



Minimal vertex covers in infinite hypergraphs

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ABSTRACT

A *vertex cover* of a hypergraph is a set of vertices which intersects each hyperedge.

A hypergraph possesses *property* $C(k, \rho)$ iff $|\bigcap \mathcal{E}'| < \rho$ for each k element set \mathcal{E}' of hyperedges.

Komjáth proved that every uniform hypergraph possessing property $C(2, r)$ for some $r \in \omega$ has a minimal vertex cover. In this paper we will relax the assumption of uniformity to an assumption that the set of cardinalities of the hyperedges is a “small” set of infinite cardinals, e.g. it is countable, or it does not contain uncountably many limit cardinals.

Komjáth also proved that GCH does not decide the following statement: *If a hypergraph G possessing property $C(2, \omega)$ is μ -uniform for some $\mu \geq \omega_1$, then G has a minimal vertex cover.*

Using Shelah's Revised GCH theorem, we show that if we strengthen the assumption $\mu \geq \omega_1$ to $\mu \geq \beth_\omega$, then we can prove the statement in ZFC!

We also show that if all the hyperedges of a hypergraph are countably infinite, then instead of $C(2, r)$ the assumption $C(k, r)$ (for some $k \in \omega$) is enough to guarantee the existence of a minimal vertex cover. If every hyperedge has cardinality ω_1 , then we can only prove that $C(3, r)$ is enough.

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1. Introduction

A *hypergraph* G is a pair $\langle V, \mathcal{E} \rangle$ consisting of a set V of *vertices* and a family \mathcal{E} of *edges*, where every edge is a non-empty subset of the vertices. In this paper we are interested in hypergraphs having the property that every vertex belongs to some edge. To shorten notations, in this case the hypergraph will be identified with the family \mathcal{E} of its edges, and the elements of \mathcal{E} and $\bigcup \mathcal{E}$ are called the *edges* and the *vertices* of the hypergraph \mathcal{E} , respectively.

Given a hypergraph \mathcal{E} , a vertex set Y is a *vertex cover* of \mathcal{E} iff $Y \cap E \neq \emptyset$ for each edge E of the hypergraph. We say that a vertex cover Z is a *minimal vertex cover* of \mathcal{E} iff no proper subset of Z is a vertex cover of \mathcal{E} . Clearly Y is a minimal vertex cover if and only if its complement, $\bigcup \mathcal{E} \setminus Y$, is a maximal independent set of vertices.

Using the terminology of Erdős and Hajnal [2], we say that a hypergraph \mathcal{E} possesses *property* $C(k, \rho)$ iff $|\bigcap \mathcal{E}'| < \rho$ for each $\mathcal{E}' \in [\mathcal{E}]^k$.

Dominic van der Zypen [1,9] raised the following question: *Assume that $r \in \omega$. Does every hypergraph which possesses property $C(2, r)$ have a minimal vertex cover?*

In [5] Komjáth gave some partial answers. To formulate them precisely, we introduce some more notations.

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We define the *edge cardinality spectrum* $\|\mathcal{E}\|$ of a hypergraph \mathcal{E} as the cardinalities of the edges of the hypergraph:

$$\|\mathcal{E}\| = \{|E| : E \in \mathcal{E}\}.$$

If A is a set and S is a set of cardinals, write

$$[A]^S = \{B \subset A : |B| \in S\}.$$

If $S = \{\kappa\}$, then we will use the standard notation $[A]^\kappa$ instead of $[A]^{\{\kappa\}}$.

If λ and ρ are cardinals, S is a set of cardinals, $k \in \omega$, then we write

$$\mathbf{M}(\lambda, S, k, \rho) \rightarrow \mathbf{MinVC}$$

iff every hypergraph $\mathcal{E} \subset [\lambda]^S$ possessing property $C(k, \rho)$ has a minimal vertex cover.

Komjáth ([5, Theorem 7]) answered Der Zypen's question affirmatively for uniform hypergraphs:

(k1) $\mathbf{M}(\lambda, \{\kappa\}, 2, r) \rightarrow \mathbf{MinVC}$ for each $r < \omega \leq \kappa \leq \lambda$.

In this paper we prove a strengthening of this result. To formulate our theorem, we need to introduce the following notion. A set S of ordinals is *nowhere stationary* iff $S \cap \alpha$ is not stationary in α for any ordinal α with $\text{cf}(\alpha) > \omega$. In particular, countable sets of cardinals and sets of successor cardinals are nowhere stationary. In Section 3 we prove:

Theorem 1.1. $\mathbf{M}(\lambda, S, 2, r) \rightarrow \mathbf{MinVC}$ for each cardinal λ and $r \in \omega$ provided that S is a nowhere stationary set of infinite cardinals.

Komjáth ([5, Theorems 16 and 17]) also proved

(k2) $\mathbf{M}(\lambda, \{\kappa\}, 2, \omega) \rightarrow \mathbf{MinVC}$ for each $\omega_1 \leq \kappa \leq \lambda$ provided $V = L$,

(k3) GCH does not imply that $\mathbf{M}(\aleph_{\omega+1}, \{\omega_1\}, 2, \omega) \rightarrow \mathbf{MinVC}$.

In Section 2 we show that the assumption $V = L$ can be relaxed to GCH provided we strengthen the assumption $\kappa \geq \omega_1$ to $\kappa \geq \omega_2$.

Theorem 1.2. If GCH holds, then $\mathbf{M}(\lambda, \{\kappa\}, 2, \omega) \rightarrow \mathbf{MinVC}$ for each $\omega_2 \leq \kappa \leq \lambda$.

One might conjecture the nonexistence of positive ZFC theorems for hypergraphs which possesses property $C(2, \rho)$ for some $\rho \geq \omega$. But this is not true: in Section 2, using Shelah's Revised GCH theorem, we prove in ZFC:

Theorem 1.3. $\mathbf{M}(\lambda, \{\kappa\}, 2, \rho) \rightarrow \mathbf{MinVC}$ for each $\rho < \beth_\omega \leq \kappa \leq \lambda$.

It seems to be much harder to obtain positive results for hypergraphs possessing just property $C(k, r)$ for some $k \geq 3$ instead of $C(2, r)$. In Section 4 we could prove the following results.

Theorem 1.4. $\mathbf{M}(\lambda, \{\omega\}, k, r) \rightarrow \mathbf{MinVC}$ provided $\omega \leq \lambda$ and $k, r \in \omega$.

Theorem 1.5. $\mathbf{M}(\lambda, \{\omega_1\}, 3, r) \rightarrow \mathbf{MinVC}$ provided $\omega_1 \leq \lambda$ and $r \in \omega$.

1.1. Notations and basic observations

We use the standard notation of infinite combinatorics.

If $\mathcal{F} \subset \mathcal{P}(X)$, $x \in X$ and $Y \subset X$, write

$$\mathcal{F}(x) = \{F \in \mathcal{F} : x \in F\},$$

$$\mathcal{F}[Y] = \{F \in \mathcal{F} : F \cap Y \neq \emptyset\} \text{ and } \mathcal{F}[-Y] = \{F \in \mathcal{F} : F \cap Y = \emptyset\},$$

and

$$\mathcal{F} \restriction Y = \{F \cap Y : F \in \mathcal{F}[Y]\} \text{ and } \mathcal{F} \sqcap Y = \mathcal{F} \cap \mathcal{P}(Y).$$

Note that $\emptyset \notin \mathcal{F} \restriction Y$.

Observation 1.6. Assume that \mathcal{E} is a hypergraph and Y is a vertex cover of \mathcal{E} . Then Y is a minimal vertex cover of \mathcal{E} iff there is a function $w : Y \rightarrow \mathcal{E}$ such that $w(y) \cap Y = \{y\}$ for each $y \in Y$.

We will say that w is a *witnessing function*, or that w *witnesses the minimality of Y* .

If $\langle X_\alpha : \alpha < \delta \rangle$ is a sequence of sets, and $\beta \leq \delta$, write $X_{<\beta} = \bigcup_{\alpha < \beta} X_\alpha$. We define $X_{\leq \beta}$, $X_{> \beta}$, etc. analogously.

2. Maximizing well-ordering

Given a hypergraph \mathcal{E} , a well-ordering \leq of the vertex set of \mathcal{E} is called a *maximizing well-ordering* iff every edge $E \in \mathcal{E}$ has a \leq -maximal element.

If λ and ρ are cardinals, S is a set of cardinals, $k \in \omega$, then we write

$$\mathbf{M}(\lambda, S, k, \rho) \rightarrow \mathbf{MaxWO}$$

iff every hypergraph $\mathcal{E} \subset [\lambda]^S$ possessing property $\mathbf{C}(k, \rho)$ has a maximizing well-ordering.

In [5] Komjáth observed that Klimo [4, Theorem 5] practically proved that if a hypergraph has a maximizing well-order then it has a minimal vertex cover, i.e.

$$\mathbf{MaxWO} \rightarrow \mathbf{MinVC}$$

(†)

In [5, Theorem 7] Komjáth actually proved that $\mathbf{M}(\lambda, \{\kappa\}, 2, k) \rightarrow \mathbf{MaxWO}$ for each $\omega \leq \kappa \leq \lambda$.

Using the same approach, instead of Theorems 1.2 and 1.3 we will prove the stronger Theorems 2.1 and 2.2 below.

Theorem 2.1. *If GCH holds, then $\mathbf{M}(\lambda, \{\kappa\}, 2, \omega) \rightarrow \mathbf{MaxWO}$ for each $\omega_2 \leq \kappa \leq \lambda$.*

Theorem 2.2. *$\mathbf{M}(\lambda, \{\kappa\}, 2, \rho) \rightarrow \mathbf{MaxWO}$ for each $\rho < \beth_\omega \leq \kappa \leq \lambda$.*

Before proving Theorems 2.1 and 2.2, first we need to introduce a general stepping up method.

Definition 2.3. Let \mathcal{A} be a hypergraph, $\lambda = |\bigcup \mathcal{A}|$. We say that a continuous, increasing sequence $\langle G_\alpha : \alpha < cf(\lambda) \rangle$ with $G_\alpha \in [\bigcup \mathcal{A}]^{<\lambda}$ for $\alpha < \lambda$ is a *good cut* of \mathcal{A} iff

- (i) $G_0 = \emptyset$ and $\bigcup_{\alpha < cf(\lambda)} G_\alpha = \bigcup \mathcal{A}$,
- (ii) $\forall A \in \mathcal{A} \exists \alpha < cf(\lambda) A \subset G_{\alpha+1}$ and $|G_\alpha \cap A| < |A|$.

Definition 2.4. Given hypergraphs \mathcal{A} and \mathcal{B} , we say that \mathcal{B} is a *shrink* of \mathcal{A} , and we write $\mathcal{B} \ll \mathcal{A}$, iff $\forall B \in \mathcal{B} \exists A \in \mathcal{A}$ such that $B \subset A$ and $|A \setminus B| < |A|$.

Theorem 2.5. *Let κ be an infinite cardinal. Assume that \mathbb{A} is a collection of hypergraphs with the following properties:*

- (a) *If $\mathcal{A} \in \mathbb{A}$ and $\mathcal{B} \ll \mathcal{A}$, then $\mathcal{B} \in \mathbb{A}$.*
- (b) *If $\mathcal{A} \in \mathbb{A}$ and $\lambda = |\bigcup \mathcal{A}| > \kappa$, then \mathcal{A} has a good cut.*

(1) *If*

- (c) *every $\mathcal{A} \in \mathbb{A}$ with $|\bigcup \mathcal{A}| \leq \kappa$ has a minimal vertex cover,*

then every $\mathcal{A} \in \mathbb{A}$ has a minimal vertex cover.

(2) *If*

- (c') *every $\mathcal{A} \in \mathbb{A}$ with $|\bigcup \mathcal{A}| \leq \kappa$ has a maximizing well-order,*

then every $\mathcal{A} \in \mathbb{A}$ has a maximizing well-order.

Proof of Theorem 2.5. (1) By induction on $\lambda = |\bigcup \mathcal{A}|$.

If $\lambda = \kappa$, then (c) implies the statement.

Assume that $\lambda > \kappa$. Let $\langle G_\alpha : \alpha < cf(\lambda) \rangle$ be a good cut of \mathcal{A} by (b).

By transfinite induction on $\alpha < \lambda$ we will define $Y_\alpha \subset G_{\alpha+1} \setminus G_\alpha$ and a function $w_\alpha : Y_\alpha \rightarrow \mathcal{A} \cap G_{\alpha+1}$ such that $Y_{\leq \alpha} (= \bigcup_{\alpha' \leq \alpha} Y_{\alpha'})$ is a minimal vertex cover of $\mathcal{A} \cap G_{\alpha+1}$ witnessed by $w_{\leq \alpha}$, as follows.

Assume that $\langle Y_\zeta : \zeta < \alpha \rangle$ and $\langle w_\zeta : \zeta < \alpha \rangle$ are defined. Let

$$\mathcal{A}_\alpha = (\mathcal{A}[-Y_{<\alpha}] \cap G_{\alpha+1}) \upharpoonright (G_{\alpha+1} \setminus G_\alpha).$$

Then $\mathcal{A}_\alpha \ll \mathcal{A}$, so $\mathcal{A}_\alpha \in \mathbb{A}$ by (a). Since $|\bigcup \mathcal{A}_\alpha| \leq |G_{\alpha+1}| < \lambda$, by the inductive assumption the family \mathcal{A}_α has a minimal vertex cover Y_α witnessed by a function w_α .

After the inductive construction put $Y = \bigcup_{\alpha < \text{cf}(\lambda)} Y_\alpha$ and $w = \bigcup_{\alpha < \text{cf}(\lambda)} w_\alpha$. Then Y is a minimal vertex cover of \mathcal{A} witnessed by w .

(2) By induction on $\lambda = |\bigcup \mathcal{A}|$.

If $\lambda = \kappa$, then (c') implies the statement.

Assume that $\lambda > \kappa$. Let $\langle G_\alpha : \alpha < \text{cf}(\lambda) \rangle$ be a good cut of \mathcal{A} by (b).

For each $\alpha < \kappa$ consider the family

$$\mathcal{A}_\alpha = \{A \setminus G_\alpha : A \in \mathcal{A}, A \subset G_{\alpha+1}, |A \cap G_\alpha| < |A|\}.$$

Then $\mathcal{A}_\alpha \ll \mathcal{A}$ and so $\mathcal{A}_\alpha \in \mathbb{A}$ by (a).

Since $|\bigcup \mathcal{A}_\alpha| \leq |G_{\alpha+1}| < \lambda$, by the inductive assumption, the family \mathcal{A}_α has a maximizing well-order \leq_α . We can assume that \leq_α is a well-order of $G_{\alpha+1} \setminus G_\alpha$. Define the well-ordering \leq of $\bigcup \mathcal{A}$ as follows:

(1) if $\alpha < \beta < \lambda$, then

$$(G_{\alpha+1} \setminus G_\alpha) < (G_{\beta+1} \setminus G_\beta),$$

(2) $\leq \upharpoonright (G_{\alpha+1} \setminus G_\alpha) = \leq_\alpha$.

Then \leq is a maximizing well-order of \mathcal{A} . \square

Proof of Theorem 2.1. We want to apply Theorem 2.5(2). So let

$$\mathbb{A} = \{\mathcal{A} : \exists \kappa \in [\omega_2, \lambda] (\mathcal{A} \subset [\lambda]^\kappa \text{ has property } C(2, \omega))\}.$$

We should verify properties 2.5(a,b,c')

(a) is trivial by definition.

To show (b), assume that $\mathcal{A} \in \mathbb{A}$, $\lambda = \bigcup \mathcal{A} > \kappa$, and $\mathcal{A} \subset [\lambda]^\kappa$. Next we should recall a lemma from [3]:

Lemma 2.6 ([3, Lemma 8.5]). Assume that $\lambda \geq \omega_2$ and $\mu^\omega = \mu^+$ holds for each $\mu < \lambda$ with $\text{cf}(\mu) = \omega$. If \mathcal{A} is a hypergraph with property $C(2, \omega)$ and X is any set with $|X| < \lambda$, then

$$|\{A \in \mathcal{A} : |X \cap A| > \omega\}| \leq |X|.$$

Next, by transfinite recursion on $\zeta < \lambda$, define an increasing continuous sequence of subsets of λ , $\langle M_\zeta : \zeta < \lambda \rangle$, such that

(1) $M_0 = \emptyset$ and $\zeta \subset M_\zeta \in [\lambda]^{\kappa+|\zeta|}$ for $\zeta > 0$,

(2) if $A \in \mathcal{A}$ and $|A \cap M_\zeta| > \omega$, then $A \subset M_{\zeta+1}$.

By Lemma 2.6, if M_ζ is given, we can choose a suitable $M_{\zeta+1}$.

Let $\langle \lambda_\alpha : \alpha < \text{cf}(\lambda) \rangle$ be a strictly increasing continuous sequence of limit ordinals with $\sup_{\alpha < \text{cf}(\lambda)} \lambda_\alpha = \lambda$.

Let $G_\alpha = M_{\lambda_\alpha}$ for $\alpha < \text{cf}(\lambda)$ of \mathcal{A} . We claim that $\langle G_\alpha : \alpha < \text{cf}(\lambda) \rangle$ is a good cut.

Let $A \in \mathcal{A}$ be arbitrary. Since $|A| \geq \omega_2$, there is $\zeta < \lambda$ such that $|A \cap M_\zeta| > \omega$, and so $A \subset M_{\zeta+1}$. Thus, we can define

$$\alpha = \min\{\alpha' < \lambda : |G_{\alpha'} \cap A| \geq \omega_2\}.$$

Since $G_\alpha = M_{\lambda_\alpha}$ and λ_α is a limit ordinal, there is $\xi < \lambda_\alpha$ with $|A \cap M_\xi| > \omega$, and so $A \subset M_{\xi+1} \subset G_\alpha$.

If α is a limit ordinal then there is $\beta < \alpha$ with $\xi < \lambda_\beta$, and so $A \subset M_{\xi+1} \subset G_\beta$ which contradicts the minimality of α .

Thus, $\alpha = \beta + 1$ for some β , and so $A \subset G_{\beta+1}$ and $|G_\beta \cap A| \leq \omega_1 < \kappa$. Hence, we showed that $\langle G_\alpha : \alpha < \text{cf}(\lambda) \rangle$ is really a good cut.

So we checked property 2.5(b).

(c) is straightforward: if $\mathcal{A} \subset [\kappa]^\kappa$ is $C(2, \omega)$, then $|\mathcal{A}| \leq \kappa$ by Lemma 2.6, so we can write $\mathcal{A} = \{A_\xi : \xi < \xi\}$ for some $\xi \leq \kappa$. Now let \leq be a well-ordering of κ such that writing $A'_\xi = A_\xi \setminus A_{<\xi}$ we have

(1) $A'_\xi < A'_\eta$ for $\xi < \eta < \lambda$,

(2) the order type of A'_η is $\kappa + 1$, i.e. A'_η has a \leq -maximal element.

Then \leq maximizes \mathcal{A} .

Thus, we verified condition 2.5(2)(c') and so we can apply Theorem 2.5(2) to show that every $\mathcal{A} \in \mathbb{A}$ has a maximizing well-ordering. \square

Proof of Theorem 2.2. We want to apply Theorem 2.5(2). So let

$$\mathbb{A} = \{\mathcal{A} : \exists \rho < \beth_\omega \exists \kappa \in [\beth_\omega, \lambda] (\mathcal{A} \subset [\lambda]^\kappa \text{ has property } C(2, \rho))\}.$$

We should verify properties 2.5(a,b,c').

(a) is trivial by definition.

To show (b) assume that $\mathcal{A} \in \mathbb{A}$, $\lambda = \bigcup \mathcal{A} > \kappa$, and $\mathcal{A} \subset [\lambda]^\kappa$. Next we should recall a lemma from [8] which based on the following celebrated result of Shelah.

Shelah's Revised GCH Theorem ([6, Theorem 0.1]). If $\mu \geq \beth_\omega$, then $\mu^{[v]} = \mu$ for each large enough regular cardinal $v < \beth_\omega$.

Lemma 2.7 ([8, Lemma 3.3]). If $\lambda > \kappa \geq \beth_\omega > \rho$, and $\{A_\alpha : \alpha < \tau\} \subset [\lambda]^\kappa$ is a hypergraph with property $C(2, \rho)$, then $\tau \leq \lambda$, and there is an increasing continuous sequence $\langle G_\zeta : \zeta < cf(\lambda) \rangle$ with $G_\zeta \in [\lambda]^{<\lambda}$ for $\zeta < cf(\lambda)$ such that

$$\forall \zeta < cf(\lambda) \forall \alpha \in G_{\zeta+1} \setminus G_\zeta (|A_\alpha \cap G_\zeta| < \beth_\omega \text{ and } A_\alpha \subset G_{\zeta+1}).$$

(In [7, Lemma 3.3] we considered only the special case of the lemma when $\kappa = \beth_\omega$, but a minor modification of the argument gives the stronger result above, see [8, Lemma 3.3].)

This Lemma just claims that condition 2.5(b) holds.

To check 2.5(2)(c') we need to recall the following lemma from [7]:

Lemma 2.8 ([7, Lemma 3.2]). Fix $\rho < \beth_\omega$. Then for each $\kappa \geq \beth_\omega$ there is a regular $v(\kappa) < \beth_\omega$ such that if $\mathcal{A} \subset [\kappa]^{v(\kappa)}$ is ρ -almost disjoint, then $|\mathcal{A}| \leq \kappa$.

By this lemma, if $\mathcal{A} \subset [\kappa]^\kappa$ is $C(2, \rho)$, then $|\mathcal{A}| \leq \kappa$, so we can write $\mathcal{A} = \{A_\xi : \xi < \xi\}$ for some $\xi \leq \kappa$. Now repeat an argument from the proof of the previous theorem: let \preceq be a well-ordering of κ such that writing $A'_\xi = A_\xi \setminus A_{<\xi}$ we have

- (1) $A'_\xi < A'_\eta$ for $\xi < \eta < \lambda$
- (2) the order type of A'_η is $\kappa + 1$, i.e. A'_η has a \preceq -maximal element.

Then \preceq maximizes \mathcal{A} .

Thus, we verified condition 2.5(2)(c') and so we can apply Theorem 2.5(2) to conclude that every $\mathcal{A} \in \mathbb{A}$ has a maximizing well-ordering. \square

3. Non-uniform hypergraphs

In [5, Theorem 6] Komjáth proved that $\mathbf{M}(\lambda, \{\kappa\}, 2, r) \rightarrow \mathbf{MinVC}$ for each $r < \omega \leq \kappa \leq \lambda$.

In this section we want to prove Theorem 1.1 which relaxes the requirement concerning the edge cardinality spectrum of the hypergraphs.

Proof of Theorem 1.1. Let r be a fixed natural number.

For any infinite cardinal κ let $\Delta(\kappa)$ be the following statement:

if S is a nowhere stationary set of infinite cardinals with $\sup S \leq \kappa$, then every hypergraph $\mathcal{A} \subset [\kappa]^S$ possessing property $C(2, r)$ has a minimal vertex cover.

We prove $\Delta(\kappa)$ for each infinite κ by transfinite induction on κ .

For $\kappa = \omega$, $\mathcal{A} \subset [\omega]^\omega$, so $\Delta(\omega)$ is just a special case of [5, Theorem 6].

So assume that $\kappa > \omega$ and $\Delta(\mu)$ holds for each infinite cardinal $\mu < \kappa$.

Fix a partition $\mathcal{C}' = \{C'_\xi : \xi < \kappa\} \subset [\kappa]^\kappa$ of κ .

To prove $\Delta(\kappa)$, let S be a nowhere stationary set of cardinals with $\sup S \leq \kappa$, and let $\mathcal{A} \subset [\kappa]^S$ be a hypergraph with property $C(2, r)$. Since $C'_0 \in [\kappa]^\kappa$, we can assume that $\mathcal{A} \setminus \mathcal{C}' \subset \mathcal{P}(C'_0)$ and $\mathcal{C}' \setminus \{C'_0\} \subset \mathcal{A}$, and so $|\mathcal{A} \cap [\kappa]^\kappa| = \kappa$.

Write $B = \mathcal{A} \cap [\kappa]^{<\kappa}$ and $\mathcal{C} = \mathcal{A} \cap [\kappa]^\kappa$. Let $\langle C_\xi : \xi < \kappa \rangle$ be an enumeration of \mathcal{C} .

Step 1. Construction of a chain of cardinals.

We define an increasing sequence of cardinals $\langle \mu_\zeta : \zeta < \kappa \rangle$ as follows.

Let $\mu_0 = 0$.

If $\kappa = \mu^+$ then let $\mu_\zeta = \mu$ for each $1 \leq \zeta < \kappa$.

Assume now that κ is a limit cardinal. Since S is nowhere stationary, there is a strictly increasing continuous sequence of infinite cardinals, $\langle \nu_\alpha : \alpha < cf(\kappa) \rangle$, which is cofinal in κ , and $\nu_\alpha \notin S$ for each limit ordinal $\alpha < cf(\kappa)$. For $1 \leq \zeta < \kappa$ let

$$\mu_\zeta = \min\{\nu_\alpha : \nu_\alpha \geq \zeta\}.$$

Observe that the sequence $\langle \mu_\zeta : \zeta < \kappa \rangle$ is increasing and continuous, in particular, $\mu_{\nu_\alpha} = \nu_\alpha$ and $\mu_{\nu_\alpha+1} = \nu_{\alpha+1}$.

Step 2. The inductive construction.

By transfinite recursion on $\zeta < \kappa$ we will define an increasing continuous sequence $\langle M_\zeta : \zeta < \kappa \rangle$ of subsets of κ , a sequence $\langle Y_\zeta : \zeta < \kappa \rangle$ of pairwise disjoint subsets of κ , and a sequence $\langle w_\zeta : \zeta < \kappa \rangle$ of functions such that

- (1) $_\zeta$ $\zeta \subset M_\zeta \in [\kappa]^{\mu_\zeta}$,
- (2) $_\zeta$ if $B \in \mathcal{B}$, $|B \cap M_\zeta| \geq r$ and $|B| \leq \mu_\zeta$ then $B \subset M_\zeta$,
- (3) $_\zeta$ If $\zeta = 0$ or ζ is a limit ordinal, then $M_\zeta = M_{<\zeta}$ and $Y_\zeta = w_\zeta = \emptyset$,
- (4) $_\zeta$ if $\zeta = \eta + 1$, then $Y_\zeta \subset M_\zeta \setminus M_\eta$ and

$$w_\zeta : Y_\zeta \rightarrow (\mathcal{B}[-Y_{\leq \eta}] \cap M_\zeta) \cup \{C_\eta\},$$

- (5) $_\zeta$ $\mathcal{A}[Y_{\leq \zeta}] \supset (\mathcal{B} \cap M_\zeta) \cup \{C_\xi : \xi < \zeta\}$,
- (6) $_\zeta$ $w_{\leq \zeta}(y) \cap Y_{\leq \zeta} = \{y\}$ for each $y \in Y_{\leq \zeta}$,
- (7) $_\zeta$ if $\zeta = \eta + 1$ and $C_\eta = w_\zeta(y)$ for some $y \in Y_\zeta$, then

$$C_\eta \cap \bigcup (\mathcal{B}[-Y_{\leq \eta}] \cap [\kappa]^{\leq \mu_\zeta}) \subset M_\zeta.$$

Step 3. The inductive step.

Assume that we have constructed $\langle Y_\xi : \xi < \zeta \rangle$ and $\langle w_\xi : \xi < \zeta \rangle$ such that (1) $_\xi$ –(7) $_\xi$ hold for $\xi < \zeta$.

If $\zeta = 0$, then let $M_\zeta = Y_\zeta = w_\zeta = \emptyset$. Since $\mu_0 = 0$, (1) $_\zeta$ holds. The other requirements are trivial.

If ζ is a limit ordinal, take $M_\zeta = M_{<\zeta}$ and $Y_\zeta = w_\zeta = \emptyset$.

Then (1) $_\zeta$ holds because $\mu_\zeta = \sup\{\mu_\xi : \xi < \zeta\}$.

To check (2) $_\zeta$, assume that $B \in \mathcal{B}$, $|B \cap M_\zeta| \geq r$ and $|B| \leq \mu_\zeta$.

Since $M_\zeta = M_{<\zeta}$, there is $\xi < \zeta$ such that $|B \cap M_\xi| \geq r$.

Next we show that $|B| \leq \mu_\eta$ for some $\eta < \zeta$. We know that $|B| \leq \mu_\zeta$. If $\mu_\eta < \mu_\zeta$ for each $\eta < \zeta$, then $\mu_\zeta = \nu_\alpha$ for some limit α , and so $\mu_\zeta \notin S$, hence $|B| < \mu_\zeta$. Since $M_\zeta = \bigcup_{\eta < \zeta} M_\eta$, there is $\eta < \zeta$ such that $|B| \leq \mu_\eta$.

Let $\sigma = \max(\xi, \eta)$.

Then $|B \cap M_\sigma| \geq r$ and $|B| \leq \mu_\sigma$, therefore $B \subset M_\sigma \subset M_\zeta$ by (2) $_\sigma$. Thus (2) $_\zeta$ holds.

(3) $_\zeta$ holds by the construction.

(4) $_\zeta$ is void.

To check (5) $_\zeta$, first observe that for each $\eta < \xi$, $C_\eta \in \mathcal{A}[Y_{\leq \eta+1}]$ by (5) $_{\eta+1}$, so $\mathcal{A}[Y_{\leq \zeta}] \supset \{C_\xi : \xi < \zeta\}$.

Assume that $B \in \mathcal{B} \cap M_\zeta$, i.e. $B \subset M_\zeta$ and $B \in \mathcal{B}$.

Since B is infinite and $M_\zeta = M_{<\zeta}$, there is $\xi < \zeta$ such that $|B \cap M_\xi| \geq r$.

Next we show that $|B| \leq \mu_\eta$ for some $\eta < \zeta$. We know that $|B| \leq \mu_\zeta$. If $\mu_\eta < \mu_\zeta$ for each $\eta < \zeta$, then $\mu_\zeta = \nu_\alpha$ for some limit α , and so $\mu_\zeta \notin S$, hence $|B| < \mu_\zeta$. Since $M_\zeta = \bigcup_{\eta < \zeta} M_\eta$, there is $\eta < \zeta$ such that $|B| \leq \mu_\eta$.

Let $\sigma = \max(\xi, \eta)$.

Then $|B \cap M_\sigma| \geq r$ and $|B| \leq \mu_\sigma$ and so $B \subset M_\sigma \subset M_\zeta$ by (2) $_\sigma$. Thus, (5) $_\zeta$ holds.

(6) $_\zeta$ and (7) $_\zeta$ are clear for limit ζ because $Y_{\leq \zeta} = Y_{<\zeta}$ and $w_{\leq \zeta} = w_{<\zeta}$.

So for limit ζ we can carry out the inductive step.

Assume finally that $\zeta = \eta + 1$. First we ensure that C_η is covered.

Case 1. $C_\eta \in \mathcal{C}[Y_{\leq \eta}]$, i.e. $C_\eta \cap Y_{\leq \eta} \neq \emptyset$.

Then let

$$Y'_\zeta = \emptyset \text{ and } v'_\zeta = \emptyset.$$

Case 2. $C_\eta \in \mathcal{C}[-Y_{\leq \eta}]$, i.e. $C_\eta \cap Y_{\leq \eta} = \emptyset$.

Write

$$C'_\eta = C_\eta \cap \bigcup ((\mathcal{B}[-Y_{\leq \eta}]) \cap [\kappa]^{\leq \mu_\zeta}).$$

Case 2.1 $|C'_\eta| \leq \mu_\zeta$.

Then $C''_\eta = ((C_\eta \setminus C'_\eta) \setminus \bigcup_{\xi < \eta} C_\xi)$ has cardinality κ , so we can put

$$y_\eta = \text{Min}(C''_\eta \setminus M_\eta).$$

In this case let

$$Y'_\zeta = \{y_\eta\} \text{ and } v'_\zeta = \{(y_\eta, C_\eta)\}.$$

Case 2.2 $|C'_\eta| > \mu_\zeta$.

Then $|(C'_\eta \cup \bigcup_{\xi < \eta} C_\xi)| > \mu_\zeta$ because $|(C_\eta \cap \bigcup_{\xi < \eta} C_\xi)| \leq r|\eta| \leq r|\zeta| \leq \mu_\zeta$, so we can consider

$$y_\eta = \text{Min}((C'_\zeta \setminus \bigcup_{\xi < \eta} C_\xi) \setminus M_\eta),$$

and we can pick B_η with

$$y_\eta \in B_\eta \in \mathcal{B}[-Y_{\leq \eta}] \cap [\kappa]^{\leq \mu_\zeta}.$$

In this case let

$$Y'_\zeta = \{y_\eta\} \text{ and } v'_\zeta = \{(y_\eta, B_\eta)\}.$$

So we defined Y'_ζ and v'_ζ such that $Y_{\leq \eta} \cup Y'_\zeta$ meets C_η , i.e. $C_\eta \cap (Y_{\leq \eta} \cup Y'_\zeta) \neq \emptyset$.

Next we want to define sets M_ζ and Z_ζ , and a function v_ζ such that $Y_\zeta = Y'_\zeta \cup Z_\zeta$ and $w_\zeta = v'_\zeta \cup v_\zeta$ meet the requirements.

If we are in Case 1, i.e. $C_\eta \in \mathcal{C}[Y_{\leq \eta}]$, then let $M_\zeta^- = \zeta \cup M_\eta$.

When we are in Case 2.1., i.e. $C_\eta \in \mathcal{C}[-Y_{\leq \eta}]$ and $|C'_\eta| \leq \mu_\zeta$, then let $M_\zeta^- = \zeta \cup M_\eta \cup Y'_\eta \cup C'_\eta$.

If we are in Case 2.2., i.e. $C_\eta \in \mathcal{C}[-Y_{\leq \eta}]$ and $|C'_\eta| > \mu_\zeta$, then let $M_\zeta^- = \zeta \cup M_\eta \cup Y'_\eta \cup B_\eta$.

Since $M_\zeta^- \in [\kappa]^{\mu_\zeta}$, using standard closure arguments, we can find a set $M_\zeta \in [\kappa]^{\mu_\zeta}$ such that

- (a) $M_\zeta \supset M_\zeta^-$,
- (b) if $B \in \mathcal{B}$, $|B \cap M_\zeta| \geq r$ and $|B| \leq \mu_\zeta$, then $B \subset M_\zeta$.

Write

$$C_{\leq \eta}^w = \bigcup (\text{ran } w_{\leq \eta} \cap [\kappa]^\kappa)$$

and let

$$\mathcal{B}_\zeta = (\mathcal{B} \cap M_\zeta) \setminus (Y_{\leq \eta} \cup Y'_\zeta).$$

Consider the family

$$\mathcal{B}_\zeta^- = \mathcal{B}_\zeta \upharpoonright ((M_\zeta \setminus M_\eta) \setminus (C_\eta \cup C_{\leq \eta}^w)).$$

Since $|\mathcal{B}_\zeta^-| \leq |M_\zeta|^\kappa = \mu_\zeta$, we can apply the inductive assumption $\Delta(\mu_\xi)$ for \mathcal{B}_ζ^- provided that we can show that $\|\mathcal{B}_\zeta^-\|$ is a nowhere stationary set of infinite cardinals. The next claim will actually yield that $\|\mathcal{B}_\zeta^-\| \subset \|\mathcal{B}\|$.

Claim. $|C_{\leq \eta}^w \cap B| < |B|$ for each $B \in (\mathcal{B}[-Y_{\leq \eta}]) \cap [\kappa]^{\leq \mu_\zeta}$.

Proof of the Claim. Fix $B \in \mathcal{B}[-Y_{\leq \eta}] \cap [\kappa]^{\leq \mu_\zeta}$.

Let $\beta = \sup\{\gamma : v_\gamma < |B|\}$. Since $v_\varepsilon \notin S$ for limit ε , we have $v_\beta < |B|$.

Assume that $w_{\sigma+1}(y) = C_\sigma$ and $B \cap (C_\sigma \setminus M_\eta) \neq \emptyset$. Suppose that $\mu_{\sigma+1} = v_\varepsilon$. Since the sequence $\langle v_\alpha : \alpha < cf(\kappa) \rangle$ is continuous, by the definition of $\mu_{\sigma+1}$, ε cannot be a limit ordinal. So $\mu_{\sigma+1} = v_{\alpha+1}$ for some α .

We know

$$C_\sigma \cap \bigcup ((\mathcal{B}[-Y_{\leq \sigma+1}]) \cap [\kappa]^{\leq \mu_{\sigma+1}}) \subset M_{\sigma+1},$$

by (7) $_{\sigma+1}$, and so $|B| > \mu_{\sigma+1} = v_{\alpha+1}$.

Since $\sigma + 1 \leq \mu_{\sigma+1}$ by the definition of $\mu_{\sigma+1}$, we have $\sigma \in v_{\alpha+1}$.

Since $v_{\alpha+1} < |B|$, it follows that $v_{\alpha+1} \leq v_\beta$. Therefore

$$\{\sigma : C_\sigma \in \text{ran } w_{\leq \eta} \wedge B \cap (C_\sigma \setminus M_\eta) \neq \emptyset\} \subset v_\beta < |B|.$$

Hence, we proved the Claim. \square

If $B \in \mathcal{B}_\zeta$, then $B \notin \mathcal{B} \cap M_\eta$, so $|B \cap M_\eta| < r$ or $|B| > |M_\eta|$. Moreover, $B \subset M_\zeta$, so $|B \cap (M_\zeta \setminus M_\eta)| = |B|$. Since $|B \cap C_{\leq \eta}^w| < |B|$ by the Claim and $|B \cap C_\eta| < r$, we have $|B| = |B \cap (M_\zeta \setminus M_\eta) \setminus (C_\eta \cup C_{\leq \eta}^w)|$, thus $\|\mathcal{B}_\zeta^-\| \subset (S \cap \kappa)$.

Since $\mathcal{B}_\zeta^- \subset \mathcal{P}(M_\zeta)$ and $|M_\zeta| = \mu_\zeta$ and $\|\mathcal{B}_\zeta^-\|$ is nowhere stationary, so we can apply the inductive assumption $\Delta(\mu_\xi)$ for \mathcal{B}_ζ^- to obtain a set $Z_\zeta \subset (M_\zeta \setminus M_\eta) \setminus (C_\eta \cup C_{\leq \eta}^w)$ which is a minimal vertex cover for \mathcal{B}_ζ^- , and so it is a minimal vertex cover for \mathcal{B}_ζ . Fix a witnessing function $v_\zeta : Z_\zeta \rightarrow \mathcal{B}_\zeta$ such that $v_\zeta(y) \cap Z_\zeta = \{y\}$ for each $y \in Z_\zeta$.

Let

$$Y_\zeta = Y'_\zeta \cup Z_\zeta \text{ and } w_\zeta = v'_\zeta \cup v_\zeta.$$

Now we should check that (1) $_\zeta$ –(7) $_\zeta$ hold.

(1) $_\zeta$ and (2) $_\zeta$ hold by the choice of M_ζ .

(3) $_\zeta$ is void.

(4) $_\zeta$ is clear from the construction.

Next we check (5) $_\zeta$. We have $\mathcal{A}[Z_\zeta] \supset \mathcal{B}_\zeta$ and $C_\eta \in \mathcal{A}[Y_{\leq \eta} \cup Y'_\eta]$ by the construction, and $\mathcal{A}[Y_{\leq \eta}] \supset (\mathcal{B} \cap M_\eta) \cup \{C_\xi : \xi < \eta\}$ by the inductive assumption (5) $_\eta$, so putting together we obtain

$$\mathcal{A}[Y_{\leq \zeta}] \supset ((\mathcal{B} \cap M_\zeta) \cap [\kappa]^{\leq \mu_\zeta}) \cup \{C_\xi : \xi < \zeta\}.$$

So (5) $_\zeta$ holds.

Next we check (6) $_\zeta$. Since $Y_\zeta \subset M_\zeta \setminus M_{<\zeta} \setminus C_{\leq \eta}^w$, and $\bigcup \text{ran } w_{\leq \eta} \subset M_{<\zeta} \cup C_{\leq \eta}^w$, so (6) $_\zeta$ holds for $y \in Y_{\leq \eta}$.

If y_η is defined, then $C_\eta \cap Y_{\leq \eta} = \emptyset$, so $w_\zeta(y_\eta) \cap Y_\zeta = \{y_\eta\}$ because $Z_\zeta \cap C_\eta = \emptyset$.

If $y \in Z_\zeta$, then $w_\zeta(y) \in \mathcal{B}[-(Y_{\leq \eta} \cup Y'_\eta)]$, and $w_\zeta(y) \cap Z_\zeta = \{y\}$ so $w_\zeta(y) \cap Y_{\leq \zeta} = \{y\}$.

Thus, (6) $_\zeta$ holds.

Observe that (7) $_\zeta$ also holds: if $w_\zeta(y_\eta) = C_\eta$, then $|C'_\eta| \leq |M_\zeta|$, and so $C'_\zeta \subset M_\zeta^-$ implies $C'_\zeta \subset M_\zeta$.

Step 4. Conclusion.

After the inductive construction $Y = \bigcup_{\zeta < \kappa} Y_\zeta$ is a minimal vertex cover of \mathcal{A} . The minimality is witnessed by the function $w = \bigcup_{\zeta < \kappa} w_\zeta$. \square

4. Hypergraphs possessing property C(k, r) for $k \geq 3$

In this section we try to relax the assumptions C(2, r) to C(k, r) for some $k \geq 3$.

Instead of Theorem 1.4 we will prove the following stronger theorem:

Theorem 4.1. $\mathbf{M}(\lambda, \{\omega\}, k, r) \rightarrow \mathbf{MaxWO}$ provided $\omega \leq \lambda$ and $k, r \in \omega$

Proof. We want to apply Theorem 2.5(2) to obtain the result. So let

$$\mathbb{A} = \{\mathcal{A} : \exists \lambda \geq \omega \ (\mathcal{A} \subset [\lambda]^\omega \text{ has property C}(k, r))\}.$$

We should verify properties 2.5(a,b,c').

(a) is trivial by definition.

(b) follows from the Lemma 4.2 below.

Lemma 4.2. If $\omega \leq \kappa < \lambda$ are infinite cardinals and $\mathcal{A} \subset [\lambda]^\kappa$ has property C(k, r) for some $k, r \in \omega$, then \mathcal{A} has a good cut.

Proof. Without loss of generality we can assume that $k \geq 2$.

For $x \in [\lambda]^r$ let

$$F(x) = \{A \in \mathcal{A} : x \subset A\}.$$

Since \mathcal{A} has property C(k, r), $|F(x)| \leq k - 1$.

Let $\langle \lambda_\alpha : \alpha < \omega \cdot cf(\lambda) \rangle$ be an increasing continuous cofinal sequence in λ with $\lambda_0 = 0$. Define an increasing continuous sequence $\langle H_\eta : \eta < \omega \cdot cf(\lambda) \rangle$ such that

(1) $H_0 = \emptyset$,

(2) $H_{\eta+1} = H_\eta \cup \lambda_{\eta+1} \cup \bigcup \{F(x) : x \in [H_\eta]^r\}$.

Since $|F(x)| \leq k - 1$, we have $|H_\eta| \leq |\lambda_\eta| + \kappa$. Moreover, $|A \cap H_\eta| \geq r$ implies $A \subset H_{\eta+1}$ for $A \in \mathcal{A}$.

Let $G_\alpha = H_{\omega \cdot \alpha}$. Then $\langle G_\alpha : \alpha < cf(\lambda) \rangle$ is a good cut of \mathcal{A} . \square

Property 2.5(2)(c') follows from the next lemma.

Lemma 4.3. If \mathcal{A} is a countable hypergraph which possesses property C(k, r), then \mathcal{A} has a maximizing well-ordering \prec .

Proof of Lemma 4.3. We prove it by induction on k . If $k = 1$, then we have that for all $n \in \omega$, $|A_n| \leq r - 1$, so A_n is a finite set. Choose any well-ordering \prec on $\bigcup_{n \in \omega} A_n$, then clearly all non-empty finite sets have maximal element by \prec .

Now suppose the lemma is true for hypergraphs possessing property $C(k, r)$ and let \mathcal{A} be a hypergraph with property $C(k+1, r)$. We can assume that \mathcal{A} is infinite, and fix an injective enumeration $\{A_n : n < \omega\}$ of \mathcal{A} .

For $n < \omega$ let

$$B_n = A_n \setminus \bigcup_{i < n} A_i,$$

be the disjointification of A_n . Clearly $B_n \cap B_m = \emptyset$ if $n \neq m$ and $\bigcup_{n \in \omega} B_n = \bigcup_{n \in \omega} A_n$, but $B_n = \emptyset$ is possible even if $A_n \neq \emptyset$. Choose any n , such that $B_n \neq \emptyset$.

We will show that the system $\{A_m \cap B_n : m > n\}$ is $C(k, r)$. Choose any $m_1, \dots, m_k > n$, such that the sets $A_{m_i} \cap B_n$ are pairwise distinct for $1 \leq i \leq k$. Then the sets A_{m_i} are also pairwise distinct. For the intersection

$$(A_{m_1} \cap B_n) \cap \dots \cap (A_{m_k} \cap B_n) = A_{m_1} \cap \dots \cap A_{m_k} \cap B_n \subseteq A_{m_1} \cap \dots \cap A_{m_k} \cap A_n$$

and $A_{m_1}, \dots, A_{m_k}, A_n$ are pairwise distinct, so $|(A_{m_1} \cap B_n) \cap \dots \cap (A_{m_k} \cap B_n)| \leq |A_{m_1} \cap \dots \cap A_{m_k} \cap A_n| \leq r-1$, as \mathcal{A} has property $C(k+1, r)$.

By the inductive assumption there is a well-ordering $<'_n$ on

$$\bigcup \{A_m \cap B_n : m > n\},$$

such that for all $m > n$, where $A_m \cap B_n \neq \emptyset$, the set $A_m \cap B_n$ has a maximal element. We can make a well-ordering $<_n$ from it on B_n , so that B_n has a maximal element, and for all $m > n$, if $A_m \cap B_n \neq \emptyset$, then it has a maximal element.

Now we define the ordering $<$. For any $u, v \in \bigcup_{n \in \omega} A_n = \bigcup_{n \in \omega} B_n$ there are $i, j \in \omega$, such that $u \in B_i, v \in B_j$. If $i < j$, then $u < v$, if $j < i$, then $v < u$, and if $i = j$, then we order them by $<_i$ on B_i . Choose any $n \in \omega$, that $A_n \neq \emptyset$. If $B_n \neq \emptyset$, then it has a maximal element. Since all elements of A_n are either in B_n or in some B_i , where $i < n$, we have $\max(A_n) = \max(B_n)$. If $B_n = \emptyset$, then we have $\emptyset \neq A_n = \bigcup_{i < n} B_i \cap A_n$. Let $j < n$ be the maximal, such that $A_n \cap B_j \neq \emptyset$. Then by the construction of $<_j$, we have that $A_n \cap B_j$ has a maximal element. Since all elements of A_n are even in $A_n \cap B_j$ or in some $A_n \cap B_i$, where $i < j$, it is also maximal in A_n . Thus the well-ordering $<$ is good, so the induction step works from k to $k+1$. \square

So we can apply Theorem 2.5(2) to derive that every $\mathcal{A} \in \mathbb{A}$ has a maximizing well-order. \square

After proving Theorem 1.4, it is a natural to ask if $\mathbf{M}(\lambda, \{\kappa\}, k, r) \rightarrow \mathbf{MinVC}$ holds for $k, r < \omega < \kappa \leq \lambda$. We could do that applying Theorem 2.5(2) as soon as we are able to prove that

$$\mathbf{M}(\kappa, \{\kappa\}, k, r) \rightarrow \mathbf{MinVC} \text{ holds provided } k, r < \omega < \kappa.$$

We managed to do this only in the simplest case, when $\kappa = \omega_1$ and $k = 3$, as we will see in the proof of Theorem 1.5 below.

Definition 4.4. If \mathcal{E} is a hypergraph, let $\text{Min } \mathcal{E}$ be the family of \subset -minimal elements of \mathcal{E} .

Observe that if \mathcal{E} is $C(k, r)$, then for each $E \in \mathcal{E}$ there is $E' \in \text{Min } \mathcal{E}$ with $E' \subset E$. So a minimal vertex cover of $\text{Min } \mathcal{E}$ will be a minimal vertex cover of \mathcal{E} .

Proof of Theorem 1.5. We want to apply Theorem 2.5(1) to prove Theorem 1.5. So let

$$\mathbb{A} = \{\mathcal{A} : \exists \lambda \geq \omega_1 (\mathcal{A} \subset [\lambda]^{\omega_1} \text{ has property } C(3, r))\}.$$

Property 2.5(a) is trivial by definition, and 2.5(b) follows from Lemma 4.2.

Finally, to check 2.5(1)(c) we need to prove that

$$\mathbf{M}(\omega_1, \{\omega_1\}, 3, r) \rightarrow \mathbf{MinVC} \text{ for } r < \omega.$$

Let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\} \subset [\omega_1]^{\omega_1}$ be a hypergraph having property $C(3, r)$. For each $\alpha < \omega_1$ pick

$$y_\alpha \in D_\alpha \setminus \{D_\xi \cap D_\zeta : \xi < \zeta < \alpha\}.$$

Let $Y = \{y_\alpha : \alpha < \omega_1\}$. Then $\mathcal{D}[Y] = \mathcal{D}$. Moreover, $\mathcal{D} \upharpoonright Y$ is $C(3, r)$ and $C(2, \omega_1)$ because $y_\alpha \notin D_\zeta \cap D_\xi$ for $\alpha > \max(\zeta, \xi)$. Let

$$\mathcal{A} = \text{Min}(\mathcal{D} \upharpoonright Y).$$

Then a minimal vertex cover of \mathcal{A} will be a minimal vertex cover of \mathcal{D} . So to find a minimal vertex cover of \mathcal{D} it is enough to prove the following lemma:

Lemma 4.5. Assume that a hypergraph $\mathcal{A} \subset \mathcal{P}(\omega_1)$ possesses the following properties:

- (i) $C(3, r)$,
- (ii) $C(2, \omega_1)$,
- (iii) $A \setminus A' \neq \emptyset$ for each $\{A, A'\} \in [\mathcal{A}]^2$.

Then \mathcal{A} has a minimal vertex cover.

Proof of the Lemma 4.5. Fix a partition $\mathcal{C}' = \{C'_\xi : \xi < \omega_1\} \subset [\omega_1]^{\omega_1}$ of ω_1 .

Write $\mathcal{B} = \mathcal{A} \cap [\omega_1]^{\leq \omega}$ and $\mathcal{C} = \mathcal{A} \cap [\omega_1]^{\omega_1}$. Since $C'_0 \in [\omega_1]^{\omega_1}$, we can assume that $\mathcal{A} \setminus \mathcal{C}' \subset \mathcal{P}(C'_0)$ and $\mathcal{C}' \setminus \{C'_0\} \subset \mathcal{A}$, and so $|\mathcal{C}| = \omega_1$.

Let $\langle C_\xi : \xi < \omega_1 \rangle$ be an enumeration of \mathcal{C} .

Step 1. The inductive construction.

By transfinite recursion on $\zeta < \omega_1$ we will define an increasing, continuous sequence $\langle M_\zeta : \zeta < \omega_1 \rangle$ of countable subsets of ω_1 , a sequence $\langle Y_\zeta : \zeta < \omega_1 \rangle$ of pairwise disjoint subsets of ω_1 , and a sequence $\langle w_\zeta : \zeta < \omega_1 \rangle$ of functions such that

- (1) $_\zeta$ $\zeta \subset M_\zeta \in [\omega_1]^\omega$ for $1 \leq \zeta < \omega_1$, $M_0 = \emptyset$,
- (2) $_\zeta$ if $B \in \mathcal{B}$ and $|B \cap M_\zeta| \geq r$ then $B \subset M_\zeta$,
- (3) $_\zeta$ if $\zeta = 0$ or ζ is a limit ordinal, then $M_\zeta = M_{<\zeta}$ and $Y_\zeta = w_\zeta = \emptyset$,
- (4) $_\zeta$ if $\zeta = \eta + 1$, then $Y_\zeta \subset (M_\zeta \setminus M_\eta)$ and

$$w_\zeta : Y_\zeta \rightarrow ((\mathcal{B} \cap M_\zeta) \setminus (\mathcal{B} \cap M_\eta)) \cup \{C_\eta\},$$

- (5) $_\zeta$ $\mathcal{A}[Y_{\leq \zeta}] \supset (\mathcal{B} \cap M_\zeta) \cup \{C_\xi : \xi < \zeta\}$,
- (6) $_\zeta$ $w_{\leq \zeta}(y) \cap Y_{\leq \zeta} = \{y\}$ for each $y \in Y_{\leq \zeta}$,
- (7) $_\zeta$ if $\zeta = \eta + 1$ and $C_\eta = w_\zeta(y)$ for some $y \in Y_\zeta$, then

$$C_\eta \cap \bigcup \text{Min}((\mathcal{B}[-Y_{\leq \eta}] \upharpoonright (\omega_1 \setminus M_\eta)) = \emptyset.$$

Step 2. The inductive step.

Assume that we have constructed $\langle Y_\xi : \xi < \zeta \rangle$ and $\langle w_\xi : \xi < \zeta \rangle$ such that (1) $_\xi$ –(7) $_\xi$ hold for $\xi < \zeta$.

If $\zeta = 0$, then let $M_\zeta = Y_\zeta = w_\zeta = \emptyset$. All the requirements are trivial.

If ζ is a limit ordinal, put $M_\zeta = M_{<\zeta}$ and $Y_\zeta = w_\zeta = \emptyset$.

Then (1) $_\zeta$ is trivial.

To check (2) $_\zeta$, assume that $B \in \mathcal{B}$ and $|B \cap M_\zeta| \geq r$.

Since $M_\zeta = M_{<\zeta}$, there is $\xi < \zeta$ such that $|B \cap M_\xi| \geq r$. So $B \subset M_\xi \subset M_\zeta$ by (2) $_\xi$. Thus, (2) $_\zeta$ holds.

(3) $_\zeta$ holds by the construction.

(4) $_\zeta$ is void.

To check (5) $_\zeta$ first observe that for each $\eta < \xi$, $C_\eta \in \mathcal{A}[Y_{\leq \eta+1}]$ by (5) $_{\eta+1}$, so $\mathcal{A}[Y_{\leq \zeta}] \supset \{C_\xi : \xi < \zeta\}$.

Assume that $B \in \mathcal{B} \cap M_\zeta$, i.e. $B \subset M_\zeta$ and $B \in \mathcal{B}$. If B is finite then $B \subset M_\sigma$ for some $\sigma < \zeta$. Thus, $B \in \mathcal{B} \cap M_\sigma \subset \mathcal{A}[Y_{\leq \sigma}] \subset \mathcal{A}[Y_{\leq \zeta}]$.

So we can assume that B is infinite. Since $M_\zeta = M_{<\zeta}$, there is $\xi < \zeta$ such that $|B \cap M_\xi| \geq r$, and so $B \subset M_\xi \subset M_\zeta$ by (2) $_\xi$. Thus, (5) $_\zeta$ holds.

Conditions (6) $_\zeta$, and (7) $_\zeta$ are clear for limit ζ because $Y_{\leq \zeta} = Y_{<\zeta}$ and $w_{\leq \zeta} = w_{<\zeta}$.

So for limit ζ we can carry out the inductive step.

Assume finally that $\zeta = \eta + 1$.

First we ensure that C_η is covered.

Case 1. $C_\eta \in \mathcal{C}[Y_{\leq \eta}]$, i.e. $C_\eta \cap Y_{\leq \eta} \neq \emptyset$.

Let

$$Y'_\zeta = \emptyset, v'_\zeta = \emptyset \text{ and } M_\zeta^- = \zeta \cup M_\eta.$$

Case 2. Assume that $C_\eta \in \mathcal{C}[-Y_{\leq \eta}]$, i.e. $C_\eta \cap Y_{\leq \eta} = \emptyset$.

Let

$$\mathcal{B}'_\zeta = \text{Min}(\mathcal{B}[-Y_{\leq \eta}] \upharpoonright (\omega_1 \setminus M_\eta)) \text{ and } C'_\eta = C_\eta \cap \bigcup \mathcal{B}'_\zeta.$$

Observe that if $B \in \mathcal{B}[-Y_{\leq \eta}]$, then $B \setminus M_\eta \neq \emptyset$ by (5) $_\eta$, and so $\emptyset \notin \mathcal{B}'_\zeta$.

Case 2.1. $C'_\eta = \emptyset$.

Since $C_\eta \setminus \bigcup_{\xi < \eta} C_\xi$ has cardinality ω_1 , we can define

$$y_\eta = \min((C_\eta \setminus \bigcup_{\xi < \eta} C_\xi) \setminus M_\eta),$$

and let

$$Y'_\zeta = \{y_\eta\}, v'_\zeta = \{(y_\eta, C_\eta)\} \text{ and } M_\zeta^- = \zeta \cup M_\eta \cup Y'_\zeta.$$

Case 2.2. $C'_\zeta \neq \emptyset$.

In this case put

$$y_\eta = \min(C'_\zeta) \in \omega_1 \setminus M_\eta,$$

and pick $B_\eta \in \mathcal{B}[-Y_{\leq \eta}]$ with

$$y_\eta \in (B_\eta \setminus M_\eta) \in \mathcal{B}'_\zeta,$$

and let

$$Y'_\zeta = \{y_\eta\}, v'_\zeta = \{(y_\eta, B_\eta)\} \text{ and } M_\zeta^- = \zeta \cup M_\eta \cup Y'_\zeta \cup B_\eta.$$

So we defined Y'_ζ , v'_ζ and M'_ζ .

Next we want to define sets M_ζ and Z_ζ , and a function v_ζ such that $Y_\zeta = Y'_\zeta \cup Z_\zeta$ and $w_\zeta = v'_\zeta \cup v_\zeta$ meet the requirements.

Since \mathcal{A} possesses property $C(3, r)$, using standard closure arguments we can find a set $M_\zeta \in [\omega_1]^\omega$ such that

- (a) $M_\zeta \supset M'_\zeta$,
- (b) if $B \in \mathcal{B}$ and $|B \cap M_\zeta| \geq r$ then $B \subset M_\zeta$.

If we are in Case 1 or in Case 2.1, then let

$$\mathcal{B}_\zeta = (\mathcal{B} \cap M_\zeta)[- (Y_{\leq \eta} \cup Y'_\zeta)] \upharpoonright (\omega_1 \setminus M_\eta).$$

Then $\emptyset \notin \mathcal{B}_\zeta \subset \mathcal{P}(M_\zeta \setminus M_\eta)$ and \mathcal{B}_ζ possesses property $C(3, k)$.

If we are in Case 2.2, let

$$\mathcal{B}_\zeta = (\mathcal{B} \cap M_\zeta)[- (Y_{\leq \eta} \cup Y'_\zeta)] \upharpoonright (\omega_1 \setminus (M_\eta \cup B_\eta)). \quad (4.1)$$

Since $B_\eta \setminus M_\eta$ was a minimal element of $\mathcal{B}[-Y_{\leq \eta}] \upharpoonright (\omega_1 \setminus M_\eta)$, it follows that $\emptyset \notin \mathcal{B}_\zeta$.

By Theorem 4.1 the countable hypergraph $\mathcal{B}_\zeta \cap [M_\zeta]^\omega$ has a maximizing well-order \leq_ζ . Since $\mathcal{B}_\zeta \setminus [M_\zeta]^\omega$ contains only non-empty finite sets, and all non-empty finite sets have \leq_ζ -maximal elements, it follows that \leq_ζ is a maximizing well-order for \mathcal{B}_ζ . Thus, as it was observed by Klimo, \mathcal{B}_ζ has a minimal vertex cover Z_ζ . Then Z_ζ is a minimal vertex cover of $(\mathcal{B} \cap M_\zeta)[- (Y_{\leq \eta} \cup Y'_\zeta)]$ as well. Let us fix a function $v_\zeta : Z_\zeta \rightarrow (\mathcal{B} \cap M_\zeta)[- (Y_{\leq \eta} \cup Y'_\zeta)]$ witnessing that fact.

So we defined M_ζ , Z_ζ and v_ζ . Let

$$Y_\zeta = Y'_\zeta \cup Z_\zeta \text{ and } w_\zeta = v'_\zeta \cup v_\zeta.$$

Now we should check that $(1)_\zeta$ – $(7)_\zeta$ hold.

$(1)_\zeta$ and $(2)_\zeta$ hold by the choice of M_ζ .

$(3)_\zeta$ is void.

$(4)_\zeta$ is clear from the construction.

Next we check $(5)_\zeta$. We have $\mathcal{A}[Y_\zeta] \supset (\mathcal{B} \cap M_\zeta)[-Y_{\leq \eta}]$ and $C_\eta \in \mathcal{A}[Y_{\leq \eta} \cup Y'_\eta]$ by the construction, and $\mathcal{A}[Y_{\leq \eta}] \supset (\mathcal{B} \cap M_\eta) \cup \{C_\xi : \xi < \eta\}$ by the inductive assumption $(5)_\eta$, so putting together we obtain $\mathcal{A}[Y_{\leq \zeta}] \supset (\mathcal{B} \cap M_\zeta) \cup \{C_\xi : \xi < \zeta\}$. So $(5)_\zeta$ holds.

Next we check $(7)_\zeta$ before $(6)_\zeta$.

If $w_\zeta(y_\eta) = C_\eta$, then $C'_\eta = \emptyset$, which is just the statement of $(7)_\zeta$.

Finally, to check $(6)_\zeta$ fix $y \in Y_{\leq \zeta}$.

Case A. $y \in Y_{\leq \eta}$ and $w_{\leq \zeta}(y) \in \mathcal{C}$.

In this case $y = y_\sigma$ for some $\sigma < \eta$. Thus, $w_{\leq \eta}(y) \cap Y_{\leq \eta} = \{y\}$ by $(6)_\eta$ and $w_{\leq \eta}(y) = C_\sigma$, and so

$$\emptyset = C'_\sigma = C_\sigma \cap \bigcup \text{Min}(\mathcal{B}[-Y_{\leq \sigma}] \upharpoonright (\omega_1 \setminus M_\sigma)) \supset C_\sigma \cap \bigcup \text{Min}(\mathcal{B}[-Y_{\leq \eta}] \upharpoonright (\omega_1 \setminus M_\eta)) \supset C_\sigma \cap Z_\zeta,$$

and $y_\eta \notin C_\sigma$ provided y_η is defined. Thus,

$$C_\sigma \cap Y_{\leq \zeta} = (C_\sigma \cap Y_{\leq \eta}) \cup (C_\sigma \cap Y'_\zeta) \cup (C_\sigma \cap Z_\zeta) = \{y\} \cup \emptyset \cup \emptyset = \{y\}.$$

Case B. $y = y_\eta$ and $w_{\leq \zeta}(y) \in \mathcal{C}$.

In this case $w_{\leq \zeta}(y) = C_\eta$ and so we were in Case 2.1. Hence, $C_\eta \cap Y_{\leq \eta} = \emptyset$ and $C_\eta \cap \bigcup \mathcal{B}'_\zeta = \emptyset$. Since $Z_\zeta \subset \bigcup \mathcal{B}'_\zeta$, we have $C_\eta \cap Z_\zeta = \emptyset$. Moreover, $y_\eta \in C_\eta$. Putting together we obtain

$$C_\eta \cap Y_{\leq \zeta} = (C_\eta \cap Y_{\leq \eta}) \cup (C_\eta \cap \{y_\eta\}) \cup (C_\eta \cap Z_\zeta) = \emptyset \cup \{y_\eta\} \cup \emptyset = \{y_\eta\}.$$

Case C. $y \in Y_{\leq \eta}$ and $w_{\leq \zeta}(y) \in \mathcal{B}$.

By the inductive assumption (6) $_\eta$, $w_{\leq \zeta}(y) \cap Y_{\leq \eta} = w_{\leq \eta}(y) \cap Y_{\leq \eta} = \{y\}$. Since $Y_\zeta \cap M_\eta = \emptyset$ and $w_{\leq \eta}(y) \subset M_\eta$, we have $w_{\leq \zeta}(y) \cap Y_\zeta = \emptyset$. So

$$w_{\leq \zeta}(y) \cap Y_{\leq \zeta} = (w_{\leq \zeta}(y) \cap Y_{\leq \eta}) \cup (w_{\leq \zeta}(y) \cap Y_\zeta) = \{y\} \cup \emptyset = \{y\}.$$

Case D. $y = y_\eta$ and $w_{\leq \eta}(y) \in \mathcal{B}$.

In this case we know that we were in Case 2.2. Hence, $w_\zeta(y) = B_\eta \in \mathcal{B}[-Y_{\leq \eta}]$, and so $w_\zeta(y) \cap Y_{\leq \eta} = \emptyset$, and $B_\eta \cap \bigcup \mathcal{B}'_\zeta = \emptyset$. Since $Z_\zeta \subset \bigcup \mathcal{B}'_\zeta$, we obtain $B_\eta \cap Z_\zeta = \emptyset$. Since $y_\eta \in B_\eta$, we have

$$w_{\leq \zeta}(y) \cap Y_{\leq \zeta} = (B_\eta \cap Y_{\leq \zeta}) \cup (B_\eta \cap Y'_\zeta) \cup (B_\eta \cap Z_\zeta) = \emptyset \cup \{y_\eta\} \cup \emptyset = \{y_\eta\}.$$

Case E. $y \in Z_\zeta$.

In this case, $w_\zeta(y) \subset B$ for some $B \in \mathcal{B}[-(Y_{\leq \eta} \cup Y'_\zeta)]$, and so $w_\zeta(y) \cap (Y_{\leq \eta} \cup Y'_\zeta) = \emptyset$.

Since $w_\zeta(y) \cap Z_\zeta = v_\zeta(y) \cap Z_\zeta = \{y\}$ by the choice of Z_ζ and v_ζ , we have

$$w_\zeta(y) \cap Y_{\leq \zeta} = (w_\zeta(y) \cap (Y_{\leq \zeta} \cup Y'_\zeta)) \cup (w_\zeta(y) \cap Z_\zeta) = \emptyset \cup \{y\} = \{y\}.$$

Hence, we verified (6) $_\zeta$.

Step 4. Conclusion.

After the inductive construction $Y = \bigcup_{\zeta < \kappa} Y_\zeta$ is a minimal vertex cover for \mathcal{A} . The minimality is witnessed by the function $w = \bigcup_{\zeta < \kappa} w_\zeta$. So we proved Lemma 4.5. \square

As we observed, the lemma immediately yields that \mathcal{D} has a minimal vertex cover. Since \mathcal{D} was arbitrary, we proved that $\mathbf{M}(\omega_1, \{\omega_1\}, 3, r) \rightarrow \mathbf{MinVC}$ holds.

So we can apply Theorem 2.5(1) to complete the proof of Theorem 1.5. \square

5. Problems

- (1) The set $K = \{\aleph_\alpha : \alpha < \omega_1\}$ is the “simplest” set which is not nowhere stationary. Does $\mathbf{M}(\aleph_{\omega_1}, K, 2, 2) \rightarrow \mathbf{MinVC}$ hold?
- (2) It is easy to construct hypergraphs with minimal vertex covers and without maximizing well-orders. Assume that $k \in \omega$, λ and ρ are cardinals, and K is a set of cardinals. Does $\mathbf{M}(\lambda, K, k, \rho) \rightarrow \mathbf{MinVC}$ imply $\mathbf{M}(\lambda, K, k, \rho) \rightarrow \mathbf{MaxWO}$?
- (3) Does $\mathbf{M}(\omega_1, \{\omega_1\}, 4, 2) \rightarrow \mathbf{MinVC}$ hold?
- (4) Does $\mathbf{M}(\omega_2, \{\omega_2\}, 3, 2) \rightarrow \mathbf{MinVC}$ hold?
- (5) Does $\mathbf{M}(\omega_1, \{\omega, \omega_1\}, 3, 2) \rightarrow \mathbf{MinVC}$ hold?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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