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EXTREMAL PROBLEMS FOR HYPERGRAPH BLOWUPS OF TREES*

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Abstract. We study the extremal number for paths in r-uniform hypergraphs where two consecutive edges of the path intersect alternately in sets of sizes b and a with $a+b=r$ and all other pairs of edges have empty intersection. Our main result, which is about hypergraphs that are blowups of trees, determines asymptotically the extremal number of these (a,b)-paths that have an odd number of edges or that have an even number of edges and $a>b$. This generalizes the Erdős–Gallai theorem for graphs, which is the case of $a=b=1$. Our proof method involves a novel twist on Katona's permutation method, where we partition the underlying hypergraph into two parts, one of which is very small. We also find the asymptotics of the extremal number for the $(1,2)$-path of length 4 using the different Δ-systems method.

Key words. hypergraph trees, extremal hypergraph theory, Δ-systems

MSC codes: 05D05, 05C65, 05C05

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1. Paths.

1.1. Definitions for hypergraphs, two constructions.

An *r-uniform hypergraph,* or simply an *r-graph,* is a family of r-element subsets of a finite set. We associate an r-graph F with its edge set and call its vertex set $V(F)$. Usually we take $V(F)=[n]$, where $[n]$ is the set of first n integers, $[n]:=\{1,2,3,\dots,n\}$. We also use the notation $F\subseteq\binom{[n]}{r}$. For a hypergraph H, a vertex subset C of H that intersects all edges of H is called a *vertex cover* of H. Let $\tau(H)$ be the minimum size of a vertex cover of H. Let $\Psi_c(n,r)$ be the r-graph with vertex set $[n]$ consisting of all r-edges meeting $[c]$. Then Ψ has the maximum number of r-sets such that $\tau(\Psi)\le c$. When r and c are fixed and $n\to\infty$,

$$|\Psi_c(n,r)|=\binom{n}{r}-\binom{n-c}{r}=c\binom{n}{r-1}+o(n^{r-1}). \tag{1}$$

A *crosscut* of a hypergraph H is a set $X\subset V(H)$ such that $|e\cap X|=1$ for all $e\in H$. Not all hypergraphs have crosscuts. Let $\sigma(H)$ denote the smallest size of a crosscut

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in a hypergraph H with at least one crosscut. Clearly $\tau(H) \le \sigma(H)$ since a crosscut is a vertex cover. Here strict inequality may hold, as shown by a double star whose adjacent centers have high degrees. Define $\Psi_c^1(n,r) := \{E \subset [n] : |E| = r, |E \cap [c]| = 1\}$, so that it consists of all r-sets intersecting a fixed c-element subset of $V(H)$ at *exactly* one vertex. Then, for large enough n, Ψ^1 has the maximum number of r-sets such that $\sigma(\Psi^1) \le c$. Let us refer to this hypergraph as the *crosscut construction.* When r and c are fixed and $n \to \infty$,

$$|\Psi_c^1(n,r)| = c\binom{n-c}{r-1} = c\binom{n}{r-1} + o(n^{r-1}). \tag{2}$$

Given an r-graph F, let $\mathrm{ex}_r(n,F)$ denote the maximum number of edges in an r-graph on n vertices that does not contain a copy of F (if the uniformity is obvious from context, we may omit the subscript r). Crosscuts were introduced in [12] to get the following obvious lower bounds:

$$\mathrm{ex}(n,F) \ge |\Psi_{\tau(F)-1}(n,r)|, \text{ and when crosscut exists, } \mathrm{ex}(n,F) \ge |\Psi^1_{\sigma(F)-1}(n,r)|. \tag{3}$$

Notation. If H is a hypergraph and $e \subset V(H)$, then $\Gamma_H(e) = \{f \setminus e : e \subseteq f, f \in H\}$ and the *degree* of e is $d_H(e) = |\Gamma_H(e)|$. For an integer p, let the *p-shadow,* $\partial_p H$, be the collection of p-sets that lie in some edge of H. If H is an r-graph, then the $(r-1)$-shadow of H is simply called the *shadow* and is denoted by ∂H.

Whenever we write $f(n) \sim g(n)$, we always mean $\lim_{n\to\infty} f(n)/g(n) = 1$, while the other variables of f and g are fixed. This is the case even if the variable n is not indicated.

Aims of this paper. We have three aims. First, we want to find more Turán numbers (or estimates) of hypergraphs in the Erdős–Ko–Rado range. We are especially interested in cases when the excluded configuration is "dense," i.e., it has only a few vertices of degree one. Second, we present an asymmetric version of Katona's permutation method, when we first solve (estimate) the problem only on a well-chosen substructure. Third, we show the power of the Δ-systems method for $(1,2)$-paths of length 4. The (a,b)-blowups of trees and paths are good examples for all our aims.

1.2. Paths in graphs. A fundamental result in extremal graph theory is the Erdős–Gallai theorem [4], that is,

$$\mathrm{ex}_2(n,P_\ell) \le \frac{1}{2}(\ell-1)n, \tag{4}$$

where P_ℓ is the *ℓ-edge path.* (Warning: This is a nonstandard notation). Equality holds in (4) if and only if ℓ divides n and all connected components of G are ℓ-vertex complete graphs. The Turán function $\mathrm{ex}(n,P_\ell)$ was determined exactly for every ℓ and n by Faudree and Schelp [7] and independently by Kopylov [21]. Let $n \equiv r \pmod{\ell}$, $0 \le r < \ell$. Then $\mathrm{ex}(n,P_\ell) = \frac{1}{2}(\ell-1)n - \frac{1}{2}r(\ell-r)$. They also described the extremal graphs which are either

— vertex disjoint unions of $\lfloor n/\ell \rfloor$ complete graphs K_ℓ and a K_r; or

— ℓ is odd, $\ell = 2k-1$, and $r = k$ or $k-1$. In this case, other extremal graphs with a different structure can be obtained by taking a vertex-disjoint union of m copies of K_ℓ $(0 \le m < \lfloor n/\ell \rfloor)$ and a copy of $\Psi_{k-1}(n-m\ell,2)$, i.e., an $(n-m\ell)$-vertex graph with a $(k-1)$-set meeting all edges.

This variety of extremal graphs makes the solution difficult. We generalize these theorems for some hypergraph paths and trees.

1.3. Paths in hypergraphs. There are several ways to define a hypergraph path P. One of the most difficult cases appears to be the case when P is a *tight path* of length ℓ, namely the r-graph $Tight\,P^r_\ell$ with edges $\{1,2,\dots,r\},\{2,3,\dots,r+1\},\dots,\{\ell,\ell+1,\dots,\ell+r-1\}$. The best known results [15] for this special case are

$$\frac{\ell-1}{r}\binom{n}{r-1} \le \mathrm{ex}_r(n, Tight\,P^r_\ell) \le \begin{cases} \frac{\ell-1}{2}\binom{n}{r-1} & \text{if } r \text{ is even,} \\ \frac{1}{2}(\ell+\lfloor\frac{\ell-1}{r}\rfloor)\binom{n}{r-1} & \text{if } r \text{ is odd,} \end{cases}$$

where the lower bound holds as long as certain designs exist.

Another possibility is the r-uniform *loose path* (also called *linear path*) $Lin\,P^r_\ell$, which is obtained from P^2_ℓ by enlarging each edge with a new set of $(r-2)$ vertices such that these new $(r-2)$-sets are pairwise disjoint (so $|V(P^r_\ell)|=\ell(r-1)+1$). Recently, the authors [16, 22] determined $\mathrm{ex}_r(n, Lin\,P^r_\ell)$ exactly for large n, extending a work of Frankl [8], who solved the case $\ell=2$ by answering a question of Erdős as well as Sós [26] (see [20] for a solution for all n when $r=4$).

Here we consider so-called (a,b)-blowups of P_ℓ.

Definition. Suppose that a,b,ℓ are positive integers, $r=a+b$. The (a,b)-*path* $P_\ell(a,b)$ *of length* ℓ is an r-uniform hypergraph obtained from a (graph) path P_ℓ by blowing up its vertices to a-sets and b-sets. More precisely, an (a,b)-path $P_\ell(a,b)$ of length ℓ consists of ℓ sets of size $r=a+b$ as follows. Take $\ell+1$ pairwise disjoint sets $A_0,A_1,\dots,A_\ell$ with $|A_i|=a$ when i is even and $|A_i|=b$ when i is odd, and define the (hyper)edges of $P_\ell(a,b)$ as the sets of the form $A_{i-1}\cup A_i$ for $i=1,\dots,\ell$. If the elements of $A_0,\dots,A_\ell$ are ordered linearly, then the members of P can be represented as intervals of length r.

Paths of length 2. Two r-sets with intersection size b can be considered as a hypergraph path $P_2(a,b)$ of length two, where $a+b=r$, and $1\le a,b\le r-1$. If $H\subset\binom{[n]}{r}$ is $P_2(1,r-1)$-free, then the obvious inequality $r|H|=|\partial(H)|\le\binom{n}{r-1}$ yields the upper bound in the following result. The lower bound holds for any given r if n is sufficiently large ($n>n_0(r)$) due to the existence of designs (see Keevash [19]):

$$\frac{1}{r}\binom{n}{r-1} - O(n^{r-2}) < \mathrm{ex}_r(n, P_2(1,r-1)) \le \frac{1}{r}\binom{n}{r-1}. \tag{5}$$

The case $b=1$ was solved asymptotically by Frankl [8], and the general case was handled in [11]:

$$\mathrm{ex}_r(n, P_2(a,b)) = \Theta\left(n^{\max\{a-1,b\}}\right). \tag{6}$$

Two disjoint r-sets can be considered as a $P_2(r,0)$, so (6) also holds for $a=r$ since the maximum size of an intersecting family of r-sets is $\binom{n-1}{r-1}$ for $n\ge 2r$ by the Erdős–Ko–Rado theorem [6].

Paths of length 3. A $P_3(a,b)$-path has three r-sets, two of them are disjoint, and they cover the third in a prescribed way. For given $1\le a,b<r$, $r=a+b$ and for $n>n_2(r)$, Füredi and Özkahya [17] showed that

$$\mathrm{ex}_r(n, P_3(a,b)) = \binom{n-1}{r-1}.$$

Longer paths. While $P_{2k-1}(a,b)=P_{2k-1}(b,a)$, we have $P_{2k}(a,b)\ne P_{2k}(b,a)$ for $a\ne b$.

Our first results provide a nontrivial extension of the Erdős–Gallai theorem (4) for r-graphs.

Since the case $\ell = 2$ behaves somewhat differently (see (5) and (6)), we only discuss the case $\ell \geq 3$.

Suppose that $a+b=r$, $a,b \geq 1$, $r \geq 3$, and suppose that $\ell \in \{2k-1, 2k\}$, $\ell \geq 4$. Furthermore, suppose that these values are fixed and $n \to \infty$ or $n > n_3(r,k)$. Recall that $\Psi_{t-1}(n,r) := \{E \subset [n] : |E| = r, E \cap [k-1] \neq \emptyset\}$. We have the lower bound

$$\begin{aligned}\mathrm{ex}_r(n, P_{2k}(a,b)) &\geq \mathrm{ex}_r(n, P_{2k-1}(a,b)) \\ &\geq |\Psi_{k-1}(n,r)| = \binom{n}{r} - \binom{n-k+1}{r} = (k-1)\binom{n}{r-1} + o(n^{r-1}).\end{aligned}$$

Our main results (Theorems 6 and 7 below) imply that equality holds above for at least 75% of the cases.

THEOREM 1. *Let $a+b=r$, $a,b \geq 1$, and $\ell \geq 3$. Suppose further that* (i) *ℓ is odd, or* (ii) *ℓ is even and $a > b$, or* (iii) $(\ell, a, b) = (4, 1, 2)$.

Then

$$\mathrm{ex}_r(n, P_\ell(a,b)) = \left\lfloor \frac{\ell-1}{2} \right\rfloor \binom{n}{r-1} + o(n^{r-1}).$$

Moreover, if $a \neq b$, $a,b \geq 2$, and $\ell = 2k-1$, then $\Psi_{k-1}(n,r)$ is the only extremal family.

The proof of Theorem 1 in the case $(\ell,a,b) = (4,1,2)$ is different from the proofs for the other cases. The remaining cases (when ℓ is even, $a \leq b$, and $(\ell,a,b) \neq (4,1,2)$) are still open.

CONJECTURE 2. *If $r \geq 3$, $k \geq 2$, and $a \leq b$, then $\mathrm{ex}_r(n, P_{2k}(a,b)) = (1 + o(1))\Psi_{k-1}(n,r)$.*

2. Trees blown up, our main results.

Generalizing the Erdős–Gallai theorem (4), Ajtai et al. [1] claimed a proof of the Erdős–Sós conjecture [4], showing that if T is any tree with ℓ edges, where ℓ is large enough, then for all n,

$$\mathrm{ex}_2(n,T) \leq \frac{1}{2}(\ell-1)n.$$

A more general conjecture due to Kalai (see [12]) is about the extremal number for hypergraph trees. A hypergraph T is a *forest* if it consists of edges $e_1, e_2, \ldots, e_\ell$ ordered so that for every $1 < i \leq \ell$, there is $1 \leq i' < i$ such that $e_i \cap (\bigcup_{j<i} e_j) \subseteq e_{i'}$. A connected forest is called a *tree.* If T is r-uniform and for each $i > 1$, $|e_i \cap (\bigcup_{j<i} e_j)| = r-1$, then we say that T is a *tight tree.*

CONJECTURE 3 (Kalai). *Let T be an r-uniform tight tree with ℓ edges. Then*

$$\mathrm{ex}_r(n,T) \leq \frac{\ell-1}{r}\binom{n}{r-1}.$$

When $r=2$, this is precisely the Erdős–Sós conjecture. A simple greedy argument shows the following.

PROPOSITION 4. *If T is an r-uniform tight tree with ℓ edges and G is an r-graph on $[n]$ not containing T, then $|G| \leq (\ell-1)|\partial(G)|$.*

Here $\partial(G)$ is the family of $(r-1)$-sets that lie in some edge of G. We obtain

$$\mathrm{ex}_r(n,T) \le (\ell-1)\binom{n}{r-1}.$$

Our goal is to prove a nontrivial extension of the Erdős–Gallai theorem and the Erdős–Sós conjecture for r-graphs. To define the hypergraph trees we study in this paper, we make the following more general definition.

DEFINITION 5. *Let $s,t,a,b>0$ be integers, $r=a+b$, and let $H=H(U,V)$ denote a bipartite graph with parts $U=\{u_1,u_2,\dots,u_s\}$ and $V=\{v_1,v_2,\dots,v_t\}$. Let $U_1,\dots,U_s$ and $V_1,\dots,V_t$ be pairwise disjoint sets, such that $|U_i|=a$ and $|V_j|=b$ for all i,j. So $|\bigcup U_i\cup V_j|=as+bt$.*

The (a,b)-blowup of H, denoted by $H(a,b)$, is the r-uniform hypergraph with edge set

$$H(a,b):=\{U_i\cup V_j : u_iv_j\in E(H)\}.$$

Since deleting a vertex cover from a bipartite graph leaves an independent set, each crosscut in a connected bipartite graph is one of its parts. Consequently, $\sigma(H(a,b))=\min\{s,t\}$. Recalling from (2) that $\Psi^1_{\sigma-1}(n,r):=\{E\subset[n]:|E|=r,|E\cap[\sigma-1]|=1\}$, we obtain

$$(7)\qquad (\sigma-1)\binom{n}{r-1}+o(n^{r-1})=(\sigma-1)\binom{n-\sigma+1}{r-1}=|\Psi^1_{\sigma-1}(n,r)|\le \mathrm{ex}_r(n,H).$$

Let $\mathcal{T}_{s,t}$ denote the family of trees T with parts U and V where $|U|=s$ and $|V|=t$. We frequently say that T is a tree with $s+t$ vertices. Let $\mathcal{T}_{s,t}(a,b)$ denote the family of (a,b)-blowups of trees $T\in\mathcal{T}_{s,t}$. We frequently suppose that $a\ge b$ (but not always).

We are trying to determine when crosscut constructions are asymptotically extremal for (a,b)-blowups of trees. For other instances of hypergraph trees for which the crosscut constructions are asymptotically extremal, see [23]. Our main result is the following theorem.

THEOREM 6. *Suppose $r\ge 3$, $s,t\ge 2$, $a+b=r$, $b<a<r$. Let T be a tree on $s+t$ vertices, and let $\mathcal{T}=T(a,b)$be its (a,b)-blowup. Then (as $n\to\infty$) any $\mathcal{T}$-free n-vertex r-graph H satisfies*

$$|H|\le (t-1)\binom{n}{r-1}+o(n^{r-1}).$$

This is asymptotically sharp whenever $t\le s$.

Indeed, in the case $t\le s$ we have $\sigma(\mathcal{T})=t$ and (7) provides a matching lower bound.

A vertex x of $T\in\mathcal{T}_{s,t}$ is called a *critical leaf* if $\sigma(T\setminus x)<\sigma(T)$. In case of $t\le s$, it simply means that $\deg_T(x)=1$ and $x\in V$. (Similarly, a *critical leaf* of $\mathcal{T}=T(a,b)\in\mathcal{T}_{s,t}(a,b)$ with $t\le s$ is a b-set V_j in the part of size t whose degree in $\mathcal{T}$ is one.) If such a vertex exists, then we have a more precise upper bound.

THEOREM 7. *Suppose $r\ge 5$, $2\le t\le s$, $a+b=r$, $b<a<r-1$. Let T be a tree on $s+t$ vertices, and let $\mathcal{T}=T(a,b)$ be its (a,b)-blowup. Suppose that T has a critical leaf. Then for large enough n ($n>n_0(T)$)*

$$\mathrm{ex}(n, \mathcal{T}) \le \binom{n}{r} - \binom{n-t+1}{r}.$$

If, in addition, $\tau(\mathcal{T}) = t$, then equality holds above and the only example achieving the bound is $\Psi_{t-1}(n,r)$.

Since $\tau(\Psi_{t-1}(n,r)) = t-1$, no r-graph F with $\tau(F) \ge t$ is contained in $\Psi_{t-1}(n,r)$. Note that Theorems 6 and 7 imply Theorem 1.

3. Asymptotics. In this section, we prove the asymptotic version of our main results, i.e., Theorem 6. At a very high level, our proof can be viewed as a generalization of the Katona circle method. The idea of this method is to partition the underlying family into many well structured subfamilies and prove a good upper bound for the size of each subfamily. Alternatively, we can phrase this using an averaging argument. In the famous proof of the Erdős–Ko–Rado theorem using this method, these subfamilies comprise sets that appear as intervals in a cyclic permutation. Our situation is more complex. We take a random subset R of vertices and consider the r-sets in a subhypergraph H' that have a vertices in R and $b = r - a$ vertices outside R. This gives us a bipartite structure, and we use a greedy embedding algorithm to find the tree within these edges. Further complications arise due to b-sets with high codegree, and these are handled separately via a vertex cover L whose presence plays an important role in defining H'. One novelty in our approach is that the size of R is very small since this is needed for various estimates in the proof ($|R|$ is about $n^{1-1/3r}$, though we have some flexibility).

The next section proves various bounds in the bipartite environment described above.

3.1. Definition of templates and a lemma. Throughout this section, $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ and we suppose $\mathcal{T}$ is an (a,b)-blowup of a tree T. If H is an r-graph, then an (a,b)-*template in* H is a pair (A,B) where A is an a-uniform hypergraph on $V(H)$, B is a b-uniform matching on $V(H)$, and $V(A) \cap V(B) = \emptyset$. Define the bipartite graph

$$H_0 = H_0(A,B) = \{(e,f) \in A \times B : e \cup f \in H\}$$

and let $H_1 = H_1(A,B) = \{e \cup f : (e,f) \in H_0\} \subset H$. By construction, $|H_0| = |H_1|$. We claim that *if A and B are both matchings and $H_1(A,B)$ is $\mathcal{T}$-free, then*

$$|H_1(A,B)| \le (t-1)|A| + (s-1)|B|. \tag{8}$$

Indeed, otherwise $|H_0(A,B)| = |H_1(A,B)| > (t-1)|A| + (s-1)|B|$ and H_0 has a minimum induced subgraph $H_0'(A',B')$ satisfying $|H_0'(A',B')| > (t-1)|A'| + (s-1)|B'|$. By minimality, H_0' has minimum degree at least t in A' and minimum degree at least s in B'. This is sufficient to greedily construct a copy of T in H_0'. Since H_1 is an (a,b)-blowup of $H_0 \supseteq H_0'$, this shows that $\mathcal{T} \subset H_1$.

We now prove a version of (8) for templates, i.e., in the case when A may be not a matching.

LEMMA 8. *Let $\delta > 0$, and let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$. Let H be a $\mathcal{T}$-free r-graph containing an (a,b)-template (A,B). If $B = B^0 \sqcup B^1$ and $d_H(e) \le \delta n^b$ for every a-set $e \subset V(H) \backslash V(B^1)$, then*

$$|H_1(A,B)| \le (t-1)|A| + asn^{a-1}(\delta|B^0| + |B^1|). \tag{9}$$

Proof. Let $\beta_0 = as\delta n^{a-1}$ and $\beta_1 = asn^{a-1}$. Let $H_1 = H_1(A,B)$ and $H_0 = H_0(A,B)$, and suppose $|H_1| \geq (t-1)|A| + \beta_0|B^0| + \beta_1|B^1|$. By deleting vertices of H_0, we may assume

$$d_{H_0}(e) \geq t \text{ for all } e \in A \text{ and for } i \in \{0,1\}, d_{H_0}(e) > \beta_i \text{ for all } e \in B^i. \tag{10}$$

Suppose $\mathcal{T}$ is a blowup of a tree T, where T has a unique bipartition (U,V) with $|U| = s, |V| = t$. We call an embedding of the (a,b)-blowup of a subtree T' of T in $H_1(A,B)$ a *feasible embedding* if the a-sets corresponding to vertices in U are mapped to members of A and the b-sets corresponding to vertices in V are mapped to members of B. It suffices to prove that any feasible embedding h of the (a,b)-blowup of any proper subtree T' of T can be extended to a feasible embedding h' of the (a,b)-blowup of a subtree of T that strictly contains T'.

Let T' be given. Then there exists an edge xy in T with $x \in V(T')$ and $y \notin V(T')$. Let h be a feasible embedding of the (a,b)-blowup $\mathcal{T}'$ of T' in $H_1(A,B)$. First suppose that $x \in U$. Let e denote the image under h of the a-set in $\mathcal{T}'$ that corresponds to x. By our assumption, $e \in A$. Hence, by our earlier assumption, $d_{H_0}(e) \geq t$. Thus, $|\Gamma_{H_1}(e)| \geq t$. Since $\Gamma_{H_1}(e) \subseteq B$ is a matching of size at least t and the b-sets corresponding to $V - \{y\}$ are mapped to at most $t-1$ members of B, there exists $f \in B$ such that $f \cap V(h(\mathcal{T}')) = \emptyset$. We can extend h to a feasible embedding of $T' \cup xy$ by mapping the b-set in $\mathcal{T}$ corresponding to y to f.

Next, suppose $x \in V$. Let e denote the image under h of the b-set in $\mathcal{T}'$ that corresponds to x. If there exists $f \in \Gamma_{H_1}(e) - V(h(\mathcal{T}'))$, then $h(\mathcal{T}') \cup \{e \cup f\}$ is a feasible embedding of $T' \cup xy$. Hence, we may assume that no such f exists. If $e \in B^0$, then we estimate $d_{H_0}(e)$ by first adding $a-b$ new vertices, one from $V(h(\mathcal{T}'))$ and all outside $V(B^1)$, and then choosing the remaining a vertices. This yields

$$d_{H_0}(e) \leq |V(h(\mathcal{T}')) \cap V(A)| \cdot n^{a-b-1} \cdot \delta n^b \leq as\delta n^{a-1} = \beta_0,$$

a contradiction to (10). Note it is crucial here that $b < a$. Similarly, if $e \in B^1$, then

$$d_{H_0}(e) \leq |V(h(\mathcal{T}')) \cap V(A)| \cdot n^{a-1} \leq asn^{a-1} = \beta_1.$$

This contradicts $d_{H_0}(e) > \beta_1$ for $e \in B^1$. Hence, we have shown that each feasible embedding of $\mathcal{T}'$ can be extended. This completes the proof. ∎

3.2. Proof of Theorem 6. In a few places of the proof, we will use the following elementary fact or a slight variant of it. Let e be a fixed edge in $\binom{[n]}{p}$ and H a p-graph on at most n vertices. Let L be a copy of H in $\binom{[n]}{p}$ chosen uniformly at random among all copies of H. Then $\mathbb{P}(e \in L) = |H|/\binom{n}{p}$.

Let m be an integer satisfying $m > r^r$ and $m = o(\sqrt{n})$. Let $f(m) = m^{-1/r}n^{r-1} + m^2n^{r-2}$. We show that if H is $\mathcal{T}$-free for some $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, then

$$|H| \leq (t-1)\binom{n}{r-1} + O(f(m)).$$

In particular, taking $m = \lfloor n^{1/3} \rfloor$, we obtain

$$|H| \leq (t-1)\binom{n}{r-1} + O(n^{r-1-1/(3r)}).$$

In our arguments below, for convenience, we assume b divides n, since assuming so has no effect on the asymptotic bound we want to establish. Let $D = \{e \in \binom{V(H)}{a} :$

$d_H(e) \geq n^b/m\}$, and let L be a smallest vertex cover of D, meaning that every set in D intersects L. We claim

$$|L| = O(m). \tag{11}$$

Indeed, if $|L| \geq asm$, then D has a matching M of size sm. Each set in M forms an edge of H with at least n^b/m different b-sets, and at most $a|M|n^{b-1} = asmn^{b-1}$ of these b-sets intersect $V(M)$. By averaging, there is a matching N of b-sets disjoint from $V(M)$ such that

$$|H_0(M,N)| \geq \frac{|M|(n^b/m - asmn^{b-1})}{\binom{n-1}{b-1}} > |M| \cdot \frac{n}{m} - |M| \cdot asm.$$

Since n is large and $m = o(\sqrt{n})$, this is at least

$$(t-1)|M| + \left(\frac{n}{m} - t + 1 - asm\right)|M| \geq (t-1)|M| + (s-1)n > (t-1)|M| + (s-1)|N|.$$

By (8), we conclude that $\mathcal{T} \subset H_1(M,N) \subset H$, a contradiction. This proves (11).

Let $G = \{e \in H : |e \cap L| \leq 1\}$, so that

$$|G| \geq |H| - |L|^2 n^{r-2} \geq |H| - O(m^2 n^{r-2}). \tag{12}$$

Let $R \subset V(G) \backslash L$ be a set whose elements are chosen independently with probability $\alpha = m^{-1/r}$, and let $A = \binom{R}{a}$. Let P be a random partition of $V(G)$ into b-sets. Let B denote the set of b-sets in P that are disjoint from R, and let $H_1 = H_1(A,B)$. If $B^0 = \{e \in B : e \cap L = \emptyset\}$ and $B^1 = \{e \in B : |e \cap L| \geq 1\}$, then by (9) with $\delta = 1/m$, and using $|B^1| \leq |L|$,

$$|H_1| \leq (t-1)|A| + O(n^{a-1}|B^0|/m) + O(n^{a-1}|L|).$$

Taking expectations over all choices of R and P and using (11) and $|B^0| \leq n$, we get

$$\mathbb{E}(|H_1|) \leq (t-1)\alpha^a \binom{n}{a} + O(n^a/m). \tag{13}$$

For $i \in \{0,1\}$, let $G_i = \{e \in G : |e \cap L| = i\}$ and note that $G = G_0 \cup G_1$. We observe that for an edge $e \in G_0$,

$$\mathbb{P}(e \in H_1) = \frac{\binom{r}{b}\alpha^a(1-\alpha)^b}{\binom{n-1}{b-1}} := p_0$$

and for an edge $e \in G_1$,

$$\mathbb{P}(e \in H_1) = \frac{\binom{r-1}{b-1}\alpha^a(1-\alpha)^{b-1}}{\binom{n-1}{b-1}} := p_1.$$

Since $\alpha = m^{-1/r} < 1/r$ and $b \leq r-1$,

$$p_0 = \frac{r}{b}(1-\alpha)p_1 > \frac{r}{r-1}\left(1 - \frac{1}{r}\right)p_1 = p_1.$$

Therefore,

$$\mathbb{E}(|H_1|) \geq p_0|G_0| + p_1|G_1| = (p_0 - p_1)|G_0| + p_1|G| > p_1|G| > \frac{\alpha^a(r-1)!(1-\alpha)^{b-1}}{a!n^{b-1}}|G|, \tag{14}$$

and combining this with (13) yields

$$|G| < \frac{\mathbb{E}(|H_1|)a!n^{b-1}}{\alpha^a(r-1)!(1-\alpha)^{b-1}} = \binom{n}{a}\frac{(t-1)\alpha^a\, a!n^{b-1}}{\alpha^a(r-1)!(1-\alpha)^{b-1}} + O\left(\frac{n^{a+b-1}}{\alpha^a(1-\alpha)^{b-1}m}\right).$$

Using $(1-\alpha)^{-b+1} = 1 - O(m^{-1/r})$ and simplifying, we find that

$$\begin{aligned}|G| &< (t-1)\binom{n}{r-1} + O(\alpha n^{r-1}) + O(n^{r-1}/\alpha^a m)\\ &< (t-1)\binom{n}{r-1} + O(m^{-1/r}n^{r-1}).\end{aligned}$$

Together with (12), this gives the required bound on $|H|$.

In fact, the proof of Theorem 6 yields more than the theorem claims. We have the following fact.

COROLLARY 9. *Let $0 < \gamma < 1/t$, $b < a < r$, $a+b=r$, $t \le s$. Let n be sufficiently large, let $r^r < m \le n^\gamma$, and let $f(m) = m^{-1/r}n^{r-1} + m^2 n^{r-2}$. Let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, and let H be an n-vertex $\mathcal{T}$-free r-graph. If*

$$|H| = (t-1)\binom{n}{r-1} + O(f(m)), \tag{15}$$

then some $F \subset H$ with $|F| = |H| - O(f(m))$ has a crosscut L of size $O(m)$.

Proof. If $|H| = (t-1)\binom{n}{r-1} + O(f(m))$, then the upper and lower bounds for $E(|H_1|)$ given by (13) and (14) differ by $O(n^a/m)$. By (14), they also differ by at least $(p_0 - p_1)|G_0|$, so

$$(p_0 - p_1)|G_0| = O(n^a/m).$$

Using $p_0 > (1+1/r)p_1$, we get $p_1|G_0| = O(n^a/m)$ and this shows that $|G_0| = O(f(m))$. Setting $F = G_1$, L is a crosscut of F and $|F| = |H| - O(f(m))$. ∎

4. Stability. The aim of this section is to prove the following stability theorem. It is important throughout this section that $t \le s$, so that for $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, we have $\sigma(\mathcal{T}) = t$ and therefore $\Psi^1_{t-1}(n,r)$ does not contain $\mathcal{T}$. The following theorem says that if H is a $\mathcal{T}$-free r-graph on n vertices and $|H| \sim |\Psi_{t-1}(n,r)|$, then H is obtained by adding or deleting $o(n^{r-1})$ edges from $\Psi_{t-1}(n,r)$.

THEOREM 10. *Let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, where $b < a < r-1$, $t \le s$. Let H be a $\mathcal{T}$-free n-vertex r-graph with $|H| \sim (t-1)\binom{n}{r-1}$. If $\mathcal{T}$ has a critical leaf, then there exists a set S of $t-1$ vertices of H such that $|H - S| = o(n^{r-1})$.*

4.1. Degrees of sets. By Corollary 9, with $r^r < m = o(n^{1/(t+1)})$ there exists $F \subset H$ such that $|F| \sim |H|$ and F has a crosscut L of size $O(m)$. Our first claim says that most elements of ∂F have degree $t-1$ in F. For a hypergraph G and $S \subset V(G)$, we write $G - S$ to denote the induced subhypergraph $G[S]$.

Claim 1. There are $\binom{n}{r-1} - o(n^{r-1})$ sets $e \in \partial F - L$ such that $d_F(e) = t-1$.

Proof. Suppose ℓ sets $e \in \partial F - L$ have $d_F(e) \ge t$. By the definition of L, $\Gamma(e) \subset 2^L$ for each $e \in \partial F - L$. Let Z be a crosscut of $\mathcal{T}$ with $|Z| = t$ contained in B, and let $\mathcal{T}^* = \{e \backslash Z : e \in \mathcal{T}\}$ (note that here $\Gamma(e)$ is a 1-uniform hypergraph). Then $\mathcal{T}^*$ is an $(a, b-1)$-blowup of T. Proposition 4 implies

$$\mathrm{ex}(n, \mathcal{T}^*) < (s+t)n^{r-2}. \qquad ∎$$

By the pigeonhole principle, there exists a set $S \subset L$ with $|S| = t$ such that at least $k = \ell/|L|^t$ sets $e \in \partial F - L$ have $\Gamma_F(e) \supseteq S$. If $k > \mathrm{ex}(n, \mathcal{T}^*)$, then $\mathcal{T}^* \subset \partial F - L$ and for all $e \in \mathcal{T}^*$, $\Gamma_G(e) \supseteq S$. Now we can lift $\mathcal{T}^*$ to $\mathcal{T} \subset F$ via S. Indeed, we can greedily enlarge each of the $(b-1)$-sets that form $\mathcal{T}^*$ to a b-set by adding an element of S. This contradicts the choice of H. We therefore suppose that

$$\ell/|L|^t = k \leq \mathrm{ex}(n, \mathcal{T}^*) \leq (s+t)n^{r-2},$$

which gives $\ell \leq (s+t)|L|^t n^{r-2} = O(n^{r-2}m^t)$. As $|F| \sim |H| \sim (t-1)\binom{n}{r-1}$, and the number of $(r-1)$-sets in $V(F) - L$ is at most $\binom{n}{r-1}$, the average degree of sets in $\partial F - L$ is at least $t - 1 - o(1)$. We have already argued that at most $O(n^{r-2}m^t)$ of these sets have degree larger than $t-1$. Furthermore, none of them has degree greater than m. Hence, writing x for the number of sets in $\partial F - L$ of degree at most $t-2$, we have

$$(t-1)\binom{n}{r-1} - x + m\,O(n^{r-2}m^t) \geq (t-1)\binom{n}{r-1}(1 - o(1)).$$

Since $m\ n^{r-2}m^t = o(n^{r-1})$, we conclude that $x = o\left(\binom{n}{r-1}\right)$. This yields the claim.

4.2. Proof of Theorem 10. Let $S_1, S_2, \ldots, S_k$ be an enumeration of the $(t-1)$-element subsets of L, and let F_i denote the family of $(r-1)$-element sets e in $V(F)\backslash L$ such that $\Gamma_F(e) = S_i$. By Claim 1, $|F_1 \cup F_2 \cup \cdots \cup F_k| \sim \binom{n-|L|}{r-1}$. Suppose $k \geq 2$. By definition, for $i \neq j$, $F_i \cap F_j = \emptyset$. Therefore,

$$\sum_{i=1}^{k} |F_i| \sim \binom{n}{r-1}.$$

For each $i \in [k]$, if $|F_i| = o(n^{r-1}/k)$, let G_i be an empty $(r-1)$-graph, and if $|F_i| = \Omega(n^{r-1}/k)$, then delete edges of F_i containing a-sets or b-sets of "small" degree until we obtain either an empty $(r-1)$-graph or an $(r-1)$-graph G_i such that

$$d_{G_i}(e) > r(s+t)n^{r-2-a}\ \forall e \in \partial_a G_i \ \text{ and } \ d_{G_i}(f) > r(s+t)n^{r-2-b}\ \forall f \in \partial_b G_i. \tag{16}$$

By construction, $|G_i| \geq |F_i| - 2r(s+t)n^{r-2}$, and since $F_i = \Omega(n^{r-1}/k)$ and $k \leq |L|^t \leq O(m^t) = o(n)$, whenever G_i is nonempty we have

$$|G_i| = (1 - o(1))|F_i|.$$

We conclude that if $G = \bigcup G_i$, then $|G| = (1-o(1))|F| \sim \binom{n}{r-1}$ and

$$\sum_{i=1}^{k} |G_i| \sim \binom{n}{r-1}. \tag{17}$$

Claim 2. For $i \neq j$, $\partial_a G_i \cap \partial_a G_j = \emptyset$.

Proof. Let W be a tree obtained from the tree T by deleting a leaf vertex x with unique neighbor $y \in T$, such that x is in the part of T of size t. Suppose some a-set e is contained in $\partial_a G_i \cap \partial_a G_j$. By (16), we can greedily grow $W(a, b-1)$ in G_j such that e is the blowup of y. By adding one vertex of S_j to each $b-1$-set in $W(a, b-1)$, we obtain $W(a, b)$. Now there exists $x' \in S_i \backslash S_j$. Since $d_{G_i}(e) > r(s+t)n^{r-2-a}$, there exists an edge $f \in G_i$ containing e, such that $f \cap V(W(a, b-1)) = \emptyset$, and therefore

$f\cup\{x'\}\in F$ plus $W(a,b)$ gives the tree $T(a,b)$, with $f\backslash e$ the blowup of x. This proves the claim. ∎

Now we prove Theorem 10. Since $a\le r-2$, by Claim 2, for all $i\ne j$, $\partial_{r-2}G_i\cap\partial_{r-2}G_j=\emptyset$. Without loss of generality, suppose that for some $0\le p\le k$, $|G_1|\ge|G_2|\ge\cdots\ge|G_p|\ge 1$ and $G_i=\emptyset$ for $p+1\le i\le k$. For each $i\in[p]$, let $y_i\ge r-1$ denote the real such that $|G_i|=\binom{y_i}{r-1}$. Then $y_1\ge y_2\ge\cdots\ge y_p$. By the Lovász form of the Kruskal–Katona theorem, for each $i\in[p], |\partial_{r-2}(G_i)|\ge\binom{y_i}{r-2}$. By the disjointness of the $\partial_{r-2}(G_i)$'s, we have

$$\sum_{i=1}^{p}\binom{y_i}{r-2}\le\binom{n}{r-2}.$$

For each $i\in[p]$, since $\binom{y_i}{r-1}=\frac{y_i-r+2}{r-1}\binom{y_i}{r-2}\le\frac{y_1-r+2}{r-1}\binom{y_i}{r-2}$, by (17) we have

$$\sum_{i=1}^{p}|G_i|=\sum_{i=1}^{p}\binom{y_i}{r-1}\le\frac{y_1-r+2}{r-1}\sum_{i=1}^{p}\binom{y_i}{r-2}\le\frac{y_1-r+2}{r-1}\binom{n}{r-2}.$$

From this and the fact that $\sum_{i=1}^{p}|G_i|\ge(1-o(1))\binom{n}{r-1}$, we get $y_1\ge n-o(n)$. Hence, $|F_1|\ge|G_1|=\binom{y_1}{r-1}\ge\binom{n}{r-1}-o(n^{r-1})$. So there exists $S=S_1\subset L$ such that $(t-1)\binom{n}{r-1}-o(n^{r-1})$ edges of F consist of one vertex in S and $r-1$ vertices disjoint from S.

5. Exact results. The aim of this section is to prove the following theorem, which completes the proof of Theorem 7.

THEOREM 11. *Let $t\le s$, $b<a<r-1$ with $a+b=r$ and $\mathcal{T}\in\mathcal{T}_{s,t}(a,b)$ such that $\mathcal{T}$ has a critical leaf and $\tau(\mathcal{T})=t$. If n is large and H is a $\mathcal{T}$-free n-vertex r-graph with $|H|\ge\binom{n}{r}-\binom{n-t+1}{r}$, then $H\cong\Psi_{t-1}(n,r)$.*

To prove this, we aim to show that the $(t-1)$-set S given by Theorem 10 is a vertex cover of H. We prove the following consequence of Claim 1. Recall that Corollary 9 gives $F\subset H$ such that $|F|\sim|H|$.

Claim 3. Let $\Delta_u=(t-1)\binom{n-u}{r-1-u}$. Then, for each $\delta>0$, there exists $G\subset F$ with $|G|\sim|F|$ such that for any u-set $e\subset V(G)$ with $u<r$ and $d_G(e)>0$, either

(i) $|e\cap S|=0$ and $d_G(e)\ge(1-\delta)\Delta_u$ or

(ii) $|e\cap S|=1$ and $d_G(e)\ge r(s+t)n^{r-1-u}$.

Proof. Let K be the set of edges of F containing some $e\in\partial F-S$ with $d_F(e)=t-1$. By Claim 1, $|K|\sim|F|$. Also, every r-set in K has one point in S and $r-1$ points in $V(K)\backslash S$. Since $d_K(e)=t-1$ for all $e\in\partial K-S$, every u-set in $V(K)\backslash S$ has degree at most Δ_u in K.

We repeatedly delete edges from K as follows. Suppose at some stage of the deletion we have a hypergraph K'. If there exists a u-set e for some $u<r$ such that

(i') $|e\cap S|=0$ and $d_{K'}(e)<(1-\delta)\Delta_u$ or

(ii') $|e\cap S|=1$ and $d_{K'}(e)<r(s+t)n^{r-1-u}$,

then delete all edges of K' containing e. Let G be the hypergraph obtained at the end of this process. We shall prove $|G|\sim|K|$. To this end, suppose that $|G|=|K|-\eta(t-1)\binom{n}{r-1}$, and we show that $\eta=o(1)$ to complete the proof. Consider two cases.

Case 1. At least $\frac{\eta}{2}(t-1)\binom{n}{r-1}$ edges of K were deleted due to (ii').

In this case, there exists $u<r$ such that the set H' of edges of K deleted due to (ii') on u-sets satisfies $|H'|\ge\frac{\eta}{2r}(t-1)\binom{n}{r-1}$. Then, by (ii'), and since the number of u-sets with one vertex in S is $|S|\binom{n-|S|}{u-1}$,

$$|H'| \le |S|\binom{n-|S|}{u-1} \cdot r(s+t)n^{r-1-u} < |S|r(s+t)n^{r-2}.$$

Since $|H'| \ge \frac{\eta}{2r}\binom{n}{r-1}$ and $|S| = t-1$, this gives $\eta = o(1)$.

Case 2. At least $\frac{\eta}{2}(t-1)\binom{n}{r-1}$ edges of K were deleted due to (i').

In this case, there exists $u < r$ such that the set H' of edges of K deleted due to (i') on u-sets satisfies $|H'| \ge \frac{\eta}{2r}(t-1)\binom{n}{r-1}$. Let U_1 be the set of u-sets in $V(K)\backslash S$ on which edges of K were deleted due to (i'), and let U_2 be the remaining u-sets in $V(K)\backslash S$. Then

$$|U_1| > \frac{|H'|}{(1-\delta)\triangle_u} \ge \frac{\eta(t-1)\binom{n}{r-1}}{2r(t-1)\binom{n}{r-1-u}}.$$

If n is large enough, then this is at least $\frac{\eta}{4r\binom{r-1}{u}}\binom{n}{u}$. Let $\gamma = \frac{\eta}{4r\binom{r-1}{u}}$. Then

$$\begin{aligned}
|K|\binom{r-1}{u} &= \sum_{e\in\binom{V(K)\backslash S}{u}} d_K(e) \\
&= \sum_{e\in U_1} d_K(e) + \sum_{e\in U_2} d_K(e) \\
&\le |U_1|(1-\delta)\Delta_u + |U_2|\Delta_u \\
&\le \gamma(1-\delta)\binom{n}{u}\Delta_u + (1-\gamma)\binom{n}{u}\Delta_u \;=\; (1-\gamma\delta)\binom{n}{u}\Delta_u.
\end{aligned}$$

Here we used $|U_1| + |U_2| \le \binom{n}{u}$. Therefore,

$$|K| \le (1-\gamma\delta)\frac{\binom{n}{u}\Delta_u}{\binom{r-1}{u}} = (1-\gamma\delta)(t-1)\binom{n}{r-1}.$$

Since $|K| \sim |F| \sim (t-1)\binom{n}{r-1}$, $\gamma\delta = o(1)$. Since $\delta > 0$ and $\gamma = \frac{\eta}{4r\binom{r-1}{u}}$, this implies $\eta = o(1)$, as required. ∎

Let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ have a critical leaf with $\tau(\mathcal{T}) = t \le s$, $a+b=r$, $b<a<r-1$, and let H be a $\mathcal{T}$-free n-vertex r-graph with $|H| \ge \binom{n}{r} - \binom{n-t+1}{r}$. We aim to show that S is a vertex cover of H, which gives $H \cong \Psi_{t-1}(n,r)$, as required. To this end, let $H_i = \{e \in H : |e \cap S| = i\}$. So we have to show that $H_0 = \emptyset$.

Since $\mathcal{T}$ has a critical leaf, there is a b-set e' of $\mathcal{T}$ in the part of size t with $d_{\mathcal{T}}(e') = 1$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by deleting the edge containing e'. So $V(\mathcal{T}')$ has one part comprising $t-1$ sets, each of size b and the other part comprising s sets, each of size a. It has a crosscut of size $t-1$ by picking one vertex from each of the b-sets above.

Let $\mathcal{K}^1$ be the set of r-sets of $[n]$ that have exactly one vertex in S. A subfamily $T \subset \mathcal{K}^1$ is a *potential tree* if

1. $T \cong \mathcal{T}'$,
2. the $t-1$ vertices of S play the role of the crosscut vertices of $\mathcal{T}'$ described above,
3. e_0 is an a-set in $V(T)$ with $e_0 \in \partial_a H_0$,
4. there exists $e \in H_0$ such that $e_0 \subset e$, and
5. $T \cup e$ is a copy of $\mathcal{T}$.

Fix an a-set $e_0 \in \partial_a H_0$, and suppose $e_0 \subset e \in H_0$. If $T \subset H_1$ is a potential tree as described above, then $T \cup \{e\}$ is a copy of $\mathcal{T}$ in H, a contradiction. So, for each such potential tree T, there exists $f \in T - H_1$. Let us call this a *missing edge.* Let $m = as + bt - b$ be the number of vertices of each potential tree. The number of potential trees containing a fixed missing edge f is at most

$$\binom{n-|S|-(a+b-1)}{m-|S|-(a+b-1)} \cdot c(\mathcal{T}),$$

where $c(\mathcal{T})$ is the number of ways we can put a potential tree using f into the set M with $|M| = m$ and $S \cup f \subset M \subset [n]$ (note that $|f \cap S| = 1$).

On the other hand, each $e_0 \in \partial_a H_0$ and a subset M' with $|M'| = m$ and $S \subset M' \subset ([n] - e_0)$ carries at least one potential tree so that the total number of potential trees is at least

$$|\partial_a H_0| \binom{n-|S|-a}{m-|S|-a}.$$

It follows that the number of missing edges is at least $c|\partial_a H_0|n^{b-1}$ for some $c > 0$. Therefore,

$$|H| = |H_0| + |H_1| + |H_2| + \cdots + |H_r| \le \binom{n}{r} - \binom{n-t+1}{r} + |H_0| - c|\partial_a H_0|n^{b-1}.$$

By Proposition 4 and the fact that $\mathcal{T}$ is contained in a tight tree on $V(\mathcal{T})$, $|H_0| < c'|\partial H_0|$ for some constant c'.

Next, we observe that $\partial H_0 \cap \partial G = \emptyset$, for otherwise we will use Claim 3 to greedily build a copy of $\mathcal{T}$ using the edge of H_0 and whose remaining edges form a copy of $\mathcal{T}'$ and come from G. Indeed, at each step in this greedy process, we either have a b-set e' disjoint from S and we would like to find an r-set in G containing e' with one vertex in S (and disjoint from the current subtree), or we have an a-set e'' with one vertex in S and we would like to find an r-set in G containing e'' disjoint from S and from the current subtree. In the first case, we apply Claim 3(i) with $u = b$. Here $|S| = t - 1$ ensures that we can find the required r-set in G. In the second case, we apply Claim 3(ii) with $u = a$. The claim states that the number of r-sets in G containing e'' is at least $r(s+t)n^{r-1-a}$, and hence one of them can be used to enlarge the current subtree.

Since $|\partial G| \sim \binom{n}{r-1}$, we obtain $|\partial H_0| = o(n^{r-1})$. Writing $|\partial H_0| = \binom{x}{r-1}$ for some real x, we have $|\partial_a H_0| \ge \binom{x}{a}$ by the Kruskal–Katona theorem. Therefore,

$$|H_0| - c|\partial_a H_0|n^{b-1} \le c'|\partial H_0| - c|\partial_a H_0|n^{b-1} \le c'\binom{x}{r-1} - cn^{b-1}\binom{x}{a}.$$

Since $x = o(n)$, for large enough n the above expression is negative, unless $|\partial H_0| = |\partial_a H_0| = 0$. We have shown that if $|H| \ge \binom{n}{r} - \binom{n-t+1}{r}$, then $H_0 = \emptyset$ and $|H| = \binom{n}{r} - \binom{n-t+1}{r}$, as required.

6. (1,2)-paths of length 4.

6.1. Result and the setup of the proof. The goal of this section is to find asymptotics for the smallest case not covered by our results above, namely for $\mathrm{ex}_3(n, P_4(1,2))$. We will show that

$$\mathrm{ex}_3(n, P_4(1,2)) = \binom{n-1}{2} + O(n). \tag{18}$$

We cannot replace the term $O(n)$ in (18) with $o(n)$: Consider the 3-graph H with $V(H) = [n]$ and $E(H) = E_1 \cup E_2$, where $E_1 = \{\{1, i, j\} : 2 \le i < j \le n\}$ and $E_2 = \{\{2, 2i+1, 2i+2\} : 1 \le i \le n/2 - 1\}$. This 3-graph has $\binom{n-1}{2} + \lfloor (n-2)/2 \rfloor$ edges and does not contain $P_4(1,2)$.

The technique in this section is different from that used above. Instead of (18), we shall prove the following slightly stronger version.

THEOREM 12. *For every $P_4(1,2)$-free n-vertex 3-graph H,*

$$|H| - |\partial H| = O(n). \tag{19}$$

Again, we cannot replace $O(n)$ in (19) with $o(n)$: If n is divisible by 6 and H is the disjoint union of $n/6$ copies of K_6^3, then H contains no $P_4(1,2)$, $|H| = (20/6)n$, and $|\partial H| = (15/6)n$.

Our proof has the following three steps.

Step 1. *There is a C_1 such that for every n, every $P_4(1,2)$-free n-vertex 3-graph H can be made K_4^3-free after deleting at most $C_1 n$ edges.*

Step 2. *There is a C_2 such that for every n, from any $P_4(1,2)$-free n-vertex 3-graph H without K_4^3-subgraphs one can delete at most $C_2 n$ edges so that the remaining 3-graph H' is $(K_4^3)^-$-free or satisfies $|H'| \le |\partial H'|$.*

Step 3. *If a $P_4(1,2)$-free n-vertex 3-graph H has no $(K_4^3)^-$-subgraphs, then $|H| \le |\partial H|$.*

The three steps together imply Theorem 12. The main tool for Steps 1 and 2 is the Δ-system method introduced by Deza, Erdős, and Frankl [3]. In the next subsection, we introduce the notions needed to apply the Δ-system method and state an important lemma by Füredi [13] on the topic, and in the subsequent three subsections we prove the three steps.

6.2. Definitions for the Δ-system method and a lemma. A family of sets $\{F_1, \dots, F_s\}$ is an *s-star* or a *Δ-system of size s with kernel A* if $F_i \cap F_j = A$ for all $1 \le i < j \le s$.

For a member F of a family $\mathcal{F}$, let the *intersection structure of F relative to $\mathcal{F}$* be

$$\mathcal{I}(F, \mathcal{F}) = \{F \cap F' : F' \in \mathcal{F} \setminus \{F\}\}.$$

An r-uniform family $\mathcal{F} \subseteq \binom{[n]}{r}$ is *r-partite* if there exists a partition $(X_1, \dots, X_r)$ of the vertex set $[n]$ such that $|F \cap X_i| = 1$ for each $F \in \mathcal{F}$ and each $i \in [r]$.

For a partition $(X_1, \dots, X_r)$ of $[n]$ and a set $S \subseteq [n]$, the *pattern* $\Pi(S)$ is the set $\{i \in [r] : S \cap X_i \ne \emptyset\}$. Naturally, for a family $\mathcal{L}$ of subsets of $[n]$,

$$\Pi(\mathcal{L}) = \{\Pi(S) : S \in \mathcal{L}\} \subseteq 2^{[r]}.$$

LEMMA 13 (the intersection semilattice lemma (Füredi [13])). *For any positive integers s and r, there exists a positive constant $c(r,s)$ such that every family $\mathcal{F} \subseteq \binom{[n]}{r}$ contains a subfamily $\mathcal{F}^* \subseteq \mathcal{F}$ satisfying the following:*

1. *$|\mathcal{F}^*| \ge c(r,s)|\mathcal{F}|$.*
2. *$\mathcal{F}^*$ is r-partite, together with an r-partition $(X_1, \dots, X_r)$.*
3. *There exists a family $\mathcal{J}$ of proper subsets of $[r]$ such that $\Pi(\mathcal{I}(F, \mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$.*
4. *$\mathcal{F}^*$ is closed under intersection; i.e., for all $A, B \in \mathcal{J}$, we have $A \cap B \in \mathcal{J}$ as well.*

5. *For any $F \in \mathcal{F}^*$ and each $A \in \mathcal{I}(F, \mathcal{F}^*)$, there is an s-star in $\mathcal{F}^*$ containing F with kernel A.*

Remark 1. The proof of Lemma 13 in [13] yields that if $\mathcal{F}$ itself is r-partite with an r-partition $(X_1, \dots, X_r)$; then the r-partition in the statement can be taken the same.

Remark 2. By definition, if for some $k \in [r]$ none of the members of the family $\mathcal{J}$ of proper subsets of $[r]$ in Lemma 13 contains k, then the degree in $\mathcal{F}^*$ of each vertex in X_k is at most 1. Since $\mathcal{F}^*$ is r-partite, this yields $|\mathcal{F}^*| \le |X_k| \le n - r + 1$. Thus, if $|\mathcal{F}^*| \ge n$, then $\bigcup_{J \in \mathcal{J}} J = [r]$.

6.3. Proof of Step 1. Choose $C_1 = \frac{4}{c(4,6)}$, where $c(4,6)$ is from Lemma 13. Let H be a $P_4(1,2)$-free n-vertex 3-graph. Construct a 4-uniform family $\mathcal{E}$ of subsets of $[n]$ as follows. First, let $\mathcal{E}_0 = \emptyset$, $H_0 = H$. Then, for $j = 1, \dots,$ do the following:

(i) If H_{j-1} has no K_4^3-subgraphs, then let $\mathcal{E} = \mathcal{E}_{j-1}$ and $H' = H_{j-1}$.

(ii) Otherwise, choose some 4-set $e = i_1 i_2 i_3 i_4 \subset [n]$ with $H_{j-1}[e] = K_4^3$, let $\mathcal{E}_j = \mathcal{E}_{j-1} \cup \{e\}$, and let $H_j = H_{j-1} \setminus \{i_1 i_2 i_3, i_1 i_2 i_4, i_1 i_3 i_4, i_2 i_3 i_4\}$.

By construction, $|H'| = |H| - 4|\mathcal{E}|$. So, if $|\mathcal{E}| \le \frac{C_1}{4} n$, then Step 1 is done. Suppose $|\mathcal{E}| > C_1 \frac{n}{4} = \frac{n}{c(4,6)}$. By Lemma 13, for $r = 4$ and $s = 6$, there are a partition $(X_1, \dots, X_4)$ of $[n]$ and a family $\mathcal{E}^* \subseteq \mathcal{E}$ satisfying properties 1–5 in the lemma. In particular, $|\mathcal{E}^*| > c(4,6) \frac{n}{c(4,6)} = n$. By Remark 2, the union of the members of $\mathcal{J}$ is the whole [4].

On the other hand, by the definition of $\mathcal{E}$, no two members of it may share three vertices. It follows that $|J| \le 2$ for all $J \in \mathcal{J}$. Furthermore, if $|e_1 \cap e_2| = 1$ for some $e_1, e_2 \in \mathcal{E}$, say $e_1 = \{1,2,3,4\}$ and $e_2 = \{4,5,6,7\}$, then we have a $P_4(1,2)$ with edges $123, 234, 456, 567$, a contradiction. It follows that $|J| \ne 1$ for all $J \in \mathcal{J}$. By part 4 of Lemma 13, this means that up to symmetry, the only possibility for $\mathcal{J}$ is that

$$\mathcal{J} = \{\emptyset, \{1,2\}, \{3,4\}\}. \tag{20}$$

So, let $e_1 = x_1 x_2 x_3 x_4 \in \mathcal{E}^*$, where $x_i \in X_i$ for $i = 1,2,3,4$. By part 5 of Lemma 13 and by (20), there is $e_2 \in \mathcal{E}^*$ such that $e_1 \cap e_2 = \{x_1, x_2\}$, say $e_2 = x_1 x_2 x_3' x_4'$, where $x_3' \in X_3$ and $x_4' \in X_4$. For the same reasons, there is $e_3 \in \mathcal{E}^*$ such that $e_1 \cap e_3 = \{x_3, x_4\}$, say $e_3 = x_1' x_2' x_3 x_4$, where $x_1' \in X_1$ and $x_2' \in X_2$. But then H contains a $P_4(1,2)$ with edges $x_2' x_1' x_3, x_1' x_3 x_4, x_4 x_1 x_2, x_1 x_2 x_3'$, a contradiction. This proves Step 1.

6.4. Proof of Step 2. For Steps 2 and 3, we need a couple of new definitions. Call a 3-graph *normal* if it has no pairs of vertices of codegree exactly 1. In a normal 3-graph H, for every edge $xyz \in H$, there is a vertex $h(xy;z) \ne z$ such that $\{x, y, h(xy;z)\} \in H$. Such a vertex $h(xy;z)$ does not need to be unique: there are $d(x,y) - 1$ such vertices.

We will show Step 2 in the following form.

LEMMA 14. *Let $C_2 = \frac{200}{c(4,6)}$, where $c(4,6)$ is from Lemma 13. If H is a $P_4(1,2)$-free and K_4^3-free n-vertex 3-graph, then one can delete at most $C_2 n$ edges so that the remaining 3-graph H' is $(K_4^3)^-$-free or satisfies $|H'| \le |\partial H'|$.*

Proof. Suppose that the lemma does not hold, and H is a counterexample with the fewest edges. If our H is not normal, then deleting an edge containing a pair of codegree exactly 1 would create a smaller 3-graph H' with $|H'| - |\partial H'| \ge |H| - |\partial H| \ge 1$ that is again $P_4(1,2)$-free and $(K_4^3)^-$-free, contradicting the minimality of H. Thus, H is normal. ∎

Construct a 4-uniform family $\mathcal{E}$ of subsets of $[n]$ with a special vertex in each member as follows. First, let $\mathcal{E}_0 = \emptyset$, $H_0 = H$. Then for $j = 1, \ldots,$ do the following:

(i) If H_{j-1} has no $(K_4^3)^-$-subgraphs, then let $\mathcal{E} = \mathcal{E}_{j-1}$ and $H' = H_{j-1}$.

(ii) Otherwise, choose some 4-set $e = \{i_1, i_2, i_3, i_4\} \subset [n]$ with $H_{j-1}[e] = (K_4^3)^-$, say $i_2 i_3 i_4 \notin E(H_{j-1})$. Then let i_1 be the special vertex in e, let $\mathcal{E}_j = \mathcal{E}_{j-1} \cup \{e\}$, and let $H_j = H_{j-1} \setminus \{i_1 i_2 i_3, i_1 i_2 i_4, i_1 i_3 i_4\}$.

By construction, $|H'| = |H| - 3|\mathcal{E}|$. So, if $|\mathcal{E}| \le \frac{C_2}{3} n$, then the lemma is proved. Suppose $|\mathcal{E}| > C_2 n/3$. By a classic observation of Erdős and Kleitman, there is a 4-partite subfamily $\mathcal{E}'$ of $\mathcal{E}$ with $|\mathcal{E}'| \ge \frac{4!}{4^4} |\mathcal{E}| > \frac{C_2}{32} n$. Let (X_1, X_2, X_3, X_4) be the corresponding 4-partition of $[n]$. By symmetry, we may assume that at least $\frac{1}{4}|\mathcal{E}'|$ members of $\mathcal{E}'$ have the special vertex in X_1. Let $\mathcal{F}$ be the family of such members. In particular, $|\mathcal{F}| \ge \frac{1}{4}|\mathcal{E}'| > \frac{C_2}{2^7} n$.

By Lemma 13, for $r = 4$ and $s = 6$ and Remark 1 after it, there is a family $\mathcal{F}^* \subseteq \mathcal{F}$ satisfying properties 1–5 of the lemma (with the same partition (X_1, X_2, X_3, X_4)). In particular,

$$|\mathcal{E}^*| \ge c(4,6) \frac{C_2}{2^7} n > n.$$

By Remark 2, $\bigcup_{J \in \mathcal{J}} J = [4]$. Let us first show the following:

$$\text{If } \mathrm{J} \in \mathcal{J} \text{ is a singleton, then } \mathrm{J} = \{1\}. \tag{21}$$

Indeed, if, say, $J \cap J_1 = \{4\}$, then $\mathcal{F}^*$ contains sets $f_1 = \{x_1, x_2, x_3, x_4\}$ and $f_2 = \{x_1', x_2', x_3', x_4\}$. So, by the definition of $\mathcal{F}$, H has a path $P_4(1,2)$ with edge set $\{x_3 x_2 x_1, x_2 x_1 x_4, x_4 x_2' x_1', x_2' x_1' x_3'\}$. This proves (21).

Case 1. A member J of $\mathcal{J}$ is a triple. Since the intersection of any two members of $\mathcal{E}$ cannot be an edge of H, $J = \{2,3,4\}$, and $\mathcal{J}$ contains no other triples. Let J_1 be a member of $\mathcal{J}$ containing 1. Then $|J_1| \le 2$. By part 4 of Lemma 13, $J \cap J_1 \in \mathcal{J}$ and $|J \cap J_1| < |J_1|$. Then, by (21), the set $J \cap J_1$ is not a singleton and hence is $\emptyset$. If follows that the unique member of $\mathcal{J}$ containing 1 is $\{1\}$.

Let $y_1 \in X_1$. By part 5 of Lemma 13, $\mathcal{F}^*$ contains sets A_1, A_2 such that for $i = 1, 2$, $A_i = \{y_1, y_{i,2}, y_{i,3}, y_{i,4}\}$, forming a 2-star with kernel $\{y_1\}$. Since $J = \{2,3,4\} \in \mathcal{J}$ by the same part 5 for $i = 1, 2$ and $i' = 1, 2, 3$, $\mathcal{F}^*$ contains sets $B_{i,i'}$ such that $B_{i,i'} = \{z_{i',1}, y_{i,2}, y_{i,3}, y_{i,4}\}$, forming 3-stars with kernels $\{y_{1,2}, y_{1,3}, y_{1,4}\}$ and $\{y_{2,2}, y_{2,3}, y_{2,4}\}$. Since $1 \le i' \le 3$, we choose $z_{1,1} \ne y_1$ and then $z_{2,1} \notin \{y_1, z_{1,1}\}$. Then, by the definition of $\mathcal{F}$, H has a $P_4(1,2)$ with edge set $\{z_{1,1} y_{1,2} y_{1,3}, y_{1,2} y_{1,3} y_1, y_1 y_{2,2} y_{2,3}, y_{2,2} y_{2,3} z_{2,1}\}$, a contradiction.

Case 2. $|J| \le 2$ for each $J \in \mathcal{J}$, and there are nonempty $J_1, J_2 \in \mathcal{J}$ with $J_1 \cap J_2 = \emptyset$. If $|J_1| = |J_2| = 2$, then we may assume $J_1 = \{1,2\}$ and $J_2 = \{3,4\}$. In this case, we simply repeat the last paragraph of the proof of Step 1. Otherwise, by (21) we may assume $J_1 = \{1\}$ and $J_2 = \{3,4\}$. Then we take $y_1 \in X_1$ and sets A_1, A_2 as in Case 1. Since $J_2 = \{3,4\} \in \mathcal{J}$, for $i \in [2]$ and $i' \in [3]$, $\mathcal{F}^*$ contains sets $B_{i,i'}$ such that $B_{i,i'} = \{z_{i',1}, z_{i',2}, y_{i,3}, y_{i,4}\}$, forming 3-stars with kernels $\{y_{1,3}, y_{1,4}\}$ and $\{y_{2,3}, y_{2,4}\}$. Since $1 \le i' \le 3$, we choose $z_{1,1} \ne y_1$ and then $z_{2,1} \notin \{y_1, z_{1,1}\}$. Then, by the definition of $\mathcal{F}$, H has a $P_4(1,2)$ with edge set $\{z_{1,1} y_{1,3} y_{1,4}, y_{1,3} y_{1,4} y_1, y_1 y_{2,3} y_{2,4}, y_{2,3} y_{2,4} z_{2,1}\}$, a contradiction.

Case 3. $|J| \le 2$ for each $J \in \mathcal{J}$ and for all nonempty $J_1, J_2 \in \mathcal{J}$, $J_1 \cap J_2 \ne \emptyset$. Since the sets in $\mathcal{J}$ cover [5], by (21),

$$\{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}\} \subseteq \mathcal{J} \subseteq \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,4\}\}. \tag{22}$$

For each $v \in V(H)$, the *link graph* $H(v)$ is the simple graph G with $V(G) = \bigcup_{e\in H: v\in e} e \setminus \{v\}$ and $E(G) = \{e \setminus \{v\} : v \in e \in H\}$.

Observe that because H is normal, $\delta(H(v)) \geq 2$ for every vertex $v \in V(H)$ that lies in at least one edge of H. Indeed, if $x \in H(v)$, then there is an edge $vxy \in H$ and $xy \in H(v)$. By the normality of H, there is another edge $vxz \in H$ which implies that $xz \in H(v)$. This shows that $\deg_{H(v)}(x) \geq 2$.

Let $x_1 \in X_1$. Since $\{1\} \in \mathcal{J}$, $\mathcal{F}^*$ contains sets $A_1, \ldots, A_6$ such that $A_i \cap A_{i'} = \{1\}$ for all $1 \leq i < i' \leq 6$. This means that $H(x_1)$ has six vertex-disjoint triangles, say with vertex sets $A'_i = \{a_{i,1}, a_{i,2}, a_{i,3}\}$ for $i = 1, \ldots, 6$. Also, (22) implies that for every vertex $y \in N_{\mathcal{F}^*}(x_1) = \{x : \deg_{\mathcal{F}^*}(x, x_1) > 0\}$, we have $d_{H(x_1)}(y) \geq 12$. Indeed, (22) implies that we have at least six other edges in $\mathcal{F}^*$ containing both x_1 and y with kernel $\{x_1, y\}$, and each of these edges contains two edges of $H(x_1)$ that contain y. Thus, we have $\delta(H(v)) \geq 6$.

Let $w = h(a_{6,1}a_{6,2}; x_1)$. Since all A'_i are disjoint, we may assume that $w \notin \bigcup_{i=1}^4 A'_i$. If for some $1 \leq i \leq 4$ and $1 \leq i' < i'' \leq 3$, $h(a_{i,i'}a_{i,i''}; x_1) \notin \{a_{6,1}, a_{6,2}, x_1\}$, then H has a path $\{wa_{6,1}a_{6,2}, a_{6,1}a_{6,2}x_1, x_1a_{i,i'}a_{i,i''}, a_{i,i'}a_{i,i''}h(a_{i,i'}a_{i,i''}; x_1)\}$, a contradiction. Since $\{w, a_{6,1}, a_{6,2}\} \cap \bigcup_{i=1}^4 A'_i = \emptyset$, for similar reasons, for $1 \leq i_1 < i_2 \leq 4$ and any $1 \leq i'_1 < i''_1 \leq 3$ and $1 \leq i'_2 < i''_2 \leq 3$,

$$h(a_{i_1,i'_1}a_{i_1,i''_1}; x_1) = h(a_{i_2,i'_2}a_{i_2,i''_2}; x_1).$$

But then there is $w' \in \{w, a_{6,1}, a_{6,2}\}$ such that for each $yz \in H(x_1)$ with $w' \notin \{y, z\}$, $h(yz; x_1) = w'$.

Recall that $\delta(H(x_1)) \geq 6$, so $H(x_1) - w'$ has a cycle $y_1, \ldots, y_s, y_1$ for some $s \geq 6$. Then H has a $P_4(1,2)$ with edge set $\{y_1y_2x_1, y_2x_1y_3, y_3y_4w', y_4w'y_5\}$, a contradiction.

6.5. Proof of Step 3. Suppose there exists a $P_4(1,2)$-free and $(K_4^3)^-$-free n-vertex 3-graph H with $|H| > |\partial H|$. Then $|H| \geq 1$, so $|\partial H| \geq 3$, and hence $|H| \geq 4$.

If our H is not normal, then deleting an edge containing a pair of codegree exactly 1 would create a smaller 3-graph H' with $|H'| - |\partial H'| \geq |H| - |\partial H| \geq 1$ that is again $P_4(1,2)$-free and $(K_4^3)^-$-free, contradicting the minimality of H. Thus, H is normal. So, as in Step 2, for every edge $xyz \in H$, there is a vertex $h(xy; z) \neq z$ such that $\{x, y, h(xy; z)\} \in H$.

Since H is $(K_4^3)^-$-free,

$$\text{for each } v \in V(H), H(v) \text{ is triangle-free.} \tag{23}$$

We now prove another property:

$$\text{for each } v \in V(H), H(v) \text{ is } C_4\text{-free.} \tag{24}$$

Indeed, suppose H contains edges $vu_1u_2, vu_2u_3, vu_3u_4, vu_4u_1$. Let $h(u_iu_{i+1}; v) = x_i$ (indices count modulo 4). If $x_i = u_{i+2}$, then $H[\{v, u_i, u_{i+1}, u_{i+2}\}] \supseteq (K_4^3)^-$, a contradiction. Similarly, $x_i \neq u_{i-1}$. Thus, if $x_3 \neq x_1$, then H contains a $P_4(1,2)$ with edges $x_1u_1u_2, u_1u_2v$, $vu_3u_4, u_3u_4x_3$, a contradiction.

Therefore, $x_3 = x_1$ and $d(u_1, u_2) = d(u_3, u_4) = 2$. Similarly, $x_4 = x_2$.

Suppose first that $x_2 \neq x_1$. Let $w = h(x_1u_2; u_1)$. Since H is $(K_4^3)^-$-free, $w \neq v$. Since $x_2 \neq x_1$, $w \neq u_3$. Thus, if $w \neq u_4$, then H contains a $P_4(1,2)$ with edges $wx_1u_2, x_1u_2u_1, u_1vu_4, vu_4u_3$, a contradiction. It follows that $w = u_4$ and $d(u_2, x_1) = d(u_4, x_1) = 2$. Similarly, $h(x_1u_3; u_4) = u_1$ and $d(u_3, x_1) = d(u_1, x_1) = 2$.

But then the pairs $u_1x_1, u_2x_1, u_3x_1, u_4x_1$ are not in the shadow of $H' = H \setminus \{u_1u_2x_1, u_2u_4x_1, u_3u_4x_1, u_1u_3x_1\}$, and so $|H'| - |\partial H'| = |H| - |\partial H|$, contradicting the minimality of H.

Suppose now that $x_2 = x_1$. If the co-degree of each pair x_1u_i $(1 \le i \le 4)$ is 2, then similarly to above, the 3-graph $H'' = H \setminus \{u_1u_2x_1, u_2u_3x_1, u_3u_4x_1, u_1u_4x_1\}$ has the property $|H''| - |\partial H''| = |H| - |\partial H|$, contradicting the minimality of H. So by symmetry we may assume that there is some $w \notin \{v, u_1, u_2, u_3, u_4, x_1\}$ such that $wu_1x_1 \in H$. Then H contains a $P_4(1,2)$ with edges $wx_1u_1, x_1u_1u_2, u_2vu_3, vu_3u_4$. This contradiction proves (24).

Fix $v \in V(H)$. Since H is normal, $\delta(H(v)) \ge 2$, so $H(v)$ has cycles. Let $C = u_1, u_u, \ldots, u_s, u_1$ be a shortest cycle in $H(v)$. By (23) and (24), $s \ge 5$. We now show that

(25) $$\text{for each } 1 \le i \le s, h(u_iu_{i+1}; v) \in \{u_1, \ldots, u_s\}.$$

Indeed, suppose $w = h(u_1u_2; v) \notin \{u_1, \ldots, u_s\}$. Let $w' = h(u_1w; u_2)$. Since H is $(K_4^3)^-$-free, $w' \ne v$. If $w' \notin \{u_3, u_4\}$, then H contains a $P_4(1,2)$ with edges $w'wu_1, wu_1u_2, u_2vu_3, vu_3u_4$, a contradiction. Otherwise, suppose $w' = u_q$, where $q \in \{3,4\}$. Then H has a $P_4(1,2)$ with edges $u_2u_1w, u_1ww', w'vu_{q+1}, vu_{q+1}u_{q+2}$, unless $q+2 > s$, which yields $s = 5$ and $q = 4$. In this case, $u_4 = h(wu_1; u_2)$. Then, by symmetry, also $u_4 = h(wu_2; u_1)$. Hence, $|H[\{w, u_1, u_2, u_4\}]| \ge 3$, a contradiction. This proves (25).

Our next claim is that

(26) $$\text{for each } v \in V(H) \text{ with } d(v) > 0, H(v) \text{ is a cycle.}$$

Indeed, suppose $d(v) > 0$. Let $C = u_1, u_u, \ldots, u_s, u_1$ be a shortest cycle in $H(v)$. Suppose there is $w \in V(H(v)) - V(C)$. Since C is a shortest cycle in $H(v)$ and $s \ge 5$, w has at most one neighbor in C. Then, since $\delta(H(v)) \ge 2$, w has a neighbor $w' \notin V(C)$. Let $x = h(ww'; v)$. We may rename the vertices of C so that if $x \in V(C)$, then $x = u_1$. By (25), the vertex $y = h(u_2u_3; v)$ is in $V(C)$, and since H is $(K_4^3)^-$-free, $y \ne u_1$. Then the edges $xww', ww'v, vu_2u_3, u_2u_3y$ form a $P_4(1,2)$ in H, a contradiction. This proves (26).

Since $\sum_{v \in V(H)} |V(H(v))| = 2|\partial H|$ and $\sum_{v \in V(H)} |H(v)| = 3|H|$, inequality $|H| > |\partial H|$ yields that for some $v \in V(H)$, $|H(v)| > \frac{3}{2}|V(H(v))|$, which contradicts (26). This finishes Step 3 and hence the proof of Theorem 12.

7. Concluding remarks. In this paper, we determined for $b \le a < r$ the asymptotic behavior of $\mathrm{ex}_r(n, \mathcal{T})$ when $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ is an (a,b)-blowup of a tree T with parts of sizes s and t where $s \ge t$ and $\sigma(\mathcal{T}) = t$. The extremal problem appears to be more difficult when $s < t$, in which case the smallest crosscut of $\mathcal{T}$ has size s. We pose Conjecture 15, which covers all cases except $a = r - 1$.

CONJECTURE 15. *If $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, where $b \le a < r-1$, $\sigma = \sigma(\mathcal{T}) = \min\{s,t\}$, and n is sufficiently large, then each $\mathcal{T}$-free n-vertex r-graph H satisfies $|H| \le (\sigma - 1)\binom{n}{r-1} + o(n^{r-1})$ with equality only if H is isomorphic to a hypergraph obtained from $\Psi_{\sigma-1}(n,r)$ by adding or deleting $o(n^{r-1})$ edges.*

The case $a = r-1$. If $t > s$ (and $n \ge |V(\mathcal{T})|$), then $\Psi^1_{t-1}(n,r)$ contains $\mathcal{T}$, so Conjecture 15 does not hold. Since $\Psi^1_{s-1}(n,r)$ does not contain $\mathcal{T}$, it is natural to ask whether $\Psi^1_{s-1}(n,r)$ is (asymptotically) extremal for $\mathcal{T}$. In some cases, when $a = r-1$, this is certainly not so because certain Steiner systems do not contain a blowup of a star $K_{1,t}$ and are denser than $\Psi_{s-1}(n,r)$. More precisely, let T be a tree on $s+t$ vertices and let $\mathcal{T}=T(a,b)$ be its (a,b)-blowup. Suppose $a=r-1$, and let $\lambda=\max_{x\in U}\deg_T(x)$. Then $\mathrm{ex}(n,\mathcal{T})$ is at least the number of edges in a Steiner $(n,r,r-1,\lambda-1)$-system—an r-graph on n vertices where each $(r-1)$-set is contained in exactly $\lambda-1$ edges. In this case, $\mathrm{ex}(n,T(r-1,1))\geq\frac{\lambda-1}{r}\binom{n}{r-1}$ for infinitely many n (due to the existence of those designs [19]), whereas $\sigma(T)=s$ and it could be much less than $\frac{\lambda-1}{r}$.

No stability for $a=r-1$. It is important in the above proof that $a\neq r-1$. If $a=r-1$, then there is no stability theorem: consider, for instance, an $(r-1,1)$-blowup $\mathcal{T}$ of a path with four edges. Let H be the n-vertex r-graph constructed as follows. Let $V(H)=[n]$, let $G_1\sqcup G_2$ be a partition of the edge set of the complete $(r-1)$-graph on $\{3,4,\dots,n\}$, and let H consist of the edges $e\cup\{i\}$ such that $e\in G_i$ for $i\in\{1,2\}$. Then $|H|=\binom{n-2}{r-1}$ and H does not contain $\mathcal{T}$.

The case $a=b=r/2$. Let T be a tree on $s+t$ vertices; then for $\mathcal{T}=T(r/2,r/2)$ one can use an argument of Frankl [10] (applied by many others; see [24]) to prove that

$$\mathrm{ex}_r(n,\mathcal{T})\leq\frac{\mathrm{ex}(\lfloor 2n/r\rfloor,T)}{\binom{\lfloor 2n/r\rfloor}{2}}\binom{n}{r}\sim\frac{\mathrm{ex}(\lfloor 2n/r\rfloor,T)}{\lfloor 2n/r\rfloor}\binom{n}{r-1}. \tag{27}$$

Indeed, similarly to the idea of templates, given a $\mathcal{T}$-free r-graph H on n vertices, take a random partition of $[n]$ into $r/2$-sets (where for simplicity $r/2$ divides n) and consider only those r-edges of H which are unions of two partite sets. Then this subfamily consists of at most $\mathrm{ex}(2n/r,T)$ edges of H, out of the possible $\binom{2n/r}{2}$.

The bound is asymptotically tight, due to $\Psi^1_{t-1}(n,r)$, if $\sigma(\mathcal{T})=t$ and T has $2t-1$ edges. So the inequality (27) completes the proof of Theorem 1, showing that $\mathrm{ex}_r\left(n,P_{2k-1}\left(\frac{r}{2},\frac{r}{2}\right)\right)\sim(k-1)\binom{n}{r-1}$ (the other cases follow from Theorems 6 and 7). It also gives a better upper bound for the even length, $\mathrm{ex}_r\left(n,P_{2k}\left(\frac{r}{2},\frac{r}{2}\right)\right)\leq(1+o(1))\left(k-\frac{1}{2}\right)\binom{n}{r-1}$.

However, the proof of (27) does not reveal the extremal structure.

The case of forests. Many of our ideas can be generalized for the case of $\mathcal{T}=F(a,b)$, when F is a *forest,* but we do not have a general conjecture.

PROBLEM 16. *Given $a,b\geq 1$ and a forest F on $s+t$ vertices, determine* $\lim_{n\to\infty}\mathrm{ex}(n,F(a,b))\binom{n}{r-1}^{-1}$.

Other bipartite graphs. The class of (a,b)-blowups of bipartite graphs contains well-studied instances, including blowups of complete bipartite graphs. In particular, Füredi [14] made the following conjecture for blowups of a 4-cycle. Let $\mathcal{C}_4^r=\{C_4(a,b): a+b=r, a,b>0\}$.

CONJECTURE 17 (see [14]). *If $r\geq 3$, then* $\mathrm{ex}(n,\mathcal{C}_4^r)\sim\binom{n}{r-1}$.

The current record is due to Pikhurko and the last author [25], who showed that

$$\mathrm{ex}_r(n,\mathcal{C}_4^r)\leq\left(1+\frac{2}{\sqrt{r}}+o(1)\right)\binom{n}{r-1}$$

and $\mathrm{ex}_3(n,C_4(2,1))\leq\left(\frac{13}{9}+o(1)\right)\binom{n}{2}$. It was recently shown in [2] that if $t\geq 3$, then $\mathrm{ex}_r(n,C_{2t}(r-1,1))=\Theta_r(tn^{r-1})$. In the same paper, the authors also obtained new bounds on $\mathrm{ex}_r(n,K_{s,t}(1,r-1))$, some of which are sharp. They also obtained bounds on $\mathrm{ex}_r(n,G(1,r-1))$ for some other interesting families of bipartite graphs G.

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