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Peeling Sequences

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Abstract
Given a set of n labeled points in general position in the plane, we remove all of its points one by one. At each step, one point from the convex hull of the remaining set is erased. In how many ways can the process be carried out? The answer obviously depends on the point set. If the points are in convex position, there are exactly $n!$ ways, which is the maximum number of ways for n points. But what is the minimum number? It is shown that this number is (roughly) at least 3^n and at most 12.29^n.

Keywords Integer sequence · Convexity · Recursive construction

Mathematics Subject Classification 52C45 · 52C35

1 Introduction

A set of points in the plane is said to be in *general position* if no three of them are collinear. Let P be a set of n points in the plane in general position. Consider the following iterative process: remove points one by one until no point remains, under the provision that exactly one extreme point, that is, a vertex of the convex hull is removed in each step. If the points are in convex position, there are exactly $n!$ ways to

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do this, which is clearly the maximum number of ways for removing n points. What is the minimum number?

A *peeling sequence* for P is any permutation of the points of P that can be obtained by writing the labels of the points removed one by one. We are interested in the *minimum* number of such permutations that can be obtained, over all point sets of size n.

1.1 Definitions

Given a point set P in general position in the plane, let $g(P)$ count the number of peeling sequences for P; and $g(n)$ denote the minimum of $g(P)$ over all n-element point sets P in general position. It is easy to see that $g(n)$ is an increasing integer sequence; it is now entry A358251 in [13]. Observe that when $n \geq 3$, the last three points can be removed in any order; there are $3! = 6$ ways, whence $g(n)$ is a multiple of 6 for every $n \geq 3$.

The following bounds were obtained in an earlier writing [4] (all logarithms are in base 2): (i) every n-element point set in general position admits $\Omega(3^n)$ peeling sequences; (ii) on the other hand, there are sets with $2^{O(n \log \log n)}$ peeling sequences. Recently, a further improved upper bound, $2^{O(n \log \log \log n)}$, was presented by the first named author [5]. Here we significantly improve the upper bound; in particular, we show that it is of the form $O(a^n)$ for some $a > 1$.

The problem can be naturally generalized to point sets in higher dimensions. A set of points in the d-dimensional space $\mathbb{R}^d$ is said to be: (i) in *general position* if any at most $d + 1$ points are *affinely independent*; and (ii) in *convex position* if none of the points lies in the convex hull of the other points. Let $d \geq 2$ and let P be a point set in $\mathbb{R}^d$ in general position. We denote by $g(P)$ the number of peeling sequences of P and let $g_d(n)$ be the minimum of $g(P)$ over all n-element point sets P in general position in $\mathbb{R}^d$. In particular, $g_2(n)$ is the same as $g(n)$. For any fixed d we obtain exponential upper and lower bounds for $g_d(n)$.

1.2 Our results

We first observe that every n-element point set has $\Omega(3^n)$ peeling sequences. From the other direction, we show that if n is a power of 3, we can find suitable configurations with a small number of peeling sequences.

Theorem 1.1 *Let $n = 3^k$, for some $k \in \mathbb{N}$. There is a set of n points in general position in the plane with at most 27^n peeling sequences.*

Corollary 1.2 *For every $n \in \mathbb{N}$, there is a set of n points in general position in the plane with at most $19{,}683^n$ peeling sequences.*

We next arrive at our main result which summarizes our best lower and upper bound, respectively.

Theorem 1.3 *Every n-element point set in general position in the plane has $\Omega(3^n)$ peeling sequences. On the other hand, for every $n \geq 3$ there is a point set in general position with at most $12.29^n/100$ peeling sequences.*

The exponential lower bound from the planar case as well as the construction in the proof of Theorem 1.1 can be generalized to higher dimensions.

Theorem 1.4 *Every n-element point set in general position in $\mathbb{R}^d$ has $\Omega((d+1)^n)$ peeling sequences. On the other hand, if $n = (d+1)^k$, where $k \in \mathbb{N}$, there is a set of n points in general position in $\mathbb{R}^d$ with at most $(d+1)^{(d+1)n}$ peeling sequences.*

Corollary 1.5 *For every $n \in \mathbb{N}$, there is a set of n points in general position in $\mathbb{R}^d$ with at most $(d+1)^{(d+1)^2 n}$ peeling sequences.*

In this case we did not attempt to optimize the base of the exponential function.

1.3 Related Work

The concept of "peeling" a convex hull has been associated with dynamic convex hull algorithms and convex hull determination [10]. For a planar point set P, the *convex layers* of P are the convex polygons obtained by iterating the following procedure: compute its convex hull and remove its vertices from P [1, 2]. The process of peeling a point set has been shown very useful in obtaining robust estimators in statistics [2, 11]. A common estimator of a (unidimensional) sample is its arithmetic mean, however this estimator can be severely affected by outliers. A method that performs better is to discard the highest and lowest α-fraction of the data and take the mean of the remainder. This is the *α-trimmed mean*. The median is the special case $\alpha = 1/2$. The higher dimensional analog of trimming, called "peeling" by Tukey, consists of successively removing extreme points of the convex hull of the data until a certain fixed fraction of the points remains.

A quadratic algorithm for peeling (i.e., for convex layer decomposition) was initially proposed by Shamos [12]. A faster algorithm, running in $O(n \log^2 n)$ is due to Overmars and Van Leeuwen [10]. Finally, an optimal algorithm, running in $O(n \log n)$ time was obtained by Chazelle [2]. Computing the convex layers and studying their structure in a random setting have been studied by Dalal [3]. Har-Peled and Lidický [8] have studied the number of steps needed for peeling the integer grid G_n with n points (with, say, $n = k^2$); note that G_n is *not* in general position, however the peeling process can be executed on any point set.

It should be noted that while in the discussion above all the extreme points in a layer are removed in parallel (i.e., at the same time), in our study—of the function $g(n)$—the extreme points are removed sequentially one by one.

After some preliminaries in Sect. 2, we prove our main results, Theorems 1.1 and 1.3, in Sect. 3. Theorem 1.4 regarding higher dimensions can be found in Sect. 4. We conclude with some remarks in Sect. 5.

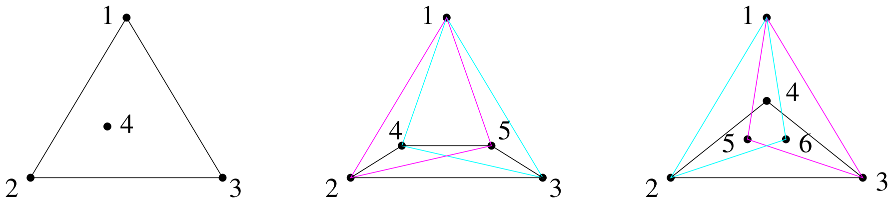

Fig. 1 Illustration for $n = 4, 5, 6$

2 Lower Bound and Small Values of *n*

As a warm-up we determine the values of $g(\cdot)$ for the first few values of n. Trivially, we have $g(1) = 1$, and $g(2) = 2$.

Proposition 2.1 *The following exact values can be observed.*

$$\begin{aligned} g(3) &= 6, \\ g(4) &= 18, \\ g(5) &= 60, \\ g(6) &= 180. \end{aligned}$$

Proof Let h denote the size of the convex hull of P. The case $n = 3$ is clear: there are $3! = 6$ permutations and each of them is a valid peeling sequence. The remaining cases are illustrated in Fig. 1.

Let now $n = 4$. If the points are in convex position, there are $4! = 24$ permutations. If the points are not in convex position, let 4 be the interior point. Then any permutation that starts with 4 is invalid, however, all remaining $4! - 3! = 18$ permutations are valid.

Let now $n = 5$. If $h \geq 4$, there are at least $4 \times g(4) = 72$ permutations. The case $h = 3$ yields the smallest number, $24 + 18 + 18 = 60$, of permutations; indeed, removing one of the extreme points yields a convex quadrilateral, while the other two removals can result in a triangle with a point inside.

Finally, let $n = 6$. If $h \geq 4$, there are at least $4 \times g(5) = 240$ permutations. The case $h = 3$ yields the smallest number: $3 \times g(5) = 3 \times 60 = 180$ permutations; indeed, removing each of the extreme points may yield the minimizer for $n = 5$ discussed above. □

2.1 Lower Bound

It is clear that $g(2) = 2$. Assume now that $n \geq 3$. Let P be any n-element point set in general position. By the assumption, $\mathrm{conv}(P)$ has at least three extreme vertices, removal of each yields a set of $n - 1$ points in general position. Any two peeling sequences resulting from the removal of two different extreme vertices are clearly different. As such, $g(\cdot)$ satisfies the recurrence $g(n) \geq 3 \cdot g(n-1)$. Consequently, $g(n) = \Omega(3^n)$. By Proposition 2.1 we also have $g(n) \geq 180 \cdot 3^{n-6}$ for every $n \geq 6$. □

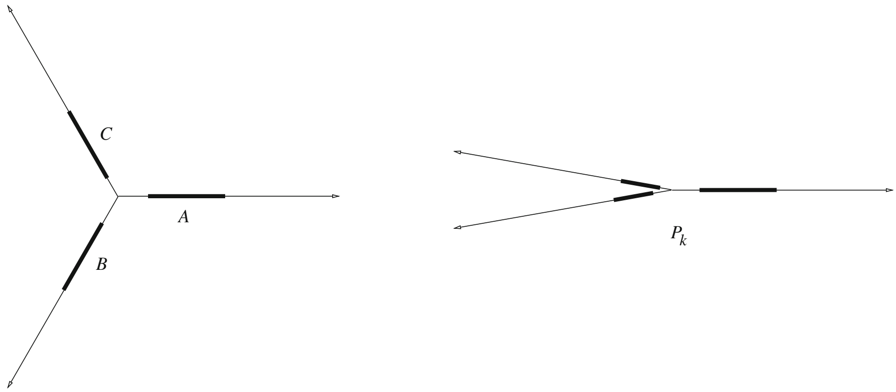

Fig. 2 The first construction: before and after the flattening step

3 Upper Bounds

3.1 First Construction and Analysis

Let $Z = X \cup Y$ be a finite point set in general position, where $X \cap Y = \emptyset$. Consider any peeling sequence, say, π, of Z. Then π naturally induces two peeling sequences, one for X and one for Y. This implies, that g is a monotone increasing function.

Proof of Theorem 1.1 We construct the sets P_k recursively for $k \in \mathbb{N}$, such that the set P_k contains $n = 3^k$ points and $g(P_k) \le 27^n$. Moreover, P_k is *flat*, that is, all points are very close (compared to the minimum distance in P_k) to a line; see Fig. 2.

The set P_0 is just one point. Suppose that we already have a construction P_{k-1} with $n/3 = 3^{k-1}$ points. Take three rays with a common origin with angles 120° between them. Place a copy of P_{k-1} along each ray. Let A, B, C denote the three copies of P_{k-1}, called *blocks* and let $P'_k = A \cup B \cup C$, a set of $n = 3^k$ points. We can assume that no two points of P'_k have the same x-coordinate, otherwise we apply a rotation. Let $P = P_k$ be a flattened copy of P'_k, that is, apply the transformation $(x, y) \longrightarrow (x, \varepsilon y)$ to P'_k; $n = |P| = 3^k$.

Let π be any peeling sequence of P, and let π' be a *prefix* of it. We say that P *yields* P' via π', written $P \overset{\pi'}{\Rightarrow} P'$ if applying π' to P yields P'. Observe that each ray "supports" $n/3$ points and the following invariant is maintained:

(I) Let π' be a *prefix* of π, and assume that $P \overset{\pi'}{\Rightarrow} P'$, where P' still has three *active* rays, i.e., none of A, B, or C, have been completely erased. Then P' has exactly 3 extreme vertices, one from each of A, B, C.

Replace every element of A (resp. B, C) with the symbol a (resp. b, c). This sequence contains $n/3$ a's, b's and c's and we call it the *simplified peeling sequence* π^*. It just tells us from which block points are peeled off at each step.

Consider now the subsequence π_A of π consisting of the elements of A. Observe that π_A is a peeling sequence of A. Clearly the same holds for B and C, or any other subset of P.

We can obtain all peeling sequences π of P in two steps: we first determine the simplified peeling sequence π^* and then we expand it to a peeling sequence. There are less than 3^n simplified peeling sequences. Let π^* be a simplified peeling sequence. Consider the last a, last b, and last c in π^* and take the first of them. Assume for simplicity that it is the a, and it is at position s. Clearly, $n/3 \le s \le n$.

Estimate now the number of ways π^* can be expanded to a peeling sequence of S. Observe that A is erased before B and C, therefore, whenever an element of A was peeled off, there was exactly one element of A, B, and C on the convex hull (in particular, there were only three extreme points). So the point that was peeled off was determined by π^*. That is, if π^* is fixed, then there is just one possible peeling order for the points of A, namely the decreasing order of distance from the origin. The same applies to the points of B and C that were peeled off by the time A is erased.

By the previous observations, for the elements of B (resp. C) we have at most $g(B) = g(C) = g(P_{k-1}) \le 27^{n/3} = 3^n$ choices. Consequently, the induction step yields

$$g(P_k) \le 3^n g^2(P_{k-1}) \le 27^n,$$

as required. $\square$

Proof of Corollary 1.2 For $3^k < n \le 3^{k+1}$, let Q_n be any subset of P_{k+1} of size n. Then by monotonicity, we have $g(n) \le g(Q_n) \le g(Q_{3^{k+1}}) = g(P_{k+1}) \le 27^{3n} = 19{,}683^n$. $\square$

Remark 3.1 A construction similar to that in the proof of Theorem 1.1 was used by Edelsbrunner and Welzl [6] to prove the existence of n-element point sets with $\Omega(n \log n)$ halving lines.

3.2 Second Construction and Analysis

3.3 Preliminaries

For $0 \le p \le 1$, denote by

$$H(p) = -p \log p - (1-p)\log(1-p),$$

the *binary entropy function*, where log stands for the logarithm in base 2. (By convention, $0 \log 0 = 0$.) We will use the following estimate in our calculation; see, e.g., [7, Lem. 3.6], or [9, Corr. 10.3]. For any integer $n \ge 1$ and $0 \le \alpha \le 1/2$, we have

$$\binom{n}{\alpha n} \le 2^{nH(\alpha)}. \tag{1}$$

Also, note that for any integer $n \ge 1$ and real x, where $n/3 \le x \le n$, we have

$$\binom{n}{\lceil x \rceil} \le 2\binom{n}{\lfloor x \rfloor}. \tag{2}$$

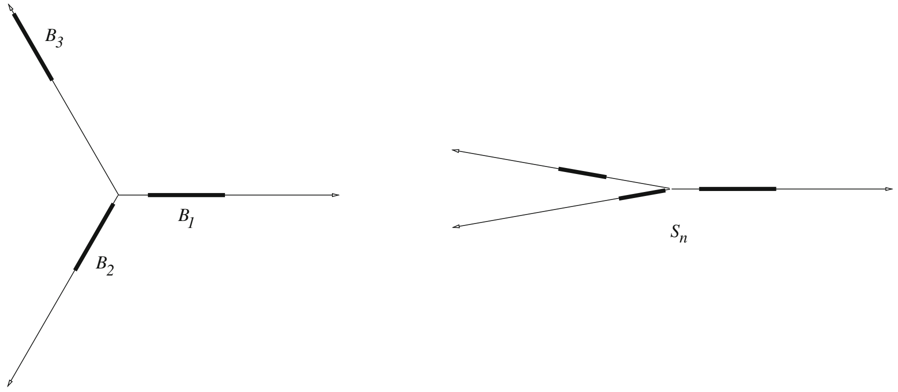

Fig. 3 The second construction

Lemma 3.2 *Let $n \geq 24$ be a positive integer. Then*

$$\binom{n}{k} \leq 2^n/6 \text{ for every } 0 \leq k \leq n. \tag{3}$$

Proof For $n = 24$ the largest binomial coefficient is $\binom{24}{12}$ and one can check that $\binom{24}{12} \leq 2^{24}/6$. Suppose that $n > 24$ and the statement holds for $n-1$. We may assume that $k \geq 1$ (since the inequality clearly holds for $k = 0$). Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \leq 2 \cdot 2^{n-1}/6 = 2^n/6$. □

A key fact in the argument is the following.

Lemma 3.3 *Let $Z = X \cup Y$ be a set of points in general position, $X \cap Y = \emptyset$, $|X| = n_1$, $|Y| = n_2$, $|Z| = n_1 + n_2 = n$. where $X \cap Y = \emptyset$. Then $g(Z) \leq \binom{n}{n_1} g(X)g(Y)$.*

Proof Let π be a peeling sequence of Z and let π^* be its simplified peeling sequence, that is, in π we replace every element of X (resp. Y) by x (resp. y). Clearly, there are $\binom{n}{n_1}$ simplified peeling sequences.

Let π_X be the subsequence of π, consisting of the elements of X. Observe that π_X is a peeling sequence of X. Define π_Y analogously, and clearly it is a peeling sequence of Y. Therefore, at most $g(X)g(Y)$ peeling sequences can have the same simplified peeling sequence, consequently, $g(Z) \leq \binom{n}{n_1} g(X)g(Y)$. □

Proof of Theorem 1.3 Write $a = 12.29$. For every n we construct the sets S_n recursively, such that S_n contains n points and $g(S_n) \leq a^n/100$ for $n \geq 3$. Moreover, S_n is *flat*, just like in the previous construction. The set S_1 is just one point; S_2 is a point pair; and S_3 is a flat (obtuse) triangle.

Suppose that $n \geq 4$ and for every $i < n$ we have already constructed S_i satisfying the requirements. In order to do a better recursion, we choose a specific variant in the previous construction; see Fig. 3. Let $n = n_1 + n_2 + n_3$, where $n_i = \lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$. Take three rays r_1, r_2, r_3 with a common origin O and with angles 120° between them, such that r_1 is horizontal. Place a copy of S_{n_1} on r_1 and call it B_1, a copy of S_{n_2} on r_2, close to O, and call it B_2, and a copy of S_{n_3} on r_3, far from O, and call it B_3. Finally, let S_n be a flattened copy of this set of n points. For simplicity we still denote the three corresponding components of S_n by B_1, B_2, B_3, respectively. Observe, that the projections of B_1, B_2, and B_3 are separated on the x-axis, they come in this order, and in S_n all points are very close to their projections. We have thereby defined the set S_n for every n and clearly it has n points.

We prove that $g(S_n) \le a^n/100$ by induction on n. The induction basis is $3 \le n \le 35$: Obviously $g(P) \le |P|!$ for every point-set P. By the special structure of S_n, we have $g(S_n) \le 3^{\lfloor n/3 \rfloor}(\lceil 2n/3 \rceil)! \le a^n/100$ for every $3 \le n \le 35$. Suppose now that $n \ge 36$ and that $g(S_i) \le a^i/100$ for every i, where $3 \le i \le n-1$. Let $n = n_1 + n_2 + n_3$, where $n_i = \lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$. Let B_1, B_2, B_3, be its three blocks, B_i is an affine copy of S_{n_i}. For any peeling sequence π of S_n, we define its simplified peeling sequence π^* so that we replace every element of B_i by i.

Now let π be a peeling sequence of S_n and let π^* be its simplified peeling sequence. Just like in the proof of Theorem 1.1, take the last 1, last 2, and last 3 in π^*, and assume that the first of them is at position s. Since one of the blocks is peeled off in step s, we have $\lfloor n/3 \rfloor \le s \le n$.

Observe that until peeling step $s+1$, all three rays were active, so there was exactly one element of B_1, B_2, and B_3 on the convex hull. Therefore, the first s elements of π are determined by π^*. We distinguish four cases based on the value of s. We estimate now the number of corresponding simplified peeling sequences, and the number of ways they can be expanded to a peeling sequence of S_n. We denote the resulting upper bound estimates by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, and then show that $\sum_{j=1}^{4} \sigma_j \le a^n/100$. More precisely, we will prove that

$$\sigma_1 \le \frac{1}{2} \cdot \frac{a^n}{100}, \quad \sigma_2 \le \frac{1}{6} \cdot \frac{a^n}{100}, \quad \sigma_3 \le \frac{1}{6} \cdot \frac{a^n}{100}, \quad \sigma_4 \le \frac{1}{6} \cdot \frac{a^n}{100}.$$

Case 1 $\lfloor n/3 \rfloor \le s \le \lceil 5n/9 \rceil$. Let σ_1 denote the number of corresponding peeling sequences of S_n. Let $\mathcal{T}$ be the corresponding set of simplified peeling sequences. By (1), (2), and (3), we have

$$\begin{aligned}
|\mathcal{T}| &\le 3\binom{\lceil 5n/9 \rceil}{\lfloor n/3 \rfloor}\binom{\lceil 2n/3 \rceil}{\lfloor n/3 \rfloor} = 3\binom{\lceil 5n/9 \rceil}{\lceil 5n/9 \rceil - \lfloor n/3 \rfloor}\binom{\lceil 2n/3 \rceil}{\lfloor n/3 \rfloor} \\
&\le 3 \cdot 2 \cdot \binom{\lceil 5n/9 \rceil}{\lfloor 2n/9 \rfloor} 2^{\lceil 2n/3 \rceil}/6 \le 2^{\lceil 5n/9 \rceil \cdot H(0.4)} 2^{\lceil 2n/3 \rceil} \\
&\le 2^{(5H(0.4)+6)n/9} \cdot 2^{8/9} \cdot 2^{2/3} \le 2^{10.855n/9} \cdot 2^{8/9} \cdot 2^{2/3} \\
&= 2^{14/9} \cdot 2^{10.855n/9}.
\end{aligned}$$

Let $\pi^* \in \mathcal{T}$. Suppose for simplicity, that at position s there is a 1. So block B_1 is finished first, and its peeling order is determined. For the other two blocks we have at most $g(S_{n_2})g(S_{n_3})$ possibilities. We have $n_1, n_2, n_3 \le \lceil n/3 \rceil$, therefore, by the induction hypothesis we have

$$\sigma_1 \le |\mathcal{T}| \cdot g^2(S_{\lceil n/3\rceil}) \le 2^{14/9} \cdot 2^{10.855n/9} \cdot \frac{a^{n-\lfloor n/3\rfloor}}{10000} \le \frac{1}{2} \cdot \frac{a^n}{100},$$

where the last inequality follows from the two inequalities:

$$2^{23/9} \cdot a^{2/3} \le 100, \text{ and } 2^{10.855/3} \le a.$$

Case 2 $\lceil 5n/9\rceil \le s \le \lfloor 2n/3\rfloor$. Let σ_2 denote the number of corresponding peeling sequences of S_n. Let $\mathcal{T}$ be the corresponding set of simplified peeling sequences. By Lemma 3.2 we have

$$|\mathcal{T}| \le 3\binom{\lfloor 2n/3\rfloor}{\lfloor n/3\rfloor}\binom{\lceil 2n/3\rceil}{\lceil n/3\rceil} \le 2^{4n/3}/12.$$

Let $\pi^* \in \mathcal{T}$. Suppose again that there is a 1 at position s, so block B_1 is finished first, and its peeling order is determined. Moreover, since $\lceil 5n/9\rceil \le s$, at least $\lceil n/9\rceil$ points of B_2 or B_3 have already been removed, say, from B_2. These removed points were the extreme points of B_2 in one direction, that is, a sub-block of B_2 has been removed. The remainder of B_2 after step s of the peeling process is a subset of (an affine image of) two sub-blocks of S_{n_2}, and by construction, both have size at most $\lceil n/9\rceil$. By Lemma 3.3 this set has at most $2^{2\lceil n/9\rceil} g(S_{\lceil n/9\rceil})^2$ peeling sequences. For B_3 we have at most $g(S_{\lceil n/3\rceil})$ possibilities. Note that $n/9 \ge 4$ and so the induction hypothesis applies. Therefore,

$$\begin{aligned}\sigma_2 &\le |\mathcal{T}| \cdot g(S_{\lceil n/3\rceil}) \cdot g^2(S_{\lceil n/9\rceil}) \cdot 2^{\lceil 2n/9\rceil}\\ &\le \frac{2^{12n/9}}{12} \cdot \frac{a^{\lceil n/3\rceil}}{100} \cdot \frac{a^{\lceil n/9\rceil}}{100} \cdot \frac{a^{\lceil n/9\rceil}}{100} \cdot 2^{2n/9} \cdot 2^{8/9}\\ &\le \frac{1}{6} \cdot 2^{14n/9} \cdot a^{5n/9} \cdot a^{22/9} \cdot \frac{1}{10^6} \le \frac{1}{6} \cdot \frac{a^n}{100}.\end{aligned}$$

The last inequality follows from the two inequalities:

$$a^{22/9} \le 10^4, \text{ and } 2^{7/2} \le a.$$

Case 3 $\lceil 2n/3\rceil \le s \le \lfloor 7n/9\rfloor$. Let σ_3 denote the number of corresponding peeling sequences and $\mathcal{T}$ the corresponding set of simplified peeling sequences. We have

$$|\mathcal{T}| \le 3\binom{\lfloor 7n/9\rfloor}{\lceil n/3\rceil}\binom{\lceil 2n/3\rceil}{\lceil n/3\rceil} \le 2^{13n/9}/12.$$

Let $\pi^* \in \mathcal{T}$. Suppose again that there is a 1 at position s. So block B_1 is finished first, and its peeling order is determined. Moreover, since $\lceil 2n/3\rceil \le s$, at least $\lceil n/3\rceil$ additional points have been removed, that is, two sub-blocks of B_2 or B_3, of at least

$\lfloor n/9 \rfloor$ points, either two sub-blocks from one of them or one sub-block from each of them. We can argue similarly to the previous cases and get the following estimate.

$$\begin{aligned}
\sigma_3 &\le |\mathcal{T}| \left(g(S_{\lceil n/3 \rceil}) \cdot g(S_{\lceil n/9 \rceil}) + g^4(S_{\lceil n/9 \rceil}) \cdot 2^{4\lceil n/9 \rceil} \right) \\
&\le \frac{2^{13n/9}}{12} \left(\frac{a^{\lceil n/3 \rceil} a^{\lceil n/9 \rceil}}{10^4} + \frac{a^{4\lceil n/9 \rceil} 2^{4\lceil n/9 \rceil}}{10^8} \right) \\
&\le \frac{2^{13n/9}}{12} \left(\frac{a^{4n/9} \cdot a^{14/9}}{10^4} + \frac{(2a)^{4n/9} \cdot (2a)^{32/9}}{10^8} \right) \\
&\le \frac{1}{6} \cdot \frac{a^n}{100}.
\end{aligned}$$

The last inequality follows from the three inequalities:

$$a^{14/9} \le 10^2, \quad (2a)^{32/9} \le 10^6, \quad \text{and } 2^{17/5} \le a.$$

Case 4 $\lceil 7n/9 \rceil \le s \le n$. Let σ_4 denote the number of corresponding peeling sequences and $\mathcal{T}$ the corresponding set of simplified peeling sequences. Obviously $|\mathcal{T}| \le 3^n$. Let $\pi^* \in \mathcal{T}$ and suppose again that block B_1 is finished first, so its peeling order is determined. Since $\lceil 7n/9 \rceil \le s$, three sub-blocks of B_2 or B_3 have been removed. The following estimate is implied.

$$\begin{aligned}
\sigma_4 &\le |\mathcal{T}| \cdot g^3(S_{\lceil n/9 \rceil}) \cdot 2^{2\lceil n/9 \rceil} \\
&\le 3^n \cdot 2^2 \cdot a^3 \cdot 2^{2n/9} \cdot a^{n/3}/100 \\
&\le \frac{1}{6} \cdot \frac{a^n}{100}.
\end{aligned}$$

The last inequality is clearly satisfied.

Finally, $g(S_n) = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 \le a^n/100$, concluding the induction step and thereby also the proof of Theorem 1.3. □

4 Higher Dimensions

In this section we prove Theorem 1.4 and its corollary.

4.1 Lower Bound

Let P be any n-element point set in general position. By the assumption, $\operatorname{conv}(P)$ has at least $d + 1$ extreme vertices, removal of each yields a set of $n - 1$ points in general position. Any two peeling sequences resulting from the removal of two different extreme vertices are clearly different. As such, $g(\cdot)$ satisfies the recurrence $g(n) \ge (d + 1) \cdot g(n - 1)$. Consequently, $g(n) = \Omega((d + 1)^n)$.

4.2 Upper bound

We proceed similarly to the planar case. We construct the sets S_k recursively for $k \in \mathbb{N}$, such that the set S_k contains $n = (d+1)^k$ points and $g(S_k) \le (d+1)^{(d+1)n}$. Moreover, S_k is *thin*, that is, all points are very close (compared to the minimum distance in S_k) to a line.

The set S_0 is just one point. Suppose that we already have a construction S_{k-1} with $n/(d+1) = (d+1)^{k-1}$ points. Take a regular simplex $\Delta \subset \mathbb{R}^d$ centered at the origin and such that $(1, 0, \dots, 0)$ is a vertex of Δ. Take the $(d+1)$ rays $r_1, \dots, r_{d+1}$ from the origin to its vertices. Place a copy of S_{k-1} along each ray. Let $B_1, \dots, B_{d+1}$ denote the copies of S_{k-1}, and let $S'_k = \cup_{i=1}^{d+1} B_i$, a set of $n = (d+1)^k$ points.

We can assume that no two points of S'_k have the same x_1-coordinate, otherwise we apply a rotation. Let $S = S_k$ be a flattened copy of S'_k, that is, apply the transformation $(x_1, x_2, \dots, x_d) \longrightarrow (x_1, \varepsilon x_2, \dots, \varepsilon x_d)$ to S'_k.

Let π be any peeling sequence of S. Observe that each ray "supports" $n/(d+1)$ points and the following invariant is maintained

(I)$_d$ Let π' be a *prefix* of π, and assume that $S \overset{\pi'}{\Rightarrow} S'$, where S' still has $d+1$ active rays, i.e., none of its blocks have been completely erased. Then S' has exactly $d+1$ extreme vertices, one from each block.

To obtain the simplified peeling sequence as before, replace every element of B_i with i. To obtain all peeling sequences π of S, first we determine the simplified peeling sequence π^* and then we expand it to a peeling sequence. There are less than $(d+1)^n$ simplified peeling sequences. Let π^* be a simplified peeling sequence. Consider the last i for every $1 \le i \le d+1$ and take the first of them. Assume for simplicity that it is 1, and it is at position s. Clearly, $n/d \le s \le n$.

Since B_1 was finished first, the peeling order of its points is determined. For the elements of B_i, $2 \le i \le d+1$ we have at most $g(B_i) = g(S_{k-1}) \le (d+1)^n$ choices. Consequently, the induction step yields

$$g(S_k) \le (d+1)^n g^d(S_{k-1}) \le (d+1)^{(d+1)n},$$

as required. □

Proof of Corollary 1.5 For $(d+1)^k < n < (d+1)^{k+1}$, let P_n be any subset of S_{k+1} of size n. Then by monotonicity, we have

$$g(n) \le g(P_n) \le g(P_{(d+1)^{k+1}}) \le (d+1)^{(d+1)^2 n},$$

as claimed. □

5 Concluding Remarks

Observe that the convex layer decomposition of a point set P—mentioned in Sect. 1—yields a set of peeling sequences naturally derived from it: remove the points from one layer, one by one, before moving to the next layer. Indeed, this is so, since any point of the layer under removal is still extreme at that step. In general, if there are m layers and their sizes are $h_1, h_2, \ldots, h_m$ counting from outside, where $h_1, \ldots, h_{m-1} \geq 3$, $h_m \geq 1$, and $\sum_{i=1}^{m} h_i = n$, then there are $h_1!h_2! \cdots h_m!$ peeling sequences given by the convex layer decomposition.

Apart from the case of points in convex position, the set of peeling sequences corresponding to layer by layer removal of the points is a strict subset of the set of peeling sequences of P. It is worth noting that this subset can be much smaller than the whole set. For example, consider a point set with $n/3$ layers, where each layer is a triangle (and so the point set is the vertex set of $n/3$ nested triangles). Then there are $6^{n/3} = 1.817\ldots^n$ peeling sequences given by the convex layer decomposition, whereas the total number of peeling sequences is $\Omega(3^n)$.

Our estimates on the growth rate of $g(n)$ are now closer, but a substantial gap remains. With a more complicated case analysis it should be possible to further improve the upper bound to $O(c^n)$ for some $c < 12.29$; however, getting close to the lower bound seems to require new ideas. A natural question is whether the trivial lower bound of $\Omega(3^n)$ can be improved.

Problem 5.1 *Is there a constant $\delta > 0$ such that $g(n) = \Omega((3+\delta)^n)$?*

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