains H^* as a subgraph. It means that not only the vertices, but also the edges of H^* are in F^* . For example, the analogue of Proposition 1.4 also holds in this setting, and immediately extends to the case $\ell - k$ is even. Indeed, in the proof of Proposition 1.4 we showed $\Omega(n^{k-2})$ copies of P_k that cannot be extended to longer paths.

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