



Paths are Turán-good

Dániel Gerbner¹

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Abstract

We show that among K_{k+1} -free n -vertex graphs, the Turán graph contains the most copies of any path.

Keywords Path · Generalized Turán · Turán graph · Turán-good

Mathematics Subject Classification 05C35

1 Introduction

A fundamental theorem of extremal graph theory is due to Turán [10], who showed that among n -vertex K_{k+1} -free graphs, the most edges are contained in the Turán graph $T(n, k)$, which is the complete k -partite graph with each part of order $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. In general, the most number of edges in n -vertex F -free graphs is denoted by $\text{ex}(n, F)$. Simonovits [9] proved that if F has a color-critical edge, i.e., an edge whose deletion decreases the chromatic number of F , then for sufficiently large n we have $\text{ex}(n, F) = |E(T(n, k))|$.

In so-called *generalized Turán problems*, we count copies of a subgraph H instead of edges. Let $\mathcal{N}(H, G)$ denote the number of copies of H in G . We denote by $\text{ex}(n, H, F)$ the largest $\mathcal{N}(H, G)$ among n -vertex F -free graphs G . The first result concerning $\text{ex}(n, H, F)$ with $H \neq K_2$ is due to Zykov [11], who showed that $\text{ex}(n, K_r, K_{k+1}) = \mathcal{N}(K_r, T(n, k))$.

One of the most studied question in this topic is the following: When is the Turán graph extremal? It was systematically studied first by Győri, Pach and Simonovits [5]. Gerbner and Palmer [4] introduced the following definition. Given a graph F with $\chi(F) = k + 1$, we say that H is *F-Turán-good* if $\text{ex}(n, H, F) = \mathcal{N}(H, T(n, k))$ for sufficiently large n . They made the following conjecture.

Conjecture 1.1 *Each path is K_{k+1} -Turán-good for every k .*

✉ Dániel Gerbner
gerbner.daniel@renyi.hu

¹ Alfréd Rényi Institute of Mathematics, Budapest, Hungary

We denote by P_ℓ the path on ℓ vertices. We say that H is a *linear forest* if each connected component of H is a path. Results from [5] imply that each path is K_3 -Turán-good and P_3 is K_{k+1} -Turán-good for every k . Gerbner and Palmer [4] showed that P_3 is F -Turán-good for any F with a color-critical edge and $\chi(F) > 3$. This was extended by Gerbner [2] to the case $\chi(F) = 3$. He also showed that P_4 is K_4 -Turán-good and matchings are K_{k+1} -Turán-good. Results of Gerbner [1] imply that each path is F -Turán-good for a large class of 3-chromatic graphs with color-critical edges, including odd cycles. Murphy and Nir [7] showed that P_4 is K_{k+1} -Turán-good. Qian, Xie and Ge [8] showed that P_5 is K_{k+1} -Turán-good. Hei, Hou and Liu [6] came close to prove Conjecture 1.1. They showed that P_ℓ for $\ell \leq 6$ is F -Turán-good for any F with a color-critical edge and any path is K_{k+1} -Turán-good for large enough k . Moreover, they reduced the problem in the remaining cases to complete multipartite graphs, as we will describe in Sect. 2.

We complete the proof of Conjecture 1.1. We prove the following more general theorem.

Theorem 1.2 *Let F be a graph with a color-critical edge and H be a linear forest. Then H is F -Turán-good.*

2 Proof

The main result of Hei, Hou and Liu [6] is the following theorem.

Theorem 2.1 (Hei, Hou and Liu [6]) *Let F be a graph with a color-critical edge and H be a graph with $\chi(H) < \chi(F)$. Let us assume that there is an n -vertex F -free graph G with $\text{ex}(n, H, F) = \mathcal{N}(H, G)$ that can be turned to a complete k -partite graph by adding and deleting $o(n^2)$ edges. If every complete k -partite graph T has $\mathcal{N}(H, T) \leq \mathcal{N}(H, T(n, k))$, then H is F -Turán-good.*

We remark that they stated a weaker theorem, adding additional conditions that hold for paths but are not actually used in the proof. Even in the weaker form, the theorem implies that a stability result accompanied with optimization on complete multipartite graphs is enough to complete the proof. They also proved the stability result for paths. In fact, they proved the following more general (and more usual) stability: if G is an n -vertex F -free graph with $\mathcal{N}(P_\ell, G) = \text{ex}(n, P_\ell, F) - o(n^\ell)$, then G can be turned to a complete k -partite graph by adding and deleting $o(n^2)$ edges.

First we extend the stability result to linear forests. It follows from a result of Gerbner [3]. He showed that if stability holds for two graphs in the above sense, then the same holds for their vertex-disjoint union (in fact, he proved a stronger result: it is enough if one of the two graphs have the stability, and the other has the correct asymptotics). Therefore, Theorem 1.2 follows from the next theorem.

Theorem 2.2 *Let H be a linear forest. Then for any complete k -partite graph G we have $\mathcal{N}(H, G) \leq \mathcal{N}(H, T(n, k))$.*

Proof First we prove the case $k = 2$. We use induction on $|V(H)|$, the base cases $|V(H)| \leq 2$ are trivial. Let us assume that the statement holds for smaller linear forests and prove it for H . Let T be an n -vertex complete bipartite graph.

Assume first that H has a component $K = P_{2i+1}$ on odd vertices. Then we let H_1 be the graph obtained by deleting an endpoint of K . By induction, the number of copies of H_1 is maximized by the Turán graph. Clearly all the vertices of T not in a copy of H_1 extend that H_1 to H by adding them to the end of a P_{2i} component of H_1 , and each vertex is connected to exactly one endpoint of such a component. Let H_1 contain x such components, then there are $x(n - |V(H_1)|)$ ways to extend H_1 to H . Each copy of H is obtained the same number of times this way, thus the number of copies of H is also maximized by the Turán graph.

Assume now that H has a component $K' = P_{2i}$. Then we let H_2 be the graph obtained by deleting the last edge of K' . By induction, the number of copies of H_2 is maximized by the Turán graph, and so does the number of edges. We count copies of H in T by taking an edge uv , and a copy of H_2 on the remaining vertices, and then adding uv to the end of a P_{2i-2} component. The first two factors are maximized by the Turán graph. Clearly, u is adjacent to one endpoint of the P_{2i-2} and v is connected to the other endpoint, thus there are two ways to unite the P_{2i-2} and uv to a P_i component. Each copy of H is obtained the same number of times this way, completing the proof in the case $k = 2$.

Let us continue with the case $k > 2$. Let G be an n -vertex F -free complete k -partite graph with parts $A_1, \dots, A_k$. Let us assume that $|A_1| < |A_2| - 1$. Then we move $\lfloor (|A_2| - |A_1|)/2 \rfloor$ vertices from A_2 to A_1 to obtain G' . We will show that the number of copies of H does not decrease this way. Clearly, after repeating this enough times with all the parts, we obtain the Turán graph.

A copy of H intersects $A_1 \cup A_2$ in a linear forest H' , and intersects $V(G) \setminus (A_1 \cup A_2)$ in another linear forest H'' . Observe that if H' extends a copy of H' inside $A_1 \cup A_2$ to H , then the same way it extends any other copy of H' inside $A_1 \cup A_2$ to H . Therefore, it is enough to show that for any subforest H' of H , the number of copies of H' does not decrease when turning G to G' . Observe that we turned $G[A_1 \cup A_2]$ to $T(|A_1 \cup A_2|, 2)$, hence what we want is exactly the case $k = 2$. We have already dealt with that case, thus we are done. $\square$

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Declarations

Conflict of Interest The author has no relevant financial or non-financial interests to disclose.

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