

# The case of equality in geometric instances of Barthe's reverse Brascamp-Lieb inequality

Karoly J. Boroczok\*, Pavlos Kalantzopoulos†, Dongmeng Xi‡

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## Abstract

The works of Bennett, Carbery, Christ, Tao and of Valdimarsson have clarified when equality holds in the Brascamp-Lieb inequality. Here we characterize the case of equality in the Geometric case of Barthe's reverse Brascamp-Lieb inequality.

## 1 Introduction

For a proper linear subspace  $E$  of  $\mathbb{R}^n$  ( $E \neq \mathbb{R}^n$  and  $E \neq \{0\}$ ), let  $P_E$  denote the orthogonal projection into  $E$ . We say that the subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  form a Geometric Brascamp-Lieb data if they satisfy

$$\sum_{i=1}^k c_i P_{E_i} = I_n. \quad (1)$$

The name ‘‘Geometric Brascamp-Lieb data’’ coined by Bennett, Carbery, Christ, Tao [17] comes from the following theorem, originating in the work of Brascamp, Lieb [22] and Ball [3, 4] in the rank one case ( $\dim E_i = 1$  for  $i = 1, \dots, k$ ), and Lieb [75] and Barthe [8] in the general case. In the rank one case, the Geometric Brascamp-Lieb data is known as Parseval frame in coding theory and computer science (see for example Casazza, Tran, Tremain [33]).

**Theorem 1 (Brascamp-Lieb, Ball, Barthe)** *For the linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), and for non-negative  $f_i \in L_1(E_i)$ , we have*

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(P_{E_i}x)^{c_i} dx \leq \prod_{i=1}^k \left( \int_{E_i} f_i \right)^{c_i} \quad (2)$$

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\*Alfred Renyi Institute of Mathematics, Realtanoda utca 13-15, 1053, Budapest, Hungary, Supported by NKFIH 132002

†Central European University, Nador utca 9, 1051, Budapest, Hungary

‡Department of Mathematics, Shanghai University, Shanghai 200444, China, Supported by National Natural Science Foundation of China (12071277).

**Remark** This is Hölder's inequality if  $E_1 = \dots = E_k = \mathbb{R}^n$  and  $B_i = I_n$ , and hence  $\sum_{i=1}^k c_i = 1$ .

We note that equality holds in Theorem 1 if  $f_i(x) = e^{-\pi\|x\|^2}$  for  $i = 1, \dots, k$ ; and hence, each  $f_i$  is a Gaussian density. Actually, Theorem 1 is an important special case discovered by Ball [4, 5] in the rank one case and by Barthe [8] in the general case of the general Brascamp-Lieb inequality Theorem 5.

After partial results by Barthe [8], Carlen, Lieb, Loss [31] and Bennett, Carbery, Christ, Tao [17], it was Valdimarsson [93] who characterized equality in the Geometric Brascamp-Lieb inequality. In order to state his result, we need some notation. Let  $E_1, \dots, E_k$  the proper linear subspaces of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfy (1). In order to understand extremizers in (5), following Carlen, Lieb, Loss [31] and Bennett, Carbery, Christ, Tao [17], we say that a non-zero linear subspace  $V$  is a critical subspace if

$$\sum_{i=1}^k c_i \dim(E_i \cap V) = \dim V,$$

which is turn equivalent saying that

$$E_i = (E_i \cap V) + (E_i \cap V^\perp) \text{ for } i = 1, \dots, k$$

according to [17] (see also Lemma 18). We say that a critical subspace  $V$  is indecomposable if  $V$  has no proper critical linear subspace.

Valdimarsson [93] introduced the so called independent subspaces and the dependent space. We write  $J$  to denote the set of  $2^k$  functions  $\{1, \dots, k\} \rightarrow \{0, 1\}$ . If  $\varepsilon \in J$ , then let  $F_{(\varepsilon)} = \bigcap_{i=1}^k E_i^{(\varepsilon(i))}$  where  $E_i^{(0)} = E_i$  and  $E_i^{(1)} = E_i^\perp$  for  $i = 1, \dots, k$ . We write  $J_0$  to denote the subset of  $\varepsilon \in J$  such that  $\dim F_{(\varepsilon)} \geq 1$ , and such an  $F_{(\varepsilon)}$  is called independent following Valdimarsson [93]. Readily  $F_{(\varepsilon)}$  and  $F_{(\tilde{\varepsilon})}$  are orthogonal if  $\varepsilon \neq \tilde{\varepsilon}$  for  $\varepsilon, \tilde{\varepsilon} \in J_0$ . In addition, we write  $F_{\text{dep}}$  to denote the orthogonal component of  $\bigoplus_{\varepsilon \in J_0} F_{(\varepsilon)}$ . In particular,  $\mathbb{R}^n$  can be written as a direct sum of pairwise orthogonal linear subspaces in the form

$$\mathbb{R}^n = \left( \bigoplus_{\varepsilon \in J_0} F_{(\varepsilon)} \right) \oplus F_{\text{dep}}. \quad (3)$$

Here it is possible that  $J_0 = \emptyset$ , and hence  $\mathbb{R}^n = F_{\text{dep}}$ , or  $F_{\text{dep}} = \{0\}$ , and hence  $\mathbb{R}^n = \bigoplus_{\varepsilon \in J_0} F_{(\varepsilon)}$  in that case.

For a non-zero linear subspace  $L \subset \mathbb{R}^n$ , we say that a linear transformation  $A : L \rightarrow L$  is positive definite if  $\langle Ax, y \rangle = \langle x, Ay \rangle$  and  $\langle x, Ax \rangle > 0$  for any  $x, y \in L \setminus \{0\}$ .

**Theorem 2 (Valdimarsson)** *For the proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), let us assume that equality holds in the Brascamp-Lieb inequality (2) for non-negative  $f_i \in L_1(E_i)$ ,  $i = 1, \dots, k$ . If  $F_{\text{dep}} \neq \mathbb{R}^n$ , then let  $F_1, \dots, F_\ell$  be the independent subspaces, and if  $F_{\text{dep}} = \mathbb{R}^n$ , then let  $\ell = 1$  and  $F_1 = \{0\}$ . There exist  $b \in F_{\text{dep}}$  and  $\theta_i > 0$  for*

$i = 1, \dots, k$ , integrable non-negative  $h_j : F_j \rightarrow [0, \infty)$  for  $j = 1, \dots, \ell$ , and a positive definite matrix  $A : F_{\text{dep}} \rightarrow F_{\text{dep}}$  such that the eigenspaces of  $A$  are critical subspaces and

$$f_i(x) = \theta_i e^{-\langle AP_{F_{\text{dep}}}x, P_{F_{\text{dep}}}x - b \rangle} \prod_{F_j \subset E_i} h_j(P_{F_j}(x)) \quad \text{for Lebesgue a.e. } x \in E_i. \quad (4)$$

On the other hand, if for any  $i = 1, \dots, k$ ,  $f_i$  is of the form as in (4), then equality holds in (2) for  $f_1, \dots, f_k$ .

Theorem 2 explains the term "independent subspaces" because the functions  $h_j$  on  $F_j$  are chosen freely and independently from each other.

A reverse form of the Geometric Brascamp-Lieb inequality was proved by Barthe [8]. We write  $\int_{*, \mathbb{R}^n} \varphi$  to denote the inner integral for a possibly non-integrable function  $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ ; namely, the supremum (actually maximum) of  $\int_{\mathbb{R}^n} \psi$  where  $0 \leq \psi \leq \varphi$  is Lebesgue measurable.

**Theorem 3 (Barthe)** *For the non-trivial linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), and for non-negative  $f_i \in L_1(E_i)$ , we have*

$$\int_{*, \mathbb{R}^n} \sup_{x = \sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i} dx \geq \prod_{i=1}^k \left( \int_{E_i} f_i \right)^{c_i}. \quad (5)$$

**Remark** This is the Prékopa-Leindler inequality Theorem 28 if  $E_1 = \dots = E_k = \mathbb{R}^n$  and  $B_i = I_n$ , and hence  $\sum_{i=1}^k c_i = 1$ .

The function  $x \mapsto \sup_{x = \sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}$  in Theorem 3 may not be measurable if  $f_1, \dots, f_k \geq 0$  are measurable but it is measurable if  $f_1, \dots, f_k$  are Borel (and hence analytic in the set theoretic sense). We note that Theorem 3 is usually stated using outer integrals (even in [8]) but it actually holds for inner integrals if one closely follows Barthe's argument. If  $f_1, \dots, f_k$  are  $C^1$  and positive, then [8] provides an elegant argument using optimal transport. For general measurable  $f_1, \dots, f_k$ , one can approximate each  $f_i$  by non-negative linear combination of characteristic functions of compact sets from below, and hence enough to consider such functions in order to prove Theorem 3. However, the characteristic function of a compact set can be approximated from above by a decreasing sequence of positive  $C^\infty$  functions, completing the proof of Theorem 3.

We say that a function  $h : \mathbb{R}^n \rightarrow [0, \infty)$  is log-concave if  $h((1-\lambda)x + \lambda y) \geq h(x)^{1-\lambda} h(y)^\lambda$  for any  $x, y \in \mathbb{R}^n$  and  $\lambda \in (0, 1)$ ; or in other words,  $h = e^{-W}$  for a convex function  $W : \mathbb{R}^n \rightarrow (-\infty, \infty]$ . Our main result is the following characterization of equality in the Geometric Barthe's inequality (5).

**Theorem 4** *For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), if  $F_{\text{dep}} \neq \mathbb{R}^n$ , then let  $F_1, \dots, F_\ell$  be the independent subspaces, and if  $F_{\text{dep}} = \mathbb{R}^n$ , then let  $\ell = 1$  and  $F_1 = \{0\}$ .*

If equality holds in the Geometric Barthe's inequality (5) for non-negative  $f_i \in L_1(E_i)$  with  $\int_{E_i} f_i > 0$ ,  $i = 1, \dots, k$ , then

$$f_i(x) = \theta_i e^{-\langle AP_{F_{\text{dep}}} x, P_{F_{\text{dep}}} x - b_i \rangle} \prod_{F_j \subset E_i} h_j(P_{F_j}(x - w_i)) \quad \text{for Lebesgue a.e. } x \in E_i \quad (6)$$

where

- $\theta_i > 0$ ,  $b_i \in E_i \cap F_{\text{dep}}$  and  $w_i \in E_i$  for  $i = 1, \dots, k$ ,
- $h_j \in L_1(F_j)$  is non-negative for  $j = 1, \dots, \ell$ , and in addition,  $h_j$  is log-concave if there exist  $\alpha \neq \beta$  with  $F_j \subset E_\alpha \cap E_\beta$ ,
- $A : F_{\text{dep}} \rightarrow F_{\text{dep}}$  is a positive definite matrix such that the eigenspaces of  $A$  are critical subspaces.

On the other hand, if for any  $i = 1, \dots, k$ ,  $f_i$  is of the form as in (6) and equality holds for all  $x \in E_i$  in (6), then equality holds in (5) for  $f_1, \dots, f_k$ .

**Remark** An independent subspace  $F_j$  is contained in a single  $E_\alpha$  if and only if  $E_i \subset E_\alpha^\perp$  for  $i \neq \alpha$ , and hence  $F_j = E_\alpha$ . In particular, if for any  $\alpha = 1, \dots, k$ ,  $\{E_i\}_{i \neq \alpha}$  spans  $\mathbb{R}^n$  in Theorem 4, then any extremizer of the Geometric Barthe's inequality is log-concave.

The explanation for the phenomenon concerning the log-concavity of  $h_j$  in Theorem 4 is as follows (see the proof of Proposition 29). Let  $\ell \geq 1$  and  $j \in \{1, \dots, \ell\}$ , and hence  $\sum_{E_i \supset F_j} c_i = 1$ . If  $f_1, \dots, f_k$  are of the form (6), then equality in Barthe's inequality (5) yields

$$\int_{*, F_j} \sup_{\substack{x = \sum_{E_i \supset F_j} c_i x_i \\ x_i \in F_j}} h_j(x_i - P_{F_j} w_i)^{c_i} dx = \prod_{E_i \supset F_j} \left( \int_{F_j} h_j(x - P_{F_j} w_i) dx \right)^{c_i} \left( = \int_{F_j} h_j(x) dx \right).$$

Therefore, if there exist  $\alpha \neq \beta$  with  $F_j \subset E_\alpha \cap E_\beta$ , then the equality conditions in the Prékopa-Leindler inequality Proposition 28 imply that  $h_j$  is log-concave. On the other hand, if there exists  $\alpha \in \{1, \dots, k\}$  such that  $F_j \subset E_\alpha^\perp$  for  $\beta \neq \alpha$ , then we do not have any condition on  $h_j$ , and  $c_\alpha = 1$ .

For completeness, let us state and discuss the general Brascamp-Lieb inequality and its reverse form due to Barthe. The following was proved by Brascamp, Lieb [22] in the rank one case and Lieb [75] in general.

**Theorem 5 (Brascamp-Lieb Inequality)** Let  $B_i : \mathbb{R}^n \rightarrow H_i$  be surjective linear maps where  $H_i$  is  $n_i$ -dimensional Euclidean space,  $n_i \geq 1$ , for  $i = 1, \dots, k$ , and let  $c_1, \dots, c_k > 0$  satisfy  $\sum_{i=1}^k c_i n_i = n$ . For non-negative  $f_i \in L_1(H_i)$ , we have

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(B_i x)^{c_i} dx \leq C \prod_{i=1}^k \left( \int_{H_i} f_i \right)^{c_i} \quad (7)$$

where  $C$  is determined by choosing centered Gaussians  $f_i(x) = e^{-\langle A_i x, x \rangle}$ ,  $A_i$  positive definite.

**Remark** The Geometric Brascamp-Lieb Inequality is readily a special case of (7). We note that (7) is Hölder's inequality if  $H_1 = \dots = H_k = \mathbb{R}^n$  and each  $B_i = I_n$ , and hence  $C = 1$  and  $\sum_{i=1}^k c_i = 1$  in that case.

We say that two Brascamp-Lieb data  $\{(B_i, c_i)\}_{i=1, \dots, k}$  and  $\{(B'_i, c'_i)\}_{i=1, \dots, k'}$  as in Theorem 5 are called equivalent if  $k' = k$ ,  $c'_i = c_i$ , and there exists linear isomorphism  $\Phi_i : H_i \rightarrow H'_i$  for  $i = 1, \dots, k$  such that  $B'_i = \Phi_i \circ B_i$ . It was proved by Carlen, Lieb, Loss [31] in the rank one case, and by Bennett, Carbery, Christ, Tao [17] in general that there exists a set of extremizers  $f_1, \dots, f_k$  for (7) if and only if the Brascamp-Lieb data  $\{(B_i, c_i)\}_{i=1, \dots, k}$  is equivalent to some Geometric Brascamp-Lieb data. Therefore, Valdimarsson's Theorem 2 provides a full characterization of the equality case in Theorem 5, as well.

The following reverse version of the Brascamp-Lieb inequality was proved by Barthe in [7] in the rank one case, and in [8] in general.

**Theorem 6 (Barthe's Inequality)** *Let  $B_i : \mathbb{R}^n \rightarrow H_i$  be surjective linear maps where  $H_i$  is  $n_i$ -dimensional Euclidean space,  $n_i \geq 1$ , for  $i = 1, \dots, k$ , and let  $c_1, \dots, c_k > 0$  satisfy  $\sum_{i=1}^k c_i n_i = n$ . For non-negative  $f_i \in L_1(H_i)$ , we have*

$$\int_{*, \mathbb{R}^n} \sup_{x = \sum_{i=1}^k c_i B_i^* x_i, x_i \in H_i} \prod_{i=1}^k f_i(x_i)^{c_i} dx \geq D \prod_{i=1}^k \left( \int_{H_i} f_i \right)^{c_i} \quad (8)$$

where  $D$  is determined by choosing centered Gaussians  $f_i(x) = e^{-\langle A_i x, x \rangle}$ ,  $A_i$  positive definite.

**Remark** The Geometric Barthe's Inequality (5) is readily a special case of (8). We note that (8) is the Prékopa-Leindler inequality if  $H_1 = \dots = H_k = \mathbb{R}^n$  and each  $B_i = I_n$ , and hence  $D = 1$  and  $\sum_{i=1}^k c_i = 1$  in that case.

Actually, again, Barthe [8] stated (8) with outer integrals, but his argument yields the same statement with inner inner integrals, see the discussion after Theorem 3.

Concerning extremals in Theorem 6, Lehec [69] proved that if there exists some Gaussian extremizers for Barthe's Inequality (8), then the corresponding Brascamp-Lieb data  $\{(B_i, c_i)\}_{i=1, \dots, k}$  is equivalent to some Geometric Brascamp-Lieb data; therefore, the equality case of (8) can be understood via Theorem 4 in that case.

However, it is still not known whether having any extremizers in Barthe's Inequality (8) yields the existence of Gaussian extremizers. One possible approach is to use iterated convolutions and renormalizations as in Bennett, Carbery, Christ, Tao [17] in the case of Brascamp-Lieb inequality.

There are three main methods of proofs that work for proving both the Brascamp-Lieb Inequality and its reverse form, Barthe's inequality. The paper Barthe [8] used optimal transportation to prove Barthe's Inequality ("the Reverse Brascamp-Lieb inequality") and reprove the Brascamp-Lieb Inequality simultaneously. A heat equation argument was provided in the rank one case by Carlen, Lieb, Loss [31] for the Brascamp-Lieb Inequality and by Barthe, Cordero-Erausquin [10] for Barthe's inequality. The general versions of both inequalities are proved via

the heat equation approach by Barthe, Huet [12]. Finally, simultaneous probabilistic arguments for the two inequalities are due to Lehec [69].

We note that Chen, Dafnis, Paouris [34] and Courtade, Liu [36], as well, deal systematically with finiteness conditions in Brascamp-Lieb and Barthe's inequalities. The importance of the Brascamp-Lieb inequality is shown by the fact that besides harmonic analysis, probability and convex geometry, it has been also even applied in number theory, see eg. Guo, Zhang [58]. Various versions of the Brascamp-Lieb inequality and its reverse form have been obtained by Balogh, Kristaly [6] Barthe [9], Barthe, Cordero-Erausquin [10], Barthe, Cordero-Erausquin, Ledoux, Maurey [11], Barthe, Wolff [13, 14], Bennett, Bez, Flock, Lee [15], Bennett, Bez, Buschenhenke, Cowling, Flock [16], Bobkov, Colesanti, Fragalà [19], Bueno, Pivarov [26], Chen, Dafnis, Paouris [34], Courtade, Liu [36], Duncan [40], Ghilli, Salani [46], Kolesnikov, Milman [68], Livshyts [72, 73], Lutwak, Yang, Zhang [77, 78], Maldague [79], Marsiglietti [80], Rossi, Salani [89, 90].

Concerning the proof of Theorem 4, we discuss the structure theory of a Brascamp-Lieb data, Barthe's crucial determinantal inequality (*cf.* Proposition 22) and the extremality of Gaussians (*cf.* Proposition 25) in Sections 3, 4 and 5. Section 6 explains how Barthe's proof of his inequality using optimal transportation in [8] yields the splitting along independent and dependent subspaces in the case of equality in Barthe's inequality for positive  $C^1$  probability densities  $f_1, \dots, f_k$ , and how the equality case of the Prékopa-Leindler inequality leads to the log-concavity of certain functions involved. However, one still needs to produce suitably smooth extremizers given any extremizers of Barthe's inequality. In order to achieve this, we discuss that convolution and suitable products of extremizers are also extremizers in Section 7. To show that extremizers are Gaussians on the dependent subspace, we use a version of Caffarelli's Contraction Principle in Section 8. Finally, all ingredients are pieced together to prove Theorem 4 in Section 9.

As applications of the understanding the equality case of the Brascamp-Lieb and Barthe's inequalities, we discuss the Bollobas-Thomason inequality and in its dual version in Section 2, and provide the characterization of the equality cases in Section 10.

## 2 Some applications: Equality in the Bollobas-Thomason inequality and in its dual

We write  $e_1, \dots, e_n$  to denote an orthonormal basis of  $\mathbb{R}^n$ . For a compact set  $K \subset \mathbb{R}^n$  with  $\dim \text{aff } K = m$ , we write  $|K|$  to denote the  $m$ -dimensional Lebesgue measure of  $K$ .

The starting point of this section is the classical Loomis-Whitney inequality [74].

**Theorem 7 (Loomis, Whitney)** *If  $K \subset \mathbb{R}^n$  is compact and affinely spans  $\mathbb{R}^n$ , then*

$$|K|^{n-1} \leq \prod_{i=1}^k |P_{e_i^\perp} K|, \quad (9)$$

*with equality if and only if  $K = \bigoplus_{i=1}^n K_i$  where  $\text{aff } K_i$  is a line parallel to  $e_i$ .*

Meyer [83] provided a dual form of the Loomis-Whitney inequality where equality holds for affine crosspolytopes.

**Theorem 8 (Meyer)** *If  $K \subset \mathbb{R}^n$  is compact convex with  $o \in \text{int}K$ , then*

$$|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^k |K \cap e_i^\perp|, \quad (10)$$

*with equality if and only if  $K = \text{conv}\{\pm\lambda_i e_i\}_{i=1}^n$  for  $\lambda_i > 0$ ,  $i = 1, \dots, n$ .*

We note that various Reverse and dual Loomis-Whitney type inequalities are proved by Campi, Gardner, Gronchi [76], Brazitikos *et al* [24, 25], Alonso-Gutiérrez *et al* [1, 2].

To consider a generalization of the Loomis-Whitney inequality and its dual form, we set  $[n] := \{1, \dots, n\}$ , and for a non-empty proper subset  $\sigma \subset [n]$ , we define  $E_\sigma = \text{lin}\{e_i\}_{i \in \sigma}$ . For  $s \geq 1$ , we say that the not necessarily distinct proper non-empty subsets  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  if each  $j \in [n]$  is contained in exactly  $s$  of  $\sigma_1, \dots, \sigma_k$ .

The Bollobas-Thomason inequality [18] reads as follows.

**Theorem 9 (Bollobas, Thomason)** *If  $K \subset \mathbb{R}^n$  is compact and affinely spans  $\mathbb{R}^n$ , and  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  for  $s \geq 1$ , then*

$$|K|^s \leq \prod_{i=1}^k |P_{E_{\sigma_i}} K|. \quad (11)$$

We note that additional the case when  $k = n$ ,  $s = n - 1$ , and hence when we may assume that  $\sigma_i = [n] \setminus e_i$ , is the Loomis-Whitney inequality Theorem 7.

Liakopoulos [71] managed to prove a dual form of the Bollobas-Thomason inequality. For a finite set  $\sigma$ , we write  $|\sigma|$  to denote its cardinality.

**Theorem 10 (Liakopoulos)** *If  $K \subset \mathbb{R}^n$  is compact convex with  $o \in \text{int}K$ , and  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  for  $s \geq 1$ , then*

$$|K|^s \geq \frac{\prod_{i=1}^k |\sigma_i|!}{(n!)^s} \cdot \prod_{i=1}^k |K \cap E_{\sigma_i}|. \quad (12)$$

The equality case of the Bollobas-Thomason inequality Theorem 9 based on Valdimarsson [93] has been known to the experts but we present this argument in order to have a written account. Let  $s \geq 1$ , and let  $\sigma_1, \dots, \sigma_k \subset [n]$  be an  $s$ -uniform cover of  $[n]$ . We say that  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l \subset [n]$  form a 1-uniform cover of  $[n]$  induced by the  $s$ -uniform cover  $\sigma_1, \dots, \sigma_k$  if  $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_l\}$  consists of all non-empty distinct subsets of  $[n]$  of the form  $\bigcap_{i=1}^k \sigma_i^{\varepsilon(i)}$  where  $\varepsilon(i) \in \{0, 1\}$  and  $\sigma_i^0 = \sigma_i$  and  $\sigma_i^1 = [n] \setminus \sigma_i$ . We observe that  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l \subset [n]$  actually form a 1-uniform cover of  $[n]$ ; namely,  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$  is a partition of  $[n]$ .

**Theorem 11 (Folklore)** *Let  $K \subset \mathbb{R}^n$  be compact and affinely span  $\mathbb{R}^n$ , and let  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  for  $s \geq 1$ . Then equality holds in (11) if and only if  $K = \bigoplus_{i=1}^l P_{E_{\tilde{\sigma}_i}} K$  where  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$  is the 1-uniform cover of  $[n]$  induced by  $\sigma_1, \dots, \sigma_k$ .*

Our main result in this section is the characterization of the equality case of the dual Bollobas-Thomason inequality Theorem 10 relating it to the Geometric Barthe's inequality.

**Theorem 12** *Let  $K \subset \mathbb{R}^n$  be compact convex with  $o \in \text{int} K$ , and let  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  for  $s \geq 1$ . Then equality holds in (12) if and only if  $K = \text{conv}\{K \cap F_{\tilde{\sigma}_i}\}_{i=1}^l$  where  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$  is the 1-uniform cover of  $[n]$  induced by  $\sigma_1, \dots, \sigma_k$ .*

### 3 The determinantal inequality and structure theory for rank one Geometric Brascamp-Lieb data

We first discuss the basic properties of a set of vectors  $u_1, \dots, u_k \in S^{n-1}$  and constants  $c_1, \dots, c_k > 0$  occurring in the Geometric Brascamp-Lieb inequality; namely, satisfying

$$\sum_{i=1}^k c_i u_i \otimes u_i = I_n. \quad (13)$$

This section just retells the story of Section 2 of Barthe [8] in the language of Carlen, Lieb, Loss [31] and Bennett, Carbery, Christ, Tao [17].

**Lemma 13** *For  $u_1, \dots, u_k \in S^{n-1}$  and  $c_1, \dots, c_k > 0$  satisfying (13), we have*

- (i)  $\sum_{i=1}^k c_i = n$ ;
- (ii)  $\sum_{i=1}^k c_i \langle u_i, x \rangle^2 = \|x\|^2$  for all  $x \in \mathbb{R}^n$ ;
- (iii)  $c_i \leq 1$  for  $i = 1, \dots, k$  with equality if and only if  $u_j \in u_i^\perp$  for  $j \neq i$ ;
- (iv)  $u_1, \dots, u_k$  spans  $\mathbb{R}^n$ , and  $k = n$  if and only if  $u_1, \dots, u_n$  is an orthonormal basis of  $\mathbb{R}^n$  and  $c_1 = \dots = c_n = 1$ ;
- (v) if  $L$  is a proper linear subspace of  $\mathbb{R}^n$ , then

$$\sum_{u_i \in L} c_i \leq \dim L,$$

with equality if and only if  $\{u_1, \dots, u_k\} \subset L \cup L^\perp$ .

**Remark** If  $\sum_{u_i \in L} c_i = \dim L$  in (v), then  $\text{lin}\{u_i : u_i \in L\} = L$  and  $\text{lin}\{u_i : u_i \in L^\perp\} = L^\perp$ .

*Proof:* Here (i) follows from comparing the traces of the two sides of (13), and (ii) is just an equivalent form of (13). To prove  $c_j \leq 1$  with the characterization of equality, we substitute  $x = u_j$  into (ii).



Turning to (iv),  $u_1, \dots, u_k$  spans  $\mathbb{R}^n$  by (ii). Next, let us assume that  $u_1, \dots, u_n \in S^{n-1}$  and  $c_1, \dots, c_n > 0$  satisfy (13). We consider  $w_j \in S^{n-1}$  for  $j = 1, \dots, n$  such that  $\langle w_j, u_i \rangle = 0$  if  $i \neq j$ , and hence (ii) shows that  $u_j = \pm w_j$  and  $c_j = 1$ .

For (v), if  $u_i \notin L$ , then we consider the unit vector

$$\tilde{u}_i = \frac{P_{L^\perp} u_i}{\|P_{L^\perp} u_i\|} \in L^\perp.$$

We deduce that if  $x \in L^\perp$ , then

$$\|x\|^2 = \sum_{i=1}^k c_i \langle u_i, x \rangle^2 = \sum_{u_i \notin L} c_i \langle P_{L^\perp} u_i, x \rangle^2 = \sum_{u_i \notin L} c_i \|P_{L^\perp} u_i\|^2 \langle \tilde{u}_i, x \rangle^2.$$

It follows from (i) and (ii) applied to  $\{\tilde{u}_i : u_i \notin L\}$  in  $L^\perp$  that

$$\dim L^\perp = \sum_{u_i \notin L} c_i \|P_{L^\perp} u_i\|^2 \leq \sum_{u_i \notin L} c_i.$$

In turn, we conclude the inequality in (v) by (i). Equality holds in (v) if and only if  $\|P_{L^\perp} u_i\| = 1$  whenever  $u_i \notin L$ ; therefore,  $u_1, \dots, u_k \subset L \cup L^\perp$ .  $\square$

Let  $u_1, \dots, u_k \in S^{n-1}$  and  $c_1, \dots, c_k > 0$  satisfy (13). Following Bennett, Carbery, Christ, Tao [17], we say that a non-zero linear subspace  $V$  is a critical subspace with respect to  $u_1, \dots, u_k$  and  $c_1, \dots, c_k$  if

$$\sum_{u_i \in V} c_i = \dim V.$$

In particular,  $\mathbb{R}^n$  is a critical subspace according to Lemma 13. We say that a non-empty subset  $\mathcal{U} \subset \{u_1, \dots, u_k\}$  is indecomposable if  $\text{lin } \mathcal{U}$  is an indecomposable critical subspace.

In order to understand the equality case of the rank one Brascamp-Lieb inequality, Barthe [8] indicated an equivalence relation on  $\{u_1, \dots, u_k\}$ . We say that a subset  $\mathcal{D} \subset \{u_1, \dots, u_k\}$  is minimally dependent if  $\mathcal{D}$  is dependent and no proper subset of  $\mathcal{D}$  is dependent. The following is folklore in matroid theory, was known most probably already to Tutte (see for example Theorem 7.3.6 in Recski [88]). For the convenience of the reader, we provide an argument.

**Lemma 14** *Given non-zero  $v_1, \dots, v_k$  spanning  $\mathbb{R}^n$ ,  $n \geq 1$ , we write  $v_i \bowtie v_j$  if either  $v_i = v_j$ , or there exists a minimal dependent set  $\mathcal{D} \subset \{v_1, \dots, v_k\}$  satisfying  $v_i, v_j \in \mathcal{D}$ .*

- (i)  $v_i \bowtie v_j$  if and only if there exists a subset  $\mathcal{U} \subset \{v_1, \dots, v_k\}$  of cardinality  $n - 1$  such that both  $\{v_i\} \cup \mathcal{U}$  and  $\{v_j\} \cup \mathcal{U}$  are independent;
- (ii)  $\bowtie$  is an equivalence relation on  $\{v_1, \dots, v_k\}$ ;
- (iii) if  $V_1, \dots, V_m$  are the linear hulls of the equivalence classes with respect to  $\bowtie$ , then they span  $\mathbb{R}^n$  and  $V_i \cap V_j = \{0\}$  for  $i \neq j$ .

*Proof:* We prove the lemma by induction on  $n \geq 1$  where the case  $n = 1$  readily holds. Therefore, we assume that  $n \geq 2$ .

We may readily assume that

$$\{v_1, \dots, v_k\} \cap \text{lin}\{v_i\} = \{v_i\} \quad \text{for } i = 1, \dots, k. \quad (14)$$

For (i), if  $\mathcal{D}$  is a minimal dependent set with  $v_i, v_j \in \mathcal{D}$ , then adding some  $\mathcal{V} \subset \{v_1, \dots, v_k\}$  to  $\mathcal{D} \setminus \{v_i\}$ , we obtain a basis of  $\mathbb{R}^n$ , and we may choose  $\mathcal{U} = \mathcal{V} \cup (\mathcal{D} \setminus \{v_i, v_j\})$ . On the other hand, if the suitable  $\mathcal{U}$  of cardinality  $n - 1$  exists such that both  $\{v_i\} \cup \mathcal{U}$  and  $\{v_j\} \cup \mathcal{U}$  are independent, then any dependent subset of  $\mathcal{U} \cup \{v_i, v_j\}$  contains  $v_i$  and  $v_j$ .

For (ii) and (iii), we call a non-zero linear subspace  $W \subset \mathbb{R}^n$  unsplittable with respect to  $\{v_1, \dots, v_k\}$  if  $W$  is spanned by  $W \cap \{v_1, \dots, v_k\}$ , but there exist no non-zero complementary linear subspaces  $A, B \subset W$  with  $\{v_1, \dots, v_k\} \cap W \subset A \cup B$ . Readily, there exist pairwise complementary unsplittable linear subspaces  $W_1, \dots, W_m \subset \mathbb{R}^n$  such that  $\{v_1, \dots, v_k\} \subset W_1 \cup \dots \cup W_m$ .

On the one hand, if  $v_i \in W_\alpha$  and  $v_j \in W_\beta$  for  $\alpha \neq \beta$ , then trivially  $v_i \not\propto v_j$ . Therefore all we need to prove is that if  $v_i, v_j \in W_\alpha$ , then  $v_i \propto v_j$ . By the induction on  $n$ , we may assume that  $m = 1$  and  $W_\alpha = \mathbb{R}^n$ . We may also assume that  $i = 1$  and  $j = 2$ .

The final part of argument is indirect; therefore, we suppose that

$$v_1 \not\propto v_2, \quad (15)$$

and seek a contradiction.

(15) implies that  $v_1$  and  $v_2$  are independent, and hence  $v_1 \not\propto v_2$  and (14) yield that  $L = \text{lin}\{v_1, v_2\}$  satisfies

$$\{v_1, \dots, v_k\} \cap L = \{v_1, v_2\}. \quad (16)$$

Now  $\mathbb{R}^n$  is unsplittable, thus  $n \geq 3$ .

Since  $v_1, \dots, v_k$  span  $\mathbb{R}^n$ , we may assume that  $v_1, \dots, v_n$  form a basis of  $\mathbb{R}^n$ . Let  $L_0 = \text{lin}\{v_3, \dots, v_n\}$ , and  $L_t = \text{lin}\{v_t, L_0\}$  for  $t = 1, 2$ . We may also assume that  $v_1, \dots, v_n$  is an orthonormal basis.

For any  $l > n$ , (i) and  $v_1 \not\propto v_2$  yield that

$$\text{either } v_l \in L_1, \text{ or } v_l \in L_2. \quad (17)$$

Since  $\mathbb{R}^n$  is unsplittable, there exist  $p, q > n$  such that

$$v_p \in L_1 \setminus L_0 \text{ and } v_q \in L_2 \setminus L_0. \quad (18)$$

For any  $w \notin L$ , we write

$$\text{supp } w = \{v_l : l \in \{3, \dots, n\} \ \& \ \langle w, v_l \rangle \neq 0\};$$

namely, the basis vectors where the corresponding coordinate of  $w|_L = 0$  is non-zero.

**Case 1** There exist  $v_p \in L_1 \setminus L_0$  and  $v_q \in L_2 \setminus L_0$ ,  $p, q > n$ , such that  $(\text{supp } v_p) \cap (\text{supp } v_q) \neq \emptyset$

Let  $v_s \in (\text{supp } v_p) \cap (\text{supp } v_q)$ . Now the  $n + 1$  element set

$$\{v_1, v_p, v_2, v_q\} \cup \{v_l : l \in \{3, \dots, n\} \setminus \{s\}\}$$

is dependent, and considering the  $1^{\text{st}}$ ,  $2^{\text{nd}}$  and  $s^{\text{th}}$  coordinates show that both  $v_1$  and  $v_2$  lie in any dependent subset. This fact contradicts (15).

**Case 2**  $(\text{supp } v_p) \cap (\text{supp } v_q) = \emptyset$  for any  $v_p \in L_1 \setminus L_0$  and  $v_q \in L_2 \setminus L_0$  with  $p, q > n$

Let  $\mathcal{U}_t = \cup\{\text{supp } v_p : p > n \ \& \ v_p \in L_t \setminus L_0\}$  for  $t = 1, 2$ . It follows that  $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ , thus  $n \geq 4$ . For any partition  $\mathcal{U}'_1 \cup \mathcal{U}'_2 = \{v_3, \dots, v_n\}$  (and hence  $\mathcal{U}'_1 \cap \mathcal{U}'_2 = \emptyset$ ) such that  $\mathcal{U}_1 \subset \mathcal{U}'_1$  and  $\mathcal{U}_2 \subset \mathcal{U}'_2$ , there exists some  $v_l \in L_0$  that is contained neither in  $\text{lin}(\mathcal{U}'_1 \cup \{v_1\})$  nor in  $\text{lin}(\mathcal{U}'_2 \cup \{v_2\})$  because  $\mathbb{R}^n$  is unslittable. In turn we deduce that we may reindex the vectors  $v_3, \dots, v_n$  on the one hand, and the vectors  $v_{n+1}, \dots, v_k$  on the other hand to ensure the following properties:

- $v_{n+1} \in L_1 \setminus L_0$  and  $v_{n+2} \in L_2 \setminus L_0$ ;
- there exist  $\alpha \in \{3, \dots, n-1\}$  and  $\beta \in \{n+3, \dots, k\}$  such that  $\text{supp } v_l \subset \{v_\alpha, \dots, v_n\}$  for  $l \in \{n+1, \dots, \beta\}$ , and  $v_l \in L_0$  if  $n+3 \leq l \leq \beta$ ;
- for any partition  $\mathcal{W}_1 \cup \mathcal{W}_2 = \{v_\alpha, \dots, v_n\}$  into non-empty sets, there exist  $l \in \{n+1, \dots, \beta\}$  such that  $\text{supp } v_l$  intersects both  $\mathcal{W}_1$  and  $\mathcal{W}_2$ .

We observe that  $\tilde{L}_0 = \text{lin}\{v_\alpha, \dots, v_n\}$  is unslittable with respect to

$$\{v_\alpha, \dots, v_n, v_{n+1}|L_0, v_{n+2}|L_0, v_{n+3}, \dots, v_\beta\}.$$

Therefore, this last set contains a minimal dependent subset  $\tilde{\mathcal{D}}$  with  $v_{n+1}|L_0, v_{n+2}|L_0 \in \tilde{\mathcal{D}}$  by induction; namely, the elements of  $\tilde{\mathcal{D}}$  different from  $v_{n+1}|L_0, v_{n+2}|L_0$  are vectors of the form  $v_l$  that lie in  $L_0$ . We conclude that

$$\mathcal{D} = \{v_1, v_2, v_{n+1}, v_{n+2}\} \cup \left( \tilde{\mathcal{D}} \setminus \{v_{n+1}|L_0, v_{n+2}|L_0\} \right)$$

is a minimal dependent set, contradicting (15), and proving Lemma 14.  $\square$

**Lemma 15** For  $u_1, \dots, u_k \in S^{n-1}$  and  $c_1, \dots, c_k > 0$  satisfying (13), we have

- (i) a proper linear subspace  $V \subset \mathbb{R}^n$  is critical if and only if  $\{u_1, \dots, u_k\} \subset V \cup V^\perp$ ;
- (ii) if  $V, W$  are proper critical subspaces with  $V \cap W \neq \{0\}$ , then  $V^\perp, V \cap W$  and  $V + W$  are critical subspaces;
- (iii) the equivalence classes with respect to the relation  $\boxtimes$  in Lemma 14 are the indecomposable subsets of  $\{u_1, \dots, u_k\}$ ;

(iv) the proper indecomposable critical subspaces are pairwise orthogonal, and any critical subspace is the sum of some indecomposable critical subspaces.

*Proof:* (i) directly follows from Lemma 13 (v), and in turn (i) yields (ii).

We prove (iii) and first half of (iv) simultaneously. Let  $V_1, \dots, V_m$  be the linear hulls of the equivalence classes of  $u_1, \dots, u_k$  with respect to the  $\bowtie$  of Lemma 14. We deduce from Lemma 13 (v) that each  $V_i$  is a critical subspace, and if  $i \neq j$ , then  $V_i$  and  $V_j$  are orthogonal.

Next let  $\mathcal{U} \subset \{u_1, \dots, u_k\}$  be an indecomposable set, and let  $V = \text{lin } \mathcal{U}$ . We write  $I \subset \{1, \dots, m\}$  to denote the set of indices  $i$  such that  $V_i \cap \mathcal{U} \neq \emptyset$ . Since  $V$  is a critical subspace, we deduce from Lemma 13 (v) that  $V_i \cap V$  is a critical subspace for  $i \in I$ , as well; therefore,  $I$  consists of a unique index  $p$  as  $\mathcal{U}$  is indecomposable. In particular,  $V = V_p$ .

It follows from Lemma 13 (v) that  $\{u_1, \dots, u_k\} \subset V \cup V^\perp$ ; therefore, there exists no minimally dependent subset of  $\{u_1, \dots, u_k\}$  intersecting both  $\mathcal{U}$  and its complement. We conclude that  $V = V_p$ .

Finally, the second half of (iv) follows from (i) and (ii).  $\square$

The following is the main result of this section, where the inequality is proved by Ball [3, 4], and the equality case is clarified by Barthe [8].

**Proposition 16 (Ball-Barthe Lemma)** *For  $u_1, \dots, u_k \in S^{n-1}$  and  $c_1, \dots, c_k > 0$  satisfying (13), if  $t_i > 0$  for  $i = 1, \dots, k$ , then*

$$\det \left( \sum_{i=1}^k c_i t_i u_i \otimes u_i \right) \geq \prod_{i=1}^k t_i^{c_i}. \quad (19)$$

*Equality holds in (19) if and only if  $t_i = t_j$  for any  $u_i$  and  $u_j$  lying in the same indecomposable subset of  $\{u_1, \dots, u_k\}$ .*

*Proof:* To simplify expressions, let  $v_i = \sqrt{c_i} u_i$  for  $i = 1, \dots, k$ .

In this argument,  $I$  always denotes some subset of  $\{1, \dots, k\}$  of cardinality  $n$ . For  $I = \{i_1, \dots, i_n\}$ , we define

$$d_I := \det[v_{i_1}, \dots, v_{i_n}]^2 \quad \text{and} \quad t_I := t_{i_1} \cdots t_{i_n}.$$

For the  $n \times k$  matrices  $M = [v_1, \dots, v_k]$  and  $\widetilde{M} = [\sqrt{t_1} v_1, \dots, \sqrt{t_k} v_k]$ , we have

$$MM^T = I_n \quad \text{and} \quad \widetilde{M}\widetilde{M}^T = \sum_{i=1}^k t_i v_i \otimes v_i. \quad (20)$$

It follows from the Cauchy-Binet formula that

$$\sum_I d_I = 1 \quad \text{and} \quad \det \left( \sum_{i=1}^k t_i v_i \otimes v_i \right) = \sum_I t_I d_I,$$

where the summations extend over all sets  $I \subset \{1, \dots, k\}$  of cardinality  $n$ . It follows that the discrete measure  $\mu$  on the  $n$  element subsets of  $\{1, \dots, k\}$  defined by  $\mu(\{I\}) = d_I$  is a probability measure. We deduce from inequality between the arithmetic and geometric mean that

$$\det \left( \sum_{i=1}^k t_i v_i \otimes v_i \right) = \sum_I t_I d_I \geq \prod_I t_I^{d_I}. \quad (21)$$

The factor  $t_i$  occurs in  $\prod_I t_I^{d_I}$  exactly  $\sum_{I, i \in I} d_I$  times. Moreover, the Cauchy-Binet formula applied to the vectors  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$  implies

$$\begin{aligned} \sum_{I, i \in I} d_I &= \sum_I d_I - \sum_{I, i \notin I} d_I = 1 - \det \left( \sum_{j \neq i} v_j \otimes v_j \right) \\ &= 1 - \det (\text{Id}_n - v_i \otimes v_i) = \langle v_i, v_i \rangle = c_i. \end{aligned}$$

Substituting this into (21) yields (19).

We now assume that equality holds in (19). Since equality holds in (21) when applying arithmetic and geometric mean, all the  $t_I$  are the same for any subset  $I$  of  $\{1, \dots, k\}$  of cardinality  $n$  with  $d_I \neq 0$ . It follows that  $t_i = t_j$  whenever  $u_i \bowtie u_j$ , and in turn we deduce that  $t_i = t_j$  whenever  $u_i$  and  $u_j$  lie in the same indecomposable set by Lemma 15 (i).

On the other hand, Lemma 15 (ii) yields that if  $t_i = t_j$  whenever  $u_i$  and  $u_j$  lie in the same indecomposable set, then equality holds in (19).  $\square$

Combining Lemma 15 and Proposition 16 leads to the following:

**Corollary 17** *For  $u_i \in S^{n-1}$  and  $c_i, t_i > 0$ ,  $i = 1, \dots, k$  satisfying (13), equality holds in (19) if and only if there exist pairwise orthogonal linear subspaces  $V_1, \dots, V_m$ ,  $m \geq 1$ , such that  $\{u_1, \dots, u_k\} \subset V_1 \cup \dots \cup V_m$  and  $t_i = t_j$  whenever  $u_i$  and  $u_j$  lie in the same  $V_p$  for some  $p \in \{1, \dots, m\}$ .*

## 4 Structure theory of a Brascamp-Lieb data and the determinantal inequality corresponding to the higher rank case

We build a structural theory for a Brascamp-Lieb data based on results proved or indicated in Barthe [8], Bennett, Carbery, Christ, Tao [17] and Valdimarsson [93].

For non-zero linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying the Geometric Brascamp-Lieb condition

$$\sum_{i=1}^k c_i P_{E_i} = I_n, \quad (22)$$

we connect (22) to (13). For  $i = 1, \dots, k$ , let  $\dim E_i = n_i$  and let  $u_1^{(i)}, \dots, u_{n_i}^{(i)}$  be any orthonormal basis of  $E_i$ . In addition, for  $i = 1, \dots, k$ , we consider the  $n \times n_i$  matrix

$M_i = \sqrt{c_i}[u_1^{(i)}, \dots, u_{n_i}^{(i)}]$ . We deduce that

$$c_i P_{E_i} = M_i M_i^T = \sum_{j=1}^{n_i} c_i u_j^{(i)} \otimes u_j^{(i)} \text{ for } i = 1, \dots, k; \quad (23)$$

$$I_n = \sum_{i=1}^k c_i P_{E_i} = \sum_{i=1}^k \sum_{j=1}^{n_i} c_i u_j^{(i)} \otimes u_j^{(i)} = \sum_{i=1}^k \sum_{j=1}^{n_i} c_j^{(i)} u_j^{(i)} \otimes u_j^{(i)} \quad (24)$$

and hence  $u_j^{(i)} \in S^{n-1}$  and  $c_j^{(i)} = c_i > 0$  for  $i = 1, \dots, k$  and  $j = 1, \dots, n_i$  form a Geometric Brascamp-Lieb data like in (13).

**Lemma 18** For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22),

(i) if  $x \in \mathbb{R}^n$ , then  $\sum_{i=1}^k c_i \|P_{E_i} x\|^2 = \|x\|^2$ ;

(ii) if  $V \subset \mathbb{R}^n$  is a proper linear subspace, then

$$\sum_{i=1}^k c_i \dim(E_i \cap V) \leq \dim V \quad (25)$$

where equality holds if and only if  $E_i = (E_i \cap V) + (E_i \cap V^\perp)$  for  $i = 1, \dots, k$ ; or equivalently, when  $V = (E_i \cap V) + (E_i^\perp \cap V)$  for  $i = 1, \dots, k$ .

*Proof:* For  $i = 1, \dots, k$ , let  $\dim E_i = n_i$  and let  $u_1^{(i)}, \dots, u_{n_i}^{(i)}$  be any orthonormal basis of  $E_i$  such that if  $V \cap E_i \neq \{0\}$ , then  $u_1^{(i)}, \dots, u_{m_i}^{(i)}$  is any orthonormal basis of  $V \cap E_i$  where  $m_i \leq n_i$ .

For any  $x \in \mathbb{R}^n$  and  $i = 1, \dots, k$ , we have  $\|P_{E_i} x\|^2 = \sum_{j=1}^{n_i} \langle u_j^{(i)}, x \rangle^2$ , thus Lemma 13 (ii) yields (i).

Concerning (ii), Lemma 13 (v) yields (25). On the other hand, if equality holds in (25), then  $V$  is a critical subspace for the rank one Geometric Brascamp-Lieb data  $u_j^{(i)} \in S^{n-1}$  and  $c_j^{(i)} = c_i > 0$  for  $i = 1, \dots, k$  and  $j = 1, \dots, n_i$  satisfying (24). Thus Lemma 18 (ii) follows from Lemma 13 (v).  $\square$

We say that a non-zero linear subspace  $V$  is a critical subspace with respect to the proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22) if

$$\sum_{i=1}^k c_i \dim(E_i \cap V) = \dim V.$$

In particular,  $\mathbb{R}^n$  is a critical subspace by calculating traces of both sides of (22). For a proper linear subspace  $V \subset \mathbb{R}^n$ , Lemma 18 yields that  $V$  is critical if and only if  $V^\perp$  is critical, which is turn equivalent saying that

$$E_i = (E_i \cap V) + (E_i \cap V^\perp) \text{ for } i = 1, \dots, k; \quad (26)$$

or in other words,

$$V = (E_i \cap V) + (E_i^\perp \cap V) \text{ for } i = 1, \dots, k. \quad (27)$$

We observe that (26) has the following consequence: If  $V_1$  and  $V_2$  are orthogonal critical subspaces, then

$$E_i \cap (V_1 + V_2) = (E_i \cap V_1) + (E_i \cap V_2) \text{ for } i = 1, \dots, k. \quad (28)$$

We recall that a critical subspace  $V$  is indecomposable if  $V$  has no proper critical linear subspace.

**Lemma 19** *If  $E_1, \dots, E_k$  are linear subspaces of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), and  $V, W$  are proper critical subspaces, then  $V^\perp$  and  $V + W$  are critical subspaces, and even  $V \cap W$  is critical provided that  $V \cap W \neq \{0\}$ .*

*Proof:* We may assume that  $\dim E_i \geq 1$  for  $i = 1, \dots, k$ .

The fact that  $V^\perp$  is also critical follows directly from (26).

Concerning  $V \cap W$  when  $V \cap W \neq \{0\}$ , we need to prove that if  $i = 1, \dots, k$ , then

$$(V \cap W) \cap E_i + (V \cap W)^\perp \cap E_i = E_i. \quad (29)$$

For a linear subspace  $L \subset E_i$ , we write  $L^{\perp i} = L^\perp \cap E_i$  to denote the orthogonal complement within  $E_i$ . We observe that as  $V$  and  $W$  are critical subspaces, we have  $(V \cap E_i)^{\perp i} = V^\perp \cap E_i$  and  $(W \cap E_i)^{\perp i} = W^\perp \cap E_i$ . It follows from the identity  $(V \cap W)^\perp = V^\perp + W^\perp$  that

$$\begin{aligned} E_i &\supset (V \cap W) \cap E_i + (V \cap W)^\perp \cap E_i = (V \cap E_i) \cap (W \cap E_i) + (V^\perp + W^\perp) \cap E_i \\ &\supset (V \cap E_i) \cap (W \cap E_i) + (V^\perp \cap E_i) + (W^\perp \cap E_i) \\ &= (V \cap E_i) \cap (W \cap E_i) + (V \cap E_i)^{\perp i} + (W \cap E_i)^{\perp i} \\ &= (V \cap E_i) \cap (W \cap E_i) + [(V \cap E_i) \cap (W \cap E_i)]^{\perp i} = E_i, \end{aligned}$$

yielding (29).

Finally,  $V + W$  is also critical as  $V + W = (V^\perp \cap W^\perp)^\perp$ .  $\square$

We deduce from Lemma 19 that any critical subspace can be decomposed into indecomposable ones.

**Corollary 20** *If  $E_1, \dots, E_k$  are proper linear subspaces of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfy (22), and  $W$  is a critical subspace or  $W = \mathbb{R}^n$ , then there exist pairwise orthogonal indecomposable critical subspaces  $V_1, \dots, V_m$ ,  $m \geq 1$ , such that  $W = V_1 + \dots + V_m$  (possibly  $m = 1$  and  $W = V_1$ ).*

We note that the decomposition of  $\mathbb{R}^n$  into indecomposable critical subspaces is not unique in general for a Geometric Brascamp-Lieb data. Valdimarsson [93] provides some examples, and in addition, we provide an example where we have a continuous family of indecomposable critical subspaces.

**Example 21 (Continuous family of indecomposable critical subspaces)** In  $\mathbb{R}^4$ , let us consider the following six unit vectors:  $u_1(1, 0, 0, 0)$ ,  $u_2(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0)$ ,  $u_3(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0, 0)$ ,  $v_1(0, 0, 1, 0)$ ,  $v_2(0, 0, \frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $v_3(0, 0, \frac{-1}{2}, \frac{\sqrt{3}}{2})$ , which satisfy  $u_2 = u_1 + u_3$  and  $v_2 = v_1 + v_3$ .

For any  $x \in \mathbb{R}^4$ , we have

$$\|x\|^2 = \sum_{i=1}^3 \frac{2}{3} \cdot (\langle x, u_i \rangle^2 + \langle x, v_i \rangle^2)$$

Therefore, we define the Geometric Brascamp-Lieb Data  $E_i = \text{lin}\{u_i, v_i\}$  and  $c_i = \frac{2}{3}$  for  $i = 1, 2, 3$  satisfying (1). In this case,  $F_{\text{dep}} = \mathbb{R}^4$ .

For any angle  $t \in \mathbb{R}$ , we have a two-dimensional indecomposable critical subspace

$$V_t = \text{lin}\{(\cos t)u_1 + (\sin t)v_1, (\cos t)u_2 + (\sin t)v_2, (\cos t)u_3 + (\sin t)v_3\}.$$

Next we prove the crucial determinantal inequality. Its proof is kindly provided by Franck Barthe.

**Proposition 22 (Barthe)** For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$ ,  $n \geq 1$  and  $c_1, \dots, c_k > 0$  satisfying (22), if  $A_i : E_i \rightarrow E_i$  is a positive definite linear transformation for  $i = 1, \dots, k$ , then

$$\det \left( \sum_{i=1}^k c_i A_i P_{E_i} \right) \geq \prod_{i=1}^k (\det A_i)^{c_i}. \quad (30)$$

Equality holds in (30) if and only if there exist linear subspaces  $V_1, \dots, V_m$  where  $V_1 = \mathbb{R}^n$  if  $m = 1$  and  $V_1, \dots, V_m$  are pairwise orthogonal indecomposable critical subspaces spanning  $\mathbb{R}^n$  if  $m \geq 2$ , and a positive definite  $n \times n$  matrix  $\Phi$  such that  $V_1, \dots, V_m$  are eigenspaces of  $\Phi$  and  $\Phi|_{E_i} = A_i$  for  $i = 1, \dots, k$ . In addition,  $\Phi = \sum_{i=1}^k c_i A_i P_{E_i}$  in the case of equality.

*Proof:* We may assume that  $\dim E_i \geq 1$  for  $i = 1, \dots, k$ .

For  $i = 1, \dots, k$ , let  $\dim E_i = n_i$ , let  $u_1^{(i)}, \dots, u_{n_i}^{(i)}$  be an orthonormal basis of  $E_i$  consisting of eigenvectors of  $A_i$ , and let  $\lambda_j^{(i)} > 0$  be the eigenvalue of  $A_i$  corresponding to  $u_j^{(i)}$ . In particular  $\det A_i = \prod_{j=1}^{n_i} \lambda_j^{(i)}$  for  $i = 1, \dots, k$ . In addition, for  $i = 1, \dots, k$ , we set  $M_i = \sqrt{c_i}[u_1^{(i)}, \dots, u_{n_i}^{(i)}]$  and  $B_i$  to be the positive definite transformation with  $A_i = B_i B_i$ , and hence

$$c_i A_i P_{E_i} = (M_i B_i)(M_i B_i)^T = \sum_{j=1}^{n_i} c_i \lambda_j^{(i)} u_j^{(i)} \otimes u_j^{(i)}.$$

We deduce from Lemma 16 and (24) that

$$\begin{aligned} \det \left( \sum_{i=1}^k c_i A_i P_{E_i} \right) &= \det \left( \sum_{i=1}^k \sum_{j=1}^{n_i} c_i \lambda_j^{(i)} u_j^{(i)} \otimes u_j^{(i)} \right) \\ &\geq \prod_{i=1}^k \left( \prod_{j=1}^{n_i} \lambda_j^{(i)} \right)^{c_i} = \prod_{i=1}^k (\det A_i)^{c_i}. \end{aligned} \quad (31)$$



If we have equality in (30), and hence also in (31), then Corollary 17 implies that there exist pairwise orthogonal critical subspaces  $V_1, \dots, V_m$ ,  $m \geq 1$  spanning  $\mathbb{R}^n$  and  $\lambda_1, \dots, \lambda_m > 0$  (where  $V_1 = \mathbb{R}^n$  if  $m = 1$ ) such that if  $E_i \cap V_j \neq \{0\}$ , then  $E_i \cap V_j$  is an eigenspace of  $A_i$  with eigenvalue  $\lambda_j$ . We conclude from (26) that each  $V_j$  is a critical subspace, and from Corollary 20 that each  $V_j$  can be assumed to be indecomposable. Finally, (28) yields that each  $E_i$  is spanned by the subspaces  $E_i \cap V_j$  for  $j = 1, \dots, m$ .

To show that each  $V_j$  is an eigenspace for the positive definite linear transform  $\sum_{i=1}^k c_i A_i P_{E_i}$  of  $\mathbb{R}^n$  with eigenvalue  $\lambda_j$ , we observe that

$$A_i P_{E_i} x = \lambda_j P_{E_i} x$$

for any  $i = 1, \dots, k$  and  $x \in V_j$ . It follows that if  $x \in V_j$ , then

$$\sum_{i=1}^k c_i A_i P_{E_i} x = \lambda_j \sum_{i=1}^k c_i P_{E_i} x = \lambda_j x,$$

proving that we can choose  $\Phi = \sum_{i=1}^k c_i A_i P_{E_i}$ .

On the other hand, let us assume that there exists a positive definite  $n \times n$  matrix  $\Theta$  whose eigenspaces  $W_1, \dots, W_l$  are critical subspaces (or  $l = 1$  and  $W_1 = \mathbb{R}^n$ ) and  $\Theta|_{E_i} = A_i$  for  $i = 1, \dots, k$ . In this case, for any  $i = 1, \dots, k$ , we may choose the orthonormal basis  $u_1^{(i)}, \dots, u_{n_i}^{(i)}$  of  $E_i$  in a way such that  $u_1^{(i)}, \dots, u_{n_i}^{(i)} \subset W_1 \cup \dots \cup W_l$ , and hence Corollary 17 yields that equality holds in (30).  $\square$

**Remark** While Proposition 22 has a crucial role in proving both the Brascamp-Lieb inequality (2) and Barthe's inequality (5) and their equality cases, Proposition 22 can be actually derived from say (2). In the Brascamp-Lieb inequality, choose  $f_i(z) = e^{-\pi \langle A_i z, z \rangle}$  for  $z \in E_i$  and  $i = 1, \dots, k$ , and hence  $\int_{E_i} f_i = (\det A_i)^{-\frac{1}{2}}$ . On the other hand, if  $x \in \mathbb{R}^n$ , then

$$\prod_{i=1}^k f_i(P_{E_i} x)^{c_i} = e^{-\pi \sum_{i=1}^k c_i \langle A_i P_{E_i} x, P_{E_i} x \rangle} = e^{-\pi \sum_{i=1}^k c_i \langle A_i P_{E_i} x, x \rangle} = e^{-\pi \langle \sum_{i=1}^k c_i A_i P_{E_i} x, x \rangle},$$

therefore, the Brascamp-Lieb inequality (2) yields

$$\left( \det \sum_{i=1}^k c_i A_i P_{E_i} \right)^{-\frac{1}{2}} \leq \prod_{i=1}^k (\det A_i)^{-\frac{c_i}{2}}.$$

In addition, the equality conditions in Proposition 22 can be derived from Valdimarsson's Theorem 2.

Let us show why indecomposability of the critical subspaces in Proposition 22 is useful.

**Lemma 23** *Let the linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfy (22), let  $F_{\text{dep}} \neq \mathbb{R}^n$ , and let  $F_1, \dots, F_l$  be the independent subspaces,  $l \geq 1$ . If  $V$  is an indecomposable critical subspace, then either  $V \subset F_{\text{dep}}$ , or there exists an independent subspace  $F_j$ ,  $j \in \{1, \dots, l\}$  such that  $V \subset F_j$ .*

*Proof:* It is equivalent to prove that if  $V$  is an indecomposable critical subspace and  $j \in \{1, \dots, l\}$ , then

$$V \not\subset F_j \text{ implies } F_j \subset V^\perp. \quad (32)$$

We deduce that  $V \cap F_j = \{0\}$  from the facts that  $V$  is indecomposable and  $F_j$  is a critical subspace, thus  $F_j \cap V$  is a critical subspace or  $\{0\}$ . There exists a partition  $M \cup N = \{1, \dots, k\}$  with  $M \cap N = \emptyset$  such that

$$F_j = (\cap_{i \in M} E_i) \cap (\cap_{i \in N} E_i^\perp).$$

Let  $y \in F_j$ . Since  $V$  is a critical subspace, we conclude that  $P_V y \in E_i$  for  $i \in M$  and  $P_V y \in E_i^\perp$  for  $i \in N$ , and hence  $P_V y \in V \cap (\cap_{i \in M} E_i) \cap (\cap_{i \in N} E_i^\perp) = \{0\}$ . Therefore,  $y \in V^\perp$ .  $\square$

## 5 Typical Gaussian extremizers for some Geometric Brascamp-Lieb data

This section continues to build on work done in Barthe [8], Bennett, Carbery, Christ, Tao [17] and Valdimarsson [93].

For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), we deduce from Lemma 18 (i) and (27) that if  $V$  is a critical subspace, then writing  $P_{E_i \cap V}^{(V)}$  to denote the restriction of  $P_{E_i \cap V}$  onto  $V$ , we have

$$\sum_{E_i \cap V \neq \{0\}} c_i P_{E_i \cap V}^{(V)} = I_V \quad (33)$$

where  $I_V$  denotes the identity transformation on  $V$ .

The equality case of Proposition 22 indicates why Lemma 24 is important.

**Lemma 24** *For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$ ,  $n \geq 1$  and  $c_1, \dots, c_k > 0$  satisfying (22), if  $\Phi$  is a positive definite linear transform whose eigenspaces are critical subspaces, then for any  $x \in \mathbb{R}^n$ , we have*

$$\|\Phi x\|^2 = \min_{\substack{x = \sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \sum_{i=1}^k c_i \|\Phi x_i\|^2. \quad (34)$$

*Proof:* We may assume that  $\dim E_i \geq 1$  for  $i = 1, \dots, k$ .

As the eigenspaces of  $\Phi$  are critical subspaces, we deduce that

$$\Phi(E_i) = E_i \text{ and } \Phi(E_i^\perp) = E_i^\perp. \quad (35)$$

For any  $x \in \mathbb{R}^n$ , we have  $\Phi P_{E_i} x = P_{E_i} \Phi x$  for  $i = 1, \dots, k$  by (35); therefore, Lemma 18 (i) yields

$$\langle \Phi x, \Phi x \rangle = \sum_{i=1}^k c_i \|P_{E_i} \Phi x\|^2 = \sum_{i=1}^k c_i \|\Phi P_{E_i} x\|^2. \quad (36)$$

Since  $x = \sum_{i=1}^k c_i P_{E_i} x$  by (22), we may choose  $x_i = P_{E_i} x$  in (34), and we have equality in (34) in this case. Therefore, Lemma 24 is equivalent to proving that if  $x = \sum_{i=1}^k c_i x_i$  for  $x_i \in E_i$ ,  $i = 1, \dots, k$ , then

$$\|\Phi x\|^2 \leq \sum_{i=1}^k c_i \|\Phi x_i\|^2. \quad (37)$$

**Case 1**  $\dim E_i = 1$  for  $i = 1, \dots, k$  and  $\Phi = I_n$

Let  $E_i = \mathbb{R}u_i$  for  $u_i \in S^{n-1}$ . If  $x \in \mathbb{R}^n$ , then  $P_{E_i} x = \langle u_i, x \rangle u_i$  for  $i = 1, \dots, k$ , and (36) yields that

$$\langle x, x \rangle = \sum_{i=1}^k c_i \langle u_i, x \rangle^2.$$

In addition, any  $x_i \in E_i$  is of the form  $x_i = t_i u_i$  for  $i = 1, \dots, k$  where  $\|x_i\|^2 = t_i^2$ . If  $x = \sum_{i=1}^k c_i t_i u_i$ , then the Hölder inequality yields

$$\langle x, x \rangle = \left\langle x, \sum_{i=1}^k c_i t_i u_i \right\rangle = \sum_{i=1}^k c_i t_i \langle x, u_i \rangle \leq \sqrt{\sum_{i=1}^k c_i t_i^2} \cdot \sqrt{\sum_{i=1}^k c_i \langle x, u_i \rangle^2} = \sqrt{\sum_{i=1}^k c_i t_i^2} \cdot \sqrt{\langle x, x \rangle},$$

proving (37) in this case.

**Case 2** The general case,  $E_1, \dots, E_k$  and  $\Phi$  are as in Lemma 24

Let  $V_1, \dots, V_m$ ,  $m \geq 1$ , be the eigenspaces of  $\Phi$  corresponding to the eigenvalues  $\lambda_1, \dots, \lambda_m$ . As  $V_1, \dots, V_m$  are orthogonal critical subspaces and  $\mathbb{R}^n = \bigoplus_{j=1}^m V_j$ . As  $V_1, \dots, V_m$  are orthogonal critical subspaces and  $\mathbb{R}^n = \bigoplus_{j=1}^m V_j$ , we deduce that  $x_{ij} = P_{V_j} x_i \in E_i \cap V_j$  for any  $i = 1, \dots, k$  and  $j = 1, \dots, m$ , and  $x_i = \sum_{j=1}^m x_{ij}$  for any  $i = 1, \dots, k$ . It follows that

$$x = \sum_{j=1}^m \left( \sum_{E_i \cap V_j \neq \{0\}} c_i x_{ij} \right) \text{ where}$$

$$P_{V_j} x = \sum_{E_i \cap V_j \neq \{0\}} c_i x_{ij}. \quad (38)$$

For any  $i = 1, \dots, k$ , the vectors  $\Phi x_{ij} = \lambda_j x_{ij}$  are pairwise orthogonal for  $j = 1, \dots, m$ , thus

$$\sum_{i=1}^k c_i \|\Phi x_i\|^2 = \sum_{i=1}^k \left( \sum_{j=1}^m c_i \|\Phi x_{ij}\|^2 \right) = \sum_{j=1}^m \left( \sum_{E_i \cap V_j \neq \{0\}} c_i \|\Phi x_{ij}\|^2 \right).$$

Since  $\|\Phi x\|^2 = \sum_{j=1}^m \|P_{V_j} \Phi x\|^2 = \sum_{j=1}^m \|\Phi P_{V_j} x\|^2$ , (37) follows if for any  $j = 1, \dots, m$ , we have

$$\|\Phi P_{V_j} x\|^2 \leq \sum_{E_i \cap V_j \neq \{0\}} c_i \|\Phi x_{ij}\|^2. \quad (39)$$

To prove (39), if  $E_i \cap V_j \neq \{0\}$ , then let  $\dim(E_i \cap V_j) = n_{ij}$ , and let  $u_1^{(ij)}, \dots, u_{n_{ij}}^{(ij)}$  be an orthonormal basis of  $E_i \cap V_j$ . Since  $V_j$  is a critical subspace (see (33)), if  $z \in V_j$ , then

$$z = \sum_{i=1}^k c_i P_{E_i} z = \sum_{E_i \cap V_j \neq \{0\}} c_i P_{E_i \cap V_j} z = \sum_{E_i \cap V_j \neq \{0\}} \sum_{\alpha=1}^{n_{ij}} c_i \langle u_\alpha^{(ij)}, z \rangle u_\alpha^{(ij)}. \quad (40)$$

(40) shows that the system of all  $u_1^{(ij)}, \dots, u_{n_{ij}}^{(ij)}$  when  $E_i \cap V_j \neq \{0\}$  form a rank one Brascamp-Lieb data where the coefficient corresponding to  $u_\alpha^{(ij)}$  is  $c_i$ .

According to (38), we have

$$P_{V_j} x = \sum_{E_i \cap V_j \neq \{0\}} \sum_{\alpha=1}^{n_{ij}} c_i \langle u_\alpha^{(ij)}, x_{ij} \rangle u_\alpha^{(ij)}.$$

We deduce from Case 1 applying to  $P_{V_j} x$  to the rank one Brascamp-Lieb data in  $V_j$  above that

$$\begin{aligned} \|\Phi P_{V_j} x\|^2 &= \lambda_j^2 \|P_{V_j} x\|^2 \leq \lambda_j^2 \sum_{E_i \cap V_j \neq \{0\}} \sum_{\alpha=1}^{n_{ij}} c_i \langle u_\alpha^{(ij)}, x_{ij} \rangle^2 \\ &= \lambda_j^2 \sum_{E_i \cap V_j \neq \{0\}} c_i \|x_{ij}\|^2 = \sum_{E_i \cap V_j \neq \{0\}} c_i \|\Phi x_{ij}\|^2, \end{aligned}$$

proving (39), and in turn (37) that is equivalent to Lemma 24.  $\square$

We now use Proposition 22 and Lemma 24 to exhibit the basic type of Gaussian exemizers of Barthe's inequality.

**Proposition 25** *For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$ ,  $n \geq 1$  and  $c_1, \dots, c_k > 0$  satisfying (22), if  $\Phi$  is a positive definite linear transform whose eigenspaces are critical subspaces, then*

$$\int_{\mathbb{R}^n} \left( \sup_{\substack{x = \sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k e^{-c_i \|\Phi x_i\|^2} \right) dx = \prod_{i=1}^k \left( \int_{E_i} e^{-\|\Phi x_i\|^2} dx_i \right)^{c_i}.$$

*Proof:* Let  $\tilde{\Phi} = \pi^{-\frac{1}{2}} \Phi$ . For  $i = 1, \dots, k$ , let  $A_i = \tilde{\Phi}|_{E_i}$ , and hence  $A_i : E_i \rightarrow E_i$  as the eigenspaces of  $\tilde{\Phi}$  are critical subspaces. We deduce first using Lemma 24, and then the equality case of Proposition 22 that

$$\begin{aligned} \int_{\mathbb{R}^n} \left( \sup_{\substack{x = \sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k e^{-c_i \|\Phi x_i\|^2} \right) dx &= \int_{\mathbb{R}^n} e^{-\pi \|\tilde{\Phi} x\|^2} dx = (\det \tilde{\Phi})^{-1} = \prod_{i=1}^k (\det A_i)^{-c_i} \\ &= \prod_{i=1}^k \left( \int_{E_i} e^{-\pi \|\tilde{\Phi} x_i\|^2} dx_i \right)^{c_i} = \prod_{i=1}^k \left( \int_{E_i} e^{-\|\Phi x_i\|^2} dx_i \right)^{c_i}, \end{aligned}$$

proving Proposition 25.  $\square$

## 6 Splitting smooth extremizers along independent and dependent subspaces

Optimal transportation as a tool proving geometric inequalities was introduced by Gromov in his Appendix to [84] in the case of the Brunn-Minkowski inequality. Actually, Barthe's inequality in [8] was one of the first inequalities in probability, analysis or geometry that was obtained via optimal transportation.

We write  $\nabla\Theta$  to denote the first derivative of a  $C^1$  vector valued function  $\Theta$  defined on an open subset of  $\mathbb{R}^n$ , and  $\nabla^2\varphi$  to denote the Hessian of a real  $C^2$  function  $\varphi$ . We recall that a vector valued function  $\Theta$  on an open set  $U \subset \mathbb{R}^n$  is  $C^\alpha$  for  $\alpha \in (0, 1)$  if for any  $x_0 \in U$  there exist an open neighbourhood  $U_0$  of  $x_0$  and a  $c_0 > 0$  such that  $\|\Theta(x) - \Theta(y)\| \leq c_0\|x - y\|^\alpha$  for  $x, y \in U_0$ . In addition, a real function  $\varphi$  is  $C^{2,\alpha}$  if  $\varphi$  is  $C^2$  and  $\nabla^2\varphi$  is  $C^\alpha$ .

Combining Corollary 2.30, Corollary 2.32, Theorem 4.10 and Theorem 4.13 in Villani [94] on the Brenier map based on McCann [81, 82] for the first two, and on Caffarelli [27, 28, 29] for the last two theorems, we deduce the following:

**Theorem 26 (Brenier, McCann, Caffarelli)** *If  $f$  and  $g$  are positive  $C^\alpha$  probability density functions on  $\mathbb{R}^n$ ,  $n \geq 1$ , for  $\alpha \in (0, 1)$ , then there exists a  $C^{2,\alpha}$  convex function  $\varphi$  on  $\mathbb{R}^n$  (unique up to additive constant) such that  $T = \nabla\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is bijective and*

$$g(x) = f(T(x)) \cdot \det \nabla T(x) \text{ for } x \in \mathbb{R}^n. \quad (41)$$

**Remarks** The derivative  $T = \nabla\varphi$  is the Brenier (transportation) map pushing forward the measure on  $\mathbb{R}^n$  induced by  $g$  to the measure associated to  $f$ ; namely,  $\int_{T(X)} f = \int_X g$  for any measurable  $X \subset \mathbb{R}^n$ .

In addition,  $\nabla T = \nabla^2\varphi$  is a positive definite symmetric matrix in Theorem 26, and if  $f$  and  $g$  are  $C^k$  for  $k \geq 1$ , then  $T$  is  $C^{k+1}$ .

Sometimes it is practical to consider the case  $n = 0$ , when we set  $T : \{0\} \rightarrow \{0\}$  to be the trivial map.

*Proof of Theorem 3 based on Barthe [8].* First we assume that each  $f_i$  is a  $C^1$  positive probability density function on  $\mathbb{R}^n$ , and let us consider the Gaussian density  $g_i(x) = e^{-\pi\|x\|^2}$  for  $x \in E_i$ . According to Theorem 26, if  $i = 1, \dots, k$ , then there exists a  $C^3$  convex function  $\varphi_i$  on  $E_i$  such that for the  $C^2$  Brenier map  $T_i = \nabla\varphi_i$ , we have

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \text{ for all } x \in E_i. \quad (42)$$

It follows from the Remark after Theorem 26 that  $\nabla T_i = \nabla^2\varphi_i(x)$  is positive definite symmetric matrix for all  $x \in E_i$ . For the  $C^2$  transformation  $\Theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by

$$\Theta(y) = \sum_{i=1}^k c_i T_i(P_{E_i}y), \quad y \in \mathbb{R}^n, \quad (43)$$

its differential

$$\nabla\Theta(y) = \sum_{i=1}^k c_i \nabla T_i(P_{E_i}y)$$

is positive definite by Proposition 22. It follows that  $\Theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is injective (see [8]), and actually a diffeomorphism. Therefore Proposition 22, (42) and Lemma 18 (i) imply

$$\begin{aligned} & \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i} dx \\ & \geq \int_{*,\mathbb{R}^n} \left( \sup_{\Theta(y)=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i} \right) \det(\nabla\Theta(y)) dy \\ & \geq \int_{\mathbb{R}^n} \left( \prod_{i=1}^k f_i(T_i(P_{E_i}y))^{c_i} \right) \det\left(\sum_{i=1}^k c_i \nabla T_i(P_{E_i}y)\right) dy \\ & \geq \int_{\mathbb{R}^n} \left( \prod_{i=1}^k f_i(T_i(P_{E_i}y))^{c_i} \right) \prod_{i=1}^k (\det \nabla T_i(P_{E_i}y))^{c_i} dy \tag{44} \\ & = \int_{\mathbb{R}^n} \left( \prod_{i=1}^k g_i(P_{E_i}y)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2} dy = 1. \end{aligned}$$

Finally, Barthe's inequality (5) for arbitrary non-negative integrable functions  $f_i$  follows by scaling and approximation (see Barthe [8]).  $\square$

We now prove that if equality holds in Barthe's inequality (5), then the diffeomorphism  $\Theta$  in (43) in the proof of Barthe's inequality splits along the independent subspaces and the dependent subspace. First we explain how Barthe's inequality behaves under the shifts of the functions involved. Given proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), first we discuss in what sense Barthe's inequality is translation invariant. For non-negative integrable function  $f_i$  on  $E_i$ ,  $i = 1, \dots, k$ , let us define

$$F(x) = \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}.$$

We observe that for any  $e_i \in E_i$ , defining  $\tilde{f}_i(x) = f_i(x + e_i)$  for  $x \in E_i$ ,  $i = 1, \dots, k$ , we have

$$\tilde{F}(x) = \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k \tilde{f}_i(x_i)^{c_i} = F\left(x + \sum_{i=1}^k c_i e_i\right). \tag{45}$$

**Proposition 27** *For non-trivial linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), we write  $F_1, \dots, F_l$  to denote the independent subspaces (if exist), and  $F_0$  to denote the*

dependent subspace (possibly  $F_0 = \{0\}$ ). Let us assume that equality holds in (5) for positive  $C^1$  probability densities  $f_i$  on  $E_i$ ,  $i = 1, \dots, k$ , let  $g_i(x) = e^{-\pi\|x\|^2}$  for  $x \in E_i$ , let  $T_i : E_i \rightarrow E_i$  be the  $C^2$  Brenier map satisfying

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \text{ for all } x \in E_i, \quad (46)$$

and let

$$\Theta(y) = \sum_{i=1}^k c_i T_i(P_{E_i} y), \quad y \in \mathbb{R}^n.$$

(i) For any  $i \in \{1, \dots, k\}$  there exists positive  $C^1$  integrable  $h_{i0} : F_0 \cap E_i \rightarrow [0, \infty)$  (where  $h_{i0}(0) = 1$  if  $F_0 \cap E_i = \{0\}$ ), and for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ , there exists positive  $C^1$  integrable  $h_{ij} : F_j \rightarrow [0, \infty)$  such that

$$f_i(x) = h_{i0}(P_{F_0} x) \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} h_{ij}(P_{F_j} x) \text{ for } x \in E_i.$$

(ii) For  $i = 1, \dots, k$ ,  $T_i(E_i \cap F_p) = E_i \cap F_p$  whenever  $E_i \cap F_p \neq \{0\}$  for  $p \in \{0, \dots, l\}$ , and if  $x \in E_i$ , then

$$T_i(x) = \bigoplus_{\substack{E_i \cap F_p \neq \{0\} \\ p \geq 0}} T_i(P_{F_p} x).$$

(iii) For  $i = 1, \dots, k$ , there exist  $C^2$  functions  $\Omega_i : E_i \rightarrow E_i$  and  $\Gamma_i : E_i^\perp \rightarrow E_i^\perp$  such that

$$\Theta(y) = \Omega_i(P_{E_i} y) + \Gamma_i(P_{E_i^\perp} y) \text{ for } y \in \mathbb{R}^n.$$

(iv) If  $y \in \mathbb{R}^n$ , then the eigenspaces of the positive definite matrix  $\nabla \Theta(y)$  are critical subspaces, and  $\nabla T_i(P_{E_i} y) = \nabla \Theta(y)|_{E_i}$  for  $i = 1, \dots, k$ .

*Proof:* According to (45), we may assume that

$$T_i(0) = 0 \text{ for } i = 1, \dots, k, \quad (47)$$

If equality holds in (5), then equality holds in the determinantal inequality in (44) in the proof of Barthe's inequality; therefore, we apply the equality case of Proposition 22. In particular, for any  $x \in \mathbb{R}^n$ , there exist  $m_x \geq 1$  and linear subspaces  $V_{1,x}, \dots, V_{m_x,x}$  where  $V_{1,x} = \mathbb{R}^n$  if  $m_x = 1$ , and  $V_{1,x}, \dots, V_{m_x,x}$  are pairwise orthogonal indecomposable critical subspaces spanning  $\mathbb{R}^n$  if  $m_x \geq 2$ , and there exist  $\lambda_{1,x}, \dots, \lambda_{m_x,x} > 0$  such that if  $E_i \cap V_{j,x} \neq \{0\}$ , then

$$\nabla T_i(P_{E_i} x)|_{E_i \cap V_{j,x}} = \lambda_{j,x} I_{E_i \cap V_{j,x}}; \quad (48)$$

and in addition, each  $E_i$  satisfies (cf. (28))

$$E_i = \bigoplus_{E_i \cap V_{j,x} \neq \{0\}} E_i \cap V_{j,x}. \quad (49)$$

Let us consider a fixed  $E_i$ ,  $i \in \{1, \dots, k\}$ . First we claim that if  $y \in E_i$ , then

$$\begin{aligned} \nabla T_i(y)(F_p) &= F_p && \text{if } p \geq 1 \text{ and } E_i \cap F_p \neq \{0\} \\ \nabla T_i(y)(F_0 \cap E_i) &= F_0 \cap E_i. \end{aligned} \quad (50)$$

To prove (50), we take  $y = x$  in (48). If  $p \geq 1$  and  $E_i \cap F_p \neq \{0\}$ , then  $F_p \subset E_i$ , and Lemma 23 yields that

$$\begin{aligned} \bigoplus_{F_p \cap V_{j,y} \neq \{0\}} V_{j,y} &\subset F_p \\ \bigoplus_{F_p \cap V_{j,y} = \{0\}} V_{j,y} &\subset F_p^\perp. \end{aligned}$$

Since the subspaces  $V_{j,y}$  span  $\mathbb{R}^n$ , we have

$$F_p = \bigoplus_{\substack{E_i \cap V_{j,y} \neq \{0\} \\ V_{j,y} \subset F_p}} V_{j,y};$$

therefore, (48) implies (50) if  $p \geq 1$ .

For the case of  $F_0$  in (50), it follows from (49) and Lemma 23 that if  $E_i \cap F_0 \neq \{0\}$ , then

$$E_i \cap F_0 = \bigoplus_{\substack{E_i \cap V_{j,y} \neq \{0\} \\ V_{j,y} \subset F_0}} E_i \cap V_{j,y}. \quad (51)$$

Therefore, (48) completes the proof of (50).

It follows from (50) that if  $E_i \cap F_p \neq \{0\}$ ,  $y \in E_i$ ,  $v \in E_i \cap F_p \cap S^{n-1}$  and  $w \in E_i \cap F_p^\perp \cap S^{n-1}$ , then

$$\left\langle v, \frac{\partial}{\partial t} T_i(y + tw) \Big|_{t=0} \right\rangle = 0. \quad (52)$$

In turn, (50), (52) and  $T_i(0) = 0$  (cf. (47)) imply that if  $y \in E_i$ , then

$$T_i(E_i \cap F_p) = E_i \cap F_p \text{ whenever } E_i \cap F_p \neq \{0\} \text{ for } p \geq 0, \quad (53)$$

$$T_i(y) = \bigoplus_{\substack{E_i \cap F_p \neq \{0\} \\ p \geq 0}} T_i(P_{F_p} y). \quad (54)$$

We deduce from (54) that if  $y \in E_i$ , then

$$\det \nabla T_i(y) = \prod_{\substack{E_i \cap F_p \neq \{0\} \\ p \geq 0}} \det (\nabla T_i(P_{F_p} y)|_{F_p}). \quad (55)$$

We conclude (i) from (52), (53), (54), and (55) as (46) yields that if  $y \in E_i$ , then

$$f_i(T_i(y)) = \prod_{\substack{E_i \cap F_p \neq \{0\} \\ p \geq 0}} \frac{e^{-\pi \|P_{F_p} y\|^2}}{\det (\nabla T_i(P_{F_p} y)|_{F_p})}.$$

We deduce (ii) from (53) and (54).



For (iii), it follows from Proposition 22 that for any  $x \in \mathbb{R}^n$ , the spaces  $V_{j,x}$  are eigenspaces for  $\nabla\Theta(x)$  and span  $\mathbb{R}^n$ ; therefore, (27) implies that if  $x \in \mathbb{R}^n$  and  $i \in \{1, \dots, k\}$ , then

$$\nabla\Theta(x) = \nabla\Theta(x)|_{E_i} \oplus \nabla\Theta(x)|_{E_i^\perp}.$$

Since  $\Theta(0) = 0$  by (47), for fixed  $i \in \{1, \dots, k\}$ , we conclude

$$\begin{aligned} \Theta(E_i) &= E_i; \\ \Theta(x) &= \Theta(P_{E_i}x)|_{E_i} \oplus \Theta(P_{E_i^\perp}x)|_{E_i^\perp} \quad \text{if } x \in \mathbb{R}^n. \end{aligned}$$

Finally, (iv) directly follows from Proposition 22, completing the proof of Proposition 27.  $\square$

Next we show that if the extremizers  $f_1, \dots, f_k$  in Proposition 27 are of the form as in (i), then for any given  $F_j \neq \{0\}$ , the functions  $h_{ij}$  on  $F_j$  for all  $i$  with  $E_i \cap F_j \neq \{0\}$  are also extremizers. We also need the Prékopa-Leindler inequality Theorem 28 (proved in various forms by Prékopa [86, 87], Leindler [70] and Borell [20]) whose equality case was clarified by Dubuc [38] (see the survey Gardner [45]). In turn, the Prékopa-Leindler inequality (56) is of the very similar structure like Barthe's inequality (5). Again, the inequality is usually stated using outer integrals, but being the special case of Barthe's inequality (5), the remarks after Theorem 3 concerning inner integration apply.

**Theorem 28 (Prékopa, Leindler, Dubuc)** *For  $m \geq 2$ ,  $\lambda_1, \dots, \lambda_m \in (0, 1)$  with  $\lambda_1 + \dots + \lambda_m = 1$  and integrable  $\varphi_1, \dots, \varphi_m : \mathbb{R}^n \rightarrow [0, \infty)$ , we have*

$$\int_{*\mathbb{R}^n} \sup_{x = \sum_{i=1}^m \lambda_i x_i, x_i \in \mathbb{R}^n} \prod_{i=1}^m \varphi_i(x_i)^{\lambda_i} dx \geq \prod_{i=1}^m \left( \int_{\mathbb{R}^n} \varphi_i \right)^{\lambda_i}, \quad (56)$$

*and if equality holds and the left hand side is positive and finite, then there exist a log-concave function  $\varphi$  and  $a_i > 0$  and  $b_i \in \mathbb{R}^n$  for  $i = 1, \dots, m$  such that*

$$\varphi_i(x) = a_i \varphi(x - b_i)$$

*for Lebesgue a.e.  $x \in \mathbb{R}^n$ ,  $i = 1, \dots, m$ .*

For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), we assume that  $F_{\text{dep}} \neq \mathbb{R}^n$ , and write  $F_1, \dots, F_l$  to denote the independent subspaces. We verify that if  $j \in \{1, \dots, l\}$ , then

$$\sum_{E_i \supset F_j} c_i = 1. \quad (57)$$

For this, let  $x \in F_j \setminus \{0\}$ . We observe that for any  $E_i$ , either  $F_j \subset E_i$ , and hence  $P_{E_i}x = x$ , or  $F_j \subset E_i^\perp$ , and hence  $P_{E_i}x = 0$ . We deduce from (1) that

$$x = \sum_{i=1}^k c_i P_{E_i}x = \left( \sum_{F_j \subset E_i} c_i \right) \cdot x,$$

which formula in turn implies (57).

**Proposition 29** For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (1), we write  $F_1, \dots, F_l$  to denote the independent subspaces (if exist), and  $F_0$  denote the dependent subspace (possibly  $F_0 = \{0\}$ ). Let us assume that equality holds in Barthe's inequality (5) for probability densities  $f_i$  on  $E_i$ ,  $i = 1, \dots, k$ , and for any  $i \in \{1, \dots, k\}$  there exists integrable  $h_{i0} : F_0 \cap E_i \rightarrow [0, \infty)$  (where  $h_{i0}(0) = 1$  if  $F_0 \cap E_i = \{0\}$ ), and for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ , there exists non-negative integrable  $h_{ij} : F_j \rightarrow [0, \infty)$  such that

$$f_i(x) = h_{i0}(P_{F_0}x) \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} h_{ij}(P_{F_j}x) \quad \text{for } x \in E_i. \quad (58)$$

(i) If  $F_0 \neq \{0\}$ , then  $\sum_{E_i \cap F_0 \neq \{0\}} c_i P_{E_i \cap F_0} = \text{Id}_{F_0}$  and

$$\int_{*, F_0} \sup_{x = \sum \{c_i x_i : x_i \in E_i \cap F_0 \ \& \ E_i \cap F_0 \neq \{0\}\}} \prod_{E_i \cap F_0 \neq \{0\}} h_{i0}(x_i)^{c_i} dx = \prod_{E_i \cap F_0 \neq \{0\}} \left( \int_{E_i \cap F_0} h_{i0} \right)^{c_i}.$$

(ii) If  $F_0 \neq \mathbb{R}^n$ , then there exist integrable  $\psi_j : F_j \rightarrow [0, \infty)$  for  $j = 1, \dots, l$  where  $\psi_j$  is log-concave whenever  $F_j \subset E_\alpha \cap E_\beta$  for  $\alpha \neq \beta$ , and there exist  $a_{ij} > 0$  and  $b_{ij} \in F_j$  for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$  such that  $h_{ij}(x) = a_{ij} \cdot \psi_j(x - b_{ij})$  for  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ .

*Proof:* We only present the argument in the case  $F_0 \neq \mathbb{R}^n$  and  $F_0 \neq \{0\}$ . If  $F_0 = \mathbb{R}^n$ , then the same argument works ignoring the parts involving  $F_1, \dots, F_l$ , and if  $F_0 = \{0\}$ , then the same argument works ignoring the parts involving  $F_0$ .

Since  $F_0 \oplus F_1 \oplus \dots \oplus F_l = \mathbb{R}^n$  and  $F_0, \dots, F_l$  are critical subspaces, (28) yields for  $i = 1, \dots, k$  that

$$E_i = (E_i \cap F_0) \oplus \bigoplus_{\substack{F_j \subset E_i \\ j \geq 1}} F_j; \quad (59)$$

therefore, the Fubini theorem and (58) imply that

$$\int_{E_i} f_i = \left( \int_{E_i \cap F_0} h_{i0} \right) \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} \int_{F_j} h_{ij}. \quad (60)$$

On the other hand, using again  $F_0 \oplus F_1 \oplus \dots \oplus F_l = \mathbb{R}^n$ , we deduce that if  $x = \sum_{j=0}^l z_j$  where  $z_j \in F_j$  for  $j \geq 0$ , then  $z_j = P_{F_j}x$ . It follows from (59) that for any  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i)^{c_i} &= \left( \sup_{\substack{P_{F_0}x = \sum_{i=1}^k c_i x_{0i}, \\ x_{0i} \in E_i \cap F_0}} \prod_{i=1}^k h_{i0}(x_{0i}) \right) \times \\ &\times \prod_{j=1}^l \left( \sup_{\substack{P_{F_j}x = \sum_{F_j \subset E_i} c_i x_{ji}, \\ x_{ji} \in F_j}} \prod_{F_j \subset E_i} h_{ij}(x_{ji})^{c_i} \right), \end{aligned}$$

and hence

$$\begin{aligned} \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i} dx &= \left( \int_{*,F_0} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i \cap F_0} \prod_{i=1}^k h_{i0}(x_i) dx \right) \times \\ &\times \prod_{j=1}^l \left( \int_{*,F_j} \sup_{x=\sum_{F_j \subset E_i} c_i x_i, x_i \in F_j} \prod_{F_j \subset E_i} h_{ij}(x_i)^{c_i} dx \right). \end{aligned} \quad (61)$$

As  $F_0$  is a critical subspace, we have

$$\sum_{i=1}^k c_i P_{E_i \cap F_0} = \text{Id}_{F_0},$$

and hence Barthe's inequality (5) yields

$$\int_{*,F_0} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i \cap F_0} \prod_{i=1}^k h_{i0}(x_i) dx \geq \prod_{i=1}^k \left( \int_{E_i \cap F_0} h_{i0} \right)^{c_i}. \quad (62)$$

We deduce from (57) and the Prékopa-Leindler inequality (56) that if  $j = 1, \dots, l$ , then

$$\int_{*,F_j} \sup_{x=\sum_{F_j \subset E_i} c_i x_i, x_i \in F_j} \prod_{F_j \subset E_i} h_{ij}(x_i)^{c_i} dx \geq \prod_{E_i \supset F_j} \left( \int_{F_j} h_{ij} \right)^{c_i}. \quad (63)$$

Combining (60), (61), (62) and (63) with the fact that  $f_1, \dots, f_k$  are extremizers for Barthe's inequality (5) implies that if  $j = 1, \dots, l$ , then

$$\int_{*,F_0} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i \cap F_0} \prod_{i=1}^k h_{i0}(x_i) dx = \prod_{i=1}^k \left( \int_{E_i \cap F_0} h_{i0} \right)^{c_i} \quad (64)$$

$$\int_{*,F_j} \sup_{x=\sum_{F_j \subset E_i} c_i x_i, x_i \in F_j} \prod_{F_j \subset E_i} h_{ij}(x_i)^{c_i} dx = \prod_{E_i \supset F_j} \left( \int_{F_j} h_{ij} \right)^{c_i}. \quad (65)$$

We observe that (64) is just (i). In addition, (ii) follows from the equality conditions in the Prékopa-Leindler inequality (see Theorem 28).  $\square$

## 7 Convolution and product of extremizers

Given proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), we say that the non-negative integrable functions  $f_1, \dots, f_k$  with positive integrals are extremizers if

equality holds in (5). In order to deal with positive smooth functions, we use convolutions. More precisely, Lemma 2 in Barthe [8] states the following.

**Lemma 30** *Given proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), if  $f_1, \dots, f_k$  and  $g_1, \dots, g_k$  are extremizers in Barthe's inequality (5), then  $f_1 * g_1, \dots, f_k * g_k$  are also extremizers.*

*Proof:* We may assume that  $\int_{\mathbb{R}^n} f_i = \int_{\mathbb{R}^n} g_i = 1$  for  $i = 1, \dots, k$ . We define

$$F(x) = \sup_{x = \sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k f_i(x_i)^{c_i}$$

$$G(y) = \sup_{y = \sum_{i=1}^k c_i y_i, y_i \in E_i} \prod_{i=1}^k g_i(y_i)^{c_i}.$$

Possibly  $F$  and  $G$  are not measurable but as  $f_1, \dots, f_k$  and  $g_1, \dots, g_k$  are extremizers, we have  $\int_{*, \mathbb{R}^n} F = \int_{*, \mathbb{R}^n} G = 1$ , and there exist measurable  $\tilde{F} \geq F$  and  $0 \leq \tilde{G} \leq G$  such that  $\int_{\mathbb{R}^n} \tilde{F}(x) dx = \int_{\mathbb{R}^n} \tilde{G}(x) dx = 1$ , and hence neither  $\{F > \tilde{F}\}$  nor  $\{G > \tilde{G}\}$  contains a subset of  $\mathbb{R}^n$  with positive measure. We write any point of  $\mathbb{R}^{2n}$  in the form  $(x, y)$  for  $x, y \in \mathbb{R}^n$ , and hence  $(z, y) \mapsto \tilde{F}(z)\tilde{G}(y)$  is a measurable witness for  $(z, y) \mapsto F(z)G(y)$ , and in turn  $(x, y) \mapsto \tilde{F}(x-y)\tilde{G}(y)$  is a measurable witness for  $(x, y) \mapsto F(x-y)G(y)$  in the case of inner integrals. We deduce writing  $x_i = z_i + y_i$  in (66) for  $i = 1, \dots, k$  and using Barthe's inequality

in (67) that

$$\begin{aligned}
1 &= \int_{\mathbb{R}^n} \tilde{F} * \tilde{G}(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{F}(x-y) \tilde{G}(y) dy dx \\
&= \int_{*, \mathbb{R}^{2n}} \sup_{x-y=\sum_{i=1}^k c_i z_i, z_i \in E_i} \prod_{i=1}^k f_i(z_i)^{c_i} \sup_{y=\sum_{i=1}^k c_i y_i, y_i \in E_i} \prod_{i=1}^k g_i(y_i)^{c_i} d(x, y) \\
&= \int_{*, \mathbb{R}^{2n}} \sup_{x-y=\sum_{i=1}^k c_i z_i, z_i \in E_i} \sup_{y=\sum_{i=1}^k c_i y_i, y_i \in E_i} \prod_{i=1}^k f_i(z_i)^{c_i} \prod_{i=1}^k g_i(y_i)^{c_i} d(x, y) \quad (66) \\
&= \int_{*, \mathbb{R}^{2n}} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \sup_{y=\sum_{i=1}^k c_i y_i, y_i \in E_i} \prod_{i=1}^k f_i(x_i - y_i)^{c_i} \prod_{i=1}^k g_i(y_i)^{c_i} d(x, y) \\
&\geq \int_{*, \mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \int_{*, \mathbb{R}^n} \sup_{y=\sum_{i=1}^k c_i y_i, y_i \in E_i} \prod_{i=1}^k (f_i(x_i - y_i) g_i(y_i))^{c_i} dy dx \quad (67) \\
&\geq \int_{*, \mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k \left( \int_{E_i} f_i(x_i - y_i) g_i(y_i) dy_i \right)^{c_i} dx \\
&= \int_{*, \mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i, x_i \in E_i} \prod_{i=1}^k (f_i * g_i(x_i))^{c_i} dx \geq \prod_{i=1}^k \left( \int_{\mathbb{R}^n} f_i * g_i(x_i) \right)^{c_i} = 1
\end{aligned}$$

as  $\int_{E_i} f_i * g_i = 1$  for  $i = 1, \dots, k$ . In turn, we conclude that  $f_i * g_i$ ,  $i = 1, \dots, k$ , is also an extremizer.  $\square$

Since in a certain case we want to work with Lebesgue integral instead of outer integrals, we use the following statement that can be proved via compactness argument.

**Lemma 31** *Given proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), if  $h_i$  is a positive continuous functions satisfying  $\lim_{x \rightarrow \infty} h_i(x) = 0$  for  $i = 1, \dots, k$ , then the function*

$$h(x) = \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k h_i(x_i)^{c_i}$$

of  $x \in \mathbb{R}^n$  is continuous.

Next we show that the product of a shift of a smooth extremizer and a Gaussian is also an extremizer for Barthe's inequality.

**Lemma 32** *Given proper linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^n$  and  $c_1, \dots, c_k > 0$  satisfying (22), if  $f_1, \dots, f_k$  are positive bounded  $C^1$  are extremizers in Barthe's inequality (5), and  $g_i(x) = e^{-\pi \|x\|^2}$  for  $x \in E_i$ , then there exist  $z_i \in E_i$ ,  $i = 1, \dots, k$ , such that the functions  $y \mapsto f_i(y - z_i) g_i(y)$  of  $y \in E_i$ ,  $i = 1, \dots, k$ , are also extremizers for (5).*

*Proof:* We may assume that  $f_1, \dots, f_k$  are probability densities.

Readily the functions  $\tilde{f}_1, \dots, \tilde{f}_k$  defined by  $\tilde{f}_i(y) = f_i(-y)$  for  $y \in E_i$  and  $i = 1, \dots, k$  are also extremizers. We deduce from Lemma 30 that the functions  $\tilde{f}_i * g_i$  for  $i = 1, \dots, k$  are also extremizers where each  $\tilde{f}_i * g_i$  is a probability density on  $E_i$ . According to Theorem 26, if  $i = 1, \dots, k$ , then there exists a  $C^2$  Brenier map  $S_i : E_i \rightarrow E_i$  such that

$$g_i(x) = \det \nabla S_i(x) \cdot (\tilde{f}_i * g_i)(S_i(x)) \text{ for all } x \in E_i,$$

and  $\nabla S_i(x)$  is a positive definite symmetric matrix for all  $x \in E_i$ . As in the proof of Theorem 3 above, we consider the  $C^2$  diffeomorphism  $\Theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by

$$\Theta(y) = \sum_{i=1}^k c_i S_i(P_{E_i} y), \quad y \in \mathbb{R}^n.$$

whose positive definite differential is

$$\nabla \Theta(y) = \sum_{i=1}^k c_i \nabla S_i(P_{E_i} y).$$

On the one hand, we note that if  $x = \sum_{i=1}^k c_i x_i$  for  $x_i \in E_i$ , then

$$\|x\|^2 \leq \sum_{i=1}^k c_i \|x_i\|^2$$

holds according to Barthe [8]; or equivalently,

$$\prod_{i=1}^k g_i(x_i)^{c_i} \leq e^{-\pi \|x\|^2}.$$

Since  $f_i$  is positive, bounded, continuous and in  $L_1(E_i)$  for  $i = 1, \dots, k$ , we observe that the function

$$z \mapsto \int_{\mathbb{R}^n} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - S_i(P_{E_i} \Theta^{-1} z))^{c_i} g_i(x_i)^{c_i} dx \quad (68)$$

of  $z \in \mathbb{R}^n$  is continuous.

Using also that  $\tilde{f}_1, \dots, \tilde{f}_k$  are extremizers and probability density functions, we have

$$\begin{aligned}
& \int_{*,\mathbb{R}^n} \int_{*,\mathbb{R}^n} \sup_{\substack{z=\sum_{i=1}^k c_i z_i, \\ z_i \in E_i}} \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - z_i)^{c_i} g_i(x_i)^{c_i} dx dz \\
&= \int_{*,\mathbb{R}^n} \int_{*,\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \left( \prod_{i=1}^k g_i(x_i)^{c_i} \right) \sup_{\substack{z=\sum_{i=1}^k c_i z_i, \\ z_i \in E_i}} \prod_{i=1}^k f_i(x_i - z_i)^{c_i} dz dx \\
&\leq \int_{*,\mathbb{R}^n} e^{-\pi\|x\|^2} \int_{*,\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \sup_{\substack{z=\sum_{i=1}^k c_i z_i, \\ z_i \in E_i}} \prod_{i=1}^k f_i(x_i - z_i)^{c_i} dz dx \\
&= \int_{*,\mathbb{R}^n} e^{-\pi\|x\|^2} \int_{*,\mathbb{R}^n} \sup_{\substack{z-x=\sum_{i=1}^k c_i y_i, \\ y_i \in E_i}} \prod_{i=1}^k \tilde{f}_i(y_i)^{c_i} dz dx \\
&= \int_{*,\mathbb{R}^n} e^{-\pi\|x\|^2} \int_{*,\mathbb{R}^n} \sup_{\substack{w=\sum_{i=1}^k c_i y_i, \\ y_i \in E_i}} \prod_{i=1}^k \tilde{f}_i(y_i)^{c_i} dw dx \\
&= \int_{\mathbb{R}^n} e^{-\pi\|x\|^2} dx = 1.
\end{aligned}$$

Using Lemma 31 and (68) in (69), Barthe's inequality (5) in (70) and Proposition 22 in (71),

we deduce that

$$\begin{aligned}
1 &\geq \int_{*,\mathbb{R}^n} \int_{*,\mathbb{R}^n} \sup_{\substack{z=\sum_{i=1}^k c_i z_i \\ z_i \in E_i}} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - z_i)^{c_i} g_i(x_i)^{c_i} dx dz \\
&\geq \int_{*,\mathbb{R}^n} \int_{*,\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - S_i(P_{E_i}\Theta^{-1}z))^{c_i} g_i(x_i)^{c_i} dx dz \tag{69}
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - S_i(P_{E_i}\Theta^{-1}z))^{c_i} g_i(x_i)^{c_i} dx dz \tag{70}$$

$$\begin{aligned}
&\geq \int_{\mathbb{R}^n} \prod_{i=1}^k \left( \int_{E_i} f_i(x_i - S_i(P_{E_i}\Theta^{-1}z)) g_i(x_i) dx_i \right)^{c_i} dz \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^k (\tilde{f}_i * g_i)(S_i(P_{E_i}\Theta^{-1}z))^{c_i} dz \\
&= \int_{\mathbb{R}^n} \left( \prod_{i=1}^k (\tilde{f}_i * g_i)(S_i(P_{E_i}y))^{c_i} \right) \det(\nabla\Theta(y)) dy \\
&= \int_{\mathbb{R}^n} \left( \prod_{i=1}^k (\tilde{f}_i * g_i)(S_i(P_{E_i}y))^{c_i} \right) \det\left(\sum_{i=1}^k c_i \nabla S_i(P_{E_i}y)\right) dy \tag{71} \\
&\geq \int_{\mathbb{R}^n} \left( \prod_{i=1}^k (\tilde{f}_i * g_i)(S_i(P_{E_i}y))^{c_i} \right) \prod_{i=1}^k (\det \nabla S_i(P_{E_i}y))^{c_i} dy \\
&= \int_{\mathbb{R}^n} \left( \prod_{i=1}^k g_i(P_{E_i}y)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2} dy = 1.
\end{aligned}$$

In particular, we conclude that

$$\begin{aligned}
1 &\geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{\substack{x=\sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - S_i(P_{E_i}\Theta^{-1}z))^{c_i} g_i(x_i)^{c_i} dx dz \\
&\geq \int_{\mathbb{R}^n} \prod_{i=1}^k \left( \int_{E_i} f_i(x_i - S_i(P_{E_i}\Theta^{-1}z)) g_i(x_i) dx_i \right)^{c_i} dz \geq 1.
\end{aligned}$$



Because of Barthe's inequality (5), it follows from (68) that

$$\begin{aligned} & \int_{\mathbb{R}^n} \sup_{\substack{x = \sum_{i=1}^k c_i x_i, \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i - S_i(P_{E_i} \Theta^{-1} z))^{c_i} g_i(x_i)^{c_i} dx \\ &= \prod_{i=1}^k \left( \int_{E_i} f_i(x_i - S_i(P_{E_i} \Theta^{-1} z)) g_i(x_i) dx_i \right)^{c_i} \end{aligned}$$

for any  $z \in \mathbb{R}^n$ ; therefore, we may choose  $z_i = S_i(0)$  for  $i = 1, \dots, k$  in Lemma 32.  $\square$

## 8 $h_{i0}$ is Gaussian in Proposition 27

For positive  $C^\alpha$  probability density functions  $f$  and  $g$  on  $\mathbb{R}^n$  for  $\alpha \in (0, 1)$ , the  $C^1$  Brenier map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  in Theorem 26 pushing forward the the measure on  $\mathbb{R}^n$  induced by  $g$  to the measure associated to  $f$  satisfies that  $\nabla T$  is positive definite. We deduce that

$$\langle T(y) - T(x), y - x \rangle = \int_0^1 \langle \nabla T(x + t(y - x)) \cdot (y - x), y - x \rangle dt \geq 0 \quad \text{for any } x, y \in \mathbb{R}^n. \quad (72)$$

We say that a continuous function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  has linear growth if there exists a positive constant  $c > 0$  such that

$$\|T(x)\| \leq c\sqrt{1 + \|x\|^2}$$

for  $x \in \mathbb{R}^n$ . It is equivalent saying that

$$\limsup_{\|x\| \rightarrow \infty} \frac{\|T(x)\|}{\|x\|} < \infty. \quad (73)$$

In general,  $T$  has polynomial growth, if there exists  $k \geq 1$  such that

$$\limsup_{\|x\| \rightarrow \infty} \frac{\|T(x)\|}{\|x\|^k} < \infty.$$

Proposition 33 related to Caffarelli Contraction Principle in Caffarelli [30] was proved by Emanuel Milman, see for example Colombo, Fathi [35], De Philippis, Figalli [37], Fathi, Gozlan, Prod'homme [41], Y.-H. Kim, E. Milman [62], Klartag, Putterman [66], Kolesnikov [67], Livshyts [72] for relevant results.

**Proposition 33 (Emanuel Milman)** *If a Gaussian probability density  $g$  and a positive  $C^\alpha$ ,  $\alpha \in (0, 1)$ , probability density  $f$  on  $\mathbb{R}^n$  satisfy  $f \leq c \cdot g$  for some positive constant  $c > 0$ , then the Brenier map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  pushing forward the measure on  $\mathbb{R}^n$  induced by  $g$  to the measure associated to  $f$  has linear growth.*

*Proof:* We may assume that  $g(x) = e^{-\pi\|x\|^2}$ .

We observe that  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is bijective as both  $f$  and  $g$  are positive. Let  $S$  be the inverse of  $T$ ; namely,  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the bijective Brenier map pushing forward the measure on  $\mathbb{R}^n$  induced by  $f$  to the measure associated to  $g$ . In particular, any Borel  $X \subset \mathbb{R}^n$  satisfies

$$\int_{S(X)} g = \int_X f. \quad (74)$$

We note that (73), and hence Proposition 33 is equivalent saying that

$$\liminf_{x \rightarrow \infty} \frac{\|S(x)\|}{\|x\|} > 0. \quad (75)$$

The main idea of the argument is the following observation. For any unit vector  $u$  and  $\theta \in (0, \pi)$ , we consider

$$\Xi(u, \theta) = \{y : \langle y, u \rangle \geq \|y\| \cdot \cos \theta\}.$$

Since  $S$  is surjective, and  $\langle S(z) - S(w), z - w \rangle \geq 0$  for any  $z, w \in \mathbb{R}^n$  according to (72), we deduce that

$$S(w) + \Xi(u, \theta) \subset S\left(w + \Xi\left(u, \theta + \frac{\pi}{2}\right)\right) \quad (76)$$

for any  $u \in S^{n-1}$  and  $\theta \in (0, \frac{\pi}{2})$ .

We suppose that  $T$  does not have linear growth, and seek a contradiction. According to (75), there exists a sequence  $\{x_k\}$  of points of  $\mathbb{R}^n \setminus \{0\}$  tending to infinity such that

$$\lim_{k \rightarrow \infty} \|x_k\| = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\|S(x_k)\|}{\|x_k\|} = 0.$$

In particular, we may assume that

$$\|S(x_k)\| < \frac{\|x_k\|}{8}. \quad (77)$$

For any  $k$ , we consider the unit vector  $e_k = x_k / \|x_k\|$ . We observe that  $X_k = x_k + \Xi(e_k, \frac{3\pi}{4})$  avoids the interior of the ball  $\frac{\|x_k\|}{\sqrt{2}} B^n$ ; therefore, if  $k$  is large, then

$$\int_{X_k} f \leq c \cdot n \kappa_n \int_{\|x_k\|/\sqrt{2}}^{\infty} r^{n-1} e^{-\pi r^2} dr < \int_{\|x_k\|/\sqrt{2}}^{\infty} e^{-2r^2} \sqrt{2} r dr = e^{-\|x_k\|^2} \quad (78)$$

On the other hand,  $S(x_k) + \Xi(e_k, \frac{\pi}{4})$  contains the ball

$$\tilde{B} = S(x_k) + \frac{x_k}{8} + \frac{\|x_k\|}{8\sqrt{2}} B^n \subset \frac{\|x_k\|}{2} B^n$$

where we have used (77). It follows from (74) and (76) that if  $k$  is large, then

$$\int_{X_k} f = \int_{S(X_k)} g \geq \int_{\tilde{B}} g \geq \kappa_n \left(\frac{\|x_k\|}{8\sqrt{2}}\right)^n e^{-\pi(\|x_k\|/2)^2} > e^{-\|x_k\|^2}.$$

This inequality contradicts (78), and in turn proves (75).  $\square$

Proposition 36 shows that if the whole space is the dependent subspace and the Brenier maps corresponding to the extremizers  $f_1, \dots, f_k$  in Proposition 27 have at most linear growth, then each  $f_i$  is actually Gaussian. The proof of Proposition 36 uses classical Fourier analysis, and we refer to Grafakos [50] for the main properties. For our purposes, we need only the action of a tempered distribution on the space of  $C_0^\infty(\mathbb{R}^m)$  of  $C^\infty$  functions with compact support, do not need to consider the space of Schwarz functions in general. We recall that if  $u$  is a tempered distribution on Schwarz functions on  $\mathbb{R}^n$ , then the support  $\text{supp } u$  is the intersection of all closed sets  $K$  such that if  $\varphi \in C_0^\infty(\mathbb{R}^n)$  with  $\text{supp } \varphi \subset \mathbb{R}^n \setminus K$ , then  $\langle u, \varphi \rangle = 0$ . We write  $\hat{u}$  to denote the Fourier transform of a  $u$ . In particular, if  $\theta$  is a function of polynomial growth and  $\varphi \in C_0^\infty(\mathbb{R}^n)$ , then

$$\langle \hat{\theta}, \varphi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \theta(x) \varphi(y) e^{-2\pi i \langle x, y \rangle} dx dy. \quad (79)$$

We consider the two well-known statements Lemma 34 and Lemma 35 about the support of a Fourier transform to prepare the proof of Proposition 36.

**Lemma 34** *If  $\theta$  is a measurable function of polynomial growth on  $\mathbb{R}^n$ , and there exist linear subspace  $E$  with  $1 \leq \dim E \leq n - 1$  and function  $\omega$  on  $E$  such that  $\theta(x) = \omega(P_E x)$ , then  $\text{supp } \hat{\theta} \subset E$ .*

*Proof:* We write a  $z \in \mathbb{R}^n$  in the form  $z = (z_1, z_2)$  with  $z_1 \in E$  and  $z_2 \in E^\perp$ . Let  $\varphi \in C_0^\infty(\mathbb{R}^n)$  satisfy that  $\text{supp } \varphi \subset \mathbb{R}^n \setminus E$ , and hence  $\varphi(x_1, 0) = 0$  for  $x_1 \in E$ , and the Fourier Integral Theorem in  $E^\perp$  implies

$$\varphi(x_1, z) = \int_{E^\perp} \int_{E^\perp} \varphi(x_1, x_2) e^{2\pi i \langle z - x_2, y_2 \rangle} dx_2 dy_2$$

for  $x_1 \in E$  and  $z \in E^\perp$ . It follows from (79) that

$$\begin{aligned} \langle \hat{\theta}, \varphi \rangle &= \int_{E^\perp} \int_E \int_{E^\perp} \int_E \omega(x_1) \varphi(x_1, x_2) e^{-2\pi i \langle x_1, y_1 \rangle} e^{-2\pi i \langle x_2, y_2 \rangle} dx_1 dx_2 dy_1 dy_2 \\ &= \int_E \int_E \omega(x_1) e^{-2\pi i \langle x_1, y_1 \rangle} \left( \int_{E^\perp} \int_{E^\perp} \varphi(x_1, x_2) e^{2\pi i \langle -x_2, y_2 \rangle} dx_2 dy_2 \right) dy_1 dx_1 \\ &= \int_E \int_E \omega(x_1) e^{-2\pi i \langle x_1, y_1 \rangle} \varphi(x_1, 0) dy_1 dx_1 = 0. \quad \square \end{aligned}$$

Next, Lemma 35 directly follows from Proposition 2.4.1 in Grafakos [50].

**Lemma 35** *If  $\theta$  is a continuous function of polynomial growth on  $\mathbb{R}^n$  and  $\text{supp } \hat{\theta} \subset \{0\}$ , then  $\theta$  is a polynomial.*

**Proposition 36** For linear subspaces  $E_1, \dots, E_k$  of  $\mathbb{R}^m$  and  $c_1, \dots, c_k > 0$  satisfying (1), we assume that

$$\bigcap_{i=1}^k (E_i \cup E_i^\perp) = \{0\}. \quad (80)$$

Let  $g_i(x) = e^{-\pi\|x\|^2}$  for  $i = 1, \dots, k$  and  $x \in E_i$ , let equality hold in (5) for positive  $C^1$  probability densities  $f_i$  on  $E_i$ ,  $i = 1, \dots, k$ , and let  $T_i : E_i \rightarrow E_i$  be the  $C^2$  Brenier map satisfying

$$g_i(x) = \det \nabla T_i(x) \cdot f_i(T_i(x)) \text{ for all } x \in E_i. \quad (81)$$

If each  $T_i$ ,  $i = 1, \dots, k$ , has linear growth, then there exist a positive definite matrix  $A : \mathbb{R}^m \rightarrow \mathbb{R}^m$  whose eigenspaces are critical subspaces, and  $a_i > 0$  and  $b_i \in E_i$ ,  $i = 1, \dots, k$ , such that

$$f_i(x) = a_i e^{-\langle Ax, x + b_i \rangle} \text{ for } x \in E_i.$$

*Proof:* We may assume that each linear subspace is non-zero.

We note that the condition (80) is equivalent saying that  $\mathbb{R}^m$  itself is the dependent subspace with respect to the Brascamp-Lieb data. We may assume that for some  $1 \leq l \leq k$ , we have  $1 \leq \dim E_i \leq m - 1$  if  $i = 1, \dots, l$ , and still

$$\bigcap_{i=1}^l (E_i \cup E_i^\perp) = \{0\}. \quad (82)$$

We use the diffeomorphism  $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}^m$  of Proposition 27 defined by

$$\Theta(y) = \sum_{i=1}^k c_i T_i(P_{E_i} y), \quad y \in \mathbb{R}^m.$$

It follows from (45) that we may assume

$$T_i(0) = 0 \text{ for } i = 1, \dots, k, \text{ and hence } \Theta(0) = 0. \quad (83)$$

We claim that there exists a positive definite matrix  $B : \mathbb{R}^m \rightarrow \mathbb{R}^m$  whose eigenspaces are critical subspaces, and

$$\nabla \Theta(y) = B \text{ for } y \in \mathbb{R}^m. \quad (84)$$

Let  $\Theta(y) = (\theta_1(y), \dots, \theta_m(y))$  for  $y \in \mathbb{R}^m$  and  $\theta_j \in C^2(\mathbb{R}^m)$ ,  $j = 1, \dots, m$ . Since each  $T_i$ ,  $i = 1, \dots, k$  has linear growth, it follows that  $\Theta$  has linear growth, and in turn each  $\theta_j$ ,  $j = 1, \dots, m$ , has linear growth.

According to Proposition 27 (iii), there exist  $C^2$  functions  $\Omega_i : E_i \rightarrow E_i$  and  $\Gamma_i : E_i^\perp \rightarrow E_i^\perp$  such that

$$\Theta(y) = \Omega_i(P_{E_i} y) + \Gamma_i(P_{E_i^\perp} y)$$

for  $i = 1, \dots, k$  and  $y \in \mathbb{R}^n$ . We write  $\Omega_i(x) = (\omega_{i1}(x), \dots, \omega_{im}(x))$  and  $\Gamma_i(x) = (\gamma_{i1}(x), \dots, \gamma_{im}(x))$ ; therefore,

$$\theta_j(y) = \omega_{ij}(P_{E_i} y) + \gamma_{ij}(P_{E_i^\perp} y) \quad (85)$$

for  $j = 1, \dots, m$  and  $i = 1, \dots, k$ .

Fix a  $j \in \{1, \dots, m\}$ . It follows from Lemma 34 and (85) that

$$\text{supp } \hat{\theta}_j \subset E_i \cup E_i^\perp$$

for  $i = 1, \dots, l$ . Thus (82) yields that

$$\text{supp } \hat{\theta}_j \subset \{0\},$$

and in turn we deduce from Lemma 35 that  $\theta_j$  is a polynomial. Given that  $\theta_j$  has linear growth, it follows that there exist  $w_j \in \mathbb{R}^m$  and  $\alpha_j \in \mathbb{R}$  such that  $\theta_j(y) = \langle w_j, y \rangle + \alpha_j$ . We deduce from  $\theta_j(o) = 0$  (cf. (83)) that  $\alpha_j = 0$ .

The argument so far yields that there exists an  $m \times m$  matrix  $B$  such that  $\Theta(y) = By$  for  $y \in \mathbb{R}^m$ . As  $\nabla\Theta(y) = B$  is positive definite and its eigenspaces are critical subspaces, we conclude the claim (84).

Since  $\nabla T_i(P_{E_i}y) = \nabla\Theta(y)|_{E_i}$  for  $i = 1, \dots, k$  and  $y \in \mathbb{R}^m$  by Proposition 27 (iv), we deduce that  $T_i^{-1} = B^{-1}|_{E_i}$  for  $i = 1, \dots, k$ . It follows from (81) that

$$f_i(x) = e^{-\pi\|B^{-1}x\|^2} \cdot \det(B^{-1}|_{E_i}) \quad \text{for } x \in E_i$$

for  $i = 1, \dots, k$ . Therefore, we can choose  $A = \pi B^{-2}$ .  $\square$

## 9 Proof of Theorem 4

We may assume that each linear subspace  $E_i$  is non-zero in Theorem 4. Let  $f_i$  be a probability density on  $E_i$  in a way such that equality holds for  $f_1, \dots, f_k$  in (5). For  $i = 1, \dots, k$  and  $x \in E_i$ , let  $g_i(x) = e^{-\pi\|x\|^2}$ , and hence  $g_i$  is a probability distribution on  $E_i$ , and  $g_1, \dots, g_k$  are extremizers in Barthe's inequality (5).

It follows from Lemma 30 that the convolutions  $f_1 * g_1, \dots, f_k * g_k$  are also extremizers for (5). We observe that for  $i = 1, \dots, k$ ,  $f_i * g_i$  is a bounded positive  $C^\infty$  probability density on  $E_i$ . Next we deduce from Lemma 32 that there exist  $z_i \in E_i$  and  $\gamma_i > 0$  for  $i = 1, \dots, k$  such that defining

$$\tilde{f}_i(x) = \gamma_i \cdot g_i(x) \cdot (f_i * g_i)(x - z_i) \quad \text{for } x \in E_i,$$

$\tilde{f}_1, \dots, \tilde{f}_k$  are probability densities that are extremizers for (5). We note that if  $i = 1, \dots, k$ , then  $\tilde{f}_i$  is positive and  $C^\infty$ , and there exists  $c > 1$  satisfying

$$\tilde{f}_i \leq c \cdot g_i. \tag{86}$$

Let  $\tilde{T}_i : E_i \rightarrow E_i$  be the  $C^\infty$  Brenier map satisfying

$$g_i(x) = \det \nabla \tilde{T}_i(x) \cdot \tilde{f}_i(\tilde{T}_i(x)) \quad \text{for all } x \in E_i, \tag{87}$$

We deduce from (86) and Proposition 33 that  $\tilde{T}_i$  has linear growth.

For  $i = 1, \dots, k$  and  $x \in F_0 \cap E_i$ , let  $g_{i0}(x) = e^{-\pi\|x\|^2}$ . It follows from Proposition 27 (i) that for  $i \in \{1, \dots, k\}$ , there exists positive  $C^1$  integrable  $h_{i0} : F_0 \cap E_i \rightarrow [0, \infty)$  (where  $h_{i0}(o) = 1$  if  $F_0 \cap E_i = \{0\}$ ), and for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ , there exists positive  $C^1$  integrable  $\tilde{h}_{ij} : F_j \rightarrow [0, \infty)$  such that

$$\tilde{f}_i(x) = \tilde{h}_{i0}(P_{F_0}x) \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} \tilde{h}_{ij}(P_{F_j}x) \quad \text{for } x \in E_i.$$

We deduce from Proposition 27 (ii) that  $\tilde{T}_{i0} = \tilde{T}_i|_{F_0 \cap E_i}$  is the Brenier map pushing forward the measure on  $F_0 \cap E_i$  determined  $g_{i0}$  onto the measure determined by  $\tilde{h}_{i0}$ . Since  $\tilde{T}_i$  has linear growth,  $\tilde{T}_{i0}$  has linear growth, as well, for  $i = 1, \dots, k$ .

We deduce from Proposition 29 (i) that  $\sum_{i=1}^k c_i P_{E_i \cap F_0} = \text{Id}_{F_0}$ , the Geometric Brascamp Lieb data  $E_1 \cap F_0, \dots, E_k \cap F_0$  in  $F_0$  has no independent subspaces, and  $\tilde{h}_{10}, \dots, \tilde{h}_{k0}$  are extremizers in Barthe's inequality for this data in  $F_0$ .

As  $\tilde{T}_{i0}$  has linear growth for  $i = 1, \dots, k$ , Proposition 36 yields the existence of a positive definite matrix  $\tilde{A} : F_0 \rightarrow F_0$  whose eigenspaces are critical subspaces, and  $\tilde{a}_i > 0$  and  $\tilde{b}_i \in F_0 \cap E_i$  for  $i = 1, \dots, k$ , such that

$$\tilde{f}_i(x) = \tilde{a}_i e^{-\langle \tilde{A}x, x + \tilde{b}_i \rangle} \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} \tilde{h}_{ij}(P_{F_j}x) \quad \text{for } x \in E_i.$$

Dividing by  $g_i$  and shifting, we deduce that there exist a symmetric matrix  $\bar{A} : F_0 \rightarrow F_0$  whose eigenspaces are critical subspaces, and  $\bar{a}_i > 0$  and  $\bar{b}_i \in F_0 \cap E_i$  for  $i = 1, \dots, k$ , and for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ , there exists positive  $C^1$   $\bar{h}_{ij} : F_j \rightarrow [0, \infty)$  such that

$$f_i * g_i(x) = \bar{a}_i e^{-\langle \bar{A}x, x + \bar{b}_i \rangle} \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} \bar{h}_{ij}(P_{F_j}x) \quad \text{for } x \in E_i.$$

Since  $f_i * g_i$  is a probability density on  $E_i$ , it follows that  $\bar{A}$  is positive definite and  $\bar{h}_{ij} \in L_1(E_i \cap F_j)$  for  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ .

For any  $i = 1, \dots, k$ , we write  $\hat{\varrho}$  for the Fourier transform of a function  $\varrho \in L_1(E_i)$ , thus we can take the inverse Fourier transform in the sense that  $\varrho$  is a.e. the  $L_1$  limit of

$$x \mapsto \int_{\mathbb{R}^n} \hat{\varrho}(\xi) e^{-a|\xi|^2} e^{2\pi i \langle \xi, x \rangle} d\xi$$

as  $a > 0$  tends to zero. For  $i = 1, \dots, k$ , using that  $\widehat{f_i * g_i} = \hat{f}_i \cdot \hat{g}_i$ , we deduce that the restriction of  $\hat{f}_i$  to  $F_0 \cap E_i$  is the quotient of two Gaussian densities. Since  $f_i$  is bounded and zero at infinity, we deduce that the restriction of  $\hat{f}_i$  to  $F_0 \cap E_i$  is a Gaussian density for  $i = 1, \dots, k$ , as well, with the symmetric matrix involved being positive definite. We conclude using the inverse Fourier transform above and the fact that the linear subspaces  $F_j$ ,  $j = 0, \dots, l$ , are pairwise orthogonal that there exist a symmetric matrix  $A : F_0 \rightarrow F_0$  whose eigenspaces are critical subspaces, and

$a_i > 0$  and  $b_i \in F_0 \cap E_i$  for  $i = 1, \dots, k$ , and for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ , there exists  $h_{ij} : F_j \rightarrow [0, \infty)$  such that

$$f_i(x) = a_i e^{-\langle Ax, x + b_i \rangle} \cdot \prod_{\substack{F_j \subset E_i \\ j \geq 1}} h_{ij}(P_{F_j}x) \quad \text{for a.e. } x \in E_i.$$

Since  $f_i$  is a probability density on  $E_i$ , it follows that  $A$  is positive definite and each  $h_{ij}$  is non-negative and integrable. Finally, Proposition 29 (ii) yields that there exist integrable  $\psi_j : F_j \rightarrow [0, \infty)$  for  $j = 1, \dots, l$  where  $\psi_j$  is log-concave whenever  $F_j \subset E_\alpha \cap E_\beta$  for  $\alpha \neq \beta$ , and there exist  $a_{ij} > 0$  and  $b_{ij} \in F_j$  for any  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$  such that  $h_{ij}(x) = a_{ij} \cdot \psi_j(x - b_{ij})$  for  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, l\}$  with  $F_j \subset E_i$ .

Finally, we assume that  $f_1, \dots, f_k$  are of the form as described in (6) and equality holds for all  $x \in E_i$  in (6). According to (45), we may assume that there exist a positive definite matrix  $\Phi : F_0 \rightarrow F_0$  whose proper eigenspaces are critical subspaces and a  $\tilde{\theta}_i > 0$  for  $i = 1, \dots, k$  such that

$$f_i(x) = \tilde{\theta}_i e^{-\|\Phi P_{F_0}x\|^2} \prod_{F_j \subset E_i} h_j(P_{F_j}(x)) \quad \text{for } x \in E_i. \quad (88)$$

We recall that according to (57), if  $j \in \{1, \dots, l\}$ , then

$$\sum_{E_i \supset F_j} c_i = 1. \quad (89)$$

We set  $\theta = \prod_{i=1}^k \tilde{\theta}_i^{c_i}$  and  $h_0(x) = e^{-\|\Phi x\|^2}$  for  $x \in F_0$ . On the left hand side of Barthe's inequality (5), we use first (89) and the log-concavity of  $h_j$  whenever  $j \geq 1$  and  $F_j \subset E_\alpha \cap E_\beta$  for  $\alpha \neq \beta$ , secondly Proposition 25, thirdly (89), fourth the Fubini Theorem, and finally (89) again to prove that

$$\begin{aligned} \int_{*, \mathbb{R}^n} \sup_{\substack{x = \sum_{i=1}^k c_i x_i \\ x_i \in E_i}} \prod_{i=1}^k f_i(x_i)^{c_i} dx &= \theta \int_{*, \mathbb{R}^n} \sup_{\substack{x = \sum_{i=1}^k \sum_{j=0}^l c_i x_{ij} \\ x_{ij} \in E_i \cap F_j}} \prod_{j=0}^l \prod_{i=1}^k h_j(x_{ij})^{c_i} dx \\ &= \theta \int_{*, \mathbb{R}^n} \prod_{j=0}^l \sup_{\substack{P_{F_j}x = \sum_{i=1}^k c_i x_{ij} \\ x_{ij} \in E_i \cap F_j}} \prod_{i=1}^k h_j(x_{ij})^{c_i} dx \\ &= \theta \int_{*, \mathbb{R}^n} \left( \sup_{\substack{P_{F_0}x = \sum_{i=1}^k c_i x_{i0} \\ x_{i0} \in E_i \cap F_0}} \prod_{i=1}^k e^{-c_i \|\Phi x_{i0}\|^2} \right) \times \prod_{j=1}^l h_j(P_{F_j}x) dx \\ &= \theta \left( \prod_{i=1}^k \left( \int_{F_0 \cap E_i} e^{-\|\Phi y\|^2} dy \right)^{c_i} \right) \times \prod_{j=1}^l \int_{F_j} h_j \\ &= \prod_{i=1}^k \left( \int_{E_i} f_i \right)^{c_i}, \end{aligned}$$

completing the proof of Theorem 4.  $\square$

## 10 Equality in the Bollobas-Thomason inequality and in its dual

We fix an orthonormal basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$  for the whole section. We set  $\sigma_i^0 = \sigma_i$  and  $\sigma_i^1 = [n] \setminus \sigma_i$ . When we write  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_\ell$  for the induced cover from  $\sigma_1, \dots, \sigma_k$ , we assume that the sets  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_\ell$  are pairwise distinct.

**Lemma 37** *For  $s \geq 1$ , let  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$ , and let  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_\ell$  be the 1-uniform cover of  $[n]$  induced by  $\sigma_1, \dots, \sigma_k$ . Then*

(i) *the subspaces  $E_{\sigma_i} := \text{lin}\{e_j : j \in \sigma_i\}$  satisfy*

$$\sum_{i=1}^k \frac{1}{s} P_{E_{\sigma_i}} = I_n \quad (90)$$

*i.e. form a Geometric Brascamp Lieb data;*

(ii) *For  $r \in \tilde{\sigma}_j$ ,  $j = 1, \dots, \ell$ , we have*

$$\tilde{\sigma}_j := \bigcap_{r \in \sigma_i} \sigma_i^0 \cap \bigcap_{r \notin \sigma_i} \sigma_i^1; \quad (91)$$

(iii) *the subspaces  $F_{\tilde{\sigma}_j} := \text{lin}\{e_r : r \in \tilde{\sigma}_j\}$  are the independent subspaces of the Geometric Brascamp Lieb data (90) and  $F_{\text{dep}} = \{0\}$ .*

*Proof:* Since  $\sigma_1, \dots, \sigma_k$  form a  $s$ -uniform cover, every  $e_i \in \mathbb{R}^n$  is contained in exactly  $s$  of  $E_{\sigma_1}, \dots, E_{\sigma_k}$ , yielding (i).

For (ii), the definition of  $\tilde{\sigma}_j$  directly implies (91).

For (iii), the linear subspaces  $F_{\tilde{\sigma}_1}, \dots, F_{\tilde{\sigma}_\ell}$  are pairwise orthogonal because  $\sigma_i^0 \cap \sigma_i^1 = \emptyset$  for  $i = 1, \dots, k$ . On the other hand, for any  $r \in [n]$ ,  $r \in \bigcap_{i=1}^n \sigma_i^{\varepsilon(i)}$  where  $\varepsilon(i) = 0$  if  $r \in \sigma_i$ , and  $\varepsilon(i) = 1$  if  $r \notin \sigma_i$ ; therefore,  $F_{\tilde{\sigma}_1}, \dots, F_{\tilde{\sigma}_\ell}$  span  $\mathbb{R}^n$ . In particular,  $F_{\text{dep}} = \{0\}$ .  $\square$

Let us introduce the notation that we use when handling both the Bollobas-Thomason inequality and its dual. Let  $\sigma_1, \dots, \sigma_k$  be the  $s$  cover of  $[n]$  occurring in Theorem 11 and Theorem 12, and hence  $E_i = E_{\sigma_i}$ ,  $i = 1, \dots, k$ , satisfies

$$\frac{1}{s} \sum_{i=1}^k P_{E_{\sigma_i}} = I_n. \quad (92)$$



Let  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$  be the 1-uniform cover of  $[n]$  induced by  $\sigma_1, \dots, \sigma_k$ . It follows that

$$F_j = E_{\tilde{\sigma}_j} \text{ for } j = 1, \dots, l \text{ are the independent subspaces,} \quad (93)$$

$$F_{\text{dep}} = \{0\}. \quad (94)$$

For any  $i \in \{1, \dots, k\}$ , we set

$$I_i = \{j \in \{1, \dots, l\} : F_j \subset E_i\},$$

and for any  $j \in \{1, \dots, l\}$ , we set

$$J_j = \{i \in \{1, \dots, k\} : F_j \subset E_i\}.$$

For the reader's convenience, we restate Theorem 9 and Theorem 11 as Theorem 38, and Theorem 10 and Theorem 12 as Theorem 39.

**Theorem 38** *If  $K \subset \mathbb{R}^n$  is compact and affinely spans  $\mathbb{R}^n$ , and  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  for  $s \geq 1$ , then*

$$|K|^s \leq \prod_{i=1}^k |P_{E_{\sigma_i}} K|. \quad (95)$$

*Equality holds if and only if  $K = \bigoplus_{i=1}^l P_{F_{\tilde{\sigma}_i}} K$  where  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$  is the 1-uniform cover of  $[n]$  induced by  $\sigma_1, \dots, \sigma_k$  and  $F_{\tilde{\sigma}_i}$  is the linear hull of the  $e_i$ 's with indices from  $\tilde{\sigma}_i$ .*

*Proof:* We set  $E_i := E_{\sigma_i}$  which subspaces compose a geometric data according to Lemma 37. We start with a proof of Bollobas-Thomason inequality. It follows directly from the Brascamp-Lieb inequality as

$$\begin{aligned} |K| &= \int_{\mathbb{R}^n} 1_K(x) dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^k 1_{P_{E_i}(K)}(P_{E_i}(x))^{\frac{1}{s}} dx \\ &\leq \prod_{i=1}^k \left( \int_{E_i} 1_{P_{E_i}(K)} \right)^{\frac{1}{s}} = \prod_{i=1}^k |P_{E_i}(K)|^{\frac{1}{s}} \end{aligned} \quad (96)$$

where the first inequality is from the monotonicity of the integral while the second is Brascamp-Lieb inequality Theorem 1. Now, if equality holds in (96), then on the one hand,

$$1_K(x) = \prod_{i=1}^k 1_{P_{E_i}(K)}(P_{E_i}(x))$$

and on the other hand, if  $F_1, \dots, F_l$  are the independent subspaces of the data, then they span  $\mathbb{R}^n$  according to Lemma 37; namely,  $F_{\text{dep}} = \{0\}$ . It follows from Theorem 2 that there exist integrable functions  $h_j : F_j \rightarrow \mathbb{R}$ , such that, for Lebesgue a.e.  $x_i \in E_i$

$$1_{P_{E_i}K}(x_i) = \theta_i \prod_{j \in I_i} h_j(P_{F_j}(x_i))$$

Therefore from the previous two, we have for  $x \in \mathbb{R}^n$

$$1_K(x) = \prod_{i=1}^k \theta_i \prod_{j \in I_i} h_j(P_{F_j}(P_{E_i}(x)))$$

Now, since for  $j \in I_i$  we have  $F_j \subset E_i$  we can delete the  $P_{E_i}$  on the above product. Thus, for  $\theta = \prod_{i=1}^k \theta_i$ , we have for Lebesgue a.e.  $x \in \mathbb{R}^n$

$$1_K(x) = \theta \prod_{i=1}^k \prod_{j \in I_i} h_j(P_{F_j}(x)) = \theta \prod_{j=1}^l h_j(P_{F_j}(x))^{|J_j|}. \quad (97)$$

Now, for  $x \in K$  the last product on above is constant, so

$$\theta = \frac{1}{\prod_{i=1}^l h_j(P_{F_j}(x_0))^{|J_j|}} \quad (98)$$

for some  $x_0 \in K$ . For  $j = 1, \dots, l$  we set  $\varphi_j : F_j \rightarrow \mathbb{R}^n$ , by

$$\varphi_j(x) = \frac{h_j(x + P_{F_j}(x_0))^{|J_j|}}{h_j(P_{F_j}(x_0))^{|J_j|}}.$$

We see that  $\varphi_j(o) = 1$  and also (97) and (98) yields

$$1_{K-x_0}(x) = \prod_{j=1}^l \varphi_j(P_{F_j}(x)) \quad (99)$$

For  $m \in \{1, \dots, l\}$ , taking  $x \in F_m$  in (99) (and hence  $\varphi_j(P_{F_j}(x)) = 1$  for  $j \neq m$ ) shows that

$$1_{K-x_0}(y) = \varphi_m(y),$$

for Lebesgue a.e.  $y \in F_m$ . Therefore (99) and the orthogonality of the  $F_j$ 's,

$$K - x_0 = \bigcap_{j=1}^l P_{F_j}^{-1}(P_{F_j}(K - x_0)) = \bigoplus_{j=1}^l P_{F_j}(K - x_0),$$

completing the proof of Theorem 38.  $\square$

To prove Theorem 39, we use two small observations. First if  $M$  is any convex body with  $o \in \text{int } M$ , then

$$\int_{\mathbb{R}^n} e^{-\|x\|_M} dx = \int_0^\infty e^{-r} n r^{n-1} |M| dr = n! |M|. \quad (100)$$

Secondly, if  $F_j$  are pairwise orthogonal subspaces and  $M = \text{conv} \{M_1, \dots, M_l\}$  where  $M_j \subset F_j$  is a  $\dim F_j$ -dimensional compact convex set with  $o \in \text{relint } M_j$ , then for any  $x \in \mathbb{R}^n$

$$\|x\|_M = \sum_{i=1}^l \|P_{F_j} x\|_{M_j}. \quad (101)$$

In addition, we often use the fact, for a subspace  $F$  of  $\mathbb{R}^n$  and  $x \in F$ , then  $\|x\|_K = \|x\|_{K \cap F}$ .

**Theorem 39** *If  $K \subset \mathbb{R}^n$  is compact convex with  $o \in \text{int}K$ , and  $\sigma_1, \dots, \sigma_k \subset [n]$  form an  $s$ -uniform cover of  $[n]$  for  $s \geq 1$ , then*

$$|K|^s \geq \frac{\prod_{i=1}^k |\sigma_i|!}{(n!)^s} \cdot \prod_{i=1}^k |K \cap E_{\sigma_i}|. \quad (102)$$

*Equality holds if and only if  $K = \text{conv}\{E_{\tilde{\sigma}_i} \cap K\}_{i=1}^l$  where  $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$  is the 1-uniform cover of  $[n]$  induced by  $\sigma_1, \dots, \sigma_k$ .*

*Proof:* We define

$$f(x) = e^{-\|x\|_K}, \quad (103)$$

which is a log-concave function with  $f(o) = 1$ , and satisfying (cf (100))

$$\int_{\mathbb{R}^n} f(y)^n dy = \int_{\mathbb{R}^n} e^{-n\|y\|_K} dy = \int_{\mathbb{R}^n} e^{-\|y\|_{\frac{1}{n}K}} dy = n! \left| \frac{1}{n}K \right| = \frac{n!}{n^n} \cdot |K|. \quad (104)$$

We claim that

$$n^n \int_{\mathbb{R}^n} f(y)^n dy \geq \prod_{i=1}^k \left( \int_{E_i} f(x_i) dx_i \right)^{1/s}. \quad (105)$$

Equating the traces of the two sides of (90), we deduce that,  $d_i := |\sigma_i| = \dim E_i$

$$\sum_{i=1}^k \frac{d_i}{sn} = 1. \quad (106)$$

For  $z = \sum_{i=1}^k \frac{1}{s} x_i$  with  $x_i \in E_i$ , the log-concavity of  $f$  and its definition (103), imply

$$f(z/n) \geq \prod_{i=1}^k f(x_i/d_i)^{\frac{d_i}{ns}} = \prod_{i=1}^k f(x_i)^{\frac{1}{ns}}. \quad (107)$$

Now, the monotonicity of the integral and Barthe's inequality yield

$$\int_{\mathbb{R}^n} f(z/n)^n dz \geq \int_{\mathbb{R}^n}^* \sup_{z = \sum_{i=1}^k \frac{1}{s} x_i, x_i \in E_i} \prod_{i=1}^k f(x_i)^{1/s} dz \geq \prod_{i=1}^k \left( \int_{E_i} f(x_i) dx_i \right)^{1/s}. \quad (108)$$

Making the change of variable  $y = z/n$  we conclude to (105). Computing the right hand side of (105), we have

$$\int_{E_i} f(x_i) dx_i = \int_{E_i} e^{-\|x_i\|_K} dx_i = \int_{E_i} e^{-\|x_i\|_{K \cap E_i}} dx_i = d_i! |K \cap E_i|. \quad (109)$$

Therefore, (104), (105) and (109) yield (102).

Let us assume that equality holds in (102), and hence we have two equalities in (108). We set

$$M = \text{conv}\{K \cap F_j\}_{1 \leq j \leq l}.$$

Clearly,  $K \supseteq M$ . For the other inclusion, we start with  $z \in \text{int}K$ , namely  $\|z\|_K < 1$ . Equality in the first inequality in (108) means,

$$(e^{-\|z/n\|_K})^n = \sup_{z = \sum_{i=1}^k \frac{1}{s} x_i, x_i \in E_i} \prod_{i=1}^k e^{-\|x_i\|_K 1/s},$$

or in other words,

$$\|z\|_K = \frac{1}{s} \cdot \inf_{z = \sum_{i=1}^k \frac{1}{s} x_i, x_i \in E_i} \sum_{i=1}^k \|x_i\|_K = \inf_{z = \sum_{i=1}^k y_i, y_i \in E_i} \sum_{i=1}^k \|y_i\|_K. \quad (110)$$

We deduce that there exist  $y_i \in E_i$ ,  $i = 1, \dots, k$  such that

$$z = \sum_{i=1}^k y_i \quad \text{and} \quad \sum_{i=1}^k \|y_i\|_K < 1, \quad (111)$$

Therefore, from (111), then (101) and after the triangle inequality for  $\|\cdot\|_{K \cap F_j}$ , we have

$$\|z\|_M = \left\| \sum_{i=1}^k \sum_{j \in I_i} P_{F_j} y_i \right\|_M = \sum_{i=1}^k \left\| \sum_{j \in I_i} P_{F_j} y_i \right\|_{K \cap F_j} \leq \sum_{i=1}^k \sum_{j \in I_i} \|P_{F_j} y_i\|_{K \cap F_j}. \quad (112)$$

It suffices to show that

$$K \cap E_i = \text{conv}\{K \cap F_j\}_{j \in I_i} \quad (113)$$

because then, from (112), applying (101) and (111), we have

$$\|z\|_M \leq \sum_{j=1}^l \sum_{i \in J_j} \|P_{F_j} y_i\|_{K \cap F_j} = \sum_{i=1}^k \|y_i\|_{K \cap E_i} < 1,$$

which means  $z \in M$ . Now, to show (113), we start with the equality case of Barthe's inequality which has been applied in (108). From Theorem 4, there exist  $\theta_i > 0$  and  $w_i \in E_i$  and log-concave  $h_j : F_j \rightarrow [0, \infty)$ , namely  $h_j = e^{-\varphi_j}$  for a convex function  $\varphi_j$ , such that

$$e^{-\|x_i\|_{K \cap E_i}} = \theta_i \prod_{j \in I_i} h_j(P_{F_j}(x_i - w_i)). \quad (114)$$

for Lebesgue a.e.  $x_i \in E_i$ . For  $i \in [k]$  and  $j \in I_i$  we set,  $\psi_{ij} : F_j \rightarrow \mathbb{R}$  by

$$\psi_{ij}(x) = \varphi_j(x - P_{F_j} w_i) - \varphi_j(-P_{F_j} w_i) + \frac{\ln \theta_i}{|I_i|}.$$

We see

$$\psi_{ij}(o) = 0 \text{ and } \psi_{ij} \text{ is convex on } F_j. \quad (115)$$

and also (114) yields, for  $x \in E_i$

$$e^{-\|x\|_{K \cap E_i}} = \exp \left( - \sum_{j \in I_i} \psi_{ij}(P_{F_j}x) \right). \quad (116)$$

For  $x \in F_j$ , we apply  $\lambda x$  to (116) with  $\lambda > 0$ , and we have from  $\psi_{im}(o) = 0$  for  $m \in I_i \setminus \{j\}$  that

$$\psi_{ij}(\lambda x) = \lambda \psi_{ij}(x) \text{ and } \psi_{ij}(x) > 0. \quad (117)$$

We deduce from (115) and (117) that  $\psi_{ij}$  is a norm. Therefore,  $\psi_{ij}(x) = \|x\|_{C_{ij}}$  for some  $(\dim F_j)$ -dimensional compact convex set  $C_{ij} \subset F_j$  with  $o \in \text{relint } C_{ij}$ . Now (116) becomes,

$$\|x\|_{K \cap E_i} = \sum_{j \in I_i} \|P_{F_j}x\|_{C_{ij}}$$

and hence by (101) we conclude to

$$K \cap E_i = \text{conv} \{C_{ij}\}_{j \in I_i}.$$

In particular, if  $i \in [k]$  and  $j \in I_i$ , then  $C_{ij} = (K \cap E_i) \cap F_j = K \cap F_j$ , completing the proof of (113), and in turn yielding Theorem 12.  $\square$

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