

ON NEGATIVE SPHERES IN ELLIPTIC SURFACES

ANDRÁS I. STIPSICZ AND ZOLTÁN SZABÓ

ABSTRACT. Using elliptic fibrations with specific singular fibers, we find spheres with very negative self-intersections in elliptic surfaces and in their blow-ups.

1. INTRODUCTION

Suppose that X is a simply connected, closed, smooth, oriented four-manifold. As $H_2 \cong \pi_2$ in this case, all second homology classes can be represented by a smooth map $f: S^2 \rightarrow X$, and in this dimension we can assume that f is an immersion. There are, however, serious restrictions on the homology class if we demand f to be an embedding, i.e. if the homology class can be represented by an embedded sphere. (We will call such a homology class *spherical*.)

The question of which integers appear as self-intersections of spherical classes has been studied for some time. For example, in $S^2 \times S^2$ (being diffeomorphic to the Hirzebruch surfaces \mathbb{F}_n for any even n , hence containing spheres of self-intersections n and $-n$) we have only mild homological constraints (namely the parity of the self-intersection). A similar argument shows spheres in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with arbitrarily odd negative and positive self-intersections, and similar statements hold for the blow-ups of these manifolds (which are diffeomorphic to $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ with $n \geq 2$).

If X has nontrivial Seiberg-Witten invariants [6, 8] and $b_2^+(X) > 1$, then the possibilities are much more restrictive; for example, the self-intersection of a non-torsion spherical homology class in such a four-manifold must be negative. It is an open question, however, if the negative numbers appearing as self-intersections of spheres in a four-manifold X with $b_2^+(X) > 1$ and non-trivial Seiberg-Witten invariants SW_X form a bounded set.

For the $K3$ surface, the unique smooth four-manifold (up to diffeomorphism) which is simply connected and admits a complex structure with $c_1 = 0$, so far the largest (in absolute value) negative self-intersection has been found in [2]: Finashin and Mikhalkin showed that for any even k between -2 and -86 (including the two endpoints) there is a spherical class with self-intersection k . (As the $K3$ surface is spin, self-intersections are even.) It is still an open question whether there are spherical classes in the $K3$ surface with self-intersection < -86 .

In this note we generalize the result of [2] to simply connected elliptic surface which admit a section and have $b_2^+ > 1$ (these manifolds are usually denoted by $E(n)$ with $n > 1$, and in this notation $K3$ is simply $E(2)$, cf. [3, 4]). We also extend results to blow-ups of these elliptic surfaces.

Theorem 1.1. *The four-manifold $E(n)$ with $n \geq 2$ contains a sphere with self-intersection*

$$(1.1) \quad s(n) = -44.2 \cdot n + 0.8 \cdot (5 - r)$$

where $r \in \{0, 1, 2, 3, 4\}$ is the residue of $n \bmod 5$.

This result recovers the Finashin-Mikhalkin example for $n = 2$ and generalizes it to the further $E(n)$'s.

The above examples can be modified and adapted in the blown-up elliptic surfaces. In particular, we have

Theorem 1.2. *Let $X_{n,k}$ denote the k -fold blow-up of the elliptic surface $E(n)$ (once again, with $n \geq 2$). Then $X_{n,k}$ contains a sphere of self-intersection $s(n) - 5k$.*

In some cases we find spheres in $X_{n,k}$ with self-intersection less than the value given in the above theorem (see the examples at the end of Section 3). Examining the examples we have found, we arrived to the following conjecture:

Conjecture 1.3. *There is a universal constant C with the following property. For a closed, oriented four-manifold X with $b_2^+(X) > 1$ and with nontrivial Seiberg-Witten invariant SW_X , and $S \subset X$ smoothly embedded sphere in X , the self-intersection $[S]^2$ satisfies the inequality*

$$[S]^2 \geq C \cdot b_2(X).$$

The best candidate for C so far (based on examples in elliptic surfaces and their blow-ups) is $C = -5$. Note that by [7] the condition $SW_X \neq 0$ on the Seiberg-Witten invariants is satisfied by all symplectic four-manifolds with $b_2^+ > 1$.

Remark 1.4. *We can also consider locally flat maps $f: S^2 \rightarrow X$, for which the situation is drastically different: as we disregard the smooth structure of X in this way, it is not hard to see that once X is indefinite, we can have arbitrarily large (in absolute value) positive and negative self-intersections of spheres. The argument rests on the fact that both S^2 -bundles over S^2 ($S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$) contain spheres of arbitrarily large positive and negative self-intersections, and indefinite simply connected topological manifolds admit one of the above bundles as direct summands. The question still remains interesting for definite four-manifolds, though. See [1] for self-intersections of smooth spheres in the definite smooth four-manifolds $\#n\mathbb{CP}^2$.*

The question of minimal self-intersections of spheres can be extended to arbitrary genus: fix a genus g and consider those homology classes in the closed, oriented, smooth four-manifold X with $b_2^+(X) > 1$ (and with non-trivial SW_X) which can be represented by a smoothly embedded genus- g surface. The adjunction inequality provides an upper bound for the self-intersection of such surfaces; we did not find examples of arbitrary negative self-intersections (once again, with fixed g).

Acknowledgements: AS was partially supported by the *Élvonal (Frontier) grant* KKP126683 of the NKFIH. ZSz was partially supported by NSF Grant DMS-1904628. We would like to thank William Ballinger for enlightening discussions.

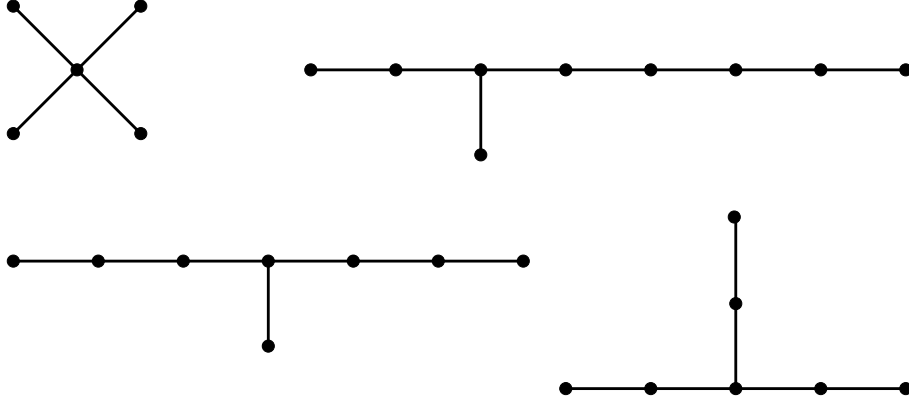


FIGURE 1. The plumbing graphs of the singular fibers I_0^* (upper left) and \tilde{E}_j (with $j+1$ vertices) for $j = 6, 7, 8$ in an elliptic fibration. In the plumbing graphs all vertices represent spheres and come with self-intersection -2 . The monodromies are: $(ab)^3$ (for I_0^*), $(ab)^5$ (for \tilde{E}_8), $(ab)^4a$ (for \tilde{E}_7) and $(ab)^4$ (for \tilde{E}_6).

2. ELLIPTIC FIBRATIONS AND MONODROMIES

Suppose that X is a compact complex surface and $f: X \rightarrow \mathbb{CP}^1$ is a holomorphic map. The map f is an *elliptic fibration* if for a generic $t \in \mathbb{CP}^1$ the fiber $f^{-1}(t)$ over t is an elliptic curve (topologically a two-dimensional torus). A *section* of an elliptic fibration is a map $p: \mathbb{CP}^1 \rightarrow X$ satisfying $f \circ p = id_{\mathbb{CP}^1}$. A complex surface is an elliptic surface if it admits an elliptic fibration.

Singular fibers in an elliptic fibration were classified by Kodaira (see [5]). In the following we will not need all possible singular elliptic fibers; we restrict our attention to those which will be useful in proving our main results. Indeed, the singular fibers \tilde{E}_j (with $j = 6, 7, 8$) and the singular fiber I_0^* can be given by the plumbing diagrams of Figure 1.

Elliptic surfaces were classified by Kodaira. Simply connected elliptic surface admitting a section can be constructed as follows: consider two cubic curves in the complex projective plane \mathbb{CP}^2 so that every intersection point is a smooth point of at least one of the curves. Consider the pencil generated by these curves. (If the two curves are given as zero sets of the cubic polynomials p_0 and p_1 , then the pencil is the family of curves defined by the cubic polynomials $p_t = t_0 p_0 + t_1 p_1$ with the projective parameter $t = [t_0 : t_1] \in \mathbb{CP}^1$.) The nine-fold (possibly infinitely close) blow-up of the pencil on \mathbb{CP}^2 provides an elliptic fibration on $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. Taking n -fold fiber sums of this surface with itself, we get $E(n)$, and up to complex deformation (according to Kodaira's classification) in this way we get all simply connected elliptic surfaces admitting sections, cf. also [4]. This construction does not provide all the elliptic *fibrations* on these smooth four-manifolds, though: not every fibration on $E(n)$ can be described as a fiber sum of fibrations on $E(1)$.

Consider a disk in \mathbb{CP}^1 containing some points with the property that their inverse image is not a regular fiber (those are called singular fibers). Assume furthermore

that the boundary of the disk does not contain any such points. Then the restriction of the fibration to this boundary circle is a torus bundle over the circle, which can be described by its monodromy, an element of the mapping class group of the torus, isomorphic to the group $\mathrm{SL}_2(\mathbb{Z})$. The monodromy of the simplest singular fiber (coming from a nodal curve) corresponds to a right-handed Dehn twist along a simple closed curve in the torus, while more complicated singular fibers (or more singular fibers in a disk) give rise to more complicated words in $\mathrm{SL}_2(\mathbb{Z})$.

The group $\mathrm{SL}_2(\mathbb{Z})$ admits a presentation as

$$\langle a, b \mid aba = bab, (ab)^6 = 1 \rangle,$$

where a and b can be chosen to be right-handed Dehn twists along two simple closed curves in the torus intersecting each other transversely in a single point. Furthermore, the \tilde{E}_8 fiber has monodromy $(ab)^5$, the \tilde{E}_6 fiber has monodromy $(ab)^4$, while the I_0^* fiber has monodromy $(ab)^3$. (More on these singular fibers see [5, pages 35-36].)

For a generic choice of cubic curves, the blow-up of the corresponding pencil provides the monodromy presentation $(ab)^6$ on $E(1)$. For $E(n)$ the corresponding presentation is $(ab)^{6n}$.

Proof of Theorem 1.1. Suppose that $n = 5k + r$, with $r \in \{0, \dots, 4\}$. Then the monodromy $(ab)^{6n}$ is equal to $(ab)^{30k+6r} = ((ab)^5)^{6k}(ab)^{6r}$. Write $6r$ as

- 0 when $r = 0$
- $3 + 3$ when $r = 1$
- $5 + 4 + 3$ when $r = 2$
- $5 + 5 + 5 + 3$ when $r = 3$, and
- $5 + 5 + 5 + 5 + 4$ when $r = 4$.

With these decompositions, the monodromy presentation equips $E(n)$ with a fibration containing $6k$ \tilde{E}_8 -fibers and some possible further \tilde{E}_8 -fibers (for $r = 2, 3, 4$, corresponding to the 5's in the decomposition of $6r$) one or two I_0^* fibers (corresponding to the 3's in the decomposition of $6r$) and possibly one further \tilde{E}_6 -fiber (for $r = 2, 4$). Together with a section of the fibration (which is a sphere of self-intersection $-n$, intersecting each fiber in a single point) these fibers give rise to a configuration of (-2) spheres (with the exception of the section, which has self-intersection $-n$) intersecting each other transversely according to a tree. Orienting them so that all intersections are negative (which is possible, as a tree is two-colorable), and then smoothing the intersections, we get a sphere with the desired self-intersection, concluding the proof. \square

Remark 2.1. *This construction essentially generalizes the proof given for the K3 surface in [2], although they decomposed 12 as $4 + 4 + 4$ (and not as $5 + 4 + 3$ as above), receiving a more symmetric diagram in [2, Figure 3].*

3. BLOWN UP ELLIPTIC SURFACES

Suppose that the four-manifold X contains a sphere of self-intersection $-m$. As $\overline{\mathbb{CP}}^2$ contains a (-4) -sphere, we can easily find in the k -fold blow-up $X \# k\overline{\mathbb{CP}}^2$ a

sphere of self-intersection $-m - 4k$ by tubing the spheres in the components of the connected sum decomposition together.

In some cases, however, this construction can be significantly improved.

Proposition 3.1. *Suppose that X contains two spheres transversely intersecting each other in a single point, and having self-intersections x and y . Then in the k -fold blow-up $X \# k\overline{\mathbb{CP}}^2$ there is an embedded sphere with self-intersection $x + y - 2 - 5k$.*

Proof. Indeed, consider the two spheres in X and blow up their (unique) intersection point. Together with the exceptional divisor we get now three embedded spheres of self-intersections $x - 1$, -1 and $y - 1$, intersecting each other along a linear graph. Blowing up edges of this graph (or, phrased differently, intersection points of the spheres), smoothing the intersection points (with appropriate orientations), a simple count verifies the claim. \square

Proof of Theorem 1.2. In an elliptic surface we construct the sphere of self-intersection $s(n)$ by finding a tree of embedded spheres, and then smoothing their intersection points. By skipping one smoothing we get the configuration of two transversely intersecting spheres as demanded by Proposition 3.1, hence the application of the proposition concludes the proof. \square

For the blown-up elliptic surfaces we can use other singular fibers to blow up, providing further negative spheres. Indeed, a cusp fiber (also called type II fiber in the table of [5, page 35]) is a singular sphere in an elliptic fibration, which admits one singular point modelled by the cone on the trefoil knot. The monodromy of a cusp fiber is equal to ab . We get a configuration of smooth curves intersecting each other transversally (and without triple intersections) after blowing up the cusp point three times. In these blow-ups the fiber will become a smooth sphere of self-intersection (-6) , connected to a sphere of self-intersection (-1) , which is connected to a (-2) - and a (-3) -sphere. In a similar way, we can use a type III fiber (with monodromy aba), which consists of two (-2) -curves tangent to each other. After two blow-ups this fiber transforms to a configuration of a (-1) -sphere, intersected by two (-4) -spheres and a (-2) -sphere. Sometimes a type IV fiber provides the best result; this fiber consists of three (-2) -spheres passing through the same point, and has monodromy $(ab)^2$. Blowing up the common intersection of the three spheres once, we get a configuration of four spheres, a central (-1) -sphere intersected by three (-3) -spheres.

Another way to get an embedded sphere proceeds as follows. Suppose that in the elliptic surface $E(n)$ we have an embedded sphere transversely intersecting a cusp fiber in a single point. The neighbourhood N_c of the cusp point is modelled by the cone on the trefoil knot. The same local model occurs near the singular point of the cuspidal cubic curve

$$C = \{[x : y : z] \in \mathbb{CP}^2 \mid x^3 = y^2 z\}.$$

Therefore we can glue $E(n) \setminus N_c$ with the complement of the similar neighbourhood of the cusp point of C — as we use an orientation reversing diffeomorphism between the boundaries, we find a sphere in $E(n) \# \overline{\mathbb{CP}}^2$, the blow-up of $E(n)$. The resulting

sphere has self-intersection -9 , still transversely intersecting the sphere in $E(n)$ in a unique point.

For $X_{n,k} = E(n) \# k\overline{\mathbb{CP}}^2$ we can follow a mixed strategy of constructing spheres: start with the monodromy presentation of $E(n)$ as $(ab)^{6n}$, separate a few cusp (or type III or IV) fibers (occupying some of the monodromy) for the blow-ups, partition the rest of the blow-ups to get \tilde{E}_8 - or \tilde{E}_6 -fibers (taking $(ab)^5$ or $(ab)^4$ in the monodromy), blow up the singular points of the cusp (or type III or IV) fibers sufficiently many times, and apply the method of Proposition 3.1 for the leftover blow-ups.

As examples, we construct spheres in $E(2) \# \overline{\mathbb{CP}}^2$, in $E(6) \# \overline{\mathbb{CP}}^2$, and in $E(6) \# 3\overline{\mathbb{CP}}^2$ using one or more of the strategies outlined above.

Example 3.2 (Spheres in $E(2) \# \overline{\mathbb{CP}}^2$). *Considering the monodromy of $E(2)$ as $(ab)^5 \cdot (ab)^5 \cdot (ab)^2$, we get an elliptic fibration on the $K3$ surface with three singular fibers, two \tilde{E}_8 and one type IV. Blowing up the singular point of the type IV fiber, we get a tree of spheres with 23 vertices, 19 of them (-2) -spheres, one (-1) and three (-3) . A simple computation shows the existence of a sphere of self-intersection -92 . (This construction gives a slightly better result than blowing up an edge in the tree of spheres giving the (-86) -sphere in the $K3$ surface, which results a (-91) -sphere.)*

Example 3.3 (Spheres in $E(6) \# \overline{\mathbb{CP}}^2$). *As it was shown in Theorem 1.1, there is a sphere of self-intersection -262 in $E(6)$; its connected sum with the (-4) -sphere in $\overline{\mathbb{CP}}^2$ gives a (-266) -sphere. As the (-262) -sphere is given by a tree, hence can be viewed as smoothing of two spheres intersecting transversally once, by blowing up the intersection point of these spheres we get a (-267) -sphere in $E(6) \# \overline{\mathbb{CP}}^2$. Consider now a slightly different construction: take a fibration on $E(6)$ with 7 \tilde{E}_8 -fibers and a cusp (as the total monodromy of $E(6)$ is $(ab)^{36} = ((ab)^5)^7 \cdot (ab)$). Use the seven \tilde{E}_8 -fibers (together with the section) to create a (-258) -sphere in $E(6)$ intersecting the unique cusp fiber in one transverse point. Now summing it with the cuspidal cubic in $\overline{\mathbb{CP}}^2$ we get a (-9) -curve, and smoothing the intersection with the appropriate orientations we finally receive a sphere in $E(6) \# \overline{\mathbb{CP}}^2$ of self-intersection -269 .*

Example 3.4 (Spheres in $E(6) \# 3\overline{\mathbb{CP}}^2$). *By presenting the (-262) -sphere in $E(6)$ with a plumbing graph and blowing up edges there, we get a (-277) -sphere in the three-fold blow-up. There is a further possibility: consider the fibration with 7 \tilde{E}_8 -fibers and a cusp fiber, and blow up the cusp three times. Simple calculation shows that the resulting sphere has self-intersection -278 . A better result can be achieved by taking the (-269) -sphere of Example 3.3 in $E(6) \# \overline{\mathbb{CP}}^2$ and blow it up twice (as done in Proposition 3.1) to get a sphere with self-intersection -279 .*

Notice that all the above examples satisfy Conjecture 1.3; the ratio of the (negative of the) self-intersection and b_2 is significantly less than 5. So far we found examples with this ratio being close to 5 only after a significant amount of blow-ups.

REFERENCES

- [1] W. Ballinger, *Configurations of spheres in $\#n\mathbb{CP}^2$* . in preparation, 2021.
- [2] S. Finashin and G. Mikhalkin, *A (-86) -sphere in the $K3$ surface*. Turkish J. Math., **21** (1997), 129–131.
- [3] R. Friedman and J. Morgan, *Smooth four-manifolds and complex surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete **27**, Springer-Verlag, Berlin, 1994.
- [4] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics **20** (1999), American Mathematical Society, Providence, RI,
- [5] J. Harer, A. Kas and R. Kirby, *Handlebody decompositions of complex surfaces*. Mem. Amer. Math. Soc. **62** (1986), no. 350.
- [6] J. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*. Mathematical Notes **44**, Princeton University Press, Princeton, NJ, 1996.
- [7] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*. Math. Res. Lett. **1** (1994), 809–822.
- [8] E. Witten, *Monopoles and four-manifolds*. Math. Res. Lett. **1** (1994), 769–796.

RÉNYI INSTITUTE OF MATHEMATICS, H-1053 BUDAPEST, REÁLTANODA UTCA 13–15, HUNGARY

Email address: `stipsicz.andras@renyi.hu`

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ, 08544

Email address: `szabo@math.princeton.edu`