The Logarithmic Minkowski conjecture and the L_p -Minkowski Problem

Károly J. Böröczky*

January 24, 2024

Abstract

The current state of art concerning the L_p -Minkowski problem as a Monge-Ampére equation on the sphere and Lutwak's Logarithmic Minkowski conjecture about the uniqueness of even solution in the p = 0 case are surveyed and connections to many related problems are discussed.

MSC 2010 Primary: 35J96, Secondary: 52A40

1 Introduction

The Minkowski problem forms the core of various areas in fully nonlinear partial differential equations and convex geometry (see Trudinger, Wang [239] and Schneider [233]), which was extended to the L_p -Minkowski theory by Lutwak [193, 194, 195] where p = 1 corresponds to the classical case. The classical Minkowski's existence theorem due to Minkowski and Aleksandrov characterizes the surface area measure S_K of a convex body K in \mathbb{R}^n , more precisely, it solves the Monge-Ampére equation

$$\det(\nabla^2 h + h \operatorname{Id}) = f$$

on the sphere S^{n-1} where a convex body K with C^2_+ boundary provides a solution if $h = h_K|_{S^{n-1}}$ for the support function h_K of K, and in this case, 1/f(u) is the Gaussian curvature at the $x \in \partial K$ where u is an exterior

^{*}Supported by NKFIH grant K 132002

[†]Alfréd Rényi Institute of Mathematics, Realtanoda u. 13-15, H-1053 Budapest, Hungary, boroczky.karoly.j@renyi.hu

normal for $u \in S^{n-1}$. The so-called log-Minkowski problem (short hand for logarithmic Minkowski problem)

$$h \det(\nabla^2 h + h \operatorname{Id}) = f \tag{1}$$

or L_0 -Minkowski problem was posed by Firey in his seminal paper [111]. It seeks to charaterize the cone volume measure $dV_K = \frac{1}{n} h_K dS_K$ of a convex body K containing the origin o, and to determine whether the even solution is unique if f is even. The latter problem is the so-called Log-Minkowski conjecture by Lutwak. However, the log-Minkowski problem has received due attention only after finding its place as part of Lutwak's L_p -Minkowski problem

$$h^{1-p} \det(\nabla^2 h + h \operatorname{Id}) = f$$

in the 1990's where the cases p = 1 and p = 0 are the classical and the logarithmic Minkowski problem.

The uniqueness of the solution of the classical Minkowski problem up to translation is the consequence of the equality case of the Brunn-Minkowski inequality (see Gardner [114] or Schneider [233])

$$V(\alpha K + \beta C)^{\frac{1}{n}} \ge \alpha V(K)^{\frac{1}{n}} + \beta V(C)^{\frac{1}{n}}$$

$$\tag{2}$$

for convex bodies K and C and $\alpha, \beta \geq 0$. For $p \geq 0$, the L_p -Minkowski problem is intimately related to the L_p version of the Brunn-Minkowski inequality/conjecture

$$\begin{aligned}
V((1-\lambda)K+_p\lambda C)^{\frac{p}{n}} &\geq (1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(C)^{\frac{p}{n}} & \text{if } p > 0; \\
V((1-\lambda)K+_0\lambda C) &\geq V(K)^{1-\lambda}V(C)^{\lambda} & \text{if } p = 0
\end{aligned} \tag{3}$$

for $\lambda \in (0,1)$ and convex bodies K, C containing the origin. Here (3) is the classical Brunn-Minkowsi inequality if p = 1, a theorem of Firey [110] if p > 1, and assuming that K and C are origin symmetric, a conjecture being the central theme of this paper if $p \in [0,1)$. Actually, the conjecture has been recently verified if $p \in (0,1)$ is close to 1; more precisely, combining Kolesnikov, Milman [169] and Chen, Huang, Li, Liu [71] proves that the L_p Brunn-Minkowski conjecture holds if $p \in [p_n, 1)$ and K, C are origin symmetric convex bodies for an explicit $p_n \in (0, 1)$.

The main goal of this survey is to inspire the resolution of the Log-Brunn-Minkowski conjecture (*cf.* (3) when p = 0), or Lutwak's essentially equivalent Log-Minkowski conjecture (the Monge-Ampére equation (1) on S^{n-1} has a unique even solution if f is even, positive and C^{∞}).

Its versatility is an intriguing aspect of the log-Minkowski conjecture; namely, uniqueness of the even solutions of a Monge-Ampére equation on the sphere is equivalent to some strengthening of the Brunn-Minkowski inequality for origin symmetric convex bodies, to an inequality for the Gaussian density, and to some spectral gap estimates for certain self-adjoint operators, and in turn to displacement convexity of certain functional of probability measures on the sphere in optimal transportation.

In this survey, we review some related aspects of the classical Brunn-Minkowski Theory in Section 2, the state of art concerning the Log-Minkowski problem and the Log-Minkowski conjecture in Section 3, Lutwak's L_p -Minkowski problem and the L_p -Minkowski conjecture in Section 4, and some variants of the L_p -Minkowski problem in Section 5.

2 Classical Brunn-Minkowski Theory

This section serves as the introduction into the relevant aspects of Brunn-Minkowski Theory on the one hand, but also introduces the basic ideas and tools used in the upcoming sections. For a thorough discussion the subject and related problems from various perpectives, see Artstein-Avidan, Giannopoulos, Milman [11, 12], Gardner [114], Leichtweiß [174], Schneider [233].

We call a compact convex set in \mathbb{R}^n with non-empty interior a convex body. The family of convex bodies in \mathbb{R}^n is denoted by \mathcal{K}^n , and we write $\mathcal{K}^n_o(\mathcal{K}^n_{(o)})$ to denote the subfamily of $K \in \mathcal{K}^n$ with $o \in K$ ($o \in \text{int } K$), and write \mathcal{K}^n_e to denote the family of origin symmetric convex bodies in \mathbb{R}^n . The support function of a compact convex set K is $h_K(u) = \max_{x \in K} \langle u, x \rangle$ for $u \in \mathbb{R}^n$, and hence h_K is convex and homogeneous (the latter property says that $h_K(\lambda u) = \lambda h_K(u)$ for $\lambda \geq 0$). In turn, for any convex and homogeneous function h on \mathbb{R}^n , there exists a unique compact convex set K such that $h = h_K$. We note that differences of support functions are dense among continous functions on the sphere; more precisely, functions of the form $(h_K - h_C)|_{S^{n-1}}$ for convex bodies K and C with C^{∞}_+ boundary in \mathbb{R}^n are dense in $C(S^{n-1})$ with respect to the L_{∞} metric.

We say that convex bodies K and C are homothetic if $K = \gamma C + z$ for $\gamma > 0$ and $z \in \mathbb{R}^n$. We write V(X) to denote Lebesgue measure of a measurable subset X of \mathbb{R}^n (with $V(\emptyset) = 0$), and \mathcal{H} to denote the (n - 1)-Hausdorff measure normalized in a way such that it coincides with the (n-1)-dimensional Lebesgue measure on n - 1-dimensional affine subspaces. For $X, Y \subset \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, the Minkowski linear combination is $\alpha X +$ $\beta Y = \{\alpha x + \beta y : x \in X, y \in Y\}$, which is convex compact if X and Y are convex compact. We write B^n to denote the unit Euclidean ball centered at the origin o, and equip the space of compact convex sets of \mathbb{R}^n with topology induced by the Hausdorff metric (sometimes called Hausdorff distance); namely, if K and C are compact convex sets, then their Hausdorff distance is

$$\delta_H(K,C) = \min\{r \ge 0 : K \subset C + rB^n \text{ and } C \subset K + rB^n\}.$$

The Brunn-Minkowski inequality says that if $\alpha, \beta > 0$ and K, C are convex bodies in \mathbb{R}^n , then

$$V(\alpha K + \beta C)^{\frac{1}{n}} \ge \alpha V(K)^{\frac{1}{n}} + \beta V(C)^{\frac{1}{n}},\tag{4}$$

with equality if and only if K and C are homothetic. We note that the Brunn-Minkowski inequality (4) also holds if K and C are bounded Borel subsets of \mathbb{R}^n (note that Minkowski linear combination of measurable subsets may not be measurable; therefore, outer measure is used in that case). The Brunn-Minkowski inequality famously yields the isoperimertric inequality; namely, that the surface area of a bounded Borel set X of given volume is minimized by balls. Naturally, one needs a suitable notion of surface area. It is the Hausdorff measure $\mathcal{H}(\partial X)$ if X is a convex body, or more generally, ∂X is the finite union of the images of Lipschitz functions defined on bounded subsets of \mathbb{R}^{n-1} (see Schneider [233] or Ambrosio, Colesanti, Villa [6]), but the right notion is finite perimeter (see Maggi [200]). Fusco, Maggi, Pratelli [113] proved an optimal stability version of the isoperimetric inequality in terms of the symmetric difference metric (whose result was extended to the Brunn-Minkowski inequality by Figalli, Maggi, Pratelli [105, 106], see Theorem 2.1 below).

Because of the homogeneity of the Lebesgue measure, an equivalent form of the Brunn-Minkowski inequality (4) is that if K, C are convex bodies in \mathbb{R}^n and $\lambda \in (0, 1)$, then

$$V((1-\lambda)K + \lambda C) \ge V(K)^{1-\lambda}V(C)^{\lambda},$$
(5)

with equality if and only if K and C are translates. A big advantage of this product form of the Brunn-Minkowski inequality is that it is dimension invariant.

The first stability forms of the Brunn-Minkowski inequality were due to Minkowski himself (see Groemer [122]). If the distance of the convex bodies K and C is measured in terms of the so-called Hausdorff distance, then

Diskant [89] and Groemer [121] provided close to optimal stability versions (see Groemer [122]). However, the natural distance is in terms of the volume of the symmetric difference, and the optimal result is due to Figalli, Maggi, Pratelli [105, 106]. To define the "homothetic distance" A(K, C) of convex bodies K and C, let $\alpha = |K|^{\frac{-1}{n}}$ and $\beta = V(C)^{\frac{-1}{n}}$, and let

$$A(K,C) = \min \left\{ V(\alpha K \Delta(x + \beta C)) : x \in \mathbb{R}^n \right\}.$$

In addition, let

$$\sigma(K,C) = \max\left\{\frac{V(C)}{V(K)}, \frac{V(K)}{V(C)}\right\}.$$

THEOREM 2.1 (Figalli, Maggi, Pratelli) For $\gamma^*(n) > 0$ depending on n and any convex bodies K and C in \mathbb{R}^n ,

$$V(K+C)^{\frac{1}{n}} \ge (V(K)^{\frac{1}{n}} + V(C)^{\frac{1}{n}}) \left[1 + \frac{\gamma^*(n)}{\sigma(K,C)^{\frac{1}{n}}} \cdot A(K,C)^2 \right].$$

Here the exponent 2 of $A(K, C)^2$ is optimal, see Figalli, Maggi, Pratelli [106]. We note that prior to [106], the only known error term in the Brunn-Minkowski inequality was of order $A(K, C)^{\eta}$ with $\eta \ge n$, coming from the estimates of Diskant [89] and Groemer [121] in terms of the Hausdorff distance.

Figalli, Maggi, Pratelli [106] proved a factor of the form $\gamma^*(n) = cn^{-14}$ for some absolute constant c > 0, which was improved to cn^{-7} by Segal [232], and subsequently to $cn^{-5.5}$ by Kolesnikov, Milman [169], Theorem 12.12. The current best known bound for $\gamma^*(n)$ is $cn^{-5}(\log n)^{-10}$, which follows by combining the general estimate of Kolesnikov-Milman [169], Theorem 12.2, with the polylogarithmic bound of Klartag, Lehec [164] on the Cheeger constant of a convex body in isotropic position improving on Yuansi Chen's work [76] on the Kannan-Lovász-Simonovits conjecture. Harutyunyan [137] conjectured that $\gamma^*(n) = cn^{-2}$ is the optimal order of the constant, and showed that it can't be of smaller order. Actually, Segal [232] observed that Dar's conjecture in [88] would imply that we may choose $\gamma^*(n) = cn^{-2}$ for some absolute constant c > 0. Here Dar's conjectured strengthening of the Brunn-Minkowski inequality states in [88] that if K and C are convex bodies in \mathbb{R}^n , and $M = \max_{x \in \mathbb{R}^n} V(K \cap (x + C))$, then

$$V(K+C)^{\frac{1}{n}} \ge M^{\frac{1}{n}} + \left(\frac{V(K)V(C)}{M}\right)^{\frac{1}{n}}.$$
 (6)

Dar's conjecture is only known to hold in the plane (see Xi, Leng [241]), and in some very specific cases in higher dimension (see Dar [88]).

The paper Eldan, Klartag [94] discusses "isomorphic" stability versions of the Brunn-Minkowski inequality under condition of the type $|\frac{1}{2}K + \frac{1}{2}C| \leq 5\sqrt{|K| \cdot |C|}$, and considers, for example, the L^2 Wasserstein distance of the uniform measures on suitable affine images of K and C.

We note that stability versions of the Brunn-Minkowski inequality have been verified even if K or C are not convex. The essentially optimal estimate Theorem 2.1 (with much worse factor $\gamma^*(n)$) is verified if K is bounded measurable and C a convex body by Barchiesi, Julin [19] (improving on the estimate in Carlen, Maggi [67]), if $n \ge 2$ and K = C is bounded Borel set by Hintum, Spink, Tiba [142], and if n = 2 and K and C are bounded Borel sets by Hintum, Spink, Tiba [143]. If $n \ge 3$ and K and C are bounded Borel sets, then only a much weaker estimate in terms of A(K, C) is known, proved by Figalli, Jerison [103, 104]. On the other hand, a better error term of order A(X, Y) holds if n = 1 according to Freiman and Christ (see Christ [79]).

It was proved by Minkowski that if K and C are convex bodies and $\alpha, \beta \geq 0$, then

$$V(\alpha K + \beta C) = \sum_{i=0}^{n} {n \choose i} V(K,C;i) \alpha^{n-i} \beta^{i}$$
(7)

where V(K, C; i) are the so-called mixed volumes. For fixed i, V(K, C; i) is positive, continuous in both variables, and satisfies $V(\alpha K, \beta C; i) = \alpha^{n-i}\beta^i V(K, C; i)$ for $\alpha, \beta > 0, V(K, C; i) = V(C, K; n - i)$ and $V(\Phi K + x, \Phi C + y; i) =$ V(K, C; i) for $x, y \in \mathbb{R}^n$ and $\Phi \in SL(n)$. Many mixed volumes have geometric meaning, like V(K, K; i) = V(K, C; 0) = V(K) and $\frac{1}{n}V(K, B^n; 1) =$ $\mathcal{H}(\partial K)$ is the surface area of K. In addition, if $i = 1, \ldots, n - 1$, then $V(K, B^n; n - i)$ is proportional with the mean *i*-dimensional projection of K according to the Kubota formula (see Leichtweiß [174], Schneider [233]).

It follows from the Brunn-Minkowski inequality (4) that the function $f(\lambda) = V((1-\lambda)K + \lambda C)^{\frac{1}{n}}$ is concave on [0,1]. Combining $f'(0) \ge f(1) - f(0)$ and (7) leads to the famous Minkowski inequality

$$V(K,C;1)^n \ge V(K)^{n-1}V(C),$$
(8)

with equality if and only if K and C are homothetic. The Minkowski inequality is actually equivalent to the Brunn-Minkowski inequality because it implies that the function $f(\lambda) = V((1-\lambda)K + \lambda C)^{\frac{1}{n}}$ is concave on [0, 1], which in turn yields (4). Considering the second derivative $f''(\lambda)$ leads to Minkowski's second inequality

$$V(K,C;1)^2 \ge V(K)V(K,C;2),$$
(9)

that is in turn also equivalent to the Brunn-Minkowski inequality (4). The rather involved equality case of (9) has been only recently clarified by van Handel, Shenfeld [135].

Actually, Minkowski defined the mixed volume $V(C_1, \ldots, C_n)$ of n convex bodies via the identity

$$V(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^m V(K_{i_1},\ldots,K_{i_n}) \cdot \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} \quad (10)$$

for $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$ where $V(C_1, \ldots, C_n) \geq 0$ is symmetric and continuous in its variables (see [233]), and V(K, C; i) means i copies of C and n - i copies of K. A far reaching generalization of Minkowski's first and second inequalities is the Alexandrov-Fenchel inequality

$$V(K_1, K_2, K_3, \dots, K_n)^2 \ge V(K_1, K_1, K_3, \dots, K_n)V(K_2, K_2, K_3, \dots, K_n)$$

(see Alexandrov [1, 5] and Schneider [233]). Equality in the Alexander-Fenchel inequality has been much better understood now due to van Handel, Shenfeld [135, 136] where [136] clarifies the case of polytopes.

Let us summarize some equivalent formulations of the Brunn-Minkowski inequality that holds for all convex bodies K and C in \mathbb{R}^n :

- $V(\alpha K + \beta C)^{\frac{1}{n}} \ge \alpha V(K)^{\frac{1}{n}} + \beta V(C)^{\frac{1}{n}} ((cf. (4));$
- $V((1-\lambda)K + \lambda C) \ge V(K)^{1-\lambda}V(C)^{\lambda}$ ((cf. (5));
- $f(\lambda) = V((1 \lambda)K + \lambda C)^{\frac{1}{n}}$ is concave on [0, 1];
- Minkowski inequality $V(K,C;1)^n \ge V(K)^{n-1}V(C)$ (cf. (8));
- Minkowski's second inequality $V(K, C; 1)^2 \ge V(K)V(K, C; 2)$ (cf. (9)).

The classical Minkowski problem is concerned with the characterization of the so-called surface area measure S_K of a convex body K. Let $\partial' K$ denote the subset of the boundary of K such that there exists a unique exterior unit normal vector $\nu_K(x)$ at any point $x \in \partial' K$. It is well-known that $\mathcal{H}(\partial K \setminus \partial' K) = 0$ and $\partial' K$ is a Borel set (see Schneider [233]). The function $\nu_K : \partial' K \to S^{n-1}$ is the spherical Gauss map that is continuous on $\partial' K$. The surface area measure S_K of K is a Borel measure on S^{n-1} satisfying that $S_K(\eta) = \mathcal{H}(\nu_K^{-1}(\eta))$ for any Borel set $\eta \subset S^{n-1}$. The surface area measure is the first variation of the volume; namely, if C is any convex body in \mathbb{R}^n , then

$$nV(K,C;1) = \lim_{\varepsilon \to 0^+} \frac{V(K+\varepsilon C) - V(K)}{\varepsilon} = \int_{S^{n-1}} h_C \, dS_K.$$
(11)

It also follows that the Minkowski inequality (8) can be written in the following form: If V(K) = V(C) holds for $K, C \in \mathcal{K}^n$, then

$$\int_{S^{n-1}} h_C \, dS_K \ge \int_{S^{n-1}} h_K \, dS_K,\tag{12}$$

with equality if and only if K and C are translates.

To consider some examples for the surface area measure, if P is a polytope with facets F_1, \ldots, F_k and exterior unit normals u_1, \ldots, u_k , then S_P is concentrated onto $\{u_1, \ldots, u_k\}$ and $S_P(u_i) = \mathcal{H}(F_i)$ for $i = 1, \ldots, k$. On the other hand, if ∂K is C_+^2 , namely, C^2 with positive Gaussian curvature, and the Gaussian curvature at the point of ∂K with exterior unit normal $u \in S^{n-1}$ is $\kappa(u) = \kappa(K, u)$, then

$$dS_K = \kappa^{-1} d\mathcal{H} = \det(\nabla^2 h + h \operatorname{Id}) d\mathcal{H}$$
(13)

on S^{n-1} where $h = h_K|_{S^{n-1}}$ and ∇h and $\nabla^2 h$ are the gradient and the Hessian of h with respect to a moving orthonormal frame. In particular, S_K is absolute continuous in this case.

We note that if ∂K is C^2_+ for $K \in \mathcal{K}^n$ and $h = h_K|_{S^{n-1}}$, then for any $u \in S^{n-1}$, the differential operator

$$D^{2}h(u) = \nabla^{2}h(u) + h(u) \operatorname{Id}$$
(14)

is the restriction of the Hessian of h_K (in \mathbb{R}^n) at λu to an operator $u^{\perp} \mapsto u^{\perp}$ for any $\lambda > 0$, and the eigenvalues of $D^2h(u)$ are the radii of curvature at $x \in \partial K$ where u the exterior unit normal is. In turn, for any given $h \in C^m(S^{n-1})$ with $m \geq 2$, $h = h_K|_{S^{n-1}}$ for $K \in \mathcal{K}^n$ with $C^m(C^m_+)$ boundary if and only if $D^2h(u)$ is positive semi-definit (positive definit) for $u \in S^{n-1}$.

Now the Minkowski problem asks for necessary and sufficient conditions for a Borel measure μ on S^{n-1} such that

$$\mu = S_K \tag{15}$$

for a convex body K. The solution, together with its uniqueness was provided by Minkowski [207, 208] if the measure μ is discrete (and hence the convex body is a polytope) or absolutely continuous. Minkowski's solution was extended to any general measure μ by Alexandrov [2, 3, 5]; namely, there exists a convex body K with $\mu = S_K$ if and only if

$$\mu(L \cap S^{n-1}) < \mu(S^{n-1}) \qquad \text{for any linear } (n-1)\text{-subspace } L \subset \mathbb{R}^n; (16)$$
$$\int_{S^{n-1}} u \, dS_K(u) = o; \tag{17}$$

moreover, $S_K = S_C$ holds for convex bodies K and C if and only if K and C are translates. Essentially complete solutions were published also by Fenchel, Jensen [99] and Lewy [176] about the same time. In particular, the Monge-Ampère equation on the sphere S^{n-1} corresponding to the Minkowski problem is

$$\det(\nabla^2 h + h \operatorname{Id}) = f \tag{18}$$

where f is a given non-negative function with positive integral. If the given Borel measure μ on S^{n-1} is not absolutely continuous with respect to the Lebesgue measure, then $h = h_K|_{S^{n-1}}$ is a solution of (18) in the Alexandrov sense if (15) holds. We note that the surface area measure S_K is actually the Monge-Ampère measure corresponding to h (see Trudinger, Wang [239] and Böröczky, Fodor [43], Section 7).

The regularity of the solution of the Minkowski problem (18) is well investigated by Nirenberg [214], Cheng and Yau [77], Pogorelov [217], showing eventually that if f is positive and C^k for $k \ge 1$, then h is C^{k+2} . Finally, Caffarelli [62, 63] proves that if f is positive and C^{α} for $\alpha \in (0, 1)$ (namely, $|f(x) - f(y)| \le C ||x - y||^{\alpha}$ for $x, y \in S^{n-1}$ and constant C > 0), then the solution h is $C^{2,\alpha}$ (see also Böröczky, Fodor [43], Section 7 on how to connect results on Monge-Ampère equations on \mathbb{R}^n to Monge-Ampère equations on S^{n-1} , and Chen, Liu, Wang [75] for an extension of [62, 63]).

Turning to proofs, one of the elegant arguments proving the Brunn-Minkowski inequality (4) is due to Hilbert, and is based on a spectral gap estimate for a differential operator (see (42) and Bonnesen, Fenchel [36]). This approach was further developed by Alexandrov [1, 5] leading to the Alexandrov-Fenchel inequality, by van Handel, Shenfeld [135, 136] to characterize equality in the Alexandrov-Fenchel inequality in certain case, and by Milman, Kolesnikov [169] leading to the L_p -Minkowski inequility Theorem 4.4 improving the Brunn-Minkowski inequality for origin symmetric convex bodies (see the end of Section 4). Another fundamental approach proving the Brunn-Minkowski inequality is initiated by Gromov's influential appendix to Milman, Schechtman [206] using ideas by Knothe [171] provided a proof of the isoperimetric inequality using optimal (mass) transport, and the argument can be readily extended to the Brunn-Minkowski inequality (4) and the Prékopa-Leindler inequality (19) below. This approach lead even to the stability version Theorem 2.1 by Figalli, Maggi, Pratelli [105, 106]. We note that the original argument of Brunn and Minkowski for (4) (see Bonnesen, Fenchel [36]) can be also considered as a version of the mass transportation approach.

For the Minkowski problem (15), the variational method seeks the minimum of $\int_{S^{n-1}} h_C d\mu$ over all convex bodies C with V(C) = 1 where μ satisfies (16) and (17). It follows from (17) that the integral is invariant under translating C, hence the existence of a minimizer C_0 can be established. The fact that S_{C_0} is proprotional to μ follows via Alexandrov's Lemma 2.2 extending (11) (see Theorem 7.5.3 in Schneider [233]).

LEMMA 2.2 (Alexandrov) Given a convex body K in \mathbb{R}^n and continuous functions $h_t, g: S^{n-1} \to \mathbb{R}$, let us assume that the Wulff shape $K_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_t(u) \ \forall u \in S^{n-1}\}$ is a convex body and $\lim_{t\to 0} \frac{h_t(u) - h_K(u)}{t} = g(u)$ uniformly in $u \in S^{n-1}$. Then

$$\lim_{t \to 0} \frac{V(K_t) - V(K)}{t} = \int_{S^{n-1}} g \, dS_K.$$

The classical functional form of the Brunn-Minkowski inequality is the Prékopa-Leindler due to Prékopa [218] and Leindler [175] in dimension one, was generalized in Prékopa [219, 220] and Borell [37] (*cf.* also Marsiglietti [202], Bueno, Pivovarov [61]), Brascamp, Lieb [56], Kolesnikov, Werner [170], Bobkov, Colesanti, Fragalà [33]). Various applications are provided and surveyed in Ball [13], Barthe [21], Fradelizi, Meyer [112] and Gardner [114]. The following multiplicative version from [13] is often more useful and is more convenient for geometric applications.

THEOREM 2.3 (Prékopa-Leindler) If $\lambda \in (0,1)$ and h, f, g are nonnegative integrable functions on \mathbb{R}^n satisfying $h((1-\lambda)x+\lambda y) \ge f(x)^{1-\lambda}g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h \ge \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}^n} g\right)^{\lambda}.$$
(19)

For a convex function $W : \mathbb{R}^n \to (-\infty, \infty]$, we say that the function $\varphi = e^{-W}$ is log-concave where $e^{-\infty} = 0$ (in other words, $\log \varphi$ is concave

for $\varphi : \mathbb{R}^n \to [0, \infty)$). According to Dubuc [91], if equality holds in (2.3) assuming $\int_{\mathbb{R}^n} h > 0$, then h is log-concave, and there exist a > 0 and $z \in \mathbb{R}^n$ such that $f(x) = a^{\lambda} h(x - \lambda z)$ and $g(x) = a^{-(1-\lambda)}h(x + (1-\lambda)z)$ for almost all $x \in \mathbb{R}^n$. Stability versions of the Prékopa-Leindler inequality in terms of the L_1 distance have been established by Böröczky, De [39] in the log-concave case, and by Böröczky, Figalli, Ramos [42] for any functions where the case of log-concave functions in one variable have been dealt with ealier by Ball, Böröczky [16]. A stability version of the Prékopa-Leindler inequality of somewhat different nature is due to Bucur, Fragalà [60].

An "isomorphic" stability result for the Prékopa-Leindler inequality, in terms of the transportation distance is obtained in Eldan [92], Lemma 5.2. By rather standard considerations, one can show that non-isomorphic stability results in terms of transportation distance imply stability in terms of L_1 distance (e.g., such implication is attained by combining Proposition 2.9 in Bubeck, Eldan, Lehec [59] and Proposition 10 in Eldan, Klartag [94]). However, the current result in [92], due to its isomorphic nature, falls short of being able to obtain a meaningful bound in terms of the L_1 distance.

Brascamp, Lieb [56] proved a local version of the Prékopa-Leindler inequality for log-concave functions (Theorem 4.2 in [56]), which is equivalent to a Poincare-type so called Brascamp-Lieb inequality Theorem 4.1 in [56]. The paper Livshyts [187] provides a stability version of this Brascamp-Lieb inequality, and Bolley, Cordero-Erausquin, Fujita, Gentil, Guillin [34] proves a more general inequality.

Our final topic in this section is the Blasshke-Sataló inequality (20). For $K \in \mathcal{K}_{(o)}^n$ and $u \in S^{n-1}$, the radial function $\varrho_K(u) > 0$ satisfies $\varrho_K(u)u \in \partial K$, and the polar (dual) $K^* \in \mathcal{K}_{(o)}^n$ of K is defined by $\varrho_{K^*}(u) = h_K(u)$ for $u \in S^{n-1}$. Next, the centroid of a convex body K in \mathbb{R}^n is $\sigma_K = \frac{1}{V(K)} \int_K x \, dx$, which is invariant under affine transformations. For $K \in \mathcal{K}^n$, we call it centered if $\sigma_K = o$, and Kannan, Lovász, Simonovits [162] prove that there exists a centered ellipsoid E such that $E \subset K \subset nE$ in this case.

According to the Blashke-Santaló inequality (see Santaló [227], Luwak [193] or Schneider [233]), if $K \in \mathcal{K}^n$ is centered, then $V(K^*)V(K) \leq V(B^n)^2$, or equivalently,

$$\int_{S^{n-1}} h_K^{-n} \, d\mathcal{H} \le \frac{nV(B^n)^2}{V(K)},\tag{20}$$

with equality if and only if K is a centered ellipsoid.

The Blaschke-Santaló inequality can be proved for example via the Brunn-Minkowski inequality (see Ball [13] in the origin symmetric case, and Meyer, Pajor [203] in general). Various equivalent formulations are discussed in the beautiful survey Lutwak [193] (see also Böröczky [38] and Schneider [233]). Stability versions of the Blaschke-Santaló inequality are verified in Böröczky [38] and Ball, Böröczky [17]. Following Ball [13], functional versions of the Blaschke-Santaló inequality have been obtained by Artstein-Avidan, Klartag, Milman [10], Fradelizi, Meyer [112], Lehec [172, 173], Kolesnikov, Werner [170], Kalantzopoulos, Saroglou [161].

Recently, various breakthrough stability results about geometric functional inequalities have been obtained. Stonger versions of the functional Blaschke-Santaló inequality is provided by Barthe, Böröczky, Fradelizi [23], of the Borell-Brascamp-Lieb inequality is provided by Ghilli, Salani [119], Rossi, Salani [222, 223] and Balogh, Kristály [18] (later even on Riemannian manifolds), of the Sobolev inequality by Figalli, Zhang [108] (extending Bianchi, Egnell [32] and Figalli, Neumayer [107]), Nguyen [213] and Wang [247], of the log-Sobolev inequality by Gozlan [120], and of some related inequalities by Caglar, Werner [65], Cordero-Erausquin [83], Kolesnikov, Kosov [166]. Another functional version of the Brunn-Minkowski inequality is provided by Artstein-Avidan, Florentin, Segal [9].

3 Cone volume measure, log-Minkowski problem, log-Brunn-Minkowski conjecture

Given a convex body K containing the origin, the cone volume measure is defined as $dV_K = \frac{1}{n} h_K dS_K$, and hence the total measure is $V_K(S^{n-1}) = V(K)$. The name originates from the fact that if P is a polytope with facets F_1, \ldots, F_k and exterior unit normals u_1, \ldots, u_k , then V_P is concentrated onto $\{u_1, \ldots, u_k\}$ and $V_P(u_i) = \frac{h_P(u_i)}{n} \cdot \mathcal{H}(F_i)$ is the volume of the cone $\operatorname{conv}\{o, F_i\}$ for $i = 1, \ldots, k$. We note that the Monge-Ampère equation on the sphere S^{n-1} corresponding to the logarithmic Minkowski problem is

$$h \det(\nabla^2 h + h \operatorname{Id}) = nf \tag{21}$$

for a non-negative measurable function f on S^{n-1} with $0 < \int_{S^{n-1}} f d\mathcal{H} < \infty$. It follows via Caffarelli [62, 63] that if f is positive and C^{α} for $\alpha \in (0, 1)$, then the solution h is $C^{2,\alpha}$, and if f is positive and C^k for integer $k \ge 1$, then the solution h is C^{k+2} . For a finite non-trivial Borel measure μ on S^{n-1} , a non negative function h on S^{n-1} that is the restriction of the support function h_K for a convex body K is the solution of (21) in the Alexandrov sense if

$$d\mu = dV_K = \frac{1}{n} h_K \, dS_K. \tag{22}$$

A characteristic feature of the cone volume measure is that it intertwines with linear transformations; more precisely, $V_{\Phi K} = |\det \Phi| \cdot (\Phi^{-t})_* V_K$ for any $K \in \mathcal{K}_o^n$ and $\Phi \in \operatorname{GL}(n, \mathbb{R})$. We note that if $u \in S^{n-1}$ is an exterior normal at an $x \in \partial K$, then $\Phi^{-t}u$ is an exterior normal at $\Phi x \in \partial(\Phi K)$, and the push forward measure $\Psi_*\mu$ on S^{n-1} for a Borel measure μ on S^{n-1} and $\Psi \in \operatorname{GL}(n)$ is defined (with a slight abuse of notation) in a way such that if $\omega \subset S^{n-1}$ is Borel, then

$$\Psi_*\mu(\omega) = \mu\left(\left\{\frac{\Psi^{-1}(u)}{\|\Psi^{-1}(u)\|} : u \in \omega\right\}\right).$$

Cone volume measure was introduced by Firey [111], and has been a widely used tool since the paper Gromov, Milman [123], see for example Barthe, Guédon, Mendelson, Naor [26], Naor [210], Paouris, Werner [216]. The still open logarithmic Minkowski problem (22) or (21) was posed by Firey [111] in 1974, who showed that if f is a positive constant function, then (21) has a unique even solution coming from the suitable centered ball. For a positive constant function f, the general uniqueness result without the evenness condition is due to Andrews [7] if n = 2, 3, and Brendle, Choi, Daskalopoulos [57] if $n \ge 4$. It is known that uniqueness of the solution may not hold if f is not a constant function (see, for example, Chen, Li, Zhu [74]). However, the celebrated "Logarithmic Minkowski conjecture" by Lutwak [194] from 1993 states that (21) has a unique even solution if f is even and positive (Conjecture 3.1 is a more restricted version).

CONJECTURE 3.1 (Log-Minkowski conjecture #1) If f is a positive even C^{∞} function in (21), then (21) has a unique even solution.

As we explain below, the following logarithmic analogue of Minkowski's inequality (12) is an intimately related form of the Log-Minkowski conjecture (see Böröczky, Lutwak, Yang, Zhang [49] for origin symmetric bodies, and by Böröczky, Kalantzopoulos [48] for centered convex bodies).

CONJECTURE 3.2 (Log-Minkowski conjecture #2) If K and C are convex bodies in \mathbb{R}^n whose centroid is the origin, then

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \ge \frac{V(K)}{n} \log \frac{V(C)}{V(K)} \tag{23}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$. In particular, the more precise form of the Logarithmic Minkowski Conjecture 3.1 is that if $K, C \in \mathcal{K}^n$ are centered, then $V_K = V_C$ implies the equality conditions in Conjecture 3.2.

We note that the choice of the right translates of K and C are important in Conjecture 3.2 according to the examples by Nayar, Tkocz [211], and that Conjecture 3.2 is invariant under applying the same non-singular linear transformation to C and K. An equivalent form of Conjecture 3.2 is that if V(C) = V(K) for centered C and K, then

$$\int_{S^{n-1}} \log h_C \, dV_K \ge \int_{S^{n-1}} \log h_K \, dV_K \tag{24}$$

where the case of equality is like in Conjecture 3.2.

Let me explain why Conjecture 3.1 is equivalent to the case of Conjecture 3.2 when $K \in \mathcal{K}_e^n$ has C_+^∞ boundary. Since V_K satisfies the strict subspace concentration condition (see below), Böröczky, Lutwak, Yang, Zhang [50] prove that the function $C \mapsto \int_{S^{n-1}} \log h_C \, dV_K$ of origin symmetric convex bodies C with V(C) = V(K) attains its minimum; moreover, whenever it attains its minimum at some $C = \widetilde{C}$, then $V_{\widetilde{C}} = V_K$. In turn, the stated equivalence follows. In addition, it follows by approximation that Conjecture 3.1 yields the inequality (23) for any pair of origin symmetric convex bodies K and C without the case of equality.

In \mathbb{R}^2 , Conjecture 3.2 is verified in Böröczky, Lutwak, Yang, Zhang [49] for origin symmetric convex bodies, but it is still open in general even in the plane. In higher dimensions, Conjecture 3.2 is proved for complex bodies (*cf.* Rotem [224]), and if there exist *n* independent linear reflections that are common symmetries of *K* and *C* (*cf.* Böröczky, Kalantzopoulos [48], and even a stability version is verified by Böröczky and De [41]). The latter type of bodies include unconditional convex bodies, which case was handled earlier by Saroglou [228]. In addition, Conjecture 3.2 is verified if *C* is origin symmetric and *K* is a zonoid by van Handel [133] (with equality case only clarified when *K* has C_+^2 boundary), or if *C* is a centered convex body and *K* is a centered ellipsoid by Guan, Ni [125]. The latter case directly follows from the Jensen inequality and the Blaschke-Santaló inequality (20), as assuming that $K = B^n$ and $V(C) = V(B^n)$, we have

$$\exp\left(\int_{S^{n-1}} \log h_C \cdot \frac{1}{V(B^n)} dV_K\right) = \exp\left(\int_{S^{n-1}} \log h_C \cdot \frac{1}{nV(B^n)} d\mathcal{H}\right)$$
$$\geq \left(\int_{S^{n-1}} h_C^{-n} \cdot \frac{1}{nV(B^n)} d\mathcal{H}\right)^{\frac{-1}{n}} \geq 1.$$

For origin symmetric K and C, Conjecture 3.2 is proved when K is close to be an ellipsoid (with equality case only clarified when K has C_{+}^{2} boundary) by a combination of the local estimates by Kolesnikov, Milman [169], and the use of the continuity method in PDE by Chen, Huang, Li, Liu [71]. Here closeness to an ellipsoid means that there exist some $c_n > 0$ depending only on n and an origin symmetric ellipsoid E such that $E \subset$ $K \subset (1+c_n)E$. Another even more recent proof of this result is due to Putterman [221]. We note that an analogues result holds for linear images of Hausdorff neighbourhoods of l_q balls for q > 2 if the dimension n is high enough according to [169] and the method of [71]. Actually, Milman [205] provides rather generous explicit curvature pinching bounds for ∂K in order to Conjecture 3.2 to hold, and proves that for any origin symmetric convex body M there exists an origin symmetric convex body K with C^{∞}_{+} boundary and $M \subset K \subset 8M$ such that Conjecture 3.2 holds for any origin symmetric convex body C. Additional local versions of Conjecture 3.2 are due to Colesanti, Livshyts, Marsiglietti [81], Kolesnikov, Livshyts [168] and Hosle, Kolesnikov, Livshyts [144]. We review Kolesnikov and Milman's approach in [169] based on the Hilbert-Brunn-Minkowski operator at the end of Section 4.

Xi, Leng [241] considered a version of Conjecture 3.2 where the convex bodies K and C in \mathbb{R}^n are translated by vectors depending in both K and C. We set $r(K, C) = \max\{t > 0 : \exists x, \ x + tC \subset K\}$ and $R(K, C) = \min\{t > 0 : \exists x, \ K \subset x + tC\}$, and say that K and C are in dilated position if $o \in K \cap C$ and

$$r(K,C) C \subset K \subset R(K,C) C.$$
(25)

We observe that $r(C, K) K \subset C \subset R(C, K) K$ in this case. Now for any convex bodies K and C there exist $z \in K$ and $w \in C$ such that K - z and C - w are in dilated position. If n = 2 and K and C are in dilated position, then Xi, Leng [241] proved (24) including the characterization of equality. Actually, [241] even verified Dar's conjecture (6) for convex planar bodies in dilated position (no need for translation in this case).

Let us discuss the existence of the solution of the logaritmic Minkowski problem (21) or (22). Following partial and related results by Andrews [7], Chou, Wang [78], He, Leng, Li [138], Henk, Schürman, Wills [141], Stancu [234], Xiong [246], the paper Böröczky, Lutwak, Yang, Zhang [50] characterized even cone volume measures by the so-called subspace concentration condition (i) and (ii) in Theorem 3.3.

THEOREM 3.3 There exists an origin symmetric convex body $K \in \mathcal{K}_e^n$ with $\mu = V_K$ for a non-trival finite even Borel measure μ on S^{n-1} if and only if

- (i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any proper linear subspace $L \subset \mathbb{R}^n$;
- (ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with $\operatorname{supp} \mu \subset L \cup L'$.

We observe that V_K satisfies (ii) if and only if K = C + C' where $C \subset L^{\perp}$ and $C' \subset L'^{\perp}$ compact convex sets. A finite Borel measure μ on S^{n-1} satisfies the strict subspace concentration condition if $\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any proper linear subspace $L \subset \mathbb{R}^n$.

Given a non-trival finite even Borel measure μ on S^{n-1} that is invariant under n reflections Φ_1, \ldots, Φ_n through n independent linear hyperplanes, Böröczky, Kalantzopoulos [48] proved that $\mu = V_K$ for a convex body K in \mathbb{R}^n invariant under Φ_1, \ldots, Φ_n if and only if μ satisfies the subspace concentration condition (i) and (ii) for any proper linear subspace $L \subset \mathbb{R}^n$ invariant under Φ_1, \ldots, Φ_n . Actually, the statement also holds if Φ_1, \ldots, Φ_n are linear reflections (see [48] for details). For a centered convex body $K \in \mathcal{K}^n$, Böröczky, Henk [46] (see Henk and Linke [139] for the case of polytopes) verified that V_K satisfies the subspace concentration condition (i) and (ii), but V_K satisfies some additional conditions, as well. On the other hand, if $V_K(L \cap S^{n-1}) \ge (1-\varepsilon) \cdot \frac{\dim L}{n} \cdot V(K)$ holds for $K \in \mathcal{K}^n$, a proper linear subspace $L \subset \mathbb{R}^n$ and a small $\varepsilon > 0$, then K is close to the sum of two complementary lower dimensional compact convex sets according to Böröczky, Henk [47]. We note that Freyer, Henk, Kipp [109] even verified certain so-called Affine Subspace Concentration Conditions for the cone volume measure of centered polytopes.

Much less is known, not even a conjecture about the characteristic properties of a cone volume measure on S^{n-1} , not even in the plane. Chen, Li, Zhu [74] proved that if a non-trival finite Borel measure μ on S^{n-1} satisfies the subspace concentration condition (i) and (ii), then μ is a cone volume measure. On the other hand, Böröczky, Hegedűs [44] characterized the restriction of a cone volume measure to a pair of antipodal points.

As Lutwak, Yang, Zhang [197] conjectured, a Borel probability measure μ on S^{n-1} satisfies the subspace concentration condition (i) and (ii) if and only if there exists an isotropic linear image $\Phi_*\mu$ for a $\Phi \in \operatorname{GL}(n)$ according to Böröczky, Lutwak, Yang, Zhang [51] (extending the work by Carlen, Cordero-Erausquin [66] in the discrete case and Klartag [163] in the strict subspace concentration condition case). Here the probability measure μ on S^{n-1} is isotropic if $n \int_{S^{n-1}} u \otimes u \, d\mu(u) = \operatorname{Id}_n$; or in other words, $||x|| = n \int_{S^{n-1}} \langle x, u \rangle^2 \, d\mu(u)$ for any $x \in \mathbb{R}^n$.

Next we turn to the logarithmic Brunn-Minkowski conjecture/inequality. For $\lambda \in (0,1)$ and $K, C \in \mathcal{K}_o^n$, we define their logarithmic or L_0 linear combination by the formula

$$(1-\lambda)K +_0 \lambda C = \{ x \in \mathbb{R}^n : \langle x, u \rangle \le h_K(u)^{1-\lambda} h_C(u)^\lambda \ \forall u \in S^{n-1} \}.$$

The L_0 linear combination is linear invariant; namely, if $\Phi \in \operatorname{GL}(n)$, then $\Phi((1-\lambda)K+_0\lambda C) = (1-\lambda)\Phi(K)+_0\lambda\Phi(C)$. Moreover, the L_0 linear combination is a convex body if $\{h_K=0\} = \{h_C=0\}$ (for example, when $K, C \in \mathcal{K}^n_{(o)}$). We note that $(1-\lambda)(\alpha K)+_0\lambda(\beta C) = \alpha^{1-\lambda}\beta^{\lambda}((1-\lambda)K+_0\lambda C))$ for $\alpha, \beta > 0$. The L_0 linear combination of polytopes is always a polytope, but the boundary of the L_0 linear combination of convex bodies with C^2_+ boundaries may contain segments, and hence may not be C^2_+ . A functional analogue of the L_0 -addition is presented by Crasta, Fragalà [87].

We observe that $(1 - \lambda)K +_0 \lambda C \subset (1 - \lambda)K + \lambda C$ for any convex bodies K and C containing the origin interior, but $(1 - \lambda)K +_0 \lambda C$ might be much smaller than $(1 - \lambda)K + \lambda C$. For example, if a > 0 is large, n = 2, $K = \left[\frac{-1}{a}, \frac{1}{a}\right] \times \left[-a, a\right]$ and $C = \left[-a, a\right] \times \left[\frac{-1}{a}, \frac{1}{a}\right]$, then

$$\frac{\frac{1}{2}K}{\frac{1}{2}K} + \frac{1}{2}C = [-1,1]^2 \\ \frac{1}{2}K + \frac{1}{2}C = \left[-\frac{1}{2}(a+\frac{1}{a}), \frac{1}{2}(a+\frac{1}{a})\right]^2.$$
(26)

Böröczky, Lutwak, Yang, Zhang [49] conjectured the following for origin symetric convex bodies, and Martin Henk proposed the version with centered convex bodies (see also [48]).

CONJECTURE 3.4 (Log-Brunn-Minkowski conjecture) If $\lambda \in (0, 1)$ and K and C are centered convex bodies in \mathbb{R}^n , then

$$V((1-\lambda)K +_0 \lambda C) \ge V(K)^{1-\lambda}V(C)^{\lambda}$$
(27)

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

The Log-Brunn-Minkowski Conjecture 3.4 is a significant strengthening of the Brunn-Minkowski inequality for centered convex bodies (see (26)). Given $K, C \in \mathcal{K}_o^n$, if (27) holds for all $\lambda \in (0, 1)$, then the Logarithmic Minkowski inequality (23) follows by considering $\frac{d}{d\lambda}V((1-\lambda)K+_0\lambda C)|_{\lambda=0^+}$ and using Alexandrov's Lemma 2.2 according to [49]. On the other hand, the argument in [49] shows that if \mathcal{F} is any family of convex bodies closed under L_0 linear combination, then the Logarithmic Minkowski inequality (23) for all $K, C \in \mathcal{F}$ is equivalent to the Logarithmic Brunn-Minkowski inequality (27) for all $K, C \in \mathcal{F}$ and $\lambda \in (0, 1)$. In particular, the equivalence holds for the family of origin symmetric convex bodies. According to Kolesnikov, Milman [169] and Putterman [221], taking the second derivative of $\lambda \mapsto$ $V((1-\lambda)K+_0\lambda C)$ for origin symmetric convex bodies K and C in \mathbb{R}^n leads to the conjectured inequality

$$\frac{V(K,C;1)^2}{V(K)} \ge \frac{n-1}{n} V(K,C;2) + \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K^2} dV_K$$
(28)

that is a strengthened from of Minkowski's second inequality (9), and is equivalent to the Log-Brunn-Minkowski conjecture without the charactherization of equality. More precisely, [169] proves that for a fixed $K \in \mathcal{K}_e^n$ with C_+^2 boundary, (28) for all smooth $C \in \mathcal{K}_e^n$ is equivalent to a local form of the Log-Brunn-Minkowski around K, and [221] verifies the global statement. Therefore, we have the following three equivalent forms of the Log-Brunn-Minkowski conjecture for origin symmetric convex bodies K and C in \mathbb{R}^n (without the charactherization of equality in the case of the third formulation):

• $V((1-\lambda)K +_0 \lambda C) \ge V(K)^{1-\lambda}V(C)^{\lambda}$ as in (27);

•
$$\int_{S^{n-1}} \log \frac{h_C}{h_K} dV_K \ge \frac{V(K)}{n} \log \frac{V(C)}{V(K)}$$
 as in (23);

•
$$\frac{V(K,C;1)^2}{V(K)} \ge \frac{n-1}{n} V(K,C;2) + \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K^2} dV_K$$
 as in (28).

Another equivalent formulation using the Hilbert-Brunn-Minkowski operator (42) is due to Kolesnikov, Milman [169], and is discussed at the end of Section 4 (see (44)). In addition, Saroglou [228] verified that the Log-Brunn-Minkowski inequality for any origin symmetric convex bodies is equivalent with the so-called *B*-property: For any origin symmetric convex body K in \mathbb{R}^n and $n \times n$ positive definite diagonal matrix Φ , the function $s \mapsto V([-1,1]^n \cap \Phi^s K)$ of $s \in \mathbb{R}$ is log-concave. Yet another equivalent formulation of the Log-Brunn-Minkowski conjecture for origin symmetric convex bodies in \mathbb{R}^n using the "strong *B*-property" is due to Nayar, Tkocz [212]: For any N > n and *n*-dimensional linear subspace L of \mathbb{R}^N , the *n*volume of $L \cap \prod_{i=1}^N [-e^{t_i}, e^{t_i}]$ is a log-concave function of $(t_1, \ldots, t_N) \in \mathbb{R}^N$. Actually, [212] proves an analogous property of the crosspolytopes.

Saroglou [229] proved that if the log-Brunn-Minkowski Conjecture (27) holds for any origin symmetric convex bodies K and C and $\lambda \in (0, 1)$, then

it holds for any even log-concave measure μ on \mathbb{R}^n ; namely,

$$\mu((1-\lambda)K +_0 \lambda C) \ge \mu(K)^{1-\lambda}\mu(C)^{\lambda}.$$
(29)

In turn, the argument in [229] shows that (29) for the Gaussian meaure $\mu = \gamma_n$ implies the log-Brunn-Minkowski Conjecture (27) for origin symmetric convex bodies. Finally, Kolesnikov [165] provides another equivalent formulation of the log-Brunn-Minkowski Conjecture for origin symmetric convex bodies in terms of displacement convexity of certain functional of probability measures on the sphere in optimal transportation.

The Log-Brunn-Minkowski Conjecture 3.4 is still open but has been verified in various cases. In \mathbb{R}^2 , Conjecture 3.4 is verified by Böröczky, Lutwak, Yang, Zhang [49] for origin symmetric convex bodies, but it is still open for general centered planar convex bodies. For unconditional convex bodies, the L_0 linear combination contains the so-called coordinatewise product (see Saroglou [228]); therefore, the corresponding inequality for the coordinatewise product by Uhrin [238], Bollobás, Leader [35] and Cordero-Erausquin, Fradelizi, Maurey [84], following from the Prékopa-Leindler inequality Theorem 2.3 yields Conjecture 3.4. The equality case of the Log-Brunn-Minkowski inequality for unconditional convex bodies was clarified by Saroglou [228] (see also [48]). The Log-Brunn-Minkowski Conjecture 3.4 for convex bodies invariant under reflections through n independent linear hyperplanes is due to Böröczky, Kalantzopoulos [48]. In addition, Conjecture 3.4 is proved for complex bodies by Rotem [224].

Conjecture 3.4 holds for origin symmetric convex bodies in a neighbourhood of a fixed centered ellipsoid E; more precisely, for origin symmetric Kand C provided $E \subset K, C \subset (1 + c_n)E$ where $c_n > 0$ depends only on n. In this form, the statement is due to Chen, Huang, Li, Liu [71] extending the local estimate by Kolesnikov, Milman [169] (an analogues result holds for linear images of l_q balls for q > 2 if the dimension n is high enough according to [169] and the method of [71]). We note that the case when Kand C are in a C^2 neighbourhood of E was handled earlier by Colesanti, Livshyts, Marsiglietti [81].

In some cases when uniqueness of the solution of the Log-Minkowski problem is known, even the stability of the solution has been established. For example, Böröczky, De [40] established this among convex bodies invariant under n given reflections through linear hyperplanes. Concerning Firey's classical result that the only origin symmetric solution of the Log-Minkowski problem (21) with constant f is the centered ball, Ivaki [156] verified a stability version. Next Böröczky, Saroglou [54] proved the uniqueness of the solution if the possibly non-even f in (21) is C^{α} close to a constant function (the case n = 3 was handled earlier by Chen, Feng, Liu [69]).

If n = 2 and K and C are in dilated position (25), then Xi, Leng [241] proved (27) for any $\lambda \in (0, 1)$ including the characterization of equality using the same method as in the case of the planar Dar conjecture. It is an intriguing question what the relation between Dar's conjecture (6) and the Log-Brunn-Minkowski Conjecture 3.4 is, whether one of them implies the other for origin symmetric convex bodies.

We note that there exist $\eta_2 > \eta_1 > 0$ depending on *n* such that if $\lambda \in (0, 1)$ and *K* and *C* are centered convex bodies in \mathbb{R}^n , then

$$\eta_1 V(K)^{1-\lambda} V(C)^{\lambda} \le V((1-\lambda)K +_0 \lambda C) \le \eta_2 V(K)^{1-\lambda} V(C)^{\lambda}, \qquad (30)$$

which estimates indicate why proving the Log-Brunn-Minkowski Conjecture 3.4 is so notoriously difficult. Conjecture 3.4 states that $\eta_1 = 1$, but here we only verify that $\eta_1 = n^{-n}$ and $\eta_2 = n^{3n/2}$ work. According to Kannan, Lovász, Simonovits [162], there exist centered ellipsoids $E' \subset K$ and $E \subset C$ such that $K \subset nE'$ and $C \subset nE$. After a linear transform, we may assume that $E' = B_2^n$ and E is unconditional. Since Conjecture 3.4 holds for the unconditional convex bodies B_2^n and E, we deduce that $\eta_1 = n^{-n}$ works in (30). For the upper bound, let a_1, \ldots, a_n be the half axes of E, and hence $C \subset \widetilde{C} = \prod_{i=1}^{n} [-na_i, na_i]$ and $K \subset \widetilde{K} = [-n, n]^n$ with $V(\widetilde{C}) \leq n^{3n/2}V(C)$ and $V(\widetilde{K}) \leq n^{3n/2}V(K)$. Since $V((1 - \lambda)\widetilde{K} +_0\lambda\widetilde{C}) = V(\widetilde{K})^{1-\lambda}V(\widetilde{C})^{\lambda}$, it follows that $\eta_2 = n^{3n/2}$ works in (30).

The validity of the Log-Minkowski (or Log-Brunn-Minkowski) Conjecture is also supported by the fact that various consequences of it has been verified. For example, the L_p -Minkowski Conjecture has been proved when $p \in (0,1)$ is close to 1 (see Theorem 4.4). Nest we turn to results about the canonical Gaussian probability measure γ_n on \mathbb{R}^n . One possible consequence of the Log-Brunn-Minkowski Conjecture 3.4 is the earlier celebrated "*B*-inequality" by Cordero-Erausquin, Fradelizi, Maurey [84] stating that $\gamma_n(e^t K)$ is a log-concave function of $t \in \mathbb{R}$ for any origin symmetric $K \in \mathcal{K}^n$. Next, the Gardner-Zvavitch conjecture in [117] stated that if K and C are origin symmetric convex bodies in \mathbb{R}^n , then

$$\gamma_n((1-\lambda)K + \lambda C)^{\frac{1}{n}} \ge (1-\lambda)\gamma_n(K)^{\frac{1}{n}} + \lambda\gamma_n(C)^{\frac{1}{n}}.$$
(31)

It was proved by Livshyts, Marsiglietti, Nayar, Zvavitch [189], that the log-Brunn-Minkowski conjecture would imply the Gardner-Zvavitch conjecture. After various attempts, the conjecture was finally verified by Eskenazis, Moschidis [96] not much before that, Kolesnikov, Livshyts [167] verified that if the exponents $\frac{1}{n}$ in (31) are changed into $\frac{1}{2n}$, then this modified Gardner-Zvavitch conjecture holds for any pair of centered convex bodies K and C.

We note that independently of the log-Brunn-Minkowski conjecture, various Brunn-Minkowski type inequalities have been proved and conjectured for the Gaussian measure, the most famous ones being the Ehrhardt inequality and the Gaussian isoperimetric inequality (see Livshyts [187]).

Colesanti, Livshyts, Marsiglietti [81] conjectured the following generalization of the Gardner-Zvavitch conjecture: If μ is an even log-concave measure on \mathbb{R}^n , then

$$\mu((1-\lambda)K+\lambda C)^{\frac{1}{n}} \ge (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda\mu(C)^{\frac{1}{n}}$$
(32)

holds for any origin symmetric convex bodies K and C. According to Livshyts, Marsiglietti, Nayar, Zvavitch [189], the Log-Brunn-Minkowski Conjecture 3.4 would imply the conjecture (32). Cordero-Erausquin, Rotem [86] proved (32) if μ is a rotationally symmetric log-concave measure. In addition, Livshyts [188] verified that (32) holds for any even log-concave measure on \mathbb{R}^n and origin symmetric convex bodies K and C if the exponents $\frac{1}{n}$ in (32) are changed into $n^{-4-o(1)}$.

4 Lutwak's L_p -Minkowski theory

The rapidly developing new L_p -Brunn-Minkowski theory (where p = 1 is the classical case and p = 0 corresponds to the cone-volume measure) initiated by Lutwak [193, 194, 195], has become main research area in modern convex geometry and geometric analysis. For $p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$, the L_p -surface area measure $S_{K,p}$ on S^{n-1} is defined by

$$dS_{K,p} = h_K^{1-p} \, dS_K \tag{33}$$

where if p > 1 and $o \in \partial K$, then we assume that $S_K(\{h_K = 0\}) = 0$. In particular, $S_{K,1} = S_K$ and $S_{K,0} = nV_K$. For $p \in \mathbb{R}$, the Monge-Ampére equation on S^{n-1} corresponding to the L_p -Minkowski problem is

$$\det(\nabla^2 h + h \operatorname{Id}) = h^{p-1} f \quad \text{if} \quad p > 1$$

$$h^{1-p} \det(\nabla^2 h + h \operatorname{Id}) = f \quad \text{if} \quad p \le 1$$
(34)

where $f \in L_1(S^{n-1})$ is non-negative with $\int_{S^{n-1}} f d\mathcal{H} > 0$, and for a finite non-trivial Borel measure μ on S^{n-1} , a convex body $K \in \mathcal{K}_o^n$ is an Alexan-

drov solution of the L_p -Minkowski problem if

$$dS_K = h_K^{p-1} d\mu \quad \text{if} \quad p > 1$$

$$h_K^{1-p} dS_K = d\mu \quad \text{if} \quad p \le 1.$$
(35)

If p > 1 and $p \neq n$, then Hug, Lutwak, Yang, Zhang [152] (improving on Chou, Wang [78]) prove that (35) has an Alexandrov solution if and only if the μ is not concentrated onto any closed hemisphere, and the solution is unique. If in addition p > n, then the unique solution of (35) satifies $o \in \text{int } K$, and hence $S_{K,p} = \mu$. However, examples in [152] show that if 1 , then it may happen that the density function <math>f is a positive continuous in (34) and $o \in \partial K$ holds for the unique Alexandrov solution. If p = n, then $S_{K,n} = S_{\lambda K,n}$ holds for $\lambda > 0$; therefore, all what is known (see [152]) is that for any measure μ not concentrated onto any closed hemisphere, there exists a convex body $K \in \mathcal{K}_o^n$ and c > 0 such that $\mu = c \cdot S_{K,n}$.

The case p = 1 is the classical Minkowski problem (see Section 2), and the case case p = 0 is the logarithmic Minkowski problem (see Section 3).

If $p \in (0, 1)$ and the measure μ is not concentrated onto any great subsphere, then Chen, Li, Zhu [73] prove that there exists an Alexandrov solution $K \in \mathcal{K}_o^n$ of (35) with $S_{K,p} = \mu$. For $p \in (0, 1)$, complete characterization of L_p surface area measures is only known if n = 2 by Böröczky, Trinh [55]; namely, a finite non-trivial Borel measure μ on S^1 is an L_p surface area measure if and only if $\operatorname{supp} \mu$ does not consists of a pair of antipodal points. Finally, let $p \in (0, 1)$ and $n \geq 3$, and let us assume that $1 \leq \dim L \leq n - 1$ where L is the linear hull of $\operatorname{supp} \mu$ in \mathbb{R}^n . If $\operatorname{supp} \mu$ is contained in a closed hemisphere centered at a point of $L \cap S^{n-1}$, then μ is an L_p surface area measure according to Bianchi, Böröczky, Colesanti, Yang [30]. On the other hand, Saroglou [231] proved that if $\mu(\omega)$ is the Lebesgue measure of $\omega \cap L$ for any Borel $\omega \subset S^{n-1}$, then μ is not a L_p surface area measure.

for any Borel $\omega \subset S^{n-1}$, then μ is not a L_p surface area measure. If $-n and <math>f \in L_{\frac{n}{n+p}}(S^{n-1})$ in (34), then (34) has a solution according to Bianchi, Böröczky, Colesanti, Yang [30]. If p < 0 and the μ in (35) is discrete satisfying that μ is not concentrated on any closed hemisphere and any n unit vectors in the support of μ are independent, then [252] manages to solve the L_p -Minkowski problem.

The p = -n case of the L_p -Minkowski problem is the critical case because its link with the SL(n) invariant centro-affine curvature. If $K \in \mathcal{K}^n_{(o)}$ has C^2_+ boundary, then its centro-affine curvature at $u \in S^{n-1}$ is $\kappa_0(K, u) = \frac{\kappa(K, u)}{h_K(u)^{n+1}}$ where $\kappa(K, u)$ is the Gaussian curvature at the point with exterior normal u. It is well known to be SL(n) invariant in the sense that $\kappa_0(\Phi K, u) = \kappa_0(K, \frac{\Phi^t u}{\|\Phi^t u\|})$ for $\Phi \in \mathrm{SL}(n)$ (see Hug [153] or Ludwig [191]). It follows from (13) that if $K \in \mathcal{K}^n_{(o)}$ has C^2_+ boundary, then $dS_{K,-n}(u) = \kappa_0(K, u)^{-1} d\mathcal{H}(u)$; therefore, solving the L_p -Minkowski problem (34) for p = -n and positive C^{α} function f is equivalent to reconstructing a convex body $K \in \mathcal{K}^n_{(o)}$ from its centro-affine curvature function.

All in all, the centro-affine (L_{-n}) Minkowki problem is wide open. If p = -n and the f in (34) is unconditional and satisfies certain additional technical conditions, then Jian, Lu, Zhu [160] verify the existence of a solution of (34). Moreover the paper Li, Guang, Wang [130] solves a variant of the centro-affine Minkowki problem. On the other hand, Chou, Wang [78] prove an implicit condition on possible functions f in (34) such that in f^{-1} is a centro affine curvature (see also [30]), and Du [90] construct an explicit example of a positive $C^{\alpha} f$ such that (34) has no solution when p = -n.

In the super-critical case p < -n, Li, Guang, Wang [128] have recently achieved a breakthrough by proving that for any positive $C^2 f$, there exists a C^4 solution of (34). In addition, [128] verify that if p < -n and 1/c < f < c for a constant c > 1 in (34), then there exists a $C^{1,\alpha}$ Alexandrov solution $h_K|_{S^{n-1}}$ satisfying (35) where $d\mu = f d\mathcal{H}$. In their paper, Guang, Li, Wang [128] combine a flow argument with homology calculations. On the other hand, Du [90] construct a non-negative C^{α} function f that is positive everywhere but a fixed pair of antipodal points and (34) has no solution, not even in Alexandrov sense. It is not surprising that the flow argument works in the super-critical case, as Milman [204] points out the limitations of the variational argument in this case. For a discrete measure μ satisfying that μ is not concentrated on any closed hemisphere and any n unit vectors in the support of μ are independent, Zhu [252] solves the L_p -Minkowski problem (35) for p < 0.

If p > -n, then while flow arguments are also known (see e.g. Bryan, Ivaki, Scheuer [58]), the most common argument to find a solution of (34) is based on the variational method; namely, one considers the infimum of $\int_{S^{n-1}} h_C^p f \, d\mathcal{H}$ for a suitable family of convex bodies $C \in \mathcal{K}_{(0)}^n$ with V(C) =1 when f is positive and continuous (see [30] or Chou, Wang [78]). The existence of some minimizer C_0 follows via the Blaschke-Santaló inequality (20) as p > -n, and the fact that $dS_{C_0,p} = \lambda f \, d\mathcal{H}$ for some constant factor $\lambda > 0$ follows via the Alexandrov Lemma 2.2. The case of more general measures than the ones with positive continuous density functions follows by approximation. For the variational approach, it is also common to use discrete measures on S^{n-1} (corresponding to polytopes, see [152, 251, 45, 252]). Concerning the smoothness of the solution of the L_p -Minkowski problem (34), if f is positive and C^{α} and h is positive (equivalently, $o \in \operatorname{int} K$ for the corresponding convex body K), then h is $C^{2,\alpha}$ by Cafarelli [62, 63] (see [43], [29]). Assuming that f is positive and continuous, it is known that $o \in \operatorname{int} K$ if $p \ge n$ (see [152]) or if $p \le 2 - n$ (see [29]). On the other hand, if $2-n , then there there exists positive <math>C^{\alpha}$ function f on S^{n-1} such that $o \in \partial K$ holds for the Alexandrov solution of (35) with $d\mu = f d\mathcal{H}$, see [152] if 1 and [29] if <math>2-n . Additional results about the smoothness of the solution are provided by [29] in the case <math>2-n .

Now we discuss the uniqueness of the solution of the L_p -Minkowski problem (35). As we have seen, if p > 1 and $p \neq n$, then Hug, Lutwak, Yang, Zhang [152] proved that the Alexandrov solution of the L_p -Minkowski problem (35) is unique. However, if p < 1, then the solution of the L_p -Minkowski problem (34) may not be unique even if f is positive and continuous. Examples are provided by Chen, Li, Zhu [73, 74] if $p \in [0, 1)$, and Milman [204] shows that for any $C \in \mathcal{K}_{(0)}$, one finds $q \in [-n, 1)$ where q = -n exactly when C is a centered ellipsoid such that if p < q, then there exist multiple solutions of the L_p -Minkowski problem (35) with $\mu = S_{C,p}$; or in other words, there exists $K \in \mathcal{K}_{(0)}^n$ with $K \neq C$ and $S_{K,p} = S_{C,p}$. In addition, Jian, Lu, Wang [159] and Li, Liu, Lu [179] prove that for any p < 0, there exists positive even C^{∞} function f with rotational symmetry such that the L_p -Minkowski problem (34) has multiple positive even C^{∞} solutions. We note that in the case of the centro-affine Minkowski problem p = -n, Li [178] even verified the possibility of existence of infinitely many solutions without affine equivalence, and Stancu [236] proved that if an origin symmetric convex body K with C^{∞}_{+} boundary is a unique solution to the L_p -Minkowski problem (35) up to linear equivalence for p = -n with $\mu = S_{K,-n}$, then it is a unique solution for p = 0 with $\mu = S_{K,0}$.

The case when f is a constant function in the L_p -Minkowski problem (34) has received a special attention since Firey [111]. Through the work of Lutwak [194], Andrews [7], Andrews, Guan, Ni [8] and Brendle, Choi, Daskalopoulos [57], it has been clarified that the only solutions are centered balls if p > -n, centered ellipsoids if p = -n, and there are several solutions if p < -n. See Crasta, Fragalá [87], Ivaki, Milman [157] and Saroglou [230] for novel approaches. Stability versions of these results have been obtained by Ivaki [156], sometimes in the even case for certain ranges of p. Uniqueness of the solution of (34) if f is C^{α} close to 1 (without the evenness assumption) is proved if $p \in [0, 1)$ by Böröczky, Saroglou [54].

In particular, concerning uniqueness, the a major question left open is

the uniqueness of even solutions of the L_p -Minkowski problem (34) when fis an positive even C^{∞} function and $p \in [0, 1)$. In the case of p = 0, this is Lutwak's Log-Minkowski Conjecture 3.1. If $p \in (0, 1)$, it is also conjectured that the L_p -Minkowski problem (34) has a unique even solution for any positive, C^{∞} and even f. More generally, we have the following conjecture (see Böröczky, Lutwak, Yang, Zhang [49] for origin symmetric bodies).

CONJECTURE 4.1 (L_p -Minkowski Conjecture #1) If $p \in (0,1)$ and K and C are centered convex bodies in \mathbb{R}^n with $S_{K,p} = S_{C,p}$, then K = C.

Before presenting what is known about the L_p -Minkowski conjecture, let us discuss its relation to the L_p -Brunn-Minkowski theory for $p \ge 0$. More precisely, the cases p = 0 and p = 1 have been discussed in Sections 2 and 3.

For p > 0, $\alpha, \beta > 0$ and $K, C \in \mathcal{K}_o^n$, we define the L_p linear combination by the formula

$$(1-\lambda)K +_p \lambda C = \{ x \in \mathbb{R}^n : \langle x, u \rangle^p \le \alpha h_K(u)^p + \beta h_C(u)^p \ \forall u \in S^{n-1} \}.$$

The L_p linear combination is linear invariant; namely, if $\Phi \in \operatorname{GL}(n)$, then $\Phi(\alpha K +_p \beta C) = \alpha \Phi(K) +_p \beta \Phi(C)$. If $p \in (0,1)$, then the L_p linear combination of polytopes is always a polytope, but the boundary of the L_p linear combination of convex bodies with C^2_+ boundaries may contain segments, and hence may not be C^2_+ . On the other hand, if p > 1, then for any $\alpha, \beta > 0$, Minkowski's inequality yields that $h^p_{\alpha K +_p \beta C} = \alpha h^p_K + \beta h^p_C$, as the L_p linear combination was defined by Firey [110] in this case. According to Firey [110], if p > 1 and $K, C \in \mathcal{K}^n_o$, then the Brunn-Minkowski inequality yields the L_p -Brunn-Minkowski inequality

$$V(\alpha K +_p \beta C)^{\frac{p}{n}} \ge \alpha V(K)^{\frac{p}{n}} + \beta V(C)^{\frac{p}{n}}$$
(36)

for any $\alpha, \beta > 0$ with equality if and only if K and C are dilates; or equivalently,

$$V((1-\lambda)K +_p \lambda C) \ge V(K)^{1-\lambda}V(C)^{\lambda}$$
(37)

for $\lambda \in (0, 1)$ with equality if and only if K = C.

For p > 0 and $K, C \in \mathcal{K}^n_{(o)}$, analogously to the classical mixed volumes, Lutwak [193] introduced the L_p mixed volume

$$V_p(K,C) = \frac{p}{n} \lim_{t \to 0^+} \frac{V(K + p t C) - V(K)}{t} = \frac{1}{n} \int_{S^{n-1}} h_C^p \, dS_{K,p} = \int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K$$

and hence $V_1(K,C) = V(K,C;1)$. Considering the first derivative of $\lambda \mapsto V((1-\lambda)K + \lambda C)^{\frac{p}{n}}$ yields the L_p -Minkowski inequality

$$V_p(K,C) \ge V(K)^{\frac{n-p}{n}} V(C)^{\frac{p}{n}}$$
(38)

for p > 1 and $K, C \in \mathcal{K}^n_{(o)}$ with equality if and only if K and C are dilates. An equivalent form is that

$$\int_{S^{n-1}} h_C^p \, dS_{K,p} \ge \int_{S^{n-1}} h_K^p \, dS_{K,p} \tag{39}$$

if p > 1, $K, C \in \mathcal{K}^n_{(p)}$ and V(K) = V(C) with equality if and only if K = C.

We recall that the Brunn-Minkowski inequality (4) holds for bounded Borel subsets K and C of \mathbb{R}^n , as well. When p > 1, the L_p -Brunn-Minkowski inequality has been also extended to certain families of non-convex sets by Zhang [248], Ludwig, Xiao, Zhang [192] and Lutwak, Yang, Zhang [198].

If $p \in (0,1)$, then translating a cube shows that neither L_p -Brunn-Minkowski inequality, nor L_p -Minkowski inequality hold for general $K, C \in \mathcal{K}^n_{(o)}$. However, Böröczky, Lutwak, Yang, Zhang [49] conjecture that they hold for at least origin symmetric convex bodies (see Böröczky, Kalantzopoulos [48] for centered convex bodies).

CONJECTURE 4.2 (L_p -Minkowski conjecture #2) If $p \in (0,1)$, then (38); or equivalently, (39) hold for centered $K, C \in \mathcal{K}^n$.

CONJECTURE 4.3 (L_p -Brunn-Minkowski conjecture) If $p \in (0, 1)$, then (36); or equivalently, (37) hold for centered $K, C \in \mathcal{K}^n$.

The fact that the forms Conjectures 4.1 and 4.2 (including the characterization of equality) of the L_p -Minkowski conjecture are equivalent follows from (39) and the variational method as described above.

According to the Jensen inequality, $(1 - \lambda)K +_q \lambda \subset (1 - \lambda)K +_p \lambda C$ for $p > q \ge 0$. It follows for example via (37) (or via (43)) that if $0 \le q , then the <math>L_q$ -Brunn-Minkowski conjecture (or equivalently the L_q -Minkowski conjecture) yields the L_p -Brunn-Minkowski conjecture (or equivalently the L_q -Minkowski conjecture). In particular, the L_p -Brunn-Minkowski inequality for centered convex bodies on the one hand, and follows from the Log-Brunn-Minkowski Conjecture 3.4 on the other hand. In addition Kolesnikov-Milman [169] prove that knowing the L_p -Minkowski inequality (38) for some $p \in (0, 1)$ yields even the characterization of the equality case for the L_q -Minkowski inequality when $q \in (p, 1)$.

Let $p \in (0,1)$. Given $K, C \in \mathcal{K}_o^n$, if (36) holds for all $\alpha, \beta > 0$, then the L_p -Minkowski inequality (38) follows by considering $\frac{d}{d\lambda}V((1-\lambda)K+_0\lambda C)|_{\lambda=0^+}$ and using Alexandrov's Lemma 2.2 according to [49]. On the other hand, the argument in [49] shows that if \mathcal{F} is any family of convex bodies closed under L_p linear combination, then the L_p -Minkowski inequality (38) for all $K, C \in \mathcal{F}$ is equivalent to the L_p Brunn-Minkowski inequality (36) for all $K, C \in \mathcal{F}$ and $\alpha, \beta > 0$. In particular, the equivalence holds for the family of origin symmetric convex bodies. According to Kolesnikov, Milman [169] and Putterman [221], taking the second derivative of $\lambda \mapsto$ $V((1-\lambda)K+_p\lambda C)^{\frac{p}{n}}$ for origin symmetric convex bodies K and C in \mathbb{R}^n leads to the conjectured inequality

$$\frac{V(K,C;1)^2}{V(K)} \ge \frac{n-1}{n-p} V(K,C;2) + \frac{1-p}{n-p} \int_{S^{n-1}} \frac{h_C^2}{h_K^2} dV_K$$
(40)

that is again a strengthened from of Minkowski's second inequality (9), and is equivalent to the L_p Brunn-Minkowski conjecture without the charactherization of equality. More precisely, [169] proves that for a fixed $K \in \mathcal{K}_e^n$ with C_+^2 boundary, (28) for all smooth $C \in \mathcal{K}_e^n$ is equivalent to a local form of the L_p Brunn-Minkowski around K, and [221] verifies the global statement. We note that van Handel [133] presents an approach relating the equality case of (40) to the equality case of (38) for a fixed $K \in \mathcal{K}_e^n$ with C_+^2 boundary.

In summary, we have the following three equivalent forms of the L_{p} -Brunn-Minkowski conjecture for $p \in (0, 1)$ and origin symmetric convex bodies K and C in \mathbb{R}^{n} (without the charactherization of equality in the case of the third formulation):

- $V((1-\lambda)K +_p \lambda C) \ge V(K)^{1-\lambda}V(C)^{\lambda}$ for $\lambda \in (0,1)$;
- $V_p(K,C) \ge V(K)^{\frac{n-p}{n}}V(C)^{\frac{p}{n}};$
- $\frac{V(K,C;1)^2}{V(K)} \ge \frac{n-1}{n-p} V(K,C;2) + \frac{1-p}{n-p} \int_{S^{n-1}} \frac{h_C^2}{h_K^2} dV_K.$

Let us dicuss the cases when Conjectures 4.1, 4.2 and 4.3 have been verified. They have been verified in the planar n = 2 case by Böröczky, Lutwak, Yang, Zhang [49]. The most spectacular result is due to the combination of the local result by Kolesnikov, Milman [169] and the local to global approach based on Schrauder estimates in PDE by Chen, Huang, Li, Liu [71] (see Puttermann [221] for an Alexandrov-type argument for the local to global approach) is that the L_p -Minkowski and L_p -Brunn-Minkowski conjectures hold for origin symmetric convex bodies if $p \in (0, 1)$ is close to 1.

THEOREM 4.4 If $n \ge 3$ and $p \in (p_n, 1)$ where $0 < p_n < 1 - \frac{c}{n(\log n)^{10}}$ for an absolute constant c > 0, then the L_p -Brunn-Minkowski and L_p -Minkowski conjectures (36), (37), (38) and (39) hold for $K, C \in \mathcal{K}_e^n$ including the characterization of the equality cases.

The paper Kolesnikov, Milman [169] provides an explicit estimate for p_n , depending on the Cheeger or Poincaré constants subject of the celebrated Kannan, Lovász, Simonovits conjecture [162]. Our estimate for p_n comes from the upper bound $c(\log n)^5$ by Klartag, Lehec [164] for the Poincaré constant where c > 0 is a absolute constant.

Otherwise, the known cases of the L_p -Minkowski and L_p -Brunn-Minkowski conjectures for origin symmetric bodies follow from the known cases of the Log-Minkowski and Log-Brunn-Minkowski conjectures. Let $p \in (0, 1)$ and $n \geq 3$. Then (36), (37), (38) and (39) hold if K and C are invariant under reflections through fixed n independent linear hyperplanes (cf. [48]) and if K and C are origin symmetric complex bodies (cf. [224]). In addition, the L_p -Minkowski conjecture (38) and (39) hold for $K, C \in \mathcal{K}_e^n$ (together with characterization of equality if ∂K is C_+^3) if either K is a zonoid according to [133], or there exists a centered ellipsoid E with $E \subset K \subset (1 + c_n)E$ where $c_n > 0$ depends only on n according to [71] (an analogues result holds for linear images of l_q balls for q > 2 if the dimension n is high enough according to [169]).

Concerning the L_p Brunn-Minkowski conjecture, Hosle, Kolesnikov, Livshyts [144] and Kolesnikov, Livshyts [168] present certain natural generalizations and approaches.

In the final part of Section 4, we discuss how David Hilbert's elegant operator theoretic proof of the Brunn-Minkowski inequality has lead to recent new approaches initiated by Kolesnikov, Milman [169] towards the L_p -Minkowski conjecture (see also Putterman [221] and van Handel [133]). Here we present Kolesnikov and Milman's version of the Hilbert-Brunn-Minkowski operator based on [169] because this modified operator \mathcal{L}_K intertwines with linear transformations (*cf.* Theorem 5.8 in [169]).

The mixed discriminat $D_{\ell}(B_1, \ldots, B_{\ell})$ of ℓ positive definite $\ell \times \ell$ matrices can be defined via the identity

$$\det_{\ell} \left(\lambda_1 A_1 + \ldots + \lambda_m A_m \right) = \sum_{i_1, \ldots, i_{\ell} = 1}^m D_{\ell} (A_{i_1}, \ldots, A_{i_{\ell}}) \cdot \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_{\ell}} \quad (41)$$

for $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and positive definite $\ell \times \ell$ matrices A_1, \ldots, A_m where $D_{\ell}(A_{i_1}, \ldots, A_{i_{\ell}}) > 0$ is symmetric in its variables and $D_{\ell}(A, \ldots, A) = \det A$

(see van Handel, Shenfeld [134, 136] or Kolesnikov, Milman [169]). The similarity between (10) and (41) is not a coincidence as

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_n} D_{n-1} \left(D^2 h_{K_1}, \dots, D^2 h_{K_{n-1}} \right) d\mathcal{H}$$

for $K_1, \ldots, K_n \in \mathcal{K}^n$ with C^2_+ boundary.

For $K \in \mathcal{K}^n_{(o)}$ with C^2_+ boundary, Kolesnikov, Milman [169] defines the Hilbert-Brunn-Minkowski operator $\mathcal{L}_K : C^2(S^{n-1}) \to C^2(S^{n-1})$ by the formula

$$\mathcal{L}_{K}f = \frac{D_{n-1}\left(D^{2}(fh_{K}), D^{2}h_{K}, \dots, D^{2}h_{K}\right)}{D_{n-1}\left(D^{2}h_{K}, \dots, D^{2}h_{K}\right)} - f.$$
(42)

Following Hilbert's footsteps, [169] verifies that the operator \mathcal{L}_K is elliptic, and hence admits a unique self-adjoint extension in $L^2(dV_K)$, and has discrete spectrum. The operator $-\mathcal{L}_K$ is positive semi-definite, its smallest eigenvalue is $\lambda_0(\mathcal{L}_K) = 0$ whose eigenspace consists of the constant functions. As Hilbert (see also van Handel, Shenfeld [134, 136] or Kolesnikov, Milman [169]) proved, the next eigenvalue is $\lambda_1(-\mathcal{L}_K) = 1$ corresponding to the *n*-dimensional eigenspace spanned by the linear functions; moreover, this fact is equivalent to the Brunn-Minkowski inequality for any convex bodies.

If K is origin symmetric, then $-\mathcal{L}_K$ can be restricted to the space of even functions in $C^2(S^{n-1})$, and $\lambda_{1,e}(-\mathcal{L}_K) > 1$ holds for the smallest positive eigenvalue of this restricted operator because linear functions are odd. Here the linear invariance yields that $\lambda_{1,e}(-\mathcal{L}_K) = \lambda_{1,e}(-\mathcal{L}_{\Phi K})$ for $\Phi \in \mathrm{GL}(n)$. A key result in Kolesnikov, Milman [169] improves the estimate $\lambda_{1,e}(-\mathcal{L}_K) > 1$ uniformly; more precisely,

$$\lambda_{1,e}\left(-\mathcal{L}_{K}\right) \geq \frac{n-p_{n}}{n-1}$$

for any $K \in \mathcal{K}_e^n$ with C_+^2 boundary where the explicit $p_n \in (0, 1 - \frac{c}{n(\log n)^{10}})$ is the same as in Theorem 4.4. The connection to the L_p -Minkowski conjecture for fixed $p \in [0, 1)$ is another key result in Kolesnikov, Milman [169], as developed further by Putterman [221]; namely,

$$\lambda_{1,e}\left(-\mathcal{L}_{K}\right) \geq \frac{n-p}{n-1} \tag{43}$$

is equivalent saying that (40) holds for any $C \in \mathcal{K}_e^n$. In particular, given $p \in [0, 1)$, the L_p -Minkowski conjecture follows if (43) holds for all $K \in \mathcal{K}_e^n$

with C^{∞}_+ boundary, and the Logarithmic-Minkowski conjecture and (27) are equivalent saying that

$$\lambda_{1,e}\left(-\mathcal{L}_{K}\right) \geq \frac{n}{n-1} \tag{44}$$

for all $K \in \mathcal{K}_e^n$ with C_+^∞ boundary. If $K_m \in \mathcal{K}_e^n$ with C_+^2 boundary tends to a cube, then $\lambda_{1,e} (-\mathcal{L}_{K_m})$ tends to $\frac{n}{n-1}$ according to [169]; therefore, the Logarithmic-Minkowski conjecture states that cubes "minimize" $\lambda_{1,e} (-\mathcal{L}_K)$. On the other hand, [169] calculates that $\lambda_{1,e} (-\mathcal{L}_K) = \frac{2n}{n-1}$ if K is a centered Euclidean ball, and Milman [204] verifies that centered ellipsoids maximize $\lambda_{1,e} (-\mathcal{L}_K)$ among all $K \in \mathcal{K}_e^n$ with C_+^2 boundary.

5 Some variants of the L_p -Minkowski problem

We note that Livshyts [186] considers a version of the Minkowski problem with a given measure on \mathbb{R}^n acting as a weigh on the surface of the convex body.

Considering the variation of the *i*th intrinsic volume of a convex body K (or equivalently, variation of $V(B^n, K; i)$) for i = 2, ..., n-1 instead of the volume of K leads to the so-called Christoffel-Minkowski problem, which asks to determine a convex body when its (i-1)th area measure on S^{n-1} is prescribed (see Guan, Ma [125], Guan, Xia [126]). We note that for $K \in \mathcal{K}^n$ with C^2_+ boundary, S_K is then (n-1)th surface area measure, and the *j*th area measure is defined using the *j*th symmetric function of the principle radii of curvatures instead of the reciprocal of the Gaussian curvature. The L_p Christoffel-Minkowski problem is discussed by Guan, Xia [126], Hu, Ma, Shen [145] and Bryan, Ivaki, Scheuer [58] in the case p > 1, and by Bianchini, Colesanti, Pagnini, Roncoroni [31] in the case $p \in [0, 1)$ where again, p = 1 corresponds to the classical case.

The Minkowski problem on the sphere is solved by Guang, Li, Wang [129] (see [129] for related references, as well), and in the hyperbolic space, partial results, also about the hyperbolic Christoffel-Minkowski problem, are obtained by Gerhardt [118].

The Gaussian surface area measure of a $K \in \mathcal{K}^n$ is defined by Huang, Xi and Zhao [150], and [150] obtains significant results about the even Gaussian Minkowski problem. These results are extended to the not necessarily even case by Feng, Liu, Xu [102] and Shibing Chen, Shengnan Hu, Weiru Liu, Yiming Zhao [70], and the L_p -Gaussian Minkowski problem is considered by Liu [184] and Feng, Hu, Xu [101]. Uniqueness of the solution is discussed in the works above and in Ivaki, Milman [157]. Next we discuss the L_p dual Minkowski problem introduced by Lutwak, Yang, Zhang [199] that is a common generalization of the L_p -Minkowski problem and the Alexandrov problem. In order to define the dual curvature measures, let $K \in \mathcal{K}_{(o)}^n$. Recall that the radial function $\varrho_K(u) > 0$ satisfies $\varrho_K(u)u \in \partial K$ for any $u \in S^{n-1}$. For a Borel set $\omega \subset S^{n-1}$, its \mathcal{H} measurable reverse radial Gauss image $\alpha^*(\omega)$ is the set of $u \in S^{n-1}$ such that some $v \in \omega$ is an exterior normal at $\varrho_K(u)u$ (see Huang, Lutwak, Yang, Zhang [148]). Now for any $q \in \mathbb{R}$, [148] defines the qth dual curvature measure of the Borel set $\omega \subset S^{n-1}$ by

$$\widetilde{C}_{K,q}(\omega) = \frac{1}{n} \int_{\alpha^*(\omega)} \varrho_K^n d\mathcal{H}.$$

In particular, $\tilde{C}_{K,n} = V_K$ is the cone volume measure (discussed in Section 3), and $n\tilde{C}_{K,0}$ is the Alexandrov integral curvature measure of the polar K^* . The Monge-Ampére equation corresponding to the *q*th dual Minkowski problem is

$$(\|\nabla h\|^2 + h^2)^{\frac{q-n}{2}} \cdot h \det(\nabla^2 h + h \operatorname{Id}) = f.$$
(45)

The Alexandrov problem; namely, the charaterization of $\widetilde{C}_{K,0}$ has been solved by Aleksandrov [4, 5] (see also Oliker [215] and Bertrand [27] Böröczky, Lutwak, Yang, Zhang, Zhao [53]). For the L_p version of the Alexandrov problem posed by Huang, Lutwak, Yang, Zhang [149], see for example Zhao [250], Li, Sheng, Ye, Yi [180] Mui [209] and Wu, Wu, Xiang [240]. If $q \neq 0, n$, then the following results are known:

- If q < 0, then any Borel measure on S^{n-1} not concentrated on a closed hemisphere is a *q*th dual Minkowski curvature measure according to Zhao [249] and Li, Sheng, Wang [181].
- If 0 < q < n, then an even Borel measure on S^{n-1} is a *q*th dual Minkowski curvature measure if and only if

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{q} \cdot \mu(S^{n-1})$$

for any proper linear subspace L of of \mathbb{R}^n according to Böröczky, Lutwak, Yang, Zhang, Zhao [52] where one needs to add that μ is not concentrated onto a great subsphere if q < 1.

• If $q \ge n+1$ and $K \in \mathcal{K}^n$ is origin symmetric, then Henk, Pollehn [140] prove

$$\widetilde{C}_{K,q}(L \cap S^{n-1}) < \frac{q-n+\dim L}{q} \cdot \widetilde{C}_{K,q}(S^{n-1})$$

• If q > 0 and n = 2, then (45) has a solution for any measurable f provided $\frac{1}{c} < f < c$ for a c > 1 according to Chen, Li [72].

In particular, it is an intriguing open problem to characterize an even qth dual Minkowski curvature measure on S^{n-1} if q > n.

For $p, q \in \mathbb{R}$, Lutwak, Yang, Zhang [199] defines the qth L_p dual Minkowski curvature measure on S^{n-1} by $d\widetilde{C}_{K,p,q} = h_K^{-p} d\widetilde{C}_{K,q}$, and hence $\widetilde{C}_{K,0,q} = \widetilde{C}_{K,q}$ and $\widetilde{C}_{K,p,n} = \frac{1}{n} S_{K,p}$. Given a Borel measure μ on S^{n-1} , the simplest version of the qth the L_p dual Minkowski problem asks for a $K \in \mathcal{K}^n_{(o)}$ with $\mu = \widetilde{C}_{K,p,q}$, and the correspong Monge-Ampére equation is

$$h^{1-p}\det(\nabla^2 h + h\operatorname{Id}) = (\|\nabla h\|^2 + h^2)^{\frac{n-q}{2}} \cdot f$$
 (46)

Improving on [43] and Huang, Zhao [151], Chen, Li [68] and Lu, Pu [190] if p > 0 and $q \neq p$, then any Borel measure not concentrated on a closed hemisphere is a qth L_p dual Minkowski curvature measure (more precisely, if $p \leq q$, then some modification of the Monge-Ampére equation might be needed). Huang, Zhao [151] proved the same for p, q < 0 and $p \neq q$ within the category of even measures. See also Guang, Li, Wang [127] for a flow approach when p < 0 and q > n under regularity assumptions.

Uniqueness of the solution of the qth L_p dual Minkowski problem (46) is thouroughly investigated by Li, Liu, Lu [179]. The case when n = 2 and f is a constant function has been completely clarified by Li, Wan [177].

Another important related variant of the dual Minkowski problem is the so-called Chord Minkowski Problem (*cf.* Lutwak, Xi, Yang, Zhang [196]) and its L_p version by Xi, Yang, Zhang, Zhao [242] for p > 0, and by Li [182, 183] for p < 0, see also Guo, Xi, Zhao [131] and Xi, Yang, Zhang, Zhao [242]. In addition, the "Affine dual Minkowski problem" is proposed by Cai, Leng, Wu, Xi [64].

Cordero-Erausquin, Klartag [85] generalized and solved the classical Minkowski problem to the space of log-concave functions on \mathbb{R}^n . Recently, Fang, Xing, Ye [97] considered the L_p -Minkowski problem for log-concave functions, and Huang, Liu, Xi, Zhao [147] and Fang, Ye, Zhang, Zhao [98] managed to extend the dual Minkowski problem to log-concave functions on \mathbb{R}^n for q > 0.

Starting with Haberl, Lutwak, Yang, Zhang [132], Orlicz versions of the L_p -Minkowski problem have been intensively investigated; namely, the function $t \mapsto t^{1-p}$ in (34) is replaced by certain $\varphi : (0, \infty) \to (0, \infty)$, and hence (34) is replaced by

$$\varphi(h) \det(\nabla^2 h + h \operatorname{Id}) = f$$

where f is a given non-negative function on S^{n-1} . Typically, the solution is only up to a constant factor; namely, there exists some c > 0 such that $\varphi(h) \det(\nabla^2 h + h \operatorname{Id}) = c \cdot f$. The known existence results about the L_p -Minkowski problem for p > -n have been generalized to the Orlicz L_p -Minkowski problem where $\varphi(t)$ replaces t^{1-p} by Huang, He [146] if p > 1(see also Xie [243]), by Jian, Lu [158] if $p \in (0, 1)$, and by Bianchi, Böröczky, Colesanti [28] if $p \in (-n, 0)$.

Orlicz versions of these Monge-Ampère equations have been considered by Li, Sheng, Ye, Yi [180] and Feng, Hu, Liu [100] in the case of the Alexandrov problem, by Xing, Ye [244] in the case of the dual Minkowski problem, and Gardner, Hug, Weil, Xing, Ye [115, 116], Xing, Ye, Zhu [245] and Liu, Lu [185] in the case of the L_p dual Minkowski problem in general.

Acknowledgements: I would like to thank for illuminating discussions with Emanuel Milman, Erwin Lutwak, Gaoyong Zhang, Shibing Chen, Qi-Rui Li, Christos Saroglou and Martin Henk during the preparation of the manuscript. Also grateful for the hospitality of ETH Zürich during the completion of the survey.

References

- A.D. Aleksandrov: Zur Theorie der gemischten Volumina von konvexen Körpern II. Mat. Sbornik N.S., 2:1205-1238, 1937.
- [2] A.D. Aleksandrov: On the theory of mixed volumes. I. Extension of certain concepts in the theory of convex bodies. Mat. Sbornik N.S. 2 (1937), 947-972 (Russian; German summary).
- [3] A.D. Aleksandrov: On the theory of mixed volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies. Mat. Sbornik N.S. 3 (1938), 27-46 (Russian; German summary).
- [4] A.D. Aleksandrov: Existence and uniqueness of a convex surface with a given integral curvature. C. R. (Doklady) Acad. Sci. URSS (N.S.) 35 (1942), 131-134.
- [5] A.D. Aleksandrov: Selected works. Part I. Gordon and Breach Publishers, Amsterdam, 1996.
- [6] L. Ambrosio, A. Colesanti, E. Villa: Outer Minkowski content for some classes of closed sets. Math. Ann., 342 (2008), 727-748.

- [7] B. Andrews: Gauss curvature flow: the fate of rolling stone. Invent. Math., 138 (1999), 151-161.
- [8] B. Andrews, P. Guan, L. Ni: Flow by the power of the Gauss curvature. Adv. Math., 299 (2016), 174-201.
- [9] S. Artstein-Avidan, D.I. Florentin, A. Segal: Functional Brunn-Minkowski inequalities induced by polarity. Adv. Math., 364 (2020), 107006, 19 pp.
- [10] S. Artstein-Avidan, B. Klartag, V.D. Milman: On the Santaló point of a function and a functional Santaló inequality, Mathematika 54 (2004), 33-48.
- [11] S. Artstein-Avidan, A. Giannopoulos, V.D. Milman: Asymptotic geometric analysis. Part I. Mathematical Surveys and Monographs, 202. American Mathematical Society, Providence, RI, 2015.
- [12] S. Artstein-Avidan, A. Giannopoulos, V.D. Milman: Asymptotic geometric analysis. Part II. Mathematical Surveys and Monographs, 261. American Mathematical Society, Providence, RI, 2021.
- [13] K.M. Ball: Isoperimetric problems in ℓ_p and sections of convex sets. PhD thesis, University of Cambridge, 1986.
- [14] K.M. Ball: Volume ratios and a reverse isoperimetric inequality. J. London Math. Soc. 44 (1991), 351-359.
- [15] K.M. Ball: An elementary introduction to modern convex geometry. In: Flavors of geometry (Silvio Levy, ed.), Cambridge University Press, 1997, 1-58.
- [16] K.M. Ball, K.J. Böröczky: Stability of the Prékopa-Leindler inequality. Mathematika 56 (2010), 339-356.
- [17] K.M. Ball, K.J. Böröczky: Stability of some versions of the Prékopa-Leindler inequality. Monatsh. Math., 163 (2011), 1-14.
- [18] Z.M. Balogh, A. Kristály: Equality in Borell-Brascamp-Lieb inequalities on curved spaces. Adv. Math., 339 (2018), 453-494.
- [19] M. Barchiesi, V. Julin: Robustness of the Gaussian concentration inequality and the Brunn-Minkowski inequality. Calc. Var. Partial Differential Equations, 56 (2017), Paper No. 80, 12 pp.

- [20] F. Barthe: Inégalité fonctionelles et géometriques obtenues par transport des mesures, PhD thesis, Université de Marne-la-Vallée, Paris, 1997.
- [21] F. Barthe: Autour de l'inégalité de Brunn-Minkowski. Mémoire d'Habilitation, 2008.
- [22] F. Barthe: On a reverse form of the Brascamp-Lieb inequality. Invent. Math., 134 (1998), 335-361.
- [23] F. Barthe, K.J. Böröczky, M. Fradelizi: Stability of the functional forms of the Blaschke-Santaló inequality. Monatsh. Math. 173 (2014), no. 2, 135-159.
- [24] F. Barthe, D. Cordero-Erausquin: Invariances in variance estimates. Proc. Lond. Math. Soc., (3) 106 (2013), 33-64.
- [25] F. Barthe, M. Fradelizi: The volume product of convex bodies with many hyperplane symmetries. Amer. J. Math., 135 (2013), 311-347.
- [26] F. Barthe, O. Guédon, S. Mendelson, A. Naor: A probabilistic approach to the geometry of the l_p^n -ball. Ann. of Probability, 33 (2005), 480-513.
- [27] J. Bertrand: Prescription of Gauss curvature using optimal mass transport. Geom. Dedicata, 183 (2016), 81-99.
- [28] G. Bianchi, K.J. Böröczky, A. Colesanti: The Orlicz version of the L_p Minkowski problem for -n . Adv. in Appl. Math., 111 (2019),101937, 29 pp.
- [29] G. Bianchi, K.J. Böröczky, A. Colesanti: Smoothness in the L_p Minkowski problem for p < 1. J. Geom. Anal., 30 (2020), 680-705.
- [30] G. Bianchi, K.J. Böröczky, A. Colesanti, D. Yang: The L_p -Minkowski problem for -n according to Chou-Wang. Adv. Math., 341 (2019), 493-535.
- [31] C. Bianchini, A. Colesanti, D. Pagnini, A. Roncoroni: On *p*-Brunn–Minkowski inequalities for intrinsic volumes, with $0 \le p < 1$. Math Annalen, https://doi.org/10.1007/s00208-022-02454-0.
- [32] G. Bianchi, H. Egnell: A note on the Sobolev inequality. J. Funct. Anal. 100 (1991), 18-24.

- [33] S.G. Bobkov, A. Colesanti, I. Fragalà: Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities. Manuscripta Math., 143 (2014), 131-169.
- [34] F. Bolley, D. Cordero-Erausquin, Y. Fujita, I. Gentil, A. Guillin: New sharp Gagliardo-Nirenberg-Sobolev inequalities and an improved Borell-Brascamp-Lieb inequality. Int. Math. Res. Not. IMRN, 2020, 3042-3083.
- [35] B. Bollobás, I. Leader: Products of unconditional bodies. Geometric aspects of functional analysis (Israel, 1992–1994), Oper. Theory Adv. Appl., 77, Birkhauser, Basel, (1995), 13-24.
- [36] T. Bonnesen, W. Fenchel: Theory of convex bodies. BCS Associates, Moscow, ID, 1987.
- [37] C. Borell: Convex set functions in d-space. Period. Math. Hungar. 6 (1975), 111-136.
- [38] K.J. Böröczky: Stability of the Blaschke-Santaló and the affine isoperimetric inequality. Advances in Mathematics, 225 (2010), 1914-1928.
- [39] K.J. Böröczky, A. De: Stability of the Prékopa-Leindler inequality for log-concave functions. Adv. Math., 386 (2021), 107810.
- [40] K.J. Böröczky, A. De: Stable solution of the log-Minkowski problem in the case of hyperplane symmetries. J. Diff. Eq., 298 (2021), 298-322.
- [41] K.J. Böröczky, A. De: Stability of the log-Brunn-Minkowski inequality in the case of many hyperplane symmetries. arXiv:2101.02549
- [42] K.J. Böröczky, A. Figalli, J.P.G. Ramos: A quantitative stability result for the Prékopa-Leindler inequality for arbitrary measurable functions. submitted. arXiv:2201.11564
- [43] K.J. Böröczky, F. Fodor: The L_p dual Minkowski problem for p > 1 and q > 0. J. Differential Equations, 266 (2019), 7980-8033.
- [44] K.J. Böröczky, P. Hegedűs: The cone volume measure of antipodal points. Acta Mathematica Hungarica, 146 (2015), 449-465.
- [45] K.J. Böröczky, P. Hegedűs, G. Zhu: On the discrete logarithmic Minkowski problem. Int. Math. Res. Not. IMRN, (2016), 1807-1838.
- [46] K.J. Böröczky, M. Henk: Cone-volume measure of general centered convex bodies. Adv. Math., 286 (2016), 703-721.

- [47] K.J. Böröczky, M. Henk: Cone-volume measure and stability. Adv. Math., 306 (2017), 24-50.
- [48] K.J. Böröczky, P. Kalantzopoulos: Log-Brunn-Minkowski inequality under symmetry. Trans. AMS, 375 (2022), 5987-6013.
- [49] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang: The log-Brunn-Minkowski-inequality. Adv. Math., 231 (2012), 1974-1997.
- [50] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang: The Logarithmic Minkowski Problem. Journal of the American Mathematical Society, 26 (2013), 831-852.
- [51] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang: Affine images of isotropic measures. J. Diff. Geom., 99 (2015), 407-442.
- [52] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Yiming Zhao: The dual Minkowski problem for symmetric convex bodies. Adv. Math., 356 (2019), 106805.
- [53] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Yiming Zhao: The Gauss image problem. Communications on Pure and Applied Mathematics, 73 (2020), 1406-1452.
- [54] K.J. Böröczky, C. Saroglou: Uniqueness when the L_p curvature is close to be a constant for $p \in [0, 1)$. submitted. arXiv:2308.03367
- [55] K.J. Böröczky, Hai T. Trinh: The planar L_p -Minkowski problem for 0 . Adv. Applied Mathematics, 87 (2017), 58-81.
- [56] H.J. Brascamp, E.H. Lieb: On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Functional Analysis, 22 (1976), 366-389.
- [57] S. Brendle, K. Choi, P. Daskalopoulos: Asymptotic behavior of flows by powers of the Gaussian curvature. Acta Math., 219 (2017), 1-16.
- [58] P. Bryan, M. Ivaki, J. Scheuer: A unified flow approach to smooth, even Lp-Minkowski problems. Anal. PDE 12 (2019), 259-280.
- [59] S. Bubeck, R. Eldan, J. Lehec: Sampling from a log-concave distribution with projected Langevin Monte Carlo. Discrete Comput. Geom., 59 (2018), 757-783.

- [60] D. Bucur, I. Fragalà: Lower bounds for the Prékopa-Leindler deficit by some distances modulo translations. J. Convex Anal., 21 (2014), 289-305.
- [61] J.R. Bueno, P. Pivarov: A stochastic Prékopa-Leindler inequality for log-concave functions. Commun. Contemp. Math., https://doi.org/10.1142/S0219199720500194, arXiv:1912.06904
- [62] L.A. Caffarelli: Interior W^{2,p}-estimates for solutions of the Monge-Ampère equation. Ann. Math. (2), 131, 135-150 (1990).
- [63] L.A. Caffarelli: A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. Ann. Math. (2) 131 (1990), 129-134.
- [64] Xiaxing Cai, Gangsong Leng, Yuchi Wu, Dongmeng Xi: Affine dual Minkowski problems. manuscript
- [65] U. Caglar, E.M. Werner: Stability results for some geometric inequalities and their functional versions. Convexity and concentration, IMA Vol. Math. Appl., 161, Springer, New York, 2017, 541-564.
- [66] E.A. Carlen, D. Cordero-Erausquin: Subadditivity of the entropy and its relation to Brascamp-Lieb type inequalities. Geom. Funct. Anal., 19 (2009), 373-405.
- [67] E.A. Carlen, F. Maggi: Stability for the Brunn-Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry. Canad. J. Math., 69 (2017), 1036-1063.
- [68] H. Chen, Q-R. Li: The L_p dual Minkowski problem and related parabolic flows. J. Funct. Anal. 281 (2021), Paper No. 109139, 65 pp.
- [69] S. Chen, Y. Feng, W. Liu: Uniqueness of solutions to the logarithmic Minkowski problem in ℝ³. Adv. Math. 411 (2022), part A, Paper No. 108782, 18 pp.
- [70] Shibing Chen, Shengnan Hu, Weiru Liu, Yiming Zhao: On the planar Gaussian-Minkowski problem. arXiv:2303.17389
- [71] S. Chen, Y. Huang, Q.-R. Li, J. Liu: The L_p -Brunn-Minkowski inequality for p < 1. Adv. Math., 368 (2020), 107166.
- [72] S. Chen, Q.-R. Li: On the planar dual Minkowski problem. Adv. Math., 333 (2018), 87-117.

- [73] S. Chen, Q.-R. Li, G. Zhu: On the L_p Monge-Ampère equation. Journal of Differential Equations, 263 (2017), 4997-5011.
- [74] S. Chen, Q.-R. Li, G. Zhu: The Logarithmic Minkowski Problem for non-symmetric measures. Trans. Amer. Math. Soc., 371 (2019), 2623-2641.
- [75] S. Chen, J. Liu, Xu-Jia Wang: Global regularity for the Monge-Ampère equation with natural boundary condition. Ann. of Math. (2,) 194 (2021), 745-793.
- [76] Y. Chen: An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture. GAFA, 31 (2021), 34-61.
- [77] S.-Y. Cheng, S.-T. Yau: On the regularity of the solution of the *n*dimensional Minkowski problem. Comm. Pure Appl. Math. 29 (1976), 495-561.
- [78] K. S. Chou, X. J. Wang: The L_p -Minkowski problem and the Minkowski problem in centroaffine geometry. Adv. Math., 205 (2006), 33-83.
- [79] M. Christ: An approximate inverse Riesz–Sobolev inequality, preprint, available online at http://arxiv.org/abs/1112.3715, 2012.
- [80] A. Colesanti, I. Fragalá: The first variation of the total mass of logconcave functions and related inequalities. Adv. Math., 244 (2013), 708-749.
- [81] A. Colesanti, G. Livshyts, A. Marsiglietti: On the stability of Brunn-Minkowski type inequalities. J. Funct. Anal. 273 (2017), 1120-1139.
- [82] A. Colesanti, G. Livshyts: A note on the quantitative local version of the log-Brunn-Minkowski inequality. The mathematical legacy of Victor Lomonosov-operator theory, 85-98, Adv. Anal. Geom., 2, De Gruyter, Berlin, 2020.
- [83] D. Cordero-Erausquin: Transport inequalities for log-concave measures, quantitative forms, and applications. Canad. J. Math. 69 (2017), 481-501. arXiv:1504.06147
- [84] D. Cordero-Erausquin, M. Fradelizi, B. Maurey: The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems. J. Funct. Anal., 214 (2004), 410-427.

- [85] D. Cordero-Erausquin, B. Klartag: Moment measures. J. Funct. Anal., 268 (2015), 3834-3866.
- [86] D. Cordero-Erausquin, L. Rotem: Improved log-concavity for rotationally invariant measures of symmetric convex sets. Annals Prob., accepted. arXiv:2111.05110
- [87] Crasta, Fragalà: On a geometric combination of functions related to Prékopa-Leindler inequality. Mathematika, 69 (2023), 482-507.
- [88] S. Dar: A Brunn-Minkowski-type inequality. Geom. Dedicata, 77 (1999), 1-9.
- [89] V.I. Diskant: Stability of the solution of a Minkowski equation. (Russian) Sibirsk. Mat. Ž. 14 (1973), 669–673. [Eng. transl.: Siberian Math. J., 14 (1974), 466–473.]
- [90] S.-Z. Du: On the planar L_p -Minkowski problem. arXiv:2109.15280
- [91] S. Dubuc: Critères de convexité et inégalités integralés. Ann. Inst. Fourier Grenoble 27 (1) (1977), 135-165.
- [92] R. Eldan: Thin shell implies spectral gap up to polylog via a stochastic localization scheme. Geom. Funct. Anal., 23 (2013), 532-569.
- [93] R. Eldan, B. Klartag: Pointwise estimates for marginals of convex bodies. J. Funct. Anal., 254 (2008), 2275-2293.
- [94] R. Eldan, B. Klartag: Dimensionality and the stability of the Brunn-Minkowski inequality. Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (5) 13 (2014), 975-1007.
- [95] R. Eldan, J. Lehec, Y. Shenfeld: Stability of the logarithmic Sobolev inequality via the Föllmer process. Ann. Inst. Henri Poincaré Probab. Stat. 56 (2020), 2253-2269.
- [96] A. Eskenazis, G. Moschidis: The dimensional Brunn-Minkowski inequality in Gauss space. J. Funct. Anal. 280 (2021), Paper No. 108914, 19 pp.
- [97] Niufa Fang, Sudan Xing, Deping Ye: Geometry of log-concave functions: the L_p Asplund sum and the L_p Minkowski problem. Calc. Var. Partial Differential Equations, 61 (2022), Paper No. 45, 37 pp.

- [98] Niufa Fang, Deping Ye, Zengle Zhang, Yiming Zhao: The dual Orlicz curvature measures for log-concave functions and their related Minkowski problems.
- [99] W. Fenchel, B. Jessen: Mengenfunktionen und konvexe Körper, Danske Vid. Selskab. Mat.-fys. Medd. 16 (1938), 1-31.
- [100] Y. Feng, S. Hu, W. Liu: Existence and uniqueness of solutions to the Orlicz Aleksandrov problem. Calc. Var. Partial Differential Equations, 61 (2022), Paper No. 148, 23 pp.
- [101] Y. Feng, S. Hu, L. Xu: On the L_p Gaussian Minkowski problem. J. Differential Equations, 363 (2023), 350-390.
- [102] Y. Feng, W. Liu, L. Xu: Existence of non-symmetric solutions to the Gaussian Minkowski problem. J. Geom. Anal., 33 (2023), no. 3, Paper No. 89, 39 pp.
- [103] A. Figalli, D. Jerison: Quantitative stability for sumsets in \mathbb{R}^n . J. Eur. Math. Soc. (JEMS) 17 (2015), 1079-1106.
- [104] A. Figalli, D. Jerison: Quantitative stability for the Brunn-Minkowski inequality. Adv. Math. 314 (2017), 1-47.
- [105] A. Figalli, F. Maggi, A. Pratelli: A refined Brunn-Minkowski inequality for convex sets. Annales de IHP, 26 (2009), 2511-2519.
- [106] A. Figalli, F. Maggi, A. Pratelli: A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math., 182 (2010), 167-211.
- [107] A. Figalli, R. Neumayer: Gradient stability for the Sobolev inequality: the case $p \ge 2$. J. Eur. Math. Soc. (JEMS), 21 (2019), 319-354.
- [108] A. Figalli, Yi Ru-Ya Zhang: Sharp gradient stability for the Sobolev inequality. arXiv:2003.04037
- [109] A. Freyer, M. Henk, C. Kipp: Affine Subspace Concentration Conditions for Centered Polytopes. Mathematika, 69 (2023), 458-472.
- [110] W. J. Firey: *p*-means of convex bodies. Math. Scand., 10 (1962), 17-24.
- [111] W.J. Firey: Shapes of worn stones. Mathematika, 21 (1974), 1-11.

- [112] M. Fradelizi, M. Meyer: Some functional forms of Blaschke-Santaló inequality. Math. Z., 256 (2007), 379-395.
- [113] N. Fusco, F. Maggi, A. Pratelli: The sharp quantitative isoperimetric inequality. Ann. of Math. 168 (2008), no. 3, 941–980.
- [114] R.J. Gardner: The Brunn-Minkowski inequality. Bull. AMS, 29 (2002), 335-405.
- [115] R. Gardner, D. Hug, W. Weil, S. Xing, D. Ye: General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem I. Calc. Var. Partial Differential Equations 58 (2019), Paper No. 12, 35 pp.
- [116] R. Gardner, D. Hug, S. Xing, D. Ye: General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem II. Calc. Var. Partial Differential Equations 59 (2020), Paper No. 15, 33 pp.
- [117] R. Gardner, A. Zvavitch: Gaussian Brunn-Minkowski-type inequalities. Trans. Amer. Math. Soc., 362 (2010), 5333-5353.
- [118] C. Gerhardt: Minkowski type problems for convex hypersurfaces in hyperbolic space. arXiv:math/0602597
- [119] D. Ghilli, P. Salani: Quantitative Borell-Brascamp-Lieb inequalities for power concave functions. J. Convex Anal. 24 (2017), 857-888.
- [120] N. Gozlan: The deficit in the Gaussian log-Sobolev inequality and inverse Santalo inequalities. International Mathematics Research Notices IMRN, arXiv:2007.05255
- [121] H. Groemer: On the Brunn-Minkowski theorem. Geom. Dedicata, 27 (1988), 357-371.
- [122] H. Groemer: Stability of geometric inequalities. In: Handbook of convex geometry (P.M. Gruber, J.M. Wills, eds), North-Holland, Amsterdam, 1993, 125-150.
- [123] M. Gromov, V.D. Milman: Generalization of the spherical isoperimetric inequality for uniformly convex Banach Spaces. Composito Math., 62 (1987), 263-282.
- [124] P. Guan, X. Ma: The Christoffel-Minkowski problem. I. Convexity of solutions of a Hessian equation. Invent. Math., 151(3) (2003), 553-577.

- [125] P. Guan, L. Ni: Entropy and a convergence theorem for Gauss curvature flow in high dimension. J. EMS, 19 (2017), 3735-3761.
- [126] P. Guan, C. Xia: L^p Christoffel-Minkowski problem: the case 1 . Calc. Var. Partial Differential Equations, 57 (2018), Paper No. 69, 23 pp.
- [127] Q. Guang, Q-R. Li, X.-J. Wang: Flow by Gauss curvature to the L_p dual Minkowski problem. Math. Eng. 5 (2023), no. 3, Paper No. 049, 19 pp.
- [128] Q. Guang, Q-R. Li, X.-J. Wang: The L_p -Minkowski problem with super-critical exponents. arXiv:2203.05099
- [129] Q. Guang, Q-R. Li, X.-J. Wang: The Minkowski problem in the sphere. https://person.zju.edu.cn/person/attachments/2021-02/01-1612235356-841400.pdf
- [130] Q. Q-R. Li, Wang: Existence of Guang, X.-J. convex hypersurfaces with prescribed centroaffine curvature. https://person.zju.edu.cn/person/attachments/2022-02/01-1645171178-851572.pdf
- [131] Lujun Guo, Dongmeng Xi, Yiming Zhao: The L_p chord Minkowski problem in a critical interval. Mathematische Annalen, https://doi.org/10.1007/s00208-023-02721-8
- [132] C. Haberl, E. Lutwak, D. Yang, G. Zhang: The even Orlicz Minkowski problem. Adv. Math. 224(6), 2485-2510 (2010)
- [133] R. van Handel: The local logarithmic Brunn-Minkowski inequality for zonoids. arXiv:2202.09429
- [134] R. van Handel, Y. Shenfeld: Mixed volumes and the Bochner method. Proc. Amer. Math. Soc., 147(12) (2019), 5385-5402.
- [135] R. van Handel, Y. Shenfeld: The Extremals of Minkowski's Quadratic Inequality. Duke Math. J., 171 (2022), 957-1027. arXiv:1902.10029
- [136] R. van Handel, Y. Shenfeld: The extremals of the Alexandrov-Fenchel inequality for convex polytopes. Acta Math., accepted. arXiv:2011.04059
- [137] D. Harutyunyan: Quantitative anisotropic isoperimetric and Brunn-Minkowski inequalities for convex sets with improved defect estimates. ESAIM Control Optim. Calc. Var., 24 (2018), 479-494.

- [138] B. He, G. Leng, K. Li: Projection problems for symmetric polytopes. Adv. Math., 207 (2006), 73-90.
- [139] M. Henk, E. Linke: Cone-volume measures of polytopes. Adv. Math., 253 (2014), 50-62.
- [140] M. Henk, H. Pollehn: Necessary subspace concentration conditions for the even dual Minkowski problem. Adv. Math., 323 (2018), 114-141.
- [141] M. Henk, A. Schürman, J.M. Wills: Ehrhart polynomials and successive minima, Mathematika, 52 (2006), 1-16.
- [142] P. van Hintum, H. Spink, M. Tiba: Sharp Stability of Brunn-Minkowski for Homothetic Regions. J. Eur. Math. Soc. (JEMS), 24 (2022), 4207-4223.
- [143] P. van Hintum, H. Spink, M. Tiba: Sharp quantitative stability of the planar Brunn-Minkowski inequality. arXiv:1911.11945
- [144] J. Hosle, A.V. Kolesnikov, G.V. Livshyts: On the L_p -Brunn-Minkowski and dimensional Brunn-Minkowski conjectures for logconcave measures. J. Geom. Anal., 31 (2021), 5799-5836.
- [145] C. Hu, X. Ma, C. Shen: On the Christoffel-Minkowski problem of Firey's p-sum. Calc. Var. Partial Differ. Equ., 21(2) (2004), 137-155.
- [146] Q. Huang, B. He: On the Orlicz Minkowski problem for polytopes. Discrete Comput. Geom., 48 (2012), 281-297.
- [147] Y. Huang, J. Liu, D. Xi, Y. Zhao: Dual curvature measures for logconcave functions. arXiv:2210.02359
- [148] Y. Huang, E. Lutwak, D. Yang, G. Zhang: Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems. Acta Math. 216 (2016), 325-388.
- [149] Y. Huang, E. Lutwak, D. Yang, G. Zhang: The L_p -Aleksandrov problem for L_p -integral curvature. J. Differ. Geom., 110 (2018), 1-29.
- [150] Y. Huang, D. Xi, Y. Zhao: The Minkowski problem in Gaussian probability space. Adv. Math., 385 (2021), 107769.
- [151] Y. Huang, Y. Zhao: On the Lp dual Minkowski problem. Adv. Math. 332 (2018), 57-84.

- [152] D. Hug, E. Lutwak, D. Yang, G. Zhang: On the L_p Minkowski problem for polytopes. Discrete Comput. Geom., 33 (2005), 699-715.
- [153] D. Hug: Contributions to affine surface area. Manuscripta Math., 91 (1996), 283-301.
- [154] D. Hug, R. Schneider: Hölder continuity for support measures of convex bodies. Arch. Math. (Basel), 104 (2015), 83-92.
- [155] M.N. Ivaki: Deforming a convex hypersurface with low entropy by its Gauss curvature. J. Geom. Anal., 27 (2017), 1286-1294.
- [156] M. Ivaki: On the stability of the L_p -curvature. JFA, 283 (2022), 109684.
- [157] M. Ivaki, E. Milman: Uniqueness of solutions to a class of isotropic curvature problems. arXiv:2304.12839
- [158] H. Jian, J. Lu Existence of the solution to the Orlicz-Minkowski problem. Adv. Math., 344 (2019), 262-288
- [159] H. Jian, J. Lu, X-J. Wang: Nonuniqueness of solutions to the L_p -Minkowski problem. Adv. Math., 281 (2015), 845-856.
- [160] H. Jian, J. Lu, G. Zhu: Mirror symmetric solutions to the centro-affine Minkowski problem. Calc. Var. Partial Differential Equations, 55 (2016), Art. 41, 22 pp.
- [161] P. Kalantzopoulos, C. Saroglou: On a j-Santaló Conjecture. arXiv:2203.14815
- [162] R. Kannan, L. Lovász, M. Simonovits: Isoperimetric problems for convex bodies and a localization lemma. Discrete Comput. Geom., 13 (1995), 541-559.
- [163] B. Klartag: On nearly radial marginals of high-dimensional probability measures. J. Eur. Math. Soc. (JEMS), 12 (2010), 723-754.
- [164] B. Klartag, J. Lehec: Bourgain's slicing problem and KLS isoperimetry up to polylog. arXiv:2203.15551
- [165] A.V. Kolesnikov: Mass transportation functionals on the sphere with applications to the logarithmic Minkowski problem. Mosc. Math. J., 20 (2020), 67-91.

- [166] A.V. Kolesnikov, E.D. Kosov: Moment measures and stability for Gaussian inequalities. Theory Stoch. Process., 22 (2017), 47-61.
- [167] A.V. Kolesnikov, G. V. Livshyts: On the Gardner-Zvavitch conjecture: symmetry in inequalities of Brunn-Minkowski type. Adv. Math. 384 (2021), Paper No. 107689, 23 pp.
- [168] A.V. Kolesnikov, G. V. Livshyts: On the Local version of the Log-Brunn-Minkowski conjecture and some new related geometric inequalities. Int. Math. Res. Not. IMRN, 18 (2022), 14427-14453.
- [169] A.V. Kolesnikov, E. Milman: Local L_p -Brunn-Minkowski inequalities for p < 1. Memoirs of the American Mathematical Society, 277 (2022), no. 1360.
- [170] A.V. Kolesnikov, E.M. Werner: Blaschke-Santalo inequality for many functions and geodesic barycenters of measures. Adv. Math. 396 (2022), Paper No. 108110, 44 pp.
- [171] H. Knothe: Contributions to the theory of convex bodies. Michigan Math. J. 4 (1957), 39-52.
- [172] J. Lehec: A direct proof of the functional Santaló inequality. C. R. Math. Acad. Sci. Paris 347 (2009), no. 1-2, 55-58.
- [173] J. Lehec: Partitions and functional Santaló inequalities. Arch. Math. (Basel) 92 (2009), no. 1, 89-94.
- [174] K. Leichtweiß: Affine geometry of convex bodies. Johann Ambrosius Barth Verlag, Heidelberg, 1998.
- [175] L. Leindler: On a certain converse of Hölder's inequality. II. Acta Sci. Math. (Szeged) 33 (1972), 217-223.
- [176] H. Lewy: On differential geometry in the large. I. Minkowski problem. Trans. Amer. Math. Soc. 43, (1938), 258-270.
- [177] H. Li, Y. Wan: Classification of solutions for the planar isotropic L_p dual Minkowski problem. arXiv:2209.14630
- [178] Q-R. Li: Infinitely many solutions for the centro-affine Minkowski problem. Int. Math. Res. Not., IMRN, (2019) 5577-5596.
- [179] Q-R. Li, J. Liu, J. Lu: Non-uniqueness of solutions to the dual L_p -Minkowski problem. IMRN, (2022), 9114-9150.

- [180] Q-R. Li, W. Sheng, D. Ye, C. Yi: A flow approach to the Musielak-Orlicz-Gauss image problem. Adv. Math. 403 (2022), Paper No. 108379, 40 pp.
- [181] Q-R. Li, W. Sheng, X-J. Wang: Flow by Gauss curvature to the Aleksandrov and dual Minkowski problems. J EMS, 22 (2020), 893-923.
- [182] Yuanyuan Li: The L_p chord Minkowski problem for negative p. arXiv:2304.11299.
- [183] Yuanyuan Li: Nonuniqueness of solutions to the L_p chord Minkowski problem. arXiv:2304.12714.
- [184] J. Liu: The L_p -Gaussian Minkowski problem. Calc. Var. Partial Differential Equations 61 (2022), Paper No. 28, 23 pp.
- [185] Y. Liu, J. Lu: A flow method for the dual Orlicz-Minkowski problem. Trans. Amer. Math. Soc. 373 (2020), no. 8, 5833-5853.
- [186] G.V. Livshyts: An extension of Minkowski's theorem and its applications to questions about projections for measures. Advances in Mathematics, 356 (2019), 106803, 40 pp.
- [187] G.V. Livshyts: On a conjectural symmetric version of Ehrhard's inequality. submitted. arXiv:2103.11433
- [188] G.V. Livshyts: A universal bound in the dimensional Brunn-Minkowski inequality for log-concave measures. submitted. arXiv:2107.00095
- [189] G. Livshyts, A. Marsiglietti, P. Nayar, A. Zvavitch: On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities. Trans. Amer. Math. Soc. 369 (2017), no. 12, 8725-8742.
- [190] Fangxia Lu, Zhaonian Pu: The L_p dual Minkowski problem about 0 and <math>q > 0. Open Math., 19 (2021), 1648-1663.
- [191] M. Ludwig: General affine surface areas. Adv. Math., 224 (2010), 2346-2360.
- [192] M. Ludwig, J. Xiao, G. Zhang: Sharp convex Lorentz-Sobolev inequalities. Math. Ann., 350 (2011), 169-197.

- [193] E. Lutwak: Selected affine isoperimetric inequalities. In: Handbook of convex geometry, North-Holland, Amsterdam, 1993, 151-176.
- [194] E. Lutwak: The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. J. Differential Geom. 38 (1993), 131-150.
- [195] E. Lutwak: The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas. Adv. Math., 118 (1996), 244-294.
- [196] E. Lutwak, D. Xi, D. Yang, G. Zhang: Chord measures in integral geometry and their Minkowski problems. Comm. Pure Appl. Math., accepted.
- [197] E. Lutwak, D. Yang, G. Zhang: L_p John ellipsoids. Proc. London Math. Soc., (3) 90 (2005), 497-520.
- [198] E. Lutwak, D. Yang, G. Zhang: The Brunn-Minkowski-Firey inequality for nonconvex sets. Adv. in Appl. Math., 48 (2012), 407-413.
- [199] E. Lutwak, D. Yang, G. Zhang: L_p dual curvature measures. Adv. Math., 329 (2018), 85-132.
- [200] F. Maggi: Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory. Cambridge, 2012.
- [201] M. Meyer, A. Pajor: On the Blaschke-Santaló inequality. Arch. Math. (Basel) 55 (1990), 82-93.
- [202] A. Marsiglietti: Borell's generalized Prékopa-Leindler inequality: a simple proof. J. Convex Anal. 24 (2017), 807-817.
- [203] M. Meyer, A. Pajor: On the Blaschke-Santaló inequality. Arch. Math. (Basel) 55 (1990), 82-93.
- [204] E. Milman: A sharp centro-affine isospectral inequality of Szegő-Weinberger type and the L_p -Minkowski problem. J. Diff. Geom., accepted. arXiv:2103.02994
- [205] E. Milman: Centro-Affine Differential Geometry and the Log-Minkowski Problem. arXiv:2104.12408
- [206] V.D. Milman, G. Schechtman: Asymptotic Theory of Finite-Dimensional Normed Spaces. Lecture Notes in Mathematics, vol. 1200. Springer, Berlin (1986). With an appendix by M. Gromov

- [207] H. Minkowski: Allgemeine Lehrsätze über die konvexen Polyeder. Nachr. Ges. Wiss. Göttingen (1897), 198-219.
- [208] H. Minkowski: Volumen und Oberfäche. Math. Ann., 57 (1903), 447-495.
- [209] S. Mui: On the L_p Aleksandrov problem for negative p. Adv. Math. 408 (2022), Paper No. 108573, 26 pp.
- [210] A. Naor: The surface measure and cone measure on the sphere of l_p^n . Trans. Amer. Math. Soc., 359 (2007), 1045-1079.
- [211] P. Nayar, T. Tkocz: A Note on a Brunn-Minkowski Inequality for the Gaussian Measure. Proc. Amer. Math. Soc., 141 (2013), 4027-4030.
- [212] P. Nayar, T. Tkocz: On a convexity property of sections of the crosspolytope. Proc. Amer. Math. Soc., 148 (2020), 1271-1278.
- [213] V.H. Nguyen: New approach to the affine Polya-Szego principle and the stability version of the affine Sobolev inequality. Adv. Math. 302 (2016), 1080-1110.
- [214] L. Nirenberg: The Weyl and Minkowski problems in differential geometry in the large. Comm. Pure and Appl. Math., 6 (1953), 337-394.
- [215] V. Oliker: Embedding S^n into \mathbb{R}^{n+1} with given integral Gauss curvature and optimal masstransport on S^n . Adv. Math., 213 (2007), 600-620.
- [216] G. Paouris, E. Werner: Relative entropy of cone measures and L_p centroid bodies. Proc. London Math. Soc., 104 (2012), 253-286.
- [217] A.V. Pogorelov: The Minkowski multidimensional problem. V.H. Winston & Sons, Washington, D.C, 1978.
- [218] A. Prékopa: Logarithmic concave measures with application to stochastic programming. Acta Sci. Math. (Szeged) 32 (1971), 301-316.
- [219] A. Prékopa: On logarithmic concave measures and functions. Acta Sci. Math. (Szeged) 34 (1973), 335-343.
- [220] A. Prékopa: New proof for the basic theorem of logconcave measures. (Hungarian) Alkalmaz. Mat. Lapok 1 (1975), 385-389.
- [221] E. Putterman: Equivalence of the local and global versions of the L_p -Brunn-Minkowski inequality. J. Func. Anal., 280 (2021), 108956.

- [222] A. Rossi, P. Salani: Stability for Borell-Brascamp-Lieb inequalities. Geometric aspects of functional analysis, Lecture Notes in Math., 2169, Springer, Cham, (2017), 339-363.
- [223] A. Rossi, P. Salani: Stability for a strengthened Borell-Brascamp-Lieb inequality. Appl. Anal., 98 (2019), 1773-1784.
- [224] L. Rotem: A letter: The log-Brunn-Minkowski inequality for complex bodies. arXiv:1412.5321
- [225] L. Rotem: The anisotropic total variation and surface area measures. Geometric aspects of functional analysis, Lecture Notes in Math., 2327, Springer, Cham, (2023), 297-312.
- [226] J. Saint-Raymond: Sur le volume des corps convexes symétriques. Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981, Exp. No. 11, 25 pp., Publ. Math. Univ. Pierre et Marie Curie, 46, Univ. Paris VI, Paris, 1981.
- [227] L.A. Santaló: An affine invariant for convex bodies of n-dimensional space. (Spanish) Portugaliae Math., 8 (1949), 155-161.
- [228] C. Saroglou: Remarks on the conjectured log-Brunn-Minkowski inequality. Geom. Dedicata 177 (2015), 353-365.
- [229] C. Saroglou: More on logarithmic sums of convex bodies. Mathematika, 62 (2016), 818-841.
- [230] C. Saroglou: On a non-homogeneous version of a problem of Firey. Math. Ann. 382 (2022), 1059-1090.
- [231] C. Saroglou: A non-existence result for the L_p -Minkowski problem. arXiv:2109.06545
- [232] A. Segal: Remark on stability of Brunn-Minkowski and isoperimetric inequalities for convex bodies. In: Geometric aspects of functional analysis, volume 2050 of Lecture Notes in Math., Springer, Heidelberg, 2012, 381-391.
- [233] R. Schneider: Convex Bodies: The Brunn-Minkowski Theory. Cambridge University Press, 2014.
- [234] A. Stancu: The discrete planar L_0 -Minkowski problem. Adv. Math., 167 (2002), 160-174.

- [235] A. Stancu: On the number of solutions to the discrete two-dimensional L_0 -Minkowski problem. Adv. Math., 180 (2003), 290-323.
- [236] A. Stancu: Prescribing centro-affine curvature from one convex body to another. Int. Math. Res. Not. IMRN, (2022), 1016-1044.
- [237] T. Tao, V. Vu: Additive combinatorics. Cambridge University Press, 2006.
- [238] B. Uhrin: Curvilinear extensions of the Brunn-Minkowski-Lusternik inequality. Adv. Math., 109 (2) (1994), 288-312.
- [239] N.S. Trudinger, X.-J. Wang: The Monge-Ampere equation and its geometric applications. In: Handbook of geometric analysis, Adv. Lect. Math. 7, Int. Press, Somerville, MA, 2008, 467-524.
- [240] C. Wu, D. Wu, N. Xiang: The L_p Gauss image problem. Geom. Dedicata 216 (2022), no. 6, Paper No. 62.
- [241] D. Xi, G. Leng: Dar's conjecture and the log-Brunn-Minkowski inequality. J. Differential Geom., 103 (2016), 145-189.
- [242] D. Xi, D. Yang, G. Zhang, Y. Zhao: The L_p chord Minkowski problem. Adv. Nonlinear Stud. 23 (2023), no. 1, Paper No. 20220041, 22 pp.
- [243] F. Xie: The Orlicz Minkowski Problem for general measures. Proc. Amer. Math. Soc., 150 (2022), 4433-4445.
- [244] Sudan Xing, Deping Ye: On the general dual Orlicz-Minkowski problem. Indiana Univ. Math. J., 69 (2020), 621-655.
- [245] Suding Xing, Deping Ye, B. Zhu: The general dual-polar Orlicz-Minkowski problem. J. Geom. Anal. 32 (2022), Paper No. 91, 40 pp.
- [246] Ge Xiong: Extremum problems for the cone-volume functional of convex polytopes. Adv. Math., 225 (2010), 3214-3228.
- [247] Tuo Wang: The affine Polya-Szego principle: Equality cases and stability. J. Funct. Anal., 265 (2013), 1728-1748.
- [248] Gaoyong Zhang: The affine Sobolev inequality. J. Differential Geom., 53 (1999), 183-202.
- [249] Yiming Zhao: The dual Minkowski problem for negative indices. Calc. Var. Partial Differential Equations, 56 (2017), no. 2, Paper No. 18, 16 pp.

- [250] Yiming Zhao: The L_p Aleksandrov problem for origin-symmetric polytopes. Proc. Amer. Math. Soc., 147 (2019), 4477-4492.
- [251] G. Zhu: The L_p Minkowski problem for polytopes for 0 . J. Funct. Anal., 269 (2015), 1070-1094.
- [252] G. Zhu: The L_p Minkowski problem for polytopes for p < 0. Indiana Univ. Math. J. 66 (2017), 1333-1350.