POSITIVE BASES, CONES, HELLY TYPE THEOREMS

IMRE BÁRÁNY

ABSTRACT. Assume that $k \le d$ is a positive integer and $\mathcal{C}$ is a finite collection of convex bodies in $\mathbb{R}^d$. We prove a Helly type theorem: If for every subfamily $\mathcal{C}^* \subset \mathcal{C}$ of size at most $\max\{d+1, 2(d-k+1)\}$ the set $\bigcap \mathcal{C}^*$ contains a k-dimensional cone, then so does $\bigcap \mathcal{C}$. One ingredient in the proof is another Helly type theorem about the dimension of lineality spaces of convex cones.

1. INTRODUCTION AND MAIN RESULT

This paper is about Helly type properties of families of convex sets. Suppose for instance that $\mathcal{C}$ is a finite family of convex sets in $\mathbb{R}^d$ and $\bigcap \mathcal{C}$ contains a halfline. Then of course $\bigcap \mathcal{C}^*$ also contains a halfline for every subfamily $\mathcal{C}^*$ of $\mathcal{C}$. In the opposite direction assume that $\bigcap \mathcal{C}^*$ contains a halfline for every subfamily $\mathcal{C}^* \subset \mathcal{C}$ of size at most m. Does it follow that $\bigcap \mathcal{C}$ contains a halfline if $m = m(d)$ is chosen suitably? The answer is yes according to a theorem of Katchalski [6].

Theorem 1.1. *$m(d) = 2d$, that is, if $\bigcap \mathcal{C}^*$ contains a halfline for every subfamily $\mathcal{C}^* \subset \mathcal{C}$ of size at most $2d$, then so does $\bigcap \mathcal{C}$.*

A halfline is a one-dimensional cone. More generally a *cone* K with *apex* $u \in \mathbb{R}^d$ and generator set $V \subset \mathbb{R}^d$ is the set of points of the form $u + \sum_{v \in W}^n \alpha(v)v$ where $W \subset V$ is finite and $\alpha(v) \ge 0$ for every $v \in W$. So K consists of all finite and positive combinations of elements of V translated by the vector u. The cone K is *polyhedral* if V is finite. For properties of cones, their lineality spaces, their polar cones, etc. See for instance Gruber's book [4] or Schneider's [10].

Our first result extends Katchalski's theorem to k-dimensional cones where $k \in [d] = \{1, \dots, d\}$. Set $m(k,d) = \max\{d+1, 2(d-k+1)\}$.

Theorem 1.2. *Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ and $k \in [d]$. If $\bigcap \mathcal{C}^*$ contains a k-dimensional cone for every subfamily $\mathcal{C}^* \subset \mathcal{C}$ of size at most $m(k,d)$, then so does $\bigcap \mathcal{C}$.*

Note that the case $k = 0$ (which is not covered here) is Helly's theorem and we could have stated it by defining $m(0,d) = d+1$.

The following examples show that the value of $m(k,d)$ is optimal. We write ab for the scalar product of vectors $a, b \in \mathbb{R}^d$.

Example 1. Let $v_1, \dots, v_{d+1}$ be the vertices of a regular simplex in $\mathbb{R}^d$ with $\sum_1^{d+1} v_i = 0$. Define H_i as the halfplane $\{x \in \mathbb{R}^d : v_i x \le 0\}$, $i \in [d+1]$.

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Then $\bigcap_1^{d+1} H_i$ is a single point, namely the origin, so it contains no k-dimensional cone (for any $k > 0$), but for every $j \in [d+1]$ the set $\bigcap_{i \neq j} H_i$ contains a k-dimensional cone no matter what $k \in [d]$ is.

Example 2. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$ and define $H_i^+ = \{x \in \mathbb{R}^d : e_i x \geq 0\}$ and $H_i^- = \{x \in \mathbb{R}^d : e_i x \leq 0\}$ and set $\mathcal{H}_k = \{H_i^{\pm} : i \leq d-k+1\}$. Then $\bigcap \mathcal{H}_k$ is the subspace with $x_1 = \ldots, x_{d-k+1} = 0$, so it is a copy of $\mathbb{R}^{k-1}$ and can't contain a k-dimensional cone. Yet both $\bigcap(\mathcal{H}_k \setminus H_i^+)$ and $\bigcap(\mathcal{H}_k \setminus H_i^-)$ contain a k-dimensional cone (actually a k-dimensional halfspace) for every $i \in [d-k+1]$.

In these examples the convex sets in the family $\mathcal{C}$ are halfspaces of the form $\{x \in \mathbb{R}^d : ax \leq 0\}$. This is no coincidence: the proof of Theorem 1.2 begins with a reduction from the family $\mathcal{C}$ to a family of halfspaces of this form. For the case of such halfspaces duality (or polarity) leads to a Helly type theorem on the dimension of the lineality space of cones (Theorem 2.1), which is the other main result in this paper.

We remark that the long (and old and neat) survey paper by Danzer, Grünbaum, and Klee [2] contains several similar Helly type results. The same applies to the more recent books by Gruber [4], Matoušek [8], and Schneider [10]. An up-to-date survey is by Bárány and Kalai [1].

2. A Helly type theorem for the lineality space of cones

Suppose $A \subset \mathbb{R}^d$ is a finite set and define $\operatorname{pos} A$ as the cone hull of $A = \{a_1, \ldots, a_n\}$, i.e., $\operatorname{pos} A = \{\sum_1^n \alpha_i a_i : \alpha_i \geq 0 \text{ for all } i \in [n]\}$. The *lineality space* of $\operatorname{pos} A$, to be denoted by $\operatorname{lpos} A$, is the set $\operatorname{pos} A \cap (-\operatorname{pos} A)$ which is a linear subspace of $\mathbb{R}^d$, actually the (unique) maximal dimension subspace that $\operatorname{pos} A$ contains, see for instance [4]. Set $h(k,d) = \max\{d+1, 2(k+1)\}$.

Theorem 2.1. *Assume $A \subset \mathbb{R}^d$ is finite, $k \in [d]$, and $\dim \operatorname{lpos} B \leq k$ for every $B \subset A$ with $|B| \leq h(k,d)$. Then $\dim \operatorname{lpos} A \leq k$ as well.*

This is a Helly type result for the dimension of the lineality space. The value of $h(k,d)$ is optimal again as the following examples show. Define $A = \{v_1, \ldots, v_{d+1}\}$ with the v_i coming from Example 1, then $\operatorname{lpos} A = \mathbb{R}^d$ while $\dim \operatorname{lpos} B = 0$ for all $B \subset A$, $|B| \leq d$. This shows that $h(k,d)$ is optimal when $d+1 \geq 2(k+1)$. For $d+1 \leq 2(k+1)$ let A_k be the set of vectors $\{\pm e_1, \ldots, \pm e_k\}$. Now $\dim \operatorname{lpos} A_k = k$ but $\dim \operatorname{lpos} A_k \setminus \{a\} < k$ for every $a \in A_k$. These examples indicate a duality between Theorems 1.2 and 2.1 that will become more transparent later.

The proof method of Theorem 1.2 works in other cases. For instance a theorem of Katchalski [5] states the following.

Theorem 2.2. *Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ and $k \in \{0, 1, \ldots, d\}$. If $\dim \bigcap \mathcal{C}^* \geq k$ for every subfamily $\mathcal{C}^* \subset \mathcal{C}$ of size at most $m(k,d)$, then $\dim \bigcap \mathcal{C} \geq k$ as well.*

The case $k = 0$ is exactly Helly's theorem. We mention further that from this result Theorem 1.2 (and in particular Theorem 1.1) can be deduced in a few lines. This is explained in Section 5. Another example is the following result of de Santis [3].

Theorem 2.3. *Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ and $k \in [d]$. If $\bigcap \mathcal{C}^*$ contains an affine flat of dimension $n-k$ for every subfamily $\mathcal{C}^* \subset \mathcal{C}$ of size at most $k+1$, then so does $\bigcap \mathcal{C}$.*

Examples 1 and 2 show that the bounds $m(k,d)$ (in Theorem 2.2) and $k+1$ (in Theorem 2.3) are optimal. Section 5 explains how the proof method of Theorem 1.2 applies to the last two results.

3. Proof of Theorem 2.1

Define $L = \operatorname{lpos} A$ and choose a positive basis $X \subset A$ of L. This is just a subset of A with $\operatorname{pos} X = L$ but $\operatorname{pos}(X \setminus \{x\}) \neq L$ for any $x \in X$. We are going to use a theorem of Reay [9] stating that a positive basis X has a partition $X_1 \cup \ldots \cup X_r$ such that $|X_1| \geq |X_2| \geq \ldots \geq |X_r| \geq 2$ and $X_1 \cup \ldots \cup X_j$ is a positive basis for $\operatorname{lin}(X_1 \cup \ldots \cup X_j)$ whose dimension is $|X_1 \cup \ldots \cup X_j| - j$ for every $j \in [r]$. Note that this and $|X_i| \geq 2$ imply that $\dim \operatorname{lin}(X_1 \cup \ldots \cup X_j) \geq j$.

We show first that, in our case, $|X_1| \leq k+1$. As X_1 is a positive basis of $\operatorname{lin} X_1$ whose dimension is $|X_1| - 1 \leq d$ we have $|X_1| \leq d+1 \leq h(k,d)$. Then, according to the condition of the theorem, $|X_1| - 1 = \dim \operatorname{pos} X_1 \leq k$ and $|X_1| \leq k+1$ indeed.

Set $B_j = X_1 \cup \ldots \cup X_j$. We show, by induction on j, that $|B_j| \leq k+j$ or that $\dim \operatorname{lin} B_j \leq k$. (The two are equivalent since $\dim \operatorname{lin} B_j = |B_j| - j$.) The starting step $j=1$ was just fixed so we move to step $j-1 \to j$ assuming that $j \geq 2$. Since X_j has the smallest size among the sets $X_1, \ldots, X_j$ we have $|X_j| \leq \frac{1}{j-1}|B_{j-1}|$. By the induction hypothesis $|B_{j-1}| \leq k + (j-1)$. Thus

$$\begin{aligned} j &\leq \dim \operatorname{lin} B_j = |B_j| - j = |B_{j-1}| + |X_j| - j \\ &\leq k + (j-1) + \frac{k+(j-1)}{j-1} - j = \frac{jk}{j-1}, \end{aligned}$$

and $j \leq \frac{jk}{j-1}$ implying that $j - 1 \leq k$.

Claim 3.1. $\frac{jk}{j-1} + j \leq h(k,d)$.

This claim implies that $|B_j| \leq h(d,k)$ and then the condition of Theorem 2.1 gives $\dim \operatorname{lin} B_j \leq k$. Consequently $|B_j| \leq k+j$, completing the induction.

Proof. Assume first that $h(k,d) = 2(k+1)$ so $d+1 \leq 2(k+1)$. Direct computation shows that the inequality $\frac{jk}{j-1} + j \leq h(k,d)$ is equivalent to $(j-1)(j-2) \leq k(j-2)$ which is true for $j \geq 2$ because $j-1 \leq k$. Suppose next that $h(k,d) = d+1$, so $k \leq \frac{d-1}{2}$. Then $\frac{jk}{j-1} + j \leq d+1$ holds iff $\frac{j(d-1)}{2(j-1)} + j \leq d+1$. The last inequality is the same as $2j^2 - 5j + 2 \leq d(j-2)$, and here $2j^2 - 5j + 2 = (j-2)(2j-1)$ and $2j - 1 \leq 2(k+1) - 1 \leq d$ indeed. □

For $j = r$ the Claim gives $\dim L = \dim \operatorname{lin}(B_r) = \dim \operatorname{lpos} A \leq k$. □

Remark. A strange step in the proof is that the inequalities $|B_j| \leq \frac{jk}{j-1} + j \leq h(k,d)$ imply, via $\dim \operatorname{lin} B_j = |B_j| - j$, that $|B_j| \leq k + j$.

4. Proof of Theorem 1.2

As $m(k,d) \leq d+1$ Helly's theorem shows that $\bigcap \mathcal{C}$ is non-empty. We assume, after a translation if necessary, that $0 \in \bigcap \mathcal{C}$. We assume further that every $C \in \mathcal{C}$ is closed. This assumption is justified since a convex set contains a cone if and only if its closure contains a translate of this cone.

Lemma 4.1. *Let $C \subset \mathbb{R}^d$ be a closed convex set containing a cone K with apex at u. Then C contains the cone $v - u + K$ for every $v \in C$; the apex of this cone is at v.*

The **proof** is simple and is omitted. We remark though that it is enough to check the case when K is a halfline. $\square$

We agree that from now on the word "cone" means a cone with apex at the origin. In view of Lemma 4.1, $\mathcal{C}$ satisfies the condition

(*) $\bigcap \mathcal{C}^*$ contains a k-dimensional cone for every $\mathcal{C}^* \subset \mathcal{C}$ whose size is at most $m(k,d)$.

After these preparations the proof starts by reducing or changing the family $\mathcal{C}$ in two steps. For the first step we choose a k-dimensional cone $K(\mathcal{C}^*) \subset \bigcap \mathcal{C}^*$ for every $\mathcal{C}^* \subset \mathcal{C}$ satisfying $|\mathcal{C}^*| \leq m(k,d)$, and we choose $K(C^*)$ so that it has exactly k (of course linearly independent) generators. Replace each $C \in \mathcal{C}$ by the cone hull (or what is the same, convex hull), $D(C)$, of the union of all cones $K(\mathcal{C}^*)$ with $C \in \mathcal{C}^*$ and set $\mathcal{D} = \{D(C) : C \in \mathcal{C}\}$. The new system $\mathcal{D}$ consists of polyhedral cones and satisfies condition (*). Moreover $\bigcap \mathcal{D} \subset \bigcap \mathcal{C}$ because $D(C) \subset C$. So it suffices to show that $\bigcap \mathcal{D}$ contains a k-dimensional cone.

For the second reduction we observe that each $D(C)$ is the intersection of finitely many closed halfspaces, H, of the form $\{x \in \mathbb{R}^d : ax \leq 0\}$ for some $a \in \mathbb{R}^d$ ($a \neq 0$), the outer normal of H. Let $\mathcal{H}$ be the collection of these (finitely many) closed halfspaces, and let $A \subset \mathbb{R}^d$ be the set of the corresponding outer normals. Evidently $\mathcal{H}$ satisfies condition (*) and $\bigcap \mathcal{C}$ contains a k-dimensional cone if so does $\bigcap \mathcal{H} = \bigcap \mathcal{D}$.

The solution set of the system of linear inequalities

$$ax \leq 0, a \in A$$

coincides with $\bigcap \mathcal{H}$ and then $\bigcap \mathcal{H}$ is the polar of the cone $K = \operatorname{pos} A$. Let L be the lineality space of K, then K is the sum of L and the cone $L^\perp \cap \operatorname{pos} A$, see [4]. The latter cone is a pointed cone in $\mathbb{R}^d$ and in the subspace $L^\perp$ as well. It coincides with the cone hull of the orthogonal projection, to be denoted by $\operatorname{pr} A$, of A onto $L^\perp$. Using standard properties of polarity (c.f. [4]) we see that

$$\begin{aligned} \bigcap \mathcal{H} &= K^\circ = L^\perp \cap (L^\perp \cap \operatorname{pos} A)^\circ \\ &= (L^\perp \cap (L + (\operatorname{pos} \operatorname{pr} A)^\circ) = L^\perp \cap (\operatorname{pos} \operatorname{pr} A)^\circ, \end{aligned}$$

where the polars are taken in $\mathbb{R}^d$. Thus $\bigcap \mathcal{H}$ is a cone in $L^\perp$ and is full dimensional there because $L^\perp \cap \operatorname{pos} A = \operatorname{pos} \operatorname{pr} A$ is a pointed cone. Then $\bigcap \mathcal{H}$ contains a k-dimensional cone if and only if $\dim L^\perp = \dim(\operatorname{lpos} A)^\perp$ is at least k.

Condition (*) for $\mathcal{H}$ says that $\bigcap \mathcal{H}^*$ contains a k-dimensional cone for every $\mathcal{H}^* \subset \mathcal{H}$ with $|\mathcal{H}^*| \le m(k,d)$. Writing $B \subset A$ for the outer normals of the halfspaces in $\mathcal{H}^*$ we have $\bigcap \mathcal{H}^* = (\operatorname{pos} B)^\circ$. The previous argument applies to B (instead of A) and gives that $\bigcap \mathcal{H}^*$ contains a k-dimensional cone if and only if $\dim(\operatorname{lpos} B)^\perp \ge k$.

Then for every $B \subset A$ with $|B| \le m(k,d)$, we have $\dim(\operatorname{lpos} B)^\perp \ge k$, or, what is the same $\dim(\operatorname{lpos} B) \le d-k$. Theorem 3 applies now and shows that $\dim(\operatorname{lpos} A) \le d-k$, and equivalently $\dim L^\perp \ge k$. Thus $\bigcap \mathcal{H}$ contains a k-dimensional cone. □

A byproduct of this proof is an interesting corollary for systems of homogeneous inequalities.

Corollary 4.1. *Assume A is a finite set of nonzero vectors in $\mathbb{R}^d$. The system $ax \le 0,\ a \in A$ has at least k linearly independent solutions if and only if for every $B \subset A$ whose size is at most $m(k,d)$ the system $ax \le 0,\ a \in B$ has at least k linearly independent solutions.*

5. Proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2. We only give a sketch. Again each $C \in \mathcal{C}$ is closed and contains the origin. For every subfamily $\mathcal{C}^*$ of size at most $m(k,d)$ we choose a set of k linearly independent vectors from $\bigcap \mathcal{C}^*$. They together with the origin show that $\bigcap \mathcal{C}^*$ is indeed at least k-dimensional.

Next each $C \in \mathcal{C}$ is replaced by the cone hull, $D(C)$, of those k tuples of vectors that were chosen for some $\mathcal{C}^*$ with $C \in \mathcal{C}^*$. Let $\mathcal{D}$ be the family of all $D(C)$. This time the condition on $\mathcal{C}$ implies that for every $\mathcal{D}^* \subset \mathcal{D}$ of size at most $m(k,d)$, the set $\bigcap \mathcal{D}^*$ contains a k-dimensional cone. Then, according to Theorem 1.2, there is a k-dimensional cone in $\bigcap \mathcal{D}$ implying that there are linearly independent vectors $v_1, \dots, v_k \in \bigcap \mathcal{D}$. It is not hard to check (we omit the details) that for a small enough $r > 0$ the vectors $rv_i \in \bigcap \mathcal{C}$ for all $i \in [k]$. So $\bigcap \mathcal{C}$ is also at least k-dimensional. □

We explain next how Katchalski's theorem (Theorem 2.2) implies Theorem 2.1. We assume again that every $C \in \mathcal{C}$ is closed and contains the origin. The same reduction as in the proof of Theorem 2.1 works again: for each $\mathcal{C}^* \subset \mathcal{C}$ whose size is at most $m(k,d)$ we choose the same cone $K(C^*) \subset \bigcap \mathcal{C}^*$ whose generators are k linearly independent vectors from $\bigcap \mathcal{C}^*$. Replace each $C \in \mathcal{C}$ by the cone hull, $D(C)$, of the union of all cones $K(\mathcal{C}^*)$ with $C \in \mathcal{C}^*$ and set $\mathcal{D} = \{D(C) : C \in \mathcal{C}\}$. As we have seen $D(C) \subset C$ for every $C \in \mathcal{C}$. Then $\dim \bigcap \mathcal{D}^* \ge k$ for every $\mathcal{D}^* \subset \mathcal{D}$ whose size is at most $m(k,d)$. Thus Theorem 2.2 implies that $\dim \bigcap \mathcal{D} \ge k$ and then there are linearly independent vectors $v_1, \dots, v_k \in \bigcap \mathcal{D} \subset \bigcap \mathcal{C}$. The cone hull of these vectors is a cone of dimension at least k in $\bigcap \mathcal{C}$ finishing the proof.

There is a related result by Kołodziejczyk [7] stating a condition, similar to (*), guaranteeing that $\bigcap \mathcal{C}$ contains an k-dimensional affine halfspace. The condition is that $\bigcap \mathcal{C}^*$ contains an k-dimensional affine halfspace for every $\mathcal{C}^* \subset \mathcal{C}$ whose size is at most $m(k,d)$. This theorem can also be proved by the same method.

Proof of Theorem 2.3. We assume again that each $C \in \mathcal{C}$ is closed and that the origin lies in every $C \in \mathcal{C}$. Check that if C contains an affine flat F, then it also contains the subspace $F - f$ where $f \in F$ is arbitrary. The condition of the theorem is then modified to the following: For every subfamily $\mathcal{C}^* \subset \mathcal{C}$ whose size is at most $k+1$, $\bigcap \mathcal{C}^*$ contains an $(n-k)$-dimensional subspace. We have to show that $\bigcap \mathcal{C}$ contains an $(n-k)$-dimensional subspace.

Now comes the reduction in two steps. For each such $\mathcal{C}^*$ choose such a subspace $F(\mathcal{C}^*)$ and replace every $C \in \mathcal{C}$ by the convex hull, $D(C)$, of all subspaces $F(\mathcal{C}^*)$ with $C \in C^*$. It is easy to check that every $D(C)$ is a subspace in $\mathbb{R}^d$. The new system $\mathcal{D} = \{D(C) : C \in \mathcal{C}\}$ satisfies the previous condition, namely, that for every subfamily $\mathcal{D}^* \subset \mathcal{D}$ whose size is at most $k+1$, $\bigcap \mathcal{D}^*$ contains an $(n-k)$-dimensional subspace. It suffices to show that $\bigcap \mathcal{D}$ contains a $(n-k)$-dimensional subspace.

Every $D \in \mathcal{D}$ is a subspace and one can choose $(1 + \dim D)$ closed halfspaces whose intersection is D. Fix these halfspaces for every D and let $\mathcal{H}$ be the collection of these halfspaces. So $H \in \mathcal{H}$ is of the form $\{x \in \mathbb{R}^d : ax \le 0\}$ where $a \in \mathbb{R}^d$ is the outer normal of the halfspace H. Write A for the set of all outer normals in $\mathcal{H}$.

As $\bigcap \mathcal{H} = \bigcap \mathcal{D}$, our target is to show that $\bigcap \mathcal{H}$ contains an $(n-k)$-dimensional subspace. The condition is that the intersection of any $k+1$ of these halfspaces contains an $(n-k)$-dimensional subspace. This happens if and only if the outer normals of these $k+1$ halfspaces are linearly dependent. So the condition says that there is no linearly independent $(k+1)$-element subset in A, in other words, $\dim \operatorname{lin} A \le k$ which implies in turn that $\bigcap \mathcal{H}$ contains an $(n-k)$-dimensional subspace. $\square$

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Imre Bárány

Alfréd Rényi Institute of Mathematics
13-15 Reáltanoda Street, Budapest, 1053 Hungary,
e-mail: `imbarany@gmail.com`, and

Department of Mathematics, University College London
Gower Street, London WC1E 6BT, UK