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Traces of links and simply connected 4-manifolds

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Abstract

We study the set $\widehat{\mathcal{S}}_M$ of framed smoothly slice links which lie on the boundary of the complement of a 1-handlebody in a closed, simply connected, smooth 4-manifold M. We show that $\widehat{\mathcal{S}}_M$ is well defined and describe how it relates to exotic phenomena in dimension four. In particular, in the case when X is a smooth 4-manifold-with-boundary, with a handle decompositions with no 1-handles and homeomorphic to but not smoothly embeddable in D^4, our results tell us that X is exotic if and only if there is a link $L \hookrightarrow S^3$ which is smoothly slice in X, but not in D^4. Furthermore, we extend the notion of high genus 2-handles attachment, introduced by Hayden and Piccirillo, to prove that exotic 4-disks that are smoothly embeddable in D^4, and therefore possible counterexamples to the smooth 4-dimensional Schönflies conjecture, cannot be distinguished from D^4 only by comparing the slice genus functions of links.

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1 | INTRODUCTION

The smooth 4-dimensional Poincaré conjecture 4SPC (asserting that any smooth 4-manifold homeomorphic to the 4-sphere S^4 is diffeomorphic to it) is one of the most important and well-studied problems in topology. The difficulty of the problem stems from the fact that

currently we do not have smooth invariants directly applicable to 4-manifolds homotopy equivalent to S^4.

Indeed, the smooth invariants distinguishing exotic structures on closed 4-manifolds seem to govern the minimal genera of surfaces representing fixed second homology classes; obviously this distinction can only work if the manifolds at hand have rich second homology. The idea can be modified to manifolds with boundary: Consider knots (or links) in the boundary 3-manifold and study their 'slice' genera, that is, the genera of smooth surfaces embedded in the 4-manifold with the fixed knot or link as boundary. This approach can be used even for closed manifolds by deleting a small open disk neighbourhood of a point and working with the resulting 4-manifold with S^3 boundary. The first question along these lines is the existence of a knot or link which has different slice genus for different smooth structures on the 4-manifold. The next question is how to detect these different genera. In this work we will focus on the first question above, and see some conditions under which this strategy cannot work, and others where the desired knot or link does exist. (There are important recent developments regarding invariants of knots and links in S^3 and in more general 3-manifolds potentially detecting exoticness in this way, see [12], for example.)

It is known that the existence of an exotic 4-sphere (a counterexample to 4SPC) is equivalent to the existence of an exotic 4-disk, that is, a smooth 4-manifold with boundary S^3 which is homeomorphic to the 4-disk D^4, but not diffeomorphic to it. For such 4-manifolds we have the hope to be able to distinguish them based on the feature that knots (and links) might have different slice genera in them, see [4, 12, 13]. This observation leads us to the following definition. (We fix the convention that every manifold is oriented and diffeomorphisms, denoted by $\cong$, between manifolds always preserve the given orientations.)

Definition 1.1. Suppose that X is a compact, simply connected, smooth 4-manifold with $\partial X = S^3$. Let $\mathcal{S}_X$ denote the set of smoothly slice links in X, that is, the set of those links in S^3, which bound smoothly, disjointly and properly embedded 2-disks in X.

In trying to understand potential exotic D^4s, we can divide them into two groups (as it has been done for exotic $\mathbb{R}^4$s).

Definition 1.2. Suppose that X is a possibly exotic 4-disk. X is ***small***, if there is a smooth embedding $f : X \hookrightarrow D^4$; otherwise X is ***large***.

Our first observation shows that small exotic 4-disks cannot be detected using slice links.

Proposition 1.3. *Suppose that X is a small exotic 4-disk. Then $\mathcal{S}_X$ and $\mathcal{S}_{D^4}$ are equal.*

Remark 1.4. The existence of a small exotic 4-disk is equivalent to the failure of the well-studied smooth 4-dimensional Schönflies conjecture, asserting that a smoothly embedded 3-sphere in S^4 bounds a smoothly embedded D^4.

Based on the result of Proposition 1.3 one can hope that large exotic D^4s can be detected by their set of slice links. Indeed, this result applies for a special class of exotic disks. Recall that a 4-manifold is *geometrically simply connected* if it admits a handle decomposition without 1-handles. Obviously, geometrically simply connected manifolds are simply connected, but the converse does not hold. For example, a compact, contractible 4-manifold X with boundary

$\partial X \neq S^3$ (according to a result of Casson, see [8]) is never geometrically simply connected. The question whether simple connectivity implies geometrical simple connectivity for closed 4-manifolds is wide open.

Suppose that L is an n-component link. Let $S^3_{(0,\dots,0)}(L)$ denote the 3-manifold we get by performing 0-surgery on each component of L. Let $X(L)$ denote the corresponding surgery *trace*, that is, the 4-manifold given by attaching 0-framed 2-handles along the components of L.

There is a simple way to construct an exotic 4-sphere once two knots K_1 and K_2 with diffeomorphic 0-surgery are given, where one of the knots (say K_1) is smoothly slice, while K_2 is not. Indeed, glue the complement of the slice disk of K_1 to the 0-trace $X(K_2)$; the application of the trace embedding lemma (cf. Lemma 2.3) shows that the result is exotic. This construction, which is used in [13], admits a natural generalization to links, where Lemma 2.3 provides the simple extension of the usual trace embedding lemma to links. As the next results show, this construction is sufficient to produce all geometrically simply connected, large, exotic 4-disks.

Theorem 1.5. *Suppose that $L \hookrightarrow S^3$ is an n-component link such that $S^3_{(0,\dots,0)}(L) \cong \#^n S^1 \times S^2 = Y$ and consider the smooth 4-manifold given by $S = X(L) \cup_Y \natural^n S^1 \times D^3$. Then S is a geometrically simply connected homotopy 4-sphere. Furthermore, every geometrically simply connected exotic 4-sphere S is constructed in this way and S is exotic if and only if $X(L)$ is not diffeomorphic to $\natural^n S^2 \times D^2$.*

As it was indicated above, for a geometrically simply connected large exotic 4-disk X the set $\mathcal{S}_X$ is sufficient to verify its exoticness.

Theorem 1.6. *A geometrically simply connected 4-disk X is small if and only if $\mathcal{S}_X = \mathcal{S}_{D^4}$. Consequently, a geometrically simply connected exotic 4-disk X is large if and only if there is a link $L \hookrightarrow S^3$ that is smoothly slice in X, but not in D^4. In particular, a large, geometrically simply connected exotic 4-disk X (if it exists) can be distinguished from the usual 4-disk D^4 through $\mathcal{S}_X$.*

A condition for the existence of a large exotic D^4 is given in terms of sliceness in the 4-manifold $\natural^k S^2 \times D^2$ (see Theorem 4.4).

The definition of the set $\mathcal{S}_X$ can be extended to contain links with various framings, and even in other 3-manifolds (such as $\#^k S^1 \times S^2$) naturally associated to X, providing the set of framed links $\widehat{\mathcal{S}}_X$. We will discuss these extensions (and the precise description of $\widehat{\mathcal{S}}_X$) in Section 4, leading us to the following result.

Theorem 1.7. *If the smooth 4-dimensional Poincaré conjecture holds then the set $\widehat{\mathcal{S}}_X$ always determines the diffeomorphism type of a closed, simply connected, smooth 4-manifold X.*

A further refinement of the set of slice links is to consider a (slice) genus function on the set of links in the boundary of a 4-manifold. This naturally extends the usual genus function of a closed 4-manifold to the setting of 4-manifolds with boundary, and this concept can be interesting even in the case when the manifold at hand has no second homology. This approach requires the extension of the trace embedding lemma to be relevant in this context; in particular, we will attach higher genus handles (as it has been already considered in [7]) along framed knots and links, and will consider higher order traces (when the core surfaces of the handles have potentially multiple boundary components).

Equipped with these tools, we can show that even the *slice genus* g_4 and the *maximal 4-dimensional Euler characteristic* χ_4, the latter defined as the maximum Euler characteristic of a compact, oriented properly and smoothly embedded surface Σ, such that $L = \partial\Sigma$ and every connected component of Σ has boundary in L, cannot detect small exotic 4-disks.

Theorem 1.8. *If X is a small exotic 4-disk and L is a link in S^3 then $g_4(L) = g_4^X(L)$ and $\chi_4(L) = \chi_4^X(L)$, where g_4^X and χ_4^X denote the genus functions in X, with the convention that for $X = D^4$ we omit the superscript.*

The paper is composed as follows: In Section 2 we list a few preliminary results, and show that small exotic 4-disks cannot be detected using sliceness of links. In Section 3 we focus on geometrically simply connected exotic 4-disks, and in Section 4 we extend the notion of S_X to framed links and show that (assuming 4SPC) these sets characterize closed, simply connected, smooth 4-manifolds. Finally, in Section 5 we discuss the genus function, and show that even this invariant cannot distinguish a small exotic 4-disk from D^4.

2 | PRELIMINARIES

It follows from the Cerf–Palais lemma [2, 16] that in a closed, connected, smooth (resp. topological) n-manifold M all smooth (resp. locally flat) embeddings $D^n \hookrightarrow M$ are isotopic. This statement then easily implies that for a smooth embedding $\iota : D^4 \hookrightarrow S^4$ we have $S^4 \setminus \iota(\mathring{D}^4) \cong D^4$. This result shows that for every pair of embedded D^4 in a given homotopy 4-sphere there is a diffeomorphism that sends one into the other. This observation immediately leads to a relation between exotic 4-disks and homotopy 4-spheres; here, for an exotic 4-disk X, we denote $\widehat{X}$ the exotic 4-sphere which is gotten by gluing a 4-handle to X.

Proposition 2.1. *There is a one-to-one correspondence between exotic 4-disks and homotopy 4-spheres, in the sense that two exotic 4-disks X_1 and X_2 are diffeomorphic if and only if the same is true for $\widehat{X_1}$ and $\widehat{X_2}$.*

Proof. If $X_1 \cong X_2$ then clearly $\widehat{X_1} \cong \widehat{X_2}$ because there is a unique way to glue a 4-handle to a manifold with S^3 as boundary. The other implication follows from our observation above. □

In accordance with the identification given in Proposition 2.1, we also say that a homotopy 4-sphere $\widehat{X}$ is small or large when the corresponding X is. If $X \hookrightarrow S^4$ is a small exotic 4-disk then the same is true for $S^4 \setminus \mathring{X}$. Hence, we can put the structure of an abelian group on the set of small homotopy 4-spheres up to diffeomorphism, where the group operation is given by the boundary sum $\natural$. Very little is known about this group, except that it has at most a countable number of elements, see [6].

The following is a simple, yet useful generalization of Proposition 2.1.

Lemma 2.2. *Suppose that X_1 and X_2 are two compact smooth 4-manifolds with $\partial X_1 \cong \partial X_2 \cong \#^n S^1 \times S^2$ for some n; and denote the manifold obtained by gluing $\natural^n S^1 \times D^3$ to X_i for $i = 1, 2$ by M_i. Then, if M_i are simply connected and satisfy $M_1 \cong M_2$, then $X_1 \cong X_2$.*

Proof. The complements of X_1 and of X_2 in $M = M_1 (\cong M_2)$ are both neighbourhoods of bouquets of circles (of the same number, as this number is equal to n). By an isotopy of M we can arrange that the two bouquets have the same 0-cell. As M is simply connected, the circles of the bouquets are homotopic to each other. In this dimension, however, homotopy implies isotopy [14], ultimately providing an isotopy from X_1 to X_2, verifying the claim. □

The trace embedding lemma is one of the most crucial connections between sliceness properties of knots/links and exotic structures. The version of this lemma for knots is rather well known; here we discuss a straightforward extension to links.

Lemma 2.3 (Trace embedding lemma for links). *A link L in S^3 is smoothly slice in a possibly exotic* 4*-disk X if and only if $X(L^*) \hookrightarrow \widehat{X}$, where $\widehat{X}$ is the homotopy* 4*-sphere obtained by attaching a* 4*-handle to X, L^* is the mirror image of L and $X(L^*)$ is the* 0*-trace of L^*.*

Proof. Suppose that L is smoothly slice in X, which means that each component L_i of L bounds a properly embedded disk D_i in X and $D_i \cap D_j = \emptyset$ for $i \neq j$. Take $\widehat{X}$ and one of its handle decompositions in the way that $\widehat{X} = D^4 \cup_{S^3} X$. Hence, we can view L^* in $\partial D^4 = S^3$ and then $D^4 \cup \nu(D_1) \cup \ldots \cup \nu(D_n) \cong X(L^*) \hookrightarrow X$. Note that since the D_is are disjoint, we can assume the same for the tubular neighbourhoods $\nu(D_i)$s.

We now assume that $X(L^*) \hookrightarrow \widehat{X}$. We have that $\widehat{X} \cong X(L^*) \cup_f W$, where $W = \widehat{X} \setminus X(\mathring{L}^*)$ and $f : \partial X(L^*) \to \partial X(L^*)$ is the orientation-reversing diffeomorphism which acts as gluing map. Moreover, we can consider the handle decomposition on $X(L^*)$ given by $D^4 \cup \{\text{2-handles}\}$, where the 2-handles are attached along L^* with framing 0.

We see that the link L^* sits in $S^3 = \partial D^4$ and it bounds the cores of the 2-handles inside $X(L^*)$, which are embedded disks with boundary on S^3. Since $X \cong \widehat{X} \setminus \mathring{D}^4$, we obtain precisely that $L = (L^*)^*$ bounds a collection of disjoint properly embedded disks in X, hence L is slice. □

Remark 2.4. There is also a locally flat version of the trace embedding lemma which states that L is topologically slice if and only if $X(L)$ is a locally flat topological submanifold of S^4. Its proof proceeds in the exact same way as the smooth case.

With these preparations at hand, we are ready to verify that the set of slice links will not help in detecting small exotic 4-disks.

Proposition 2.5. *If two possibly exotic* 4*-disks are such that $X_1 \hookrightarrow X_2$ then $\mathcal{S}_{X_1} \subset \mathcal{S}_{X_2}$.*

Proof. We can apply Proposition 2.1 to restate the trace embedding lemma as L is smoothly slice in X if and only if $X(L^*) \hookrightarrow X$: in fact, since $X(L^*)$ is not closed it cannot coincide with $\widehat{X}$; and then there is a neighbourhood of a point (a smooth 4-disk) in $\widehat{X} \setminus X(L^*)$. Therefore, we have $X(L^*) \hookrightarrow X_1 \hookrightarrow X_2$ whenever L is smoothly slice in X_1. □

Proof of Proposition 1.3. We have $D^4 \hookrightarrow X \hookrightarrow D^4$, implying $\mathcal{S}_{D^4} \subset \mathcal{S}_X \subset \mathcal{S}_{D^4}$, which concludes the proof. □

3 | TRACES OF LINKS AND HOMOTOPY 4-SPHERES

It is known [14] that every compact, simply connected, smooth n-manifold with $n \geqslant 5$ is geometrically simply connected; in other words, it admits a handle decomposition where there are no 1-handles. On the other hand, whether the same holds in dimension four is still unknown, even for homotopy 4-spheres and disks. Next we turn to the proof of our result on geometrically simply connected possibly exotic 4-disks.

Proof of Theorem 1.5. First, we observe that S is necessarily a homotopy 4-sphere. In fact, since it is closed and simply connected, by Freedman's classification result [3] we only need to check that $H_2(S;\mathbb{Z}) \cong \{0\}$, but this follows from the observation that

$$b_2(S) = \chi(S) - 2 = \#|\text{2-handles}| - \#|\text{3-handles}| = n - n = 0.$$

If S is exotic and $X(L) \cong \natural^n S^2 \times D^2$ then we have a contradiction because S would have a Kirby presentation consisting of just some $(2,3)$-cancelling pairs. Let us assume now that $S \cong S^4$. Then we can apply Lemma 2.2 to prove that $X(L)$ is diffeomorphic to $\natural^n S^2 \times D^2$ and this shows the other implication.

Finally, given a geometrically simply connected homotopy 4-sphere, we take L as the framed link which presents its 2-handles. Then L has as many components as there are 3-handles because of Euler characteristic, and framings zero as $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}^{|L|}$. □

This result leads us to examine exoticness on $S^2 \times D^2$ as well.

Theorem 3.1. *There is a one-to-one correspondence between exotic* 4*-spheres and exotic* $S^2 \times D^2$*s, up to diffeomorphism. In particular, the smooth* 4*-dimensional Poincaré conjecture is equivalent to the existence of an exotic* $S^2 \times D^2$*.*

Proof. Given an exotic $S^2 \times D^2$ (say X), we obtain a homotopy 4-sphere S by gluing $S^1 \times D^3$ to X; and S is exotic because of Lemma 2.2. If $X \cong X'$ then obviously $S \cong S'$, while the converse is again true by Lemma 2.2.

To see that this identification is surjective we start by an exotic homotopy 4-sphere S; then we consider a handle decomposition of S and we take X as the manifold obtained by removing one 3-handle and one 4-handle from S. The fact that such an X is homeomorphic to $S^2 \times D^2$ follows from the argument we used in the previous paragraph. □

Our goal now is to prove the converse of Proposition 2.5 for geometrically simply connected exotic 4-disks.

Proposition 3.2 (Fake $(2,3)$-cancelling pair). *Given a compact, simply connected, smooth* 4*-manifold* X *with* $\partial X = S^3$*, admitting a handle decomposition given by* $X = W \cup_{\#^k S^1 \times S^2} \{k\ 3-$ *handles*$\}$*, we have that* $X\ \natural^k S^2 \times D^2$ *is diffeomorphic to* W*.*

Proof. It is a direct application of Lemma 2.2. In fact, both W and $X\ \natural^k S^2 \times D^2$ become diffeomorphic to the closed and simply connected 4-manifold $\widehat{X}$ after attaching $\natural^k S^1 \times D^3$. □

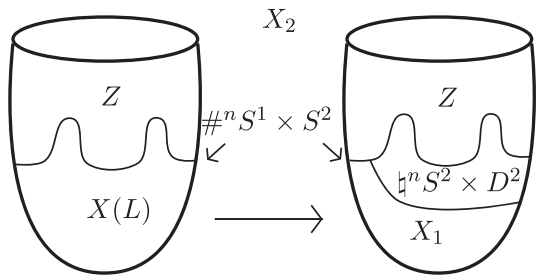

FIGURE 1 Since $X_1 \natural^n S^2 \times D^2 \cong X(L)$, we can surger $X(L)$ from X_2 and then glue it back using the diffeomorphism.

Theorem 3.3. *Two possibly exotic geometrically simply connected* 4*-disks* X_1, X_2 *satisfy* $X_1 \hookrightarrow X_2$ *if and only if* $\mathcal{S}_{X_1} \subset \mathcal{S}_{X_2}$. *In particular, a geometrically simply connected* 4*-disk* X *is small if and only if* $\mathcal{S}_X = \mathcal{S}_{D^4}$.

Proof. We only need to prove the 'if' implication because of Proposition 2.5. Let us consider an n-component link L which presents the 2-handles of X_1; since $X(L) \hookrightarrow X_1$ one has that L^* is smoothly slice in X_1, and by assumption, also in X_2. Using the trace embedding lemma again, we then obtain that $X(L) \hookrightarrow X_2$.

We saw that $X_2 = X(L) \cup_{\#^n S^1 \times S^2} Z$, where $Z = X_2 \setminus \mathring{X(L)}$. We recall that Proposition 2.1 assures us that we can take $X(L) \cap \nu(\partial X_2) = \emptyset$. Now we use Proposition 3.2 to claim that $X_1 \natural^n S^2 \times D^2 \cong X(L)$; hence, we have $X_2 = (X_1 \natural^n S^2 \times D^2) \cup_{\#^n S^1 \times S^2} Z$ which means $X_1 \hookrightarrow X_2$. See Figure 1. □

A few observations are in place. The first is that the latter statement necessarily requires links: In fact, in the proof we use the fact that a geometrically simply connected exotic 4-disk has a Kirby presentation which consists of a 0-framed link L. Such an L cannot be a knot because by Gabai's theorem [5] it would be the unknot. The second observation is that, in the proof of Theorem 3.3, we did not actually use that every link in $\mathcal{S}_{X_1}$ is contained in $\mathcal{S}_{X_2}$, but just that a link which presents the 2-handles of X_1 is. This is useful to prove the following corollary.

Corollary 3.4. *Every geometrically simply connected large exotic* 4*-disk* X *is obtained by attaching* n 3*-handles on* $X(J)$ *for some* n*-component link* J*, not smoothly slice in* D^4 *and such that* $S^3_{(0,\dots,0)}(J) \cong \#^n S^1 \times S^2$.

Proof. We use the previous observation in the following way: Assume the link J is also smoothly slice in D^4, then we can mimic the proof of Theorem 3.3 to show that $X \hookrightarrow D^4$. This is a contradiction because X is large. □

We finally show an even deeper relation between exotic 4-disks and exotic boundary sums of $S^2 \times D^2$s. First, we fix the notation that two such manifolds X_1 and X_2 are called *stably diffeomorphic* if $X_2 \cong X_1 \natural^n S^2 \times D^2$ for some n.

Theorem 3.5. *The number of geometrically simply connected exotic* 4*-spheres, up to diffeomorphism, coincides with the number of exotic* $\natural^i S^2 \times D^2$ *for any possible* $i \geqslant 1$ *which arise as* 0*-traces of links in* S^3*, up to stable diffeomorphism. Equivalently, this is the number of diffeomorphism types of* $S^2 \times D^2$*s which are stably diffeomorphic to* 0*-traces of links in* S^3.

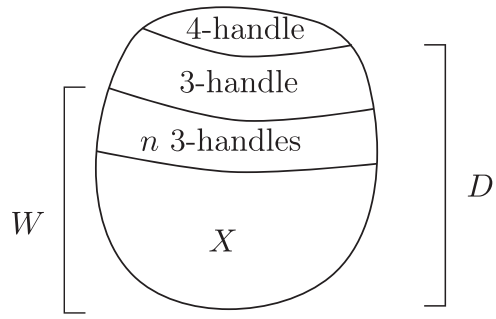

FIGURE 2 The 4-manifold S as in the proof of Lemma 3.6.

In contrast, by Wall's theorem [17] (see also [6] for more details), for every homotopy 4-sphere S there is an $N \geqslant 0$ such that

$$S \#^N(S^2 \times S^2) \cong \#^N(S^2 \times S^2).$$

The proof of Theorem 3.5 requires a preliminary lemma.

Lemma 3.6. *Every exotic $\natural^{n+1}S^2 \times D^2$ for some $n \geqslant 1$ is stably diffeomorphic to an exotic $S^2 \times D^2$.*

Proof. Let X denote our exotic $\natural^{n+1}S^2 \times D^2$. We construct a manifold W by attaching $\natural^{n+1}S^1 \times D^3$ to X, obtaining a homotopy 4-sphere S, and then removing $S^1 \times D^3$. By Lemma 2.2 there is a unique way to do this. Moreover, we call D the homotopy 4-disk $S \setminus \{4-\text{handle}\}$ (well defined from Proposition 2.1).

Observing that D is obtained from X (resp. W) by attaching $n+1$ (resp. one) 3-handles (Figure 2) and applying Proposition 3.2 twice, we have that

$$X \cong D \natural^{n+1}S^2 \times D^2 \quad \text{and} \quad W \cong D \natural S^2 \times D^2 .$$

Hence, we conclude that $X \cong W \natural^n S^2 \times D^2$ and W is an exotic $S^2 \times D^2$ because (by our hypothesis) X was exotic. □

We can now move to the proof of Theorem 3.5.

Proof of Theorem 3.5. Theorems 1.5 and 3.1 provide the identification between stable diffeomorphism types of exotic boundary sums of $S^2 \times D^2$, arising from 0-traces of links, and diffeomorphism types of geometrically simply connected exotic 4-spheres. The fact that this correspondence is well defined, and follows from Kirby calculus; moreover, we see that injectivity is a consequence of Lemma 2.2 while surjectivity follows from Theorem 1.5. To conclude, the last statement follows directly from Theorem 3.1 and Lemma 3.6. □

We obtain the following corollary.

Corollary 3.7. *Suppose that X is a possibly exotic $S^2 \times D^2$. Then X is geometrically simply connected if and only if it is stably diffeomorphic to the* 0*-trace of a link in S^3.*

Proof. We apply Theorem 3.5. If X is geometrically simply connected, then the same is true for the homotopy 4-sphere S obtained by gluing $S^1 \times D^3$ on ∂X. Conversely, if S is geometrically simply connected then there is an i such that $X \natural^i S^2 \times D^2$ is diffeomorphic to the 0-trace of a link. We saw in the proof of Lemma 3.6 that then X has a handle decomposition without 1-handles. □

Note that no exotic $S^2 \times D^2$ can be the 0-trace of a knot, but some stabilize to the 0-trace of a link in the case a geometrically simply connected exotic 4-disk exists.

4 | EMBEDDINGS OF SIMPLY CONNECTED 4-MANIFOLDS

If a smooth 4-manifold X does not admit a handle decomposition without 1-handles, then it does not seem sufficient to consider the set $\mathcal{S}_X$ of links smoothly slice in X in order to characterize X. We proceed in a more general way: First, given a closed, simply connected, smooth 4-manifold M, by Lemma 2.2 we have that the submanifold $M_k = M \setminus (\natural^k S^1 \times \mathring{D}^3)$ is well defined for every $k \geqslant 0$. Therefore, we can define the set $\widehat{\mathcal{S}}_{M_k}$ of *smoothly slice framed links* in M_k for every k. The elements of this set are N-component links $\vec{L} \hookrightarrow \#^k S^1 \times S^2$, equipped with a framing t_i for each component, such that the corresponding slice disk D_i in M_k has tubular neighbourhood whose relative Euler number agrees with t_i for $i = 1, \dots, N$. Note that $M_k \cong M_0 \natural^k S^2 \times D^2$ for every $k \geqslant 0$.

We then consider $\widehat{\mathcal{S}}_M = \bigcup_{k \geqslant 0} \widehat{\mathcal{S}}_{M_k}$ for every closed, simply connected, smooth 4-manifold M.

Remark 4.1. Note that the set $\mathcal{S}_X$, introduced earlier, coincides with $\widehat{\mathcal{S}}_{M_0} = \widehat{\mathcal{S}}_X$ where X is a possibly exotic D^4 and M the homotopy 4-sphere obtained by gluing a 4-handle to X.

We use a similar construction to extend the notion of the *trace of a framed link* $\vec{L}$ in $\#^k S^1 \times S^2$: In fact, since we can view $\vec{L}$ as embedded in $\partial(\natural^k S^1 \times D^3)$, we define $X(\vec{L})$ as the 4-manifold obtained by attaching 2-handles along $\vec{L}$, with the given framing. The notion of the trace of a framed link is well-defined, as diffeomorphic framed links in $\#^k S^1 \times S^2$ possess diffeomorphic traces. This follows from a result of Laudenbach and Poénaru in [11] telling us that every self-diffeomorphism of $\#^k S^1 \times S^2$ extends to $\natural^k S^1 \times D^3$.

We recall that the mirror image $\vec{L}^*$ of a framed link $\vec{L}$ is the mirror image of L, equipped with the framings of $\vec{L}$, after reversing their signs. We then have the following version of the trace embedding lemma.

Lemma 4.2 (Trace embedding lemma for framed links). *Let us assume that M_k is obtained from a 4-manifold M as explained before. Then a framed link $\vec{L} \hookrightarrow \#^k S^1 \times S^2$ is smoothly slice (as a framed link) in M_k if and only if $X(\vec{L}^*) \hookrightarrow M$.*

Proof. The proof proceeds in the same way as the one of Lemma 2.3. Suppose that $\vec{L}$ is smoothly slice in M_k for some $k \geqslant 0$, with slice disks $D_1, \dots, D_n$. Take a handle decomposition of M such that $M = \natural^k S^1 \times D^3 \cup_{\#^k S^1 \times S^2} M_k$. Thus we can view L^* as a link in $\partial(\natural^k S^1 \times D^3) = \#^k S^1 \times S^2$. The manifold $\natural^k S^1 \times D^3 \cup \nu(D_1) \cup \dots \cup \nu(D_n)$ is diffeomorphic to $X(\vec{L}^*)$, since each $\nu(D_i)$ can be seen as a 2-handles attached to $\natural^k S^1 \times D^3$ with framing reversed with respect to the one of $\vec{L}$.

We now assume that $X(\vec{L}^*) \hookrightarrow M$. Considering the 4-manifold upside down we obtain that L bounds a collection of mutually disjoint embedded disks in M_k; moreover, the tubular

neighbourhoods of these disks have relative Euler numbers which coincide with the framings of $\vec{L}$, because they equal the attaching framings of the 2-handles in $X(\vec{L}^*)$ with reversed signs. □

We can then prove the following generalization of Theorem 3.3.

Theorem 4.3. *Let us consider two closed, simply connected, smooth 4-manifolds M and N. Then N splits as a connected sum $M\#M'$ for some M' if and only if $\widehat{\mathcal{S}}_M \subset \widehat{\mathcal{S}}_N$. In particular, one has $\widehat{\mathcal{S}}_S = \widehat{\mathcal{S}}_{S^4}$ if and only if S is a small homotopy 4-sphere.*

Proof. If $\vec{L} \in \widehat{\mathcal{S}}_M$ then by the trace embedding lemma we have $X(\vec{L}^*) \hookrightarrow M \setminus \mathring{D}^4 \hookrightarrow N$, implying $\vec{L} \in \widehat{\mathcal{S}}_N$.

Assume now $\widehat{\mathcal{S}}_M \subset \widehat{\mathcal{S}}_N$. If we call $\vec{L}$ the framed link which presents the 2-handles in a Kirby diagram of M then $\vec{L} \hookrightarrow \natural^k S^1 \times D^3$ and $M = X(\vec{L}) \cup \natural^t S^1 \times D^3$; thus $\vec{L}^* \in \widehat{\mathcal{S}}_M \subset \widehat{\mathcal{S}}_N$. Since the trace embedding lemma implies $X(\vec{L}) \hookrightarrow N$, we conclude as in the proof of Theorem 3.3. $\widehat{\mathcal{S}}_S = \widehat{\mathcal{S}}_{S^4}$ then implies that $S_0 \subset S^4$, therefore S is a small homotopy 4-sphere. □

A corollary of this result is that the smooth 4-dimensional Schönflies theorem is equivalent to claim that S^4 is determined by its set of framed smoothly slice links. In addition, we can rewrite the smooth 4-dimensional Poincaré conjecture as follows.

Theorem 4.4. *There are no exotic 4-spheres if and only if for every n-component framed link $\vec{L} \hookrightarrow \#^k S^1 \times S^2$ with $n \geqslant 2k$, such that $S^3(\vec{L}) \cong \#^{n-k} S^1 \times S^2$ and $X(\vec{L})$ is simply connected, one has $X(\vec{L}) \cong \natural^{n-k} S^2 \times D^2$.*

Furthermore, there exists a large exotic D^4 if and only if we can find an $\vec{L}$ as before which is not smoothly slice in $\natural^k S^2 \times D^2$.

Proof. Every homotopy 4-sphere has a handle decomposition with k 1-handles, n 2-handles and $n-k$ 3-handles. By possibly multiplying the corresponding Morse function with (-1), we can assume that $n-k \geqslant k$.

The fact that all exotic spheres are diffeomorphic to S^4 is equivalent to claim that the 4-manifold given by $D^4 \cup \{1\text{-handles}\} \cup \{2\text{-handles}\}$ is always diffeomorphic to $\natural^{n-k} S^2 \times D^2$, from Lemma 2.2. It is then immediate to see that the trace of every framed link $\vec{L} \hookrightarrow \#^k S^1 \times S^2$, satisfying the hypothesis, always gives rise to such a manifold.

The second statement is then a consequence of the first one, Theorem 4.3 and its proof. □

If M has a Kirby presentation with exactly k 1-handles, we do not actually need to check the entire $\widehat{\mathcal{S}}_M$.

Proposition 4.5. *Let us assume that M is a 4-manifold as before and k is the minimum number of 1-handles in a Kirby diagram for M. Then we have that $\widehat{\mathcal{S}}_M$ is determined by $\widehat{\mathcal{S}}_{M_k}$, in the sense that if there is another manifold N with $\widehat{\mathcal{S}}_{M_k} \subset \widehat{\mathcal{S}}_{N_k}$ then $\widehat{\mathcal{S}}_M \subset \widehat{\mathcal{S}}_N$.*

Proof. Take a framed link in $\#^k S^1 \times S^2$ that presents the 2-handles in a Kirby diagram for M. Applying the same argument in the proof of Theorem 4.3 yields $M_0 = M \setminus \mathring{D}^4 \hookrightarrow N$, but then Theorem 4.3 also leads to $\widehat{\mathcal{S}}_M \subset \widehat{\mathcal{S}}_N$. □

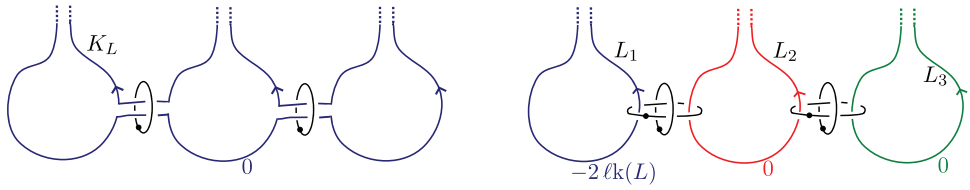

FIGURE 3 The 0-trace of the knot K_L, the knotification of L, seen on the boundary of $\sharp^{\ell-1}S^1\times D^3$ (left) and an equivalent Kirby diagram of it (right).

When M and N are such that $\widehat{\mathcal{S}}_M=\widehat{\mathcal{S}}_N$, from Theorem 4.3 we have $N\cong M\#S$, where S is a homotopy 4-sphere. Hence, we obtain Theorem 1.7.

Proof of Theorem 1.7. If $M\cong N$ then obviously one has $\widehat{\mathcal{S}}_M=\widehat{\mathcal{S}}_N$. Conversely, because of the observation above, when $\widehat{\mathcal{S}}_M=\widehat{\mathcal{S}}_N$ we have $N\cong M\#S$, but the manifold S has to be diffeomorphic to S^4, since we are assuming that 4SPC holds. □

5 | HIGH-ORDER TRACES AND APPLICATIONS

Inspired by the higher genus traces defined by Hayden and Piccirillo in [7], we generalize this concept to the link setting in the way that it can be applied to give an alternative characterization of the slice genus. We start by describing the two main constructions, leaving the general case for later: The first 4-manifold, which here we denote with $X^{g,1}(K)$, is the genus g 2-handles attached along the knot K in S^3 (with framing 0) appearing in [7]; while the second one consists of attaching a planar (genus zero) 2-handles with ℓ boundary component along an ℓ-component link in S^3, and we call it $X^{0,\ell}(L)$.

We recall that the *knotification* of an ℓ-component link L with $\ell>1$ is the knot $K_L\hookrightarrow\#^{\ell-1}S^1\times S^2$, which is shown on the left in Figure 3, obtained by adding $\ell-1$ oriented bands between the components of L, realizing $\ell-1$ merge moves; and then doing the same number of 0-surgeries along the boundaries of small disks, each with a unique ribbon intersection with one of the bands. Note that K_L is null-homologous by construction.

Ozsváth and Szabó proved in [15] that this operation is well defined; in other words, the diffeomorphism type of the knot K_L is independent of the choice of the bands we use to perform the merge moves. For more details about knotification of links see also [10].

High genus 2-handles

We recall that $X^{g,1}(K)$ is gotten by gluing $F_g\times D^2$, where $F_g=(\#^gT^2)\setminus\mathring{D}^2$ is the punctured connected sum of g tori, together with an identification $f:\nu(K)\to\partial F_g\times D^2$ where $K\hookrightarrow S^3=\partial D^4$ is a knot. Obviously, in order to specify the diffeomorphism f we also need to fix a framing: We take the latter to be zero. The resulting 4-manifold has a Kirby diagram as in Figure 4.

Planar 2-handles

We now describe the construction of $X^{0,\ell}(L)$ and we show that it is closely related to the concept of knotification of a link L. We again start from D^4 and we want to glue $G_\ell\times D^2$, where $G_\ell=S^2\setminus\{\ell\text{ disks}\}$ is a 2-sphere with ℓ punctures, along L, an ℓ-component link in ∂D^4. The attaching region of the handle $G_\ell\times D^2$ is $\partial G_\ell\times D^2\cong A_1\sqcup\ldots\sqcup A_\ell$ where $A_i\cong S^1\times D^2$ for each i; hence,

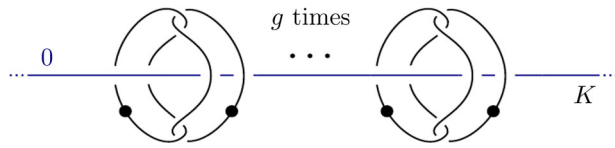

FIGURE 4 A Kirby diagram of $X^{g,1}(K)$. The picture is taken from [7].

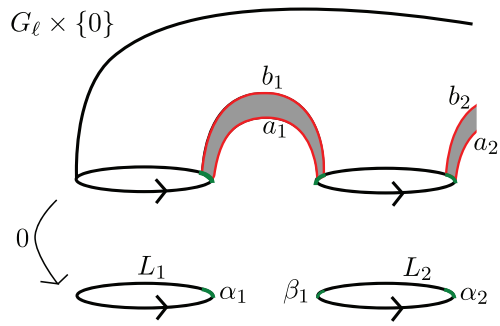

FIGURE 5 Attaching a planar 2-handles along an ℓ-component link L. The surface $\overline{G}_\ell$ is G_ℓ without the grey bands.

this time we need ℓ gluing maps $f_i : \nu(L_i) \to A_i$ with framings $t_1, \dots, t_\ell$ such that

$$t_1 + \cdots + t_\ell = -2 \cdot \ell\, \mathrm{k}(L)\,,$$

where here we take $\ell\, \mathrm{k}(L) = \sum_{i<j} \ell\, \mathrm{k}(L_i, L_j)$.

It is important to observe that the core of the handle (the surface $G_\ell \times \{0\}$) comes with an orientation that induces coherent orientations on each attaching sphere: Such orientations have to agree with the ones of the components of L; this means that the manifold $X^{0,\ell}(L)$ depends on the relative orientation of L. Note that $X^{0,1}(K)$ is the 0-trace of the knot K.

Proposition 5.1. *The 4-manifold $X^{0,\ell}(L)$ for $\ell > 1$ is diffeomorphic to $X(K_L)$, the trace of the 0-framed knot $K_L \hookrightarrow \#^{\ell-1} S^1 \times S^2$.*

Proof. We split the handle attachment in two parts. We start by attaching the grey band in Figure 5, which means gluing $I \times I \times D^2$ along $\partial I \times I \times D^2$ with framings zero; this procedure needs to be repeated $\ell - 1$ times. This requires the choice of $\ell - 1$ pairs of oriented arcs α_i and β_i inside the two components of L where we are performing the $\ell - 1$ merge moves; one has $\nu(\alpha_i) \cong \nu(\beta_i) \cong I \times D^2$ as attaching regions. As explained before, the orientations need to agree accordingly.

The 4-manifold obtained by this bridge construction is exactly $\natural^{\ell-1} S^1 \times D^3$; and in order to complete the original handle attachment we glue a standard 4-dimensional 2-handles $\overline{G}_\ell \times D^2$ along a framed knot J in $\#^{\ell-1} S^1 \times S^2 = \partial(\natural^{\ell-1} S^1 \times D^3)$. The knot J consists of the arcs $L \setminus (\alpha_1 \cup \dots \cup \alpha_{\ell-1} \cup \beta_1 \cup \dots \cup_{\ell-1})$ joined by the $\ell - 1$ pairs of arcs a_i and b_i, corresponding to $I \times \partial I \times D^2$; hence, we have that J is the knotification of L since its construction matches the one of K_L in [15]. In addition, its framing t is determined as follows:

$$t = \overline{e}_J(\overline{G}_\ell) = \overline{e}_L(G_\ell) = t_1 + \cdots + t_\ell + 2 \cdot \ell\, \mathrm{k}(L) = 0\,,$$

where $\overline{e}$ is the relative normal Euler number. □

High-order 2-handles

Mixing the constructions explained in the previous two paragraphs gives rise to what we are going to call high-order traces of a link. Let us start from a partition of an ℓ-component link L in S^3 into sublinks: $\mathcal{P} = \{P_1, \dots, P_k\}$. We say that $\mathcal{P}$ is a **weighted partition** of L if each sublink $P_i \in \mathcal{P}$, whose number of components is ℓ_i, has a non-negative integer g_i associated to it.

We denote by **high-order** 0-**trace** of L **with partition** $\mathcal{P}$ the 4-manifold $X^{\mathcal{P}}(L)$ obtained in the following way: We begin the attachment of 0-framed planar 2-handles along each sublink P_i, obtaining a 0-framed link with components $K_{P_1}, \dots, K_{P_k}$ on the boundary of $\natural^{\ell-k} S^1 \times D^3$; then we attach a 0-framed genus g_i 2-handles along K_{P_i} for $i = 1, \dots, k$. The resulting 4-manifold is well defined because the ordering of the gluings is unimportant; moreover, in the case when $k = 1$ we just denote the trace with $X^{g,\ell}(L)$. A Kirby diagram for $X^{\mathcal{P}}(L)$ can be easily produced by combining the ones in Figures 4 and 3.

Trace embedding lemma

In accordance with other results in the previous section, here we prove a third version of the trace embedding lemma. We recall that if X is a compact smooth 4-manifold and $\partial X = S^3$ then $\widehat{X} = X \cup \{4 - \text{handle}\}$; in addition, when L bounds a compact oriented surface Σ, such that all the connected components of Σ have boundary in L, we say that $\mathcal{P}_\Sigma$ is the weighted partition on L *induced by* Σ in the natural way.

Lemma 5.2 (Trace embedding lemma for high-order traces). *A link L in S^3 bounds a compact, oriented, smooth surface Σ, properly embedded in a possibly exotic 4-disk X, if and only if $X^{\mathcal{P}_\Sigma}(L^*)$ is smoothly embedded in $\widehat{X}$.*

Proof. For the first implication we assume that $X^{\mathcal{P}_\Sigma}(L^*) \hookrightarrow \widehat{X}$. Hence, we proceed as in other versions of the lemma and we write $\widehat{X} = D^4 \cup_{S^3} X$ with $L^* \hookrightarrow S^3 = \partial D^4$ and the high-order 2-handles inside X. On each sublink $P_1^*, \dots, P_k^*$ of L^* in the weighted partition $\mathcal{P}_\Sigma$, the core of the corresponding high-order 2-handles, whose attaching sphere is precisely P_i^*, is diffeomorphic to the surface $\Sigma_i = F_{g_i} \# G_{\ell_i - 1}$. Since all the high-order 2-handles we attached are disjoint, we have that $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_k \hookrightarrow X$ and its boundary is the link $(L^*)^* = L$.

We now prove the converse. Let us consider $\widehat{X} = D^4 \cup_{S^3} X$ and take $L^* \hookrightarrow S^3 = \partial D^4$. We have that each sublink P_i^* of L^* bounds a connected component Σ_i of Σ, with genus g_i and ℓ_i boundary components according to $\mathcal{P}_\Sigma$, for $i = 1, \dots, k$. Since we know that $H_2(X, \partial X; \mathbb{Z}) \cong \{0\}$, we obtain that each Σ_i bounds an embedded 3-homology class in X; thus showing that the relative Euler number $\overline{e}$ of $\nu(\Sigma_i)$ is zero. We can then conclude that

$$t_1 + \dots + t_{\ell_i} + 2 \cdot \ell \, \mathrm{k}(P_i) = \overline{e}_{P_i}(\Sigma_i) = 0$$

and then Σ_i is glued with framing $-2 \cdot \ell \, \mathrm{k}(P_i)$ to D^4 for any i.

We have proved that $(D^4 \cup \nu(\Sigma_i)) \cong X^{g_i, \ell_i}(P_i^*)$ for every $i = 1, \dots, k$. Hence, since the 4-manifold $X^{\mathcal{P}_\Sigma}(L^*)$ is constructed by performing k consecutive high-order 2-handles attachments along the P_i^*s, the latter just appears to be diffeomorphic to $D^4 \cup \nu(\Sigma)$ which is embedded in $\widehat{X}$. □

Remark 5.3. As for Lemma 2.3, the same proof also shows that L bounds a compact oriented Σ, properly locally flat embedded in D^4, if and only if $X^{\mathcal{P}_\Sigma}(L)$ is a locally flat submanifold of S^4.

5.1 | Slice genera of links in homotopy 4-spheres

The slice genus g_4 for a knot K has a unique natural definition: The minimal genus of a compact, oriented, connected surface F, properly and smoothly embedded in D^4, such that $K = \partial F$. This is not the case for links; there are many versions of slice genera of a link L. In this paper we are going to recall the most important ones, but focusing only on the two that are studied more often. We also recall that a knot K is *H-slice*, in a compact 4-manifold W with boundary, when K bounds a null-homologous properly embedded disk in W.

Consider an ℓ-component link. We denote by $g_4(L)$ what it is usually called the slice genus of L, which has exactly the same definition of g_4 for knots. Moreover, we define $g_4^X(L)$ the same invariant, but where surfaces are taken in an exotic 4-disk X instead of D^4. We recall that L is said to bound a planar smooth surface in X when $g_4^X(L) = 0$.

In addition, it follows from standard results that every null-homologous knot J in $\#^n S^1 \times S^2$ bounds a null-homologous properly embedded surface in $\natural^n S^2 \times D^2$, for every smooth structure and $n \geqslant 0$. Hence, the knot J is smoothly slice, in a possibly exotic $\natural^n S^2 \times D^2$, if and only if it is H-slice; and in this case the only possible value for the framing is zero.

Proposition 5.4. *An ℓ-component link L in S^3 bounds a planar smooth surface in X if and only if its knotification K_L is smoothly H-slice in $X \natural^\ell S^2 \times D^2$.*

Proof. Using the trace embedding lemma for high-order traces, we have that L bounds a planar smooth surface in X if and only if $X^{0,\ell}(L) \hookrightarrow \widehat{X}$. According to Proposition 5.1, one also has $X^{0,\ell}(L) \cong X(K_L)$ and, since K_L is null-homologous, according to what we said before we just need to apply the trace embedding lemma for framed links. □

The proof of the 'only if' implication in the latter proposition already appeared in [10]. We now recall the definition of the maximal 4-dimensional Euler characteristic of L as the maximum of $\chi(\Sigma)$, where Σ is a compact oriented surface, properly and smoothly embedded in X, such that $L = \partial\Sigma$ and every connected component of Σ has boundary in L. We denote this invariant with $\chi_4^X(L)$; note that the difference between the surfaces F and Σ is that the second one is not necessarily connected. $\chi_4^X(L)$ takes integer values which can be at most ℓ, with equality if and only if L is smoothly slice in X.

Remark 5.5. Another version of the slice genus which sometimes appears in the literature is g_4^*, which is defined as the minimal genus of Σ as before, but together with the condition that each connected component of Σ has exactly one boundary component. The invariant g_4^* is less used because it can only be defined for links with zero linking matrix, nonetheless Theorem 1.8 holds for it too.

Furthermore, we also mention the fact that some authors prefer to renormalize $\chi_4(L)$ as follows:

$$2G_4(L) - \ell = -\chi_4(L)\,,$$

because in this way one has $G_4(L) \geqslant 0$ as for $g_4(L)$. With the latter convention, we have $G_4(L) = 0$ if and only if L is smoothly slice.

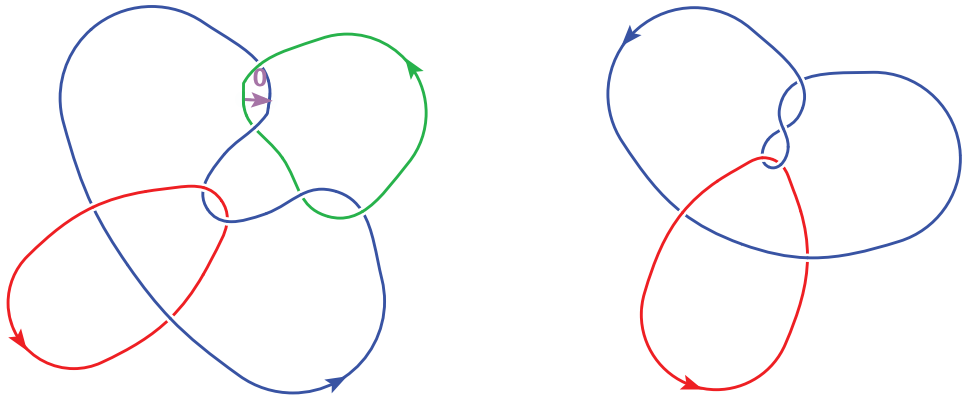

FIGURE 6 A diagram for the Borromean link B (left) and the Whitehead link W (right). An untwisted band is highlighted: This represents a merge move connecting B to W.

In Proposition 1.3 we showed that if X is a small exotic 4-disk then $S_X = S_{D^4}$, in other words the sets of slice links in X and D^4 coincide. Now we prove that the same is true for the slice genus and the maximal 4-dimensional Euler characteristic.

Proof of Theorem 1.8. The strategy of the proof is to argue that if X_1 and X_2 are two possibly exotic 4-disks such that $X_1 \hookrightarrow X_2$ and $X^{P_\Sigma}(L) \hookrightarrow X_1$ then obviously one has $X^{P_\Sigma}(L) \hookrightarrow X_2$ for every weighted partition of L. Since by assumption we have $D^4 \hookrightarrow X \hookrightarrow D^4$, it follows that $X^{P_\Sigma}(L) \hookrightarrow X$ if and only if $X^{P_\Sigma}(L) \hookrightarrow D^4$; hence, by applying the trace embedding lemma for high-order traces we obtain that L bounds a surface Σ in X if and only if the same happens in D^4. □

5.2 | Examples of non-slice null-homologous knots in $S^1 \times S^2$

In the last subsection of the paper we want to apply Proposition 5.4 to give an example of an infinite family of null-homologous knots in $S^1 \times S^2$ which are not smoothly slice in $S^2 \times D^2$. As already observed by Kuzbary in [10], to obstruct that a knot K in $S^1 \times S^2$ is not smoothly slice in $S^2 \times D^2$ is not enough to show that K is not concordant to the unknot; in fact, the knot K_{L_1} in Figure 8 is H-slice in $S^2 \times D^2$, because it is the knotification of the Hopf link which bounds a smooth annulus in S^3, but in [10] it is proved to not be concordant to the unknot.

We start by a special case which involves two of the most studied links in S^3: the Borromean link B and the Whitehead link W; a diagram for these links appears in Figure 6. It follows from a result of Klug in [9] that $g_4(B) = 1$ with every relative orientation; this also shows that $g_4(W) = 1$: in fact, in the case W was the boundary of a smooth annulus in D^4, we could build a planar smooth surface bounded by B using the merge move pictured in Figure 6. Proposition 5.1 gives that the corresponding knotifications K_B and $K_W = K_{L_2}$ are not smoothly slice in $\#^2 S^2 \times D^2$ and $S^2 \times D^2$, respectively; such knots appear in Figures 7 and 8.

Proposition 5.6. *The knots K_{L_n} in* Figure 8 *for $n \leqslant 0$ form an infinite family of null-homologous knots in $S^1 \times S^2$ which are not smoothly slice in $S^2 \times D^2$.*

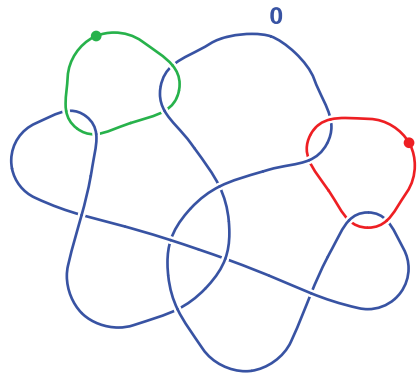

FIGURE 7 The 0-trace of $K_B \hookrightarrow \#^2 S^2 \times D^2$, the knotification of the Borromean link, with one of the possible relative orientations.

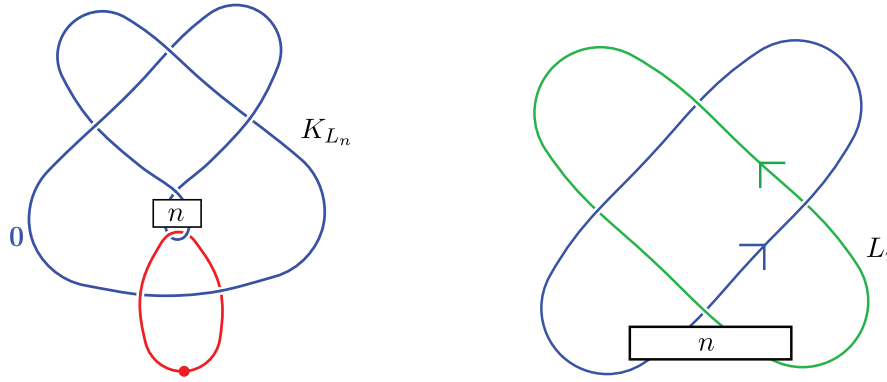

FIGURE 8 The link L_n (right) and the 0-trace of its knotification K_{L_n} (left). Here n is the number of full twists in the corresponding tangle.

Proof. By construction the knot K_{L_n} is the knotification of the link L_n which is shown on the right of Figure 8. When $n \leqslant 0$ we see that L_n is a non-split, alternating, 2-component link: We can then easily compute the τ-invariant, as in [1], from the signature; and we get $\tau(L_n) = 2$ for every $n \leqslant 0$. Therefore, the slice genus bound from [1] yields

$$2 = \tau(L_n) \leqslant g_4(L_n) + \ell - 1 = g_4(L_n) + 1$$

which means $g_4(L_n) \geqslant 1$. Since L_n never bounds a planar smooth surface in D^4 for $n \leqslant 0$, we can apply Proposition 5.1 to argue that K_{L_n} is not smoothly slice in $S^2 \times D^2$. □

It is actually possible to prove that $g_4(L_n) = 1$ for each $n \leqslant 0$, because L_n can be turned into a positive trefoil after a merge move, and this, in light of the trace embedding lemma for high-order traces, tells us that K_{L_n} bounds a null-homologous smooth torus in $S^2 \times D^2$.

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JOURNAL INFORMATION

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REFERENCES

1. A. Cavallo, *The concordance invariant tau in link grid homology*, Algebr. Geom. Topol. **18** (2018), no. 4, 1917–1951.
2. J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380.
3. M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.
4. M. Freedman, R. Gompf, S. Morrison, and K. Walker, *Man and machine thinking about the smooth 4-dimensional Poincaré conjecture*, Quantum Topol. **1** (2010), no. 2, 171–208.
5. D. Gabai, *Foliations and the topology of 3-manifolds III*, J. Differential Geom. **26** (1987), no. 3, 479–536.
6. R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999, pp. xvi+558.
7. K. Hayden and L. Piccirillo, *The trace embedding lemma and spinelessness*, arXiv:1912.13021.
8. R. Kirby, *Problems in low-dimensional topology*, AMS/IP Stud. Adv. Math. 2.2 Geometric topology (Athens, GA, 1993), Amer. Math. Soc., Providence, RI, 1997, pp. 35–473.
9. M. Klug, *A relative version of Rochlin's theorem*, arXiv:2011.12418.
10. M. Kuzbary, *Link Concordance and Groups*, Diss., Rice University, 2019. https://hdl.handle.net/1911/105956.
11. F. Laudenbach and V. Poénaru, *A note on 4-dimensional handlebodies*, Bull. Soc. Math. France **100** (1972), 337–344.
12. C. Manolescu, M. Marengon, and L. Piccirillo, *Relative genus bounds in indefinite four-manifolds*, arXiv:2012.12270.
13. C. Manolescu and L. Piccirillo, *From zero surgeries to candidates for exotic definite four-manifolds*, arXiv:2102.04391.
14. J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, NJ, 1965, p. 116.
15. P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116.
16. R. Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **34** (1960), 305–312.
17. C. T. C. Wall, *On simply-connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141–149.