

Stability of the isodiametric problem on the sphere and in the hyperbolic space



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ABSTRACT

We prove a stability version of the isodiametric inequality on the sphere and in the hyperbolic space.

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1. Introduction

Let \mathcal{M}^n be either the Euclidean space \mathbb{R}^n , hyperbolic space H^n or spherical space S^n for $n \geq 2$. We write $V_{\mathcal{M}^n}$ to denote the n -dimensional volume (Lebesgue measure) on

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\mathcal{M}^n , and $d_{\mathcal{M}^n}(x, y)$ to denote the geodesic distance between $x, y \in \mathcal{M}^n$. For $x, y \in \mathcal{M}^n$ where $x \neq -y$ if $\mathcal{M}^n = S^n$, we write $[x, y]_{\mathcal{M}^n}$ to denote the geodesic segment between x and y whose length is $d_{\mathcal{M}^n}(x, y)$. For pairwise different points $x, y, z \in \mathcal{M}^n$ (which are pairwise not antipodal in the spherical case), let $\angle(x, y, z)$ be the angle of the geodesic segments $[x, y]$ and $[y, z]$ at y .

For a bounded set $X \subset \mathcal{M}^n$, its diameter $\text{diam}_{\mathcal{M}^n} X$ is the supremum of the geodesic distances $d_{\mathcal{M}^n}(x, y)$ for $x, y \in X$. For $D > 0$ and $n \geq 2$, we are considering the maximal volume of a subset of \mathcal{M}^n of diameter at most D . For any $z \in \mathcal{M}^n$ and $r > 0$, let

$$B_{\mathcal{M}^n}(z, r) = \{x \in \mathcal{M}^n : d_{\mathcal{M}^n}(x, z) \leq r\}$$

be the n -dimensional ball centered at z where it is natural to assume $r < \pi$ if $\mathcal{M}^n = S^n$. When it is clear from the context what space we consider, we drop the subscript referring to the ambient space. In order to speak about the volume of a ball of radius r , we fix a reference point $z_0 \in \mathcal{M}^n$ where $z_0 = o$ the origin if $\mathcal{M}^n = \mathbb{R}^n$. The volume of the unit ball $B_{\mathbb{R}^n}(o, 1)$ is denoted by κ_n .

The isoperimetric inequality requires the definition of surface area. For any compact set $X \subset \mathcal{M}^n$ and $\varrho \geq 0$, we consider the parallel domain

$$X^{(\varrho)} = \{z \in \mathcal{M}^n : \exists x \in X \text{ with } d_{\mathcal{M}^n}(x, z) \leq \varrho\} = \bigcup \{B_{\mathcal{M}^n}(x, \varrho) : x \in X\}.$$

If in addition, if $\text{int } X \neq \emptyset$ and ∂X is $(n-1)$ -rectifiable; namely, it is the Lipschitz image of some compact subset of \mathbb{R}^{n-1} , or if ∂X is a set of positive reach, then the surface area of X can be interpreted as the outer Minkowski content

$$S_{\mathcal{M}^n}(X) = \lim_{\varepsilon \rightarrow 0^+} \frac{V_{\mathcal{M}^n}(X^{(\varepsilon)}) - V_{\mathcal{M}^n}(X)}{\varepsilon}$$

(see Ambrosio, Colesanti, Villa [2]). We note that if ∂X is $(n-1)$ -rectifiable (for example, X is convex), then $S_{\mathcal{M}^n}(X)$ coincides with the $(n-1)$ -dimensional Hausdorff measure of ∂X (cf. [2]).

Theorem 1.1 (*Isoperimetric inequality*). *If \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n and $X \subset \mathcal{M}^n$ is compact and $V_{\mathcal{M}^n}(X) = V_{\mathcal{M}^n}(B(z_0, r))$ for $r > 0$, then*

$$V_{\mathcal{M}^n}(X^{(\varrho)}) \geq V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(z_0, r + \varrho)) \text{ for } \varrho > 0.$$

Remark. It follows that if X has outer Minkowski content, then $S_{\mathcal{M}^n}(X) \geq S_{\mathcal{M}^n}(B(z_0, r))$.

The isoperimetric inequality was known to the ancient Greeks in the Euclidean plane, and the Euclidean case in any dimension was proved by the work of Steiner, Schwarz, Weierstrass and Minkowski in the 19th century (see Gruber [30]). The isoperimetric inequality in the spherical and hyperbolic spaces is due to E. Schmidt [39]. We provide

the elegant argument by Benyamini [7] because it works simultaneously in all spaces of constant curvature, and also the core ideas are essential ingredients for the isodiametric problem, which is our main focus.

Various stability versions of the isoperimetric inequality have been provided starting with Minkowski. In terms of the volume difference, see Fusco, Maggi, Pratelli [24] and Section 10 in the Euclidean case, Bögelein, Duzaar, Scheven [12] in the hyperbolic case, and Bögelein, Duzaar, N. Fusco [13] in the spherical case. Essentially optimal stability version of the isoperimetric inequality in \mathbb{R}^n in the terms of Hausdorff distance has been verified by Fuglede [23].

The main topic of the paper, the isodiametric inequality is proved by Bieberbach [10] in \mathbb{R}^2 and by P. Urysohn [44] in \mathbb{R}^n , $n \geq 3$, by W. Barthel, H. Pabel [6] in n -dimensional normed spaces and by Schmidt [40,41] and Böröczky, Sagmeister [14] in the spherical space S^n and the hyperbolic space H^n . The isodiametric problem for bisections in the Euclidean plane is solved by Cañete, Merino [16].

Theorem 1.2 (*Isodiametric inequality*). *If \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n , $D > 0$ (with $D < \pi$ if $\mathcal{M}^n = S^n$) and $X \subset \mathcal{M}^n$ is measurable and bounded with $\text{diam} X \leq D$, then*

$$V(X) \leq V(B(z_0, D/2)),$$

and equality holds if and only if the closure of X is a ball of radius $D/2$.

In this paper, we provide a new, conceptually more natural proof of the Isodiametric Inequality Theorem 1.2, and then we prove the following stability version. The convex hull of an $X \subset \mathcal{M}^n$ is denoted by $\text{conv}_{\mathcal{M}^n} X$ (see Section 5).

Theorem 1.3. *For $n \geq 2$, if \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n , $D > 0$ (where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$) and $X \subset \mathcal{M}^n$ is measurable with $\text{diam} X \leq D$ and*

$$V_{\mathcal{M}^n}(X) \geq (1 - \varepsilon) V_{\mathcal{M}^n} \left(B_{\mathcal{M}^n} \left(z_0, \frac{D}{2} \right) \right)$$

for $\varepsilon \in [0, \varepsilon_{\mathcal{M}^n}(D))$, then there exists a $c \in \mathcal{M}^n$ such that

$$B \left(c, \frac{D}{2} - \gamma_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}} \right) \subset \text{conv}_{\mathcal{M}^n} X \subset B \left(c, \frac{D}{2} + \gamma_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}} \right)$$

where $\varepsilon_{\mathcal{M}^n}(D) > 0$ depends on D and \mathcal{M}^n and

$$\gamma_{\mathcal{M}^n}(D) = \begin{cases} e^{21n} \cdot D & \text{if } \mathcal{M}^n = H^n \text{ and } D \leq 2, \text{ or } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n; \\ n \cdot e^{7D+8} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 1. \end{cases}$$

In addition, $V_{\mathcal{M}^n}((\text{conv}_{\mathcal{M}^n} X) \setminus X) \leq \varepsilon \cdot V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(z_0, \frac{D}{2}))$.

Remark. The order of the error term in Theorem 1.3 is not far from being optimal as if either $\mathcal{M}^n = S^n$ and $D \leq \frac{\pi}{2}$, or $\mathcal{M}^n = R^n$, or $\mathcal{M}^n = H^n$, and X is obtained from $B_{\mathcal{M}^n}(z_0, \frac{D}{2})$ by cutting of a cap of volume $\varepsilon \cdot B_{\mathcal{M}^n}(z_0, \frac{D}{2})$ for small $\varepsilon > 0$, then the radius of any ball contained in X is at most $\frac{D}{2} - \theta \cdot \varepsilon^{\frac{2}{n+1}}$ for $\theta > 0$ depending on \mathcal{M}^n and D .

The exact value of $\varepsilon_{\mathcal{M}^n}(D)$ is stated in Theorem 9.2.

The real importance of Theorem 1.3 lies at the Spherical and Hyperbolic cases as the known stability results about the Brunn-Minkowski inequality in the Euclidean space (see Section 10 for a review) directly yield a stability version of the Isodiametric inequality with better error term (see Theorem 10.1 in Section 10). Actually, for n -dimensional normed spaces (including the Euclidean space), the stability version of the isodiametric inequality is proved by Diskant [19]. In addition, Hernández Cifre, Martínez Fernández [31] proved Theorem 1.3 in a more precise form for centrally symmetric subsets of S^2 of diameter less than $\pi/2$.

Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $D > 0$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$. We say that a subset $Z \subset \mathcal{M}^n$ of diameter D is complete if $Z \subset Z' \subset \mathcal{M}^n$ and $\text{diam } Z' = D$ imply $Z = Z'$. Any $X \subset \mathcal{M}^n$ of diameter D is contained in a complete set $Z \subset \mathcal{M}^n$ of diameter D (see Section 5); therefore, we discuss intensively properties of complete sets. We note that complete sets of diameter D are also called convex bodies of constant width (cf. Section 5).

For related surveys on spherical convex bodies, see Schramm [43], Lassak [35] and Lassak, Musielak [36]. In addition, Groemer [27] surveys properties of complete sets in Minkowski spaces. For properties of convex bodies of constant width in the hyperbolic space, see Böröczky, Sagmeister [15].

For some related results in the hyperbolic space, see Alfonseca, Cordier, Florentin [1], Gallego, Reventos, Solanes, Teufel [26], Jerónimo-Castro, Jimenez-Lopez [32].

It follows from the Isodiametric Inequality Theorem 1.2 that among convex bodies of constant width D in \mathcal{M}^n , balls have the maximum volume. However, the problem of the minimum volume of convex bodies of constant width D in \mathcal{M}^n is open if $n \geq 3$ even in the Euclidean case.

If $n = 2$, then a Reuleaux triangle in a surface of constant curvature is the intersection of three circular discs of radius $D > 0$ whose centers are vertices of a regular triangle of side length D (where we assume $D < \frac{\pi}{2}$ in the spherical case). It is a convex domain of constant width D . The Blaschke–Lebesgue Theorem, due to Blaschke [11] and Lebesgue [37] states that amongst bodies of constant width in the Euclidean plane, the Reuleaux triangle has the minimal area (see Eggleston [20] for a particularly simple proof). The spherical version of the theorem was proved by Leichtweiss [38] based on some ideas of Blaschke. A new proof of the spherical case was recently published by K. Bezdek [9]. After results by Araújo [3] and Leichtweiss [38] if the boundary is piecewise smooth, Böröczky, Sagmeister [15] proved that the Reuleaux triangle has the minimal area amongst bodies

of constant width in the hyperbolic plane. We note that a stability version of the extremal property of the Reuleaux triangle is proved in [15] in all spaces of constant curvature.

2. Spaces of constant curvature

Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n (see [8] or [45]). Our focus is on the spherical- and hyperbolic space, and we assume that S^n is embedded into \mathbb{R}^{n+1} the standard way, and H^n is embedded into \mathbb{R}^{n+1} using the hyperboloid model. We write $\langle \cdot, \cdot \rangle$ to denote the standard scalar product in \mathbb{R}^{n+1} , and write $z^\perp = \{x \in \mathbb{R}^{n+1} : \langle x, z \rangle = 0\}$ for a $z \in \mathbb{R}^{n+1} \setminus \{o\}$. Fix an $e \in S^n$. In particular, we have

$$\begin{aligned} S^n &= \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = 1\} \\ H^n &= \{x + te : x \in e^\perp \text{ and } t \geq 1 \text{ and } t^2 - \langle x, x \rangle = 1\}. \end{aligned}$$

For H^n , we also consider the following symmetric bilinear form \mathcal{B} on \mathbb{R}^{n+1} : If $x = x_0 + te \in \mathbb{R}^{n+1}$ and $y = y_0 + se \in \mathbb{R}^{n+1}$ for $x_0, y_0 \in e^\perp$ and $t, s \in \mathbb{R}$, then

$$\mathcal{B}(x, y) = ts - \langle x_0, y_0 \rangle.$$

In particular,

$$\mathcal{B}(x, x) = 1 \text{ for } x \in H^n. \quad (1)$$

We note that the geodesic distance of $x, y \in \mathcal{M}^n$ where \mathcal{M}^n is either H^n or S^n is

$$\begin{aligned} d_{S^n}(x, y) &= \arccos \langle x, y \rangle \text{ if } \mathcal{M}^n = S^n; \\ d_{H^n}(x, y) &= \operatorname{arccosh}(\mathcal{B}(x, y)) \text{ if } \mathcal{M}^n = H^n. \end{aligned}$$

In particular, the isometries of \mathbb{R}^n are the maps of the form $x \mapsto Ax + b$ where $A \in O(n)$ and $b \in \mathbb{R}$, the isometries of $S^n \subset \mathbb{R}^{n+1}$ are the maps of the form $x \mapsto Ax$ where $A \in O(n+1)$, and the isometries of $H^n \subset \mathbb{R}^{n+1}$ are the maps of the form $x \mapsto Ax$ where $A \in \operatorname{GL}(n+1, \mathbb{R})$ leaves $\mathcal{B}(\cdot, \cdot)$ invariant and $\langle Ae, e \rangle > 0$. The isometry group of each \mathcal{M}^n acts transitively, and the subgroup fixing a $z \in \mathcal{M}^n$ is isomorphic to $O(n-1)$.

Again let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n using the models as above for H^n and S^n . For $z \in \mathcal{M}^n$, we define the tangent space T_z as

$$\begin{aligned} T_z &= \{x \in \mathbb{R}^{n+1} : \mathcal{B}(x, z) = 0\} \text{ if } \mathcal{M}^n = H^n \\ T_z &= z^\perp \subset \mathbb{R}^{n+1} \text{ if } \mathcal{M}^n = S^n \\ T_z &= \mathbb{R}^n \text{ if } \mathcal{M}^n = \mathbb{R}^n. \end{aligned}$$

We observe that T_z is an n -dimensional real vector space equipped with the scalar product $-\mathcal{B}(\cdot, \cdot)$ if $\mathcal{M}^n = H^n$, and with the scalar product $\langle \cdot, \cdot \rangle$ if $\mathcal{M}^n = S^n$ or $\mathcal{M}^n = \mathbb{R}^n$.

For $z \in \mathcal{M}^n$ and unit vector $u \in T_z$, the geodesic line ℓ passing through z and determined by u consists of the points

$$p_t = \begin{cases} z \cosh t + u \sinh t & \text{if } \mathcal{M}^n = H^n \\ z \cos t + u \sin t & \text{if } \mathcal{M}^n = S^n \\ z + tu & \text{if } \mathcal{M}^n = \mathbb{R}^n \end{cases} \quad (2)$$

for $t \in \mathbb{R}$. Here the map $t \mapsto p_t$ is bijective onto ℓ and satisfies $d_{\mathcal{M}^n}(z, p_t) = |t|$ for $t \in \mathbb{R}$ if $\mathcal{M}^n = H^n$ or $\mathcal{M}^n = \mathbb{R}^n$, and for $t \in (-\pi, \pi]$ if $\mathcal{M}^n = S^n$. If $t > 0$ provided $\mathcal{M}^n = H^n$ or $\mathcal{M}^n = \mathbb{R}^n$, or $0 < t < \pi$ provided $\mathcal{M}^n = S^n$, then we say that u points towards p_t along the geodesic segment

$$[z, p_t]_{\mathcal{M}^n} = \{p_s : 0 \leq s \leq t\}$$

of length t .

A hyperplane H in \mathcal{M}^n passing through the point $z \in \mathcal{M}^n$ and having unit normal $u \in T_z$, and the corresponding half-spaces H^+ and H^- where H^+ has u as the exterior unit normal are defined as follows: $H^- = \mathcal{M}^n \setminus \text{int} H^+$, and

$$\begin{aligned} H &= \{x \in H^n : \mathcal{B}(x, u) = 0\} & H^+ &= \{x \in H^n : -\mathcal{B}(x, u) \geq 0\} & \text{if } \mathcal{M}^n = H^n \\ H &= \{x \in S^n : \langle x, u \rangle = 0\} & H^+ &= \{x \in S^n : \langle x, u \rangle \geq 0\} & \text{if } \mathcal{M}^n = S^n \\ H &= \{x \in \mathbb{R}^n : \langle x, u \rangle = \langle z, u \rangle\} & H^+ &= \{x \in \mathbb{R}^n : \langle x, u \rangle \geq \langle z, u \rangle\} & \text{if } \mathcal{M}^n = \mathbb{R}^n. \end{aligned} \quad (3)$$

The reflection σ_H through H is the unique isometry of \mathcal{M}^n different from the identity fixing the points of H . In particular, for $x \in \mathcal{M}^n$ where $x \neq \pm u$ in the case $\mathcal{M}^n = S^n$, H is the hyperplane perpendicularly bisecting the segment $[x, \sigma_H x]_{\mathcal{M}^n}$ (H going through the midpoint of $[x, \sigma_H x]_{\mathcal{M}^n}$).

Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n . An important tool to obtain convex bodies with extremal properties is the Blaschke Selection Theorem. First we impose a metric on compact subsets. For a compact set $C \subset \mathcal{M}^n$ and $z \in \mathcal{M}^n$, we set $d_{\mathcal{M}^n}(z, C) = \min_{x \in C} d_{\mathcal{M}^n}(z, x)$. For any non-empty compact set $C_1, C_2 \subset \mathcal{M}^n$, we define their Hausdorff distance

$$\delta_{\mathcal{M}^n}(C_1, C_2) = \max \left\{ \max_{x \in C_2} d_{\mathcal{M}^n}(x, C_1), \max_{y \in C_1} d_{\mathcal{M}^n}(y, C_2) \right\}.$$

The Hausdorff distance is a metric on the space of compact subsets in \mathcal{M}^n . We say that a sequence $\{C_m\}$ of compact subsets of \mathcal{M}^n is bounded if there is a ball containing every C_m . For compact sets $C_m, C \subset \mathcal{M}^n$, we write $C_m \rightarrow C$ to denote if the sequence $\{C_m\}$ tends to C in terms of the Hausdorff distance.

The following statement characterizing limits of compact sets with respect to the Hausdorff distance well-known (see *e.g.* Böröczky, Sagmeister [14]).

Lemma 2.1. For compact sets $C_m, C \subset \mathcal{M}^n$ where \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , we have $C_m \rightarrow C$ if and only if

- (i): assuming $x_m \in C_m$, the sequence $\{x_m\}$ is bounded and any accumulation point of $\{x_m\}$ lies in C ; and
- (ii): for any $y \in C$, there exist $x_m \in C_m$ for each m such that $\lim_{m \rightarrow \infty} x_m = y$.

The space of compact subsets of \mathcal{M}^n is locally compact according to the Blaschke Selection Theorem (see R. Schneider [42]).

Theorem 2.2 (Blaschke). If \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , then any bounded sequence of compact subsets of \mathcal{M}^n has a convergent subsequence.

For convergent sequences of compact subsets of \mathcal{M}^n , we have the following (see Böröczky, Sagmeister [14]).

Lemma 2.3. Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let the sequence $\{C_m\}$ of compact subsets of \mathcal{M}^n tend to C .

- (i): $\text{diam}_{\mathcal{M}^n} C = \lim_{m \rightarrow \infty} \text{diam}_{\mathcal{M}^n} C_m$
- (ii): $V_{\mathcal{M}^n}(C) \geq \limsup_{m \rightarrow \infty} V_{\mathcal{M}^n}(C_m)$

Recall that for any compact set $X \subset \mathcal{M}^n$ and $\varrho \geq 0$, the parallel domain is

$$X^{(\varrho)} = \{z \in \mathcal{M}^n : \exists x \in X \text{ with } d_{\mathcal{M}^n}(x, z) \leq \varrho\} = \bigcup \{B(x, \varrho) : x \in X\}.$$

The triangle inequality and considering $x, y \in X$ with $d_{\mathcal{M}^n}(x, y) = \text{diam}_{\mathcal{M}^n} X$ lead to the following fundamental property of parallel domains.

Lemma 2.4. For $\varrho > 0$ and a compact $X \subset \mathcal{M}^n$ where \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and $2\varrho + \text{diam } X < \pi$ if $\mathcal{M}^n = S^n$, we have

$$\text{diam } X^{(\varrho)} = 2\varrho + \text{diam } X.$$

We discuss further properties of parallel domains based on Benyamini [7].

Lemma 2.5. If $\varrho \geq 0$ and \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and the sequence $\{C_m\}$ of compact subsets of \mathcal{M}^n tends to C , then

- (i): $\{C_m^{(\varrho)}\}$ tends to $C^{(\varrho)}$;
- (ii): $\text{diam}_{\mathcal{M}^n} C^{(\varrho)} = \lim_{m \rightarrow \infty} \text{diam}_{\mathcal{M}^n} C_m^{(\varrho)}$;
- (iii): $V_{\mathcal{M}^n}(C^{(\varrho)}) \geq \limsup_{m \rightarrow \infty} V_{\mathcal{M}^n}(C_m^{(\varrho)})$;

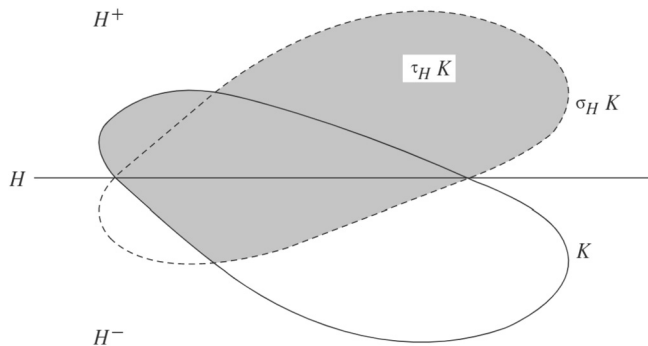


Fig. 1. Two point symmetrization.

(iv): for any $\varepsilon > 0$, $V_{\mathcal{M}^n}(C^{(\varrho)}) \leq \liminf_{m \rightarrow \infty} V_{\mathcal{M}^n}(C_m^{(\varrho+\varepsilon)})$.

Proof. We deduce (i) from Lemma 2.1, and in turn (ii) from Lemma 2.4 and (iii) from Lemma 2.3 (ii).

For (iv), we only observe that $C^{(\varrho)} \subset C_m^{(\varrho+\varepsilon)}$ if m is large. \square

3. Two-point symmetrization and a proof of Theorem 1.2 without the equality case

Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , let H^+ be a closed half-space bounded by the $(n-1)$ -dimensional subspace H in \mathcal{M}^n , and let $X \subset \mathcal{M}^n$ be compact. We write H^- to denote the other closed half-space of \mathcal{M}^n determined by H and $\sigma_H X$ to denote the reflected image of X through the $(n-1)$ -subspace H .

The two-point symmetrization $\tau_{H^+} X$ of X with respect to H^+ is a rearrangement of X by replacing $(H^- \cap X) \setminus \sigma_H X$ by its reflected image through H where readily this reflected image is disjoint from X . In particular, $\tau_{H^+} X$ can be defined by the properties Lemma 3.2 (i) and (ii) below. Naturally, interchanging the role of H^+ and H^- results in taking the reflected image of $\tau_{H^+} X$ through H . Since this operation does not change any relevant property of the new set, we simply use the notation $\tau_H X$ (see Fig. 1).

Two-point symmetrization appeared first in Wolontis [46]. It is applied to prove the isoperimetric inequality in the spherical space by Benyamini [7], and the spherical analogue of the Blaschke–Santaló inequality by Gao, Hug, Schneider [25] where a crucial step is verified by Aubrun, Fradelizi [5].

Two-point symmetrization does not lead to an object “more symmetric”, however, the definition directly yields that balls are invariant under two-point symmetrization (see Lemma 3.1).

Lemma 3.1. Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n and let H^+ be a half-space of \mathcal{M}^n .

(i): If $z \in H^+$ and $r > 0$ where $r < \pi$ provided $\mathcal{M}^n = S^n$, then $\tau_H B_{\mathcal{M}^n}(z, r) = B_{\mathcal{M}^n}(z, r)$;

(ii): if $Y \subset Z \subset \mathcal{M}^n$ compact, then $\tau_H Y \subset \tau_H Z$.

The following are additional simple properties of two-point symmetrization (see Böröczky, Sagmeister [14]).

Lemma 3.2. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , let H^+ be a half-space of \mathcal{M}^n , and let $X \subset \mathcal{M}^n$ be compact such that $\text{diam}_{\mathcal{M}^n}(X) < \pi$ if $\mathcal{M}^n = S^n$. Then $\tau_H X = \tau_{H^+} X$ is compact and satisfies*

- (i): $(\tau_H X) \cap H^+ = (X \cup \sigma_H X) \cap H^+$;
- (ii): $(\tau_H X) \cap H^- = (X \cap \sigma_H X) \cap H^-$;
- (iii): $V_{\mathcal{M}^n}(\tau_H X) = V_{\mathcal{M}^n}(X)$;
- (iv): $\text{diam}_{\mathcal{M}^n}(\tau_H X) \leq \text{diam}_{\mathcal{M}^n}(X)$.

Benyamini [7] proved the following property of parallel sets. Since [7] is hard to access, we provide a simple argument.

Lemma 3.3. *If $\varrho > 0$, \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and H^+ is a half-space of \mathcal{M}^n , then $V_{\mathcal{M}^n}((\tau_H X)^{(\varrho)}) \leq V_{\mathcal{M}^n}(X^{(\varrho)})$ for any compact $X \subset \mathcal{M}^n$.*

Proof. According to Lemma 3.2 (iii) applied to $X^{(\varrho)}$, it is sufficient to prove that

$$(\tau_H X)^{(\varrho)} \subset \tau_H \left(X^{(\varrho)} \right). \quad (4)$$

Let $z \in (\tau_H X)^{(\varrho)}$, and hence there exists $y \in \tau_H X$ such that $d(y, z) \leq \varrho$. Since the role of X and $\sigma_H X$ are symmetric in the definition of two-point symmetrization, we may assume that $y \in X$, and hence $z \in X^{(\varrho)}$.

If $z \in H^+$, then Lemma 3.2 (i) applied to $X^{(\varrho)}$ yields that $z \in \tau_H(X^{(\varrho)})$.

If $z \notin H^+$, then we use that

$$X^{(\varrho)} \cap \sigma_H(X^{(\varrho)}) \subset \tau_H(X^{(\varrho)})$$

according to Lemma 3.2 (i) and (ii) applied to $X^{(\varrho)}$. Therefore (4), and in turn the lemma follows if

$$z \in \sigma_H \left(X^{(\varrho)} \right) = (\sigma_H X)^{(\varrho)}. \quad (5)$$

If $y \in \sigma_H X$, then (5) readily holds. If $y \notin \sigma_H X$, then $y \in H^+$ by Lemma 3.2 (i), thus $d(z, \sigma_H y) \leq d(z, y) \leq \varrho$, proving (5), and in turn (4). \square

For any compact $X \neq \mathcal{M}^n$, let \mathcal{F}_X be the smallest closed subset of the space of compact subsets of \mathcal{M}^n equipped with the Hausdorff metric containing X and being

closed under two-point symmetrization. Benyamini [7] verified Lemma 3.4 whose proof we present for the convenience of reader.

Lemma 3.4. *If \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and $X \subset \mathcal{M}^n$, $X \neq \mathcal{M}^n$, is compact, then each $Y \in \mathcal{F}_X$ satisfies*

- (i): $\text{diam}_{\mathcal{M}^n} Y^{(\varrho)} \leq 2\varrho + \text{diam}_{\mathcal{M}^n} X$ for any $\varrho \geq 0$;
- (ii): $V_{\mathcal{M}^n}(Y^{(\varrho)}) \leq V_{\mathcal{M}^n}(X^{(\varrho)})$ for any $\varrho > 0$;
- (iii): $V_{\mathcal{M}^n}(Y) = V_{\mathcal{M}^n}(X)$.

Proof. For any ordinal ξ , we define the subset \mathcal{F}^ξ of the space of compact subsets of \mathcal{M}^n by transfinite recursion. Let $\mathcal{F}^0 = \{X\}$. For any ordinal ξ , we define

$$\mathcal{F}^{\xi+1} = \bigcup \{ \{C, \tau_{H^+}C\} : C \in \mathcal{F}^\xi \text{ and } H^+ \subset \mathcal{M}^n \text{ half-space} \}.$$

Finally, if ξ is a limit ordinal, then

$$\mathcal{F}^\xi = \bigcup \left\{ \lim_{m \rightarrow \infty} C_m : C_m \in \mathcal{F}^{\xi_m} \text{ where } \xi_m < \xi \text{ and } \lim_{m \rightarrow \infty} C_m \text{ exists} \right\}.$$

In particular, $\mathcal{F}^\alpha \subset \mathcal{F}^\xi$ if $\alpha < \xi$.

It follows from the definition above that if $C \in \mathcal{F}^\xi$ for a countable ordinal ξ and $C \neq X$, then

- either there exists $\alpha < \xi$, $Z \in \mathcal{F}^\alpha$ and a half-space H^+ such that $C = \tau_{H^+}Z$;
- or there exist $\xi_m < \xi$ for $m \in \mathbb{N}$ and $C_m \in \mathcal{F}^{\xi_m}$ such that $\lim_{m \rightarrow \infty} C_m = C$.

We claim that

$$\mathcal{F}_X = \bigcup \{ \mathcal{F}^\xi : \xi \text{ countable ordinal} \}. \quad (7)$$

Readily, $\mathcal{F} = \bigcup \{ \mathcal{F}^\xi : \xi \text{ countable ordinal} \} \subset \mathcal{F}_X$, and \mathcal{F} is closed under two-point symmetrization. To show that \mathcal{F} is a closed subset of the space of compact subsets of \mathcal{M}^n , let $\lim_{m \rightarrow \infty} C_m = C$ for $C_m \in \mathcal{F}$ and compact $C \subset \mathcal{M}^n$. Then $C_m \in \mathcal{F}^{\xi_m}$ for some countable ordinals ξ_m , and hence $\bigcup_{m \in \mathbb{N}} \xi_m$ is also countable. Let ξ be the smallest (and hence countable) ordinal at least $\bigcup_{m \in \mathbb{N}} \xi_m$, thus $C \in \mathcal{F}^\xi$, proving (7).

We deduce from (6), (7) and transfinite induction that each $Y \in \mathcal{F}_X$ satisfies (i) and

$$V_{\mathcal{M}^n}(Y) \geq V_{\mathcal{M}^n}(X) \quad (8)$$

where (i) follows from Lemma 2.5 (ii), Lemma 3.2 (iv) and Lemma 2.4, and $V_{\mathcal{M}^n}(Y) \geq V_{\mathcal{M}^n}(X)$ follows from Lemma 2.3 (ii) and Lemma 3.2 (iii)

For (ii), we again use transfinite induction, so let $Y \in \mathcal{F}^\xi$ with $Y \neq X$ for a countable ordinal ξ . If $Y = \tau_{H^+}Z$ where $Z \in \mathcal{F}^\alpha$ for $\alpha < \xi$ and H^+ is a half-space, then

$V_{\mathcal{M}^n}(Y^{(\varrho)}) \leq V_{\mathcal{M}^n}(Z^{(\varrho)}) \leq V_{\mathcal{M}^n}(X^{(\varrho)})$ follows from Lemma 3.3. Otherwise there exist $\xi_m < \xi$ and $C_m \in \mathcal{F}^{\xi_m}$ for $m \in \mathbb{N}$ such that $\lim_{m \rightarrow \infty} C_m = Y$. For a $\varepsilon > 0$, it follows from transfinite induction that $V_{\mathcal{M}^n}(C_m^{(\varrho+\varepsilon)}) \leq V_{\mathcal{M}^n}(X^{(\varrho+\varepsilon)})$, thus Lemma 2.5 (iv) yields

$$V_{\mathcal{M}^n}(Y^{(\varrho)}) \leq \liminf_{m \rightarrow \infty} V_{\mathcal{M}^n}(C_m^{(\varrho+\varepsilon)}) \leq V_{\mathcal{M}^n}(X^{(\varrho+\varepsilon)}).$$

Letting ε tending to zero, we deduce (ii).

For (iii), it follows from (ii) that

$$V_{\mathcal{M}^n}(Y) = \lim_{\varrho \rightarrow 0^+} V_{\mathcal{M}^n}(Y^{(\varrho)}) \leq \lim_{\varrho \rightarrow 0^+} V_{\mathcal{M}^n}(X^{(\varrho)}) = V_{\mathcal{M}^n}(X).$$

Therefore (8) implies (iii). \square

Lemma 3.5. *If \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and $X \subset \mathcal{M}^n$ is compact with $X \neq \mathcal{M}^n$ and $V_{\mathcal{M}^n}(X) > 0$, then there exists $B_{\mathcal{M}^n}(z, r) \in \mathcal{F}_X$ with $V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(z, r)) = V_{\mathcal{M}^n}(X)$ for some $r > 0$ and $z \in \mathcal{M}^n$.*

Proof. We choose $z \in \mathcal{M}^n$ and $r > 0$ such that $V(X \cap B) > 0$ and $V(B) = V(X)$ for $B = B(z, r)$, and let

$$\tilde{F} = \{C \in \mathcal{F}_X : C \cap B \neq \emptyset\},$$

which is a closed subset of \mathcal{F}_X .

Setting $D = \text{diam}(X)$, we have $Y \subset B(z, r + D)$ for $Y \in \tilde{F}$. Let

$$v = \sup\{V(Y \cap B) : Y \in \tilde{F}\},$$

and let $C_m \in \tilde{F}$, $m \in \mathbb{N}$ such that $\lim_{m \rightarrow \infty} V(C_m \cap B) = v$. Since $C_m \subset B(z, r + D)$ by Lemma 3.2 (iv), we may assume according to the Blaschke Selection Theorem 2.2 that C_m tends to a $C \in \tilde{F}$, and $C_m \cap B$ tends to a compact set Z . It follows from Lemma 2.1 that $Z \subset C$, thus Lemma 2.3 (ii) implies

$$V(C \cap B) \geq V(Z) \geq \limsup_{m \rightarrow \infty} V(C_m \cap B) = v,$$

thus $V(C \cap B) = v$.

We suppose that $C \neq B$, and hence $V(W) < V(B)$ holds for $W = C \cap B$, and seek a contradiction. We choose an $x \in (\text{int } B) \setminus C$. Since $V(C) = V(B)$, we have $V(C \setminus B) > 0$, therefore there exists a density point $y \in C \setminus B$; namely,

$$\lim_{t \rightarrow 0^+} \frac{V[(C \setminus B) \cap B(y, t)]}{V[B(y, t)]} = 1.$$

Let H be the hyperplane in \mathcal{M}^n such that $\sigma_H x = y$, and let H^+ be the half-space determined by H containing x . Since $x \in B$ and $y \notin B$, we have $z \in H^+$. We choose

$\varrho > 0$ such that $B(x, \varrho) \subset (H^+ \cap \text{int } B) \setminus C$ and $B(y, \varrho) \cap B = \emptyset$, and in particular, we have $V(B(y, \varrho) \cap C) > 0$ by the choice of y .

It follows from Lemma 3.1 that $\tau_{H^+}B = B$ and $\tau_{H^+}W \subset B \cap \tau_{H^+}C$, and Lemma 3.2 (iii) yields $V(\tau_{H^+}W) = v$. In addition, $B(y, \varrho) \cap B = \emptyset$ and $V(B(y, \varrho) \cap C) > 0$ imply that $B(x, \varrho) \cap \tau_{H^+}W = \emptyset$ and $V(B(x, \varrho) \cap \tau_{H^+}C) > 0$, therefore $V(B \cap \tau_{H^+}C) > v$. Since $\tau_{H^+}C \in \tilde{F}$, we have arrived at a contradiction, proving that $B = C \in \mathcal{F}_X$. Finally, $V(B) = V(X)$ by Lemma 3.4 (iii). \square

Proofs of Theorem 1.1 and Theorem 1.2 without the characterization of equality. For any compact set $X \subset \mathcal{M}^n$, $X \neq \mathcal{M}^n$, the family \mathcal{F}_X contains a ball $B_{\mathcal{M}^n}(z, r)$ of the same volume as X by Lemma 3.5. In addition, Lemma 3.4 implies that $\text{diam}_{\mathcal{M}^n} B_{\mathcal{M}^n}(z, r) \leq \text{diam}_{\mathcal{M}^n} X$ and $V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(z, r + \varrho)) \leq V_{\mathcal{M}^n}(X^{(e)})$ for any $\varrho > 0$. \square

4. D -hull, D -maximal sets and the uniqueness in Theorem 1.2

Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $D > 0$ where $D < \pi$ if $\mathcal{M}^n = S^n$. If $\text{diam}_{\mathcal{M}^n} X \leq D$ holds for $X \subset \mathcal{M}^n$, then we define its D -hull to be

$$D\text{-hull } X = \bigcap \{B_{\mathcal{M}^n}(z, D) : z \in \mathcal{M}^n \text{ and } X \subset B_{\mathcal{M}^n}(z, D)\}. \quad (9)$$

Lemma 4.1. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $D > 0$ where $D < \pi$ if $\mathcal{M}^n = S^n$. If $\text{diam}_{\mathcal{M}^n} X \leq D$ holds for $X \subset \mathcal{M}^n$, then $\text{diam}_{\mathcal{M}^n} D\text{-hull } X \leq D$.*

Proof. First we claim that if $p \in D\text{-hull } Z$ for a set $Z \subset \mathcal{M}^n$ with $\text{diam } Z \leq D$, then

$$d(x, p) \leq D \text{ for } x \in Z. \quad (10)$$

Here (10) follows from the fact that $p \in B(x, D)$ for any $x \in Z$.

Next let $y, z \in D\text{-hull } X$. We observe that $\text{diam } X' \leq D$ for $X' = X \cup \{y\}$ by (10), and $D\text{-hull } X' = D\text{-hull } X$ by the choice of y . In particular, $z \in D\text{-hull } X'$, thus (10) yields that $d(y, z) \leq D$. \square

For $D > 0$ where $D < \pi$ if $\mathcal{M}^n = S^n$, we say that a compact set $C \subset \mathcal{M}^n$ is D -maximal if $\text{diam}_{\mathcal{M}^n} C \leq D$ and

$$V_{\mathcal{M}^n}(C) = \sup\{V_{\mathcal{M}^n}(X) : X \subset \mathcal{M}^n \text{ compact and } \text{diam}_{\mathcal{M}^n} X \leq D\}.$$

We observe that a ball in \mathcal{M}^n of diameter D is a D -maximal set.

Lemma 4.2. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , let $D > 0$ where $D < \pi$ if $\mathcal{M}^n = S^n$ and let $X \subset \mathcal{M}^n$ with $\text{diam}_{\mathcal{M}^n} X \leq D$.*

(i): *There exists a D -maximal set Z in \mathcal{M}^n containing X .*

(ii): For any D -maximal set C in \mathcal{M}^n and $z \in \partial_{\mathcal{M}^n} C$, there exists $y \in \partial_{\mathcal{M}^n} C$ such that $d_{\mathcal{M}^n}(z, y) = D$.

Proof. Let $\{C_m\}$ be a sequence of compact subsets of \mathcal{M}^n with $X \subset C_m$, $\text{diam } C_m \leq D$ and

$$\lim_{m \rightarrow \infty} V(C_m) = \sup\{V(Y) : Y \subset \mathcal{M}^n \text{ compact and } X \subset Y \text{ and } \text{diam } Y \leq D\}.$$

According to the Blaschke Selection Theorem (cf. Theorem 2.2), we may assume that the sequence $\{C_m\}$ tends to a compact subset $Z \subset \mathcal{M}^n$. Here Z is a D -maximal set by Lemma 2.3.

Next let C be any D -maximal set in \mathcal{M}^n , and let $z \in \partial_{\mathcal{M}^n} C$. We suppose that $\tilde{D} = \max_{x \in C} d(z, x) < D$, and seek a contradiction. As $z \in \partial C$, there exists some $y \in B_{\mathcal{M}^n}(z, \frac{1}{2}(D - \tilde{D})) \setminus C$, and hence $B(y, r) \cap C = \emptyset$ for some $r \in (0, \frac{1}{2}(D - \tilde{D}))$. Therefore $C_0 = C \cup B(z, r)$ satisfies that $V(C_0) > V(C)$ and $\text{diam } C_0 \leq D$, which is a contradiction verifying (ii). \square

For a compact set $C \subset \mathcal{M}^n$ with $C \neq S^n$, we say that $x \in \partial_{\mathcal{M}^n} C$ is strongly regular if there exist $r > 0$ and $y, z \in \mathcal{M}^n$ such that $x = B_{\mathcal{M}^n}(y, r) \cap B_{\mathcal{M}^n}(z, r) = C \cap B_{\mathcal{M}^n}(z, r)$ and $B_{\mathcal{M}^n}(y, r) \subset C$. In this case, the exterior unit normal $N_C(x) \in T_x$ to C at x is the unit vector such that $-N_C(x)$ points towards y along $[x, y]_{\mathcal{M}^n}$. We deduce from Lemma 4.2 that boundary points of parallel domains of D -maximal sets are strongly regular.

Lemma 4.3. If $\varrho > 0$, \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , $D > 0$ with $D + \varrho < \pi$ if $\mathcal{M}^n = S^n$ and $X \subset \mathcal{M}^n$ is a D -maximal set, then any $x \in \partial_{\mathcal{M}^n} X^{(\varrho)}$ is strongly regular.

The upcoming Claim 4.4 provides a tool to distinguish between points on the boundary of a ball or outside of the boundary ball, and this tool is useful to understand boundary structure of a two-point symmetrization.

Claim 4.4. For $B = B_{\mathcal{M}^n}(y_0, R)$ where \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n and $R < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, let $y_1, y_2 \in \partial_{\mathcal{M}^n} B$ and $p \in \mathcal{M}^n$ such that $y_0 \in [y_1, y_2]_{\mathcal{M}^n}$ (and hence $[y_1, y_2]_{\mathcal{M}^n}$ is a diameter of B) and the geodesic line of $[y_1, y_2]_{\mathcal{M}^n}$ does not contain p . For the hyperplane H_i in \mathcal{M}^n perpendicularly bisecting the geodesic segments $[p, y_i]_{\mathcal{M}^n}$, $i = 1, 2$, we have $\sigma_{H_1} N_B(y_1) = \sigma_{H_2} N_B(y_2)$ if and only if $p \in \partial B$.

Proof. We note that p, y_1, y_2 span a two dimensional subspace Π . Since both H_1 and H_2 are orthogonal to Π if $n \geq 3$, we may actually assume that $n = 2$ and $\Pi = \mathcal{M}^2$. In particular, H_1 and H_2 are lines in this case.

We observe that $\sigma_{H_2} \sigma_{H_1} y_1 = \sigma_{H_2} p = y_2$. Since $\sigma_{H_2} \sigma_{H_2}$ is the identity, $\sigma_{H_1} N_B(y_1) = \sigma_{H_2} N_B(y_2)$ is equivalent to $\sigma_{H_2} \sigma_{H_1} N_B(y_1) = N_B(y_2)$. As $-N_B(y_1) \in T_{y_1}$ points towards y_2 along the segment $[y_1, y_2]_{\mathcal{M}^2}$, and $-N_B(y_2) \in T_{y_2}$ points towards y_1 along

the segment $[y_2, y_1]_{\mathcal{M}^2}$, we deduce that $\sigma_{H_1} N_B(y_1) = \sigma_{H_2} N_B(y_2)$ is equivalent to $\sigma_{H_2} \sigma_{H_1}([y_1, y_2]_{\mathcal{M}^2}) = ([y_2, y_1]_{\mathcal{M}^2})$.

Now if $p \in \partial B$, then $d_{\mathcal{M}^2}(y_1, y_0) = d_{\mathcal{M}^2}(p, y_0) = d_{\mathcal{M}^2}(y_2, y_0)$; therefore, $\{y_0\} = H_1 \cap H_2$ and H_1 and H_2 are orthogonal. It follows that $\sigma_{H_2} \sigma_{H_1}$ is a rotation around y_0 of angle π , and hence $\sigma_{H_2} \sigma_{H_1}([y_1, y_2]_{\mathcal{M}^2}) = ([y_2, y_1]_{\mathcal{M}^2})$.

Finally, let $\sigma_{H_1} N_B(y_1) = \sigma_{H_2} N_B(y_2)$, thus $\sigma_{H_2} \sigma_{H_1}([y_1, y_2]_{\mathcal{M}^2}) = ([y_2, y_1]_{\mathcal{M}^2})$ yields $\sigma_{H_2} \sigma_{H_1} y_0 = y_0$. If H_1 and H_2 do not intersect, then $\mathcal{M}^2 = H^2$, and $\sigma_{H_2} \sigma_{H_1}$ has no fixed points; therefore, this case can't occur. In particular, H_1 and H_2 intersect in a point q . In this case, $\sigma_{H_2} \sigma_{H_1}$ is a non-trivial rotation around q , and hence $\sigma_{H_2} \sigma_{H_1} y_0 = y_0$ yields that $\{y_0\} = \{q\} = H_1 \cap H_2$. We deduce that $d_{\mathcal{M}^2}(y_1, y_0) = d_{\mathcal{M}^2}(p, y_0) = d_{\mathcal{M}^2}(y_2, y_0)$, thus $p \in \partial B$. \square

If \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and $x_1, x_2, x_3 \in \mathcal{M}^n$ are not contained in a geodesic line, then the triangle $[x_1, x_2, x_3]_{\mathcal{M}^n}$ with vertices x_1, x_2, x_3 is the union of all geodesic segments starting from x_1 and ending at a point of $[x_2, x_3]_{\mathcal{M}^n}$. Actually, it does not matter which vertex is x_1 , we always obtain the same object. Provided $\{i, j, k\} = \{1, 2, 3\}$, the angle of the triangle at x_i is the angle of the geodesic segments $[x_i, x_j]_{\mathcal{M}^n}$ and $[x_i, x_k]_{\mathcal{M}^n}$.

Proof of the uniqueness result in Theorem 1.2. Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $y_0 \in M^n$. We drop the index notation \mathcal{M}^n as we always have a fixed ambient space.

For $D > 0$ where $D < \pi$ if $\mathcal{M}^n = S^n$, we choose $\varrho > 0$ such that $D + 2\varrho < \pi$ if $\mathcal{M}^n = S^n$. For a D -maximal set X , which exists by Lemma 4.2 (i), we have $V(X) \geq V(B(y_0, \frac{D}{2}))$ according to Theorem 1.2. It follows from the Isoperimetric Inequality Theorem 1.1 that $V(X^{(\varrho)}) \geq V(B(y_0, \frac{D}{2} + \varrho))$, thus $\text{diam } X^{(\varrho)} \leq D + 2\varrho$ implies that $Y = X^{(\varrho)}$ is $(D + 2\varrho)$ -maximal.

Let $x_1, x_2 \in \partial X$ satisfy that $d(x_1, x_2) = D$. It follows that there exist $y_1, y_2 \in \partial Y$ such that $x_1, x_2 \in [y_1, y_2]$, and $d(x_i, y_i) = \varrho$ for $i = 1, 2$. We may assume that y_0 is the midpoint of $[y_1, y_2]$, and we claim first that

$$B\left(y_0, \frac{D}{2} + \varrho\right) \subset Y. \quad (11)$$

We suppose that (11) does not hold, and seek a contradiction. Since $B(x_i, \varrho) \subset X^{(\varrho)}$ for $i = 1, 2$, we deduce the existence of a

$$p \in \left(\partial Y \cap \text{int} B\left(y_0, \frac{D}{2} + \varrho\right)\right) \setminus [y_1, y_2].$$

For $i = 1, 2$, let H_i be the hyperplane perpendicularly bisecting the segment $[y_i, p]$, and let H_i^+ be the corresponding half-space with $p \in \text{int } H_i^+$. We choose $r \in (0, \varrho)$ such that

$$B(p, r) \subset H_1^+ \cap H_2^+.$$

For $i = 1, 2$, $-N_Y(y_i)$ points along the segment $[y_1, y_2]$. Since $d(p, y_0) < d(y_i, y_0)$ yields $\angle(y_0, p, y_i) > \angle(y_0, y_i, p)$ in the triangle $[y_0, p, y_i]$, we have that $-\sigma_{H_i} N_Y(y_i) \in T_p$ points along a segment connecting p and a point of the segment $[y_0, y_i] \subset [y_1, y_2]$. Therefore

$$\sigma_{H_1} N_Y(y_1) \neq -\sigma_{H_2} N_Y(y_2).$$

In addition, $\sigma_{H_1} N_Y(y_1) \neq \sigma_{H_2} N_Y(y_2)$ according to Claim 4.4, therefore after possibly interchanging y_1 and y_2 , we may assume that

$$N_Y(p) \neq \pm \sigma_{H_1} N_Y(y_1). \quad (12)$$

We claim that

$$p \in \partial \tau_{H_1} Y. \quad (13)$$

Let $p' \in X$ such that $p \in \partial B(p', \varrho) \subset Y$. According to Lemma 4.2 (ii), there exists a $q_0 \in Y$ such that $d(p, q) = D + 2\varrho$, and hence $Y \subset B(q_0, D + 2\varrho)$. It follows that $-N_Y(p)$ points towards the segment $[p, q_0]$ and $p' \in [p, q_0]$. On the other hand, $p \in \partial B(x'_1, \varrho) \subset \sigma_{H_1} Y$ for $x'_1 = \sigma_{H_1} x_1 \in [p, \sigma_{H_1} y_2]$, and $\sigma_{H_1} Y \subset B(\sigma_{H_1} y_2, D + 2\varrho)$. Therefore,

$$\begin{aligned} N_Y(p) &= N_{B(q_0, D+2\varrho)}(p) = N_{B(p', \varrho)}(p) \\ N_{\sigma_{H_1} Y}(p) &= \sigma_{H_1} N_Y(y_1) = N_{B(\sigma_{H_1} y_2, D+2\varrho)}(p) = N_{B(x'_1, \varrho)}(p). \end{aligned} \quad (14)$$

Since $Y \subset B(q_0, D + 2\varrho)$ and $\sigma_{H_1} Y \subset B(\sigma_{H_1} y_2, D + 2\varrho)$, $N_Y(p) \neq -\sigma_{H_1} N_Y(y_1)$ (cf. (12)) yields that $p \in \partial(Y \cup \sigma_{H_1} Y)$, which in turn implies (13).

It follows from Lemma 3.2 that $\tau_{H_1} Y$ is also D -maximal, and hence (13) and Lemma 4.2 (ii) yield the existence of a $q \in \tau_{H_1} Y$ such that $d(p, q) = D + 2\varrho$ and $\tau_{H_1} Y \subset B(q, D + 2\varrho)$. However, both $B(x'_1, \varrho) \cap H^+ \subset \tau_H Y$ and $B(p', \varrho) \cap H^+ \subset \tau_H Y$, therefore (see also (14))

$$N_Y(p) = N_{B(p', \varrho)}(p) = N_{B(q, D+2\varrho)}(p) = N_{B(x'_1, \varrho)}(p) = \sigma_{H_1} N_Y(y_1).$$

This contradicts (12), and in turn proves (11).

Since $\text{diam } Y \leq D + 2\varrho$, (11) implies $X^{(\varrho)} = Y = B(y_0, \frac{D}{2} + \varrho)$, therefore $X = B(y_0, \frac{D}{2})$. \square

Remark. The condition $N_Y(p) \neq -\sigma_{H_1} N_Y(y_1)$ in (12) is only needed to prove $p \in \partial \tau_{H_1} Y$ (cf. (13)) when $\mathcal{M}^n = S^n$ and $D \geq \frac{\pi}{2}$. Otherwise, (13) simply follows from $Y \subset B(q_0, D + 2\varrho)$ and $\sigma_{H_1} Y \subset B(\sigma_{H_1} y_2, D + 2\varrho)$.

5. Convex sets and complete sets

In this section, we start our preparation for the proof of the stability version of the Isodiametric Inequality. Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n . We call $X \subset \mathcal{M}^n$ convex if

$[x, y]_{\mathcal{M}^n} \subset X$ for any $x, y \in X$, and in addition, we also assume that X is contained in an open hemisphere if $\mathcal{M}^n = S^n$. For $Z \subset \mathcal{M}^n$ where we assume that Z is contained in an open hemisphere if $\mathcal{M}^n = S^n$, the convex hull $\text{conv}_{\mathcal{M}^n} Z$ is the intersection of all convex sets containing Z .

Lemma 5.1. *If either $\mathcal{M}^n = H^n$ or $\mathcal{M}^n = \mathbb{R}^n$ and $r > 0$, or $\mathcal{M}^n = S^n$ and $r \in (0, \frac{\pi}{2})$, then $B_{\mathcal{M}^n}(z, r)$ is convex for any $z \in \mathcal{M}^n$.*

We remark that $B_{S^n}(z, r)$ is not convex if $r \in [\frac{\pi}{2}, \pi)$.

Lemma 5.2. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $X \subset \mathcal{M}^n$ be compact and satisfy $\text{diam } X < \frac{\pi}{2}$ in the case of $\mathcal{M}^n = S^n$. Then*

- (i): $\text{diam}_{\mathcal{M}^n} \text{conv}_{\mathcal{M}^n} X = \text{diam}_{\mathcal{M}^n} X$;
- (ii): $\text{conv}_{\mathcal{M}^n} X$ is compact;
- (iii): $V_{\mathcal{M}^n}(\text{conv}_{\mathcal{M}^n} X) > V_{\mathcal{M}^n}(X)$ if $V_{\mathcal{M}^n}(\text{conv}_{\mathcal{M}^n} X) > 0$ and $\text{conv}_{\mathcal{M}^n} X \neq X$.
- (iv): For $D \geq \text{diam}_{\mathcal{M}^n} X$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, the D -hull of X is convex.

Proof. For (i), let $\text{diam } X = D$, and let $x_1, x_2 \in \text{conv}_{\mathcal{M}^n} X$. Since $\text{conv}_{\mathcal{M}^n} X \neq X$ is the intersection of all convex sets containing X and $X \subset B_{\mathcal{M}^n}(x, D)$ for $x \in X$ where $B_{\mathcal{M}^n}(x_1, D)$ is convex by Lemma 5.1, we have $X \subset B_{\mathcal{M}^n}(x_1, D)$, and in turn even $\text{conv}_{\mathcal{M}^n} X \subset B_{\mathcal{M}^n}(x_1, D)$. Therefore $d_{\mathcal{M}^n}(x_1, x_2) \leq D$, proving (i).

Turning to (ii), if $\mathcal{M}^n = \mathbb{R}^n$, then (ii) follows from Carathéodory's theorem (see Schneider [42]); namely, that $\text{conv}_{\mathbb{R}^n} X$ is the union of the convex hulls of subsets of X of cardinality at most $n + 1$. Using suitable radial projection yields the statement in the spherical and in the hyperbolic case.

For (iii), we have $V(Z) > 0$ for $Z = \text{conv}_{\mathcal{M}^n} X$ and $Z \neq X$. As Z is convex, compact and $V(Z) > 0$, it follows that $\text{int} Z \neq \emptyset$ and the closure of $\text{int} Z$ is Z . Since X is compact and $X \neq Z$, there exists some $z \in (\text{int} Z) \setminus X$. Therefore $B_{\mathcal{M}^n}(z, r) \subset (\text{int} Z) \setminus X$ for some $r > 0$, verifying that $V(Z) > V(X)$.

Finally, (iv) follows from the definition (9) of the D -hull. \square

Let $X \subset \mathcal{M}^n$ be compact, and let $z \in \partial_{\mathcal{M}^n} X$. We say that H is a supporting hyperplane at z to X if $z \in H$ and X lies in one of the closed half-spaces determined by H . In addition, a ball $B_{\mathcal{M}^n}(y, r)$ is a supporting ball at z if $z \in B_{\mathcal{M}^n}(y, r)$ but $X \cap \text{int}_{\mathcal{M}^n} B_{\mathcal{M}^n}(y, r) = \emptyset$.

Lemma 5.3. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $K \subset \mathcal{M}^n$ be convex compact.*

- (i): If $x \notin K$, and $z \in K$ is a closest point to x , then the hyperplane H passing through z and orthogonal to $[x, z]_{\mathcal{M}^n}$ is a supporting hyperplane to K .
- (ii): If $z \in \partial_{\mathcal{M}^n} K$ then there exists a supporting hyperplane H at z .

Remarks. The closest point of a compact convex set to an external point is unique in the Euclidean and the hyperbolic cases, but may not be unique in the spherical case.

We also note that if $\varrho > 0$ and K is convex in the Euclidean space \mathbb{R}^n or in the hyperbolic space H^n , then $K^{(\varrho)}$ is convex. On the other hand, in the spherical space S^n , the parallel domain of a spherical segment is not convex.

Proof. For (i), it is sufficient to prove that if $y \in K$ with $y \neq z$, and $\gamma = \angle(x, z, y)$, then

$$\gamma \geq \frac{\pi}{2}. \quad (15)$$

Let $d_{\mathcal{M}^n}(x, z) = a$ and $d_{\mathcal{M}^n}(y, z) = b$. For any $t \in [0, b]$, let $y_t \in [z, y]_{\mathcal{M}^n}$ be the point with $d_{\mathcal{M}^n}(y_t, z) = t$, and hence $d_{\mathcal{M}^n}(y_t, x) \geq d_{\mathcal{M}^n}(z, x)$. Using the Law of Cosines in the triangle $[x, y, z]_{\mathcal{M}^n}$, we deduce that if $\mathcal{M}^n = \mathbb{R}^n$, then

$$0 \leq \frac{d}{dt} d_{\mathbb{R}^n}(y_t, x)^2 \Big|_{t=0^+} = \frac{d}{dt} (a^2 + t^2 - 2at \cos \gamma) \Big|_{t=0^+} = -2a \cos \gamma,$$

if $\mathcal{M}^n = H^n$, then

$$\begin{aligned} 0 &\leq \frac{d}{dt} \cosh d_{H^n}(y_t, x) \Big|_{t=0^+} = \frac{d}{dt} (\cosh a \cdot \cosh t - \sinh a \cdot \sinh t \cdot \cos \gamma) \Big|_{t=0^+} \\ &= -\sinh a \cdot \cos \gamma, \end{aligned}$$

and if $\mathcal{M}^n = S^n$, then

$$0 \geq \frac{d}{dt} \cos d_{S^n}(y_t, x) \Big|_{t=0^+} = \frac{d}{dt} (\cos a \cdot \cos t + \sin a \cdot \sin t \cdot \cos \gamma) \Big|_{t=0^+} = \sin a \cdot \cos \gamma.$$

Therefore, $\cos \gamma \leq 0$, proving (15), and in turn verifying (i).

To prove (ii), we consider a sequence $\{x_m\}$ with $x_m \notin K$ tending to z , and a closest point $z_m \in K$ to x_m for each z_m . It follows that $\lim_{m \rightarrow \infty} z_m = z$. For each z_m , there exists a supporting hyperplane H_m to K passing through z_m according to (i), and it follows from applying the Blaschke Selection Theorem to $H_m \cap B_{\mathcal{M}^n}(z, 1)$ that some subsequence $\{H_{m'}\}$ of the sequence $\{H_m\}$ tend to a supporting hyperplane H to K at z . \square

Let $K \subset \mathcal{M}^n$ be a compact convex set, and let $p \in \partial_{\mathcal{M}^n} K$. We say that a non-zero $v \in T_p$ is an exterior normal to K at p if there exists a supporting hyperplane H of K passing through p and orthogonal to v , and $-v$ points towards the closed half-space bounded by H and containing K . According to Lemma 5.3, there exists some exterior unit normal vector $v \in T_p$ at each $p \in \partial_{\mathcal{M}^n} K$. If $p \in \partial_{\mathcal{M}^n} K$ is strongly regular in the sense of Lemma 4.3; namely, there exist $r > 0$ and $y \in \mathcal{M}^n$ such that $B_{\mathcal{M}^n}(y, r) \subset K$ and $x \in B_{\mathcal{M}^n}(y, r)$, then $N_K(x) \in T_x$ is an exterior normal also in this new sense.

We call a bounded set $K \subset \mathcal{M}^n$ *complete* if $\text{diam}_{\mathcal{M}^n}(K \cup \{y\}) > \text{diam}_{\mathcal{M}^n}(K)$ holds for each $y \in \mathcal{M}^n \setminus K$. Readily, any complete set is closed.

If $\mathcal{M}^n = S^n$, then there exist some unexpected examples of complete sets with obtuse diameter. For example, the $n + 2$ vertices of an regular Euclidean $(n + 1)$ -dimensional simplex inscribed into S^n form a complete set of diameter $\arccos \frac{-1}{n+1}$. Therefore, we will consider only complete sets of acute diameter in the spherical case.

Let \mathcal{M}^n be either \mathbb{R}^n , S^n or H^n , and let $D > 0$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$. We call a compact convex set $K \subset \mathcal{M}^n$ with non-empty interior a convex body. We say that a convex body $K \subset \mathcal{M}^n$ is a *convex body of constant width* D if $\text{diam}_{\mathcal{M}^n}(K) = D$, and for any $p \in \partial_{\mathcal{M}^n} K$ and any outer unit normal $v \in T_p$ to K at p , the geodesic segment $[p, q]_{\mathcal{M}^n}$ of length D and pointing in the direction of $-v$ is contained in K . The following statement is well-known (see e.g. Dekster [17] or Böröczky, Sagmeister [15]).

Lemma 5.4. *Let \mathcal{M}^n be either \mathbb{R}^n , S^n or H^n , and let $K \subset \mathcal{M}^n$ be a compact set of diameter $D > 0$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$. The following are equivalent:*

- (i): K is a convex body of constant width D ;
- (ii): K is complete;
- (iii): $K = \bigcap_{x \in K} B_{\mathcal{M}^n}(x, D)$.

Theorem 4.2 guarantees the existence of D -maximal sets, and Lemmas 5.2 and 5.4 yield

Corollary 5.5. *If \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and $D > 0$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, then any D -maximal set in \mathcal{M}^n is complete, and hence convex.*

For a convex body K in \mathcal{M}^n where \mathcal{M}^n be either \mathbb{R}^n , S^n or H^n , the circumradius $R(K)$ is the minimal radius of any ball of \mathcal{M}^n containing K . It is well-known (see Dekster [17] or Böröczky, Sagmeister [15]) that there exists a unique so-called circumcenter $p \in \mathcal{M}^n$ with $K \subset B_{\mathcal{M}^n}(p, R(K))$, and actually $p \in K$. In addition, the inradius $r(K)$ is the maximal radius of any n -dimensional ball in K . There might be various balls of maximal radius contained in K in the Euclidean and the hyperbolic case, as it is shown by the example when K is the parallel domain of a segment. The following nice connection between the inradius and the circumradius of complete sets in any space of constant curvature is proved by Dekster [17] (see also Böröczky, Sagmeister [15]).

Proposition 5.6. *Let \mathcal{M}^n be either \mathbb{R}^n , S^n or H^n . If $K \subset \mathcal{M}^n$ is a complete set of diameter $D > 0$ where $D < \frac{\pi}{2}$ in the case $\mathcal{M}^n = S^n$, then*

$$R(K) + r(K) = D.$$

Furthermore, K contains a unique inscribed ball of radius $r(K)$ whose center is the circumcenter c of K , and there exists some diameter $[x_1, x_2]_{\mathcal{M}^n}$ of K that contains c where $d_{\mathcal{M}^n}(x_1, c) = R(K)$ and $d_{\mathcal{M}^n}(x_2, c) = r(K)$.

Now we show that the parallel domains of convex bodies of constant width are also convex bodies of constant width.

Lemma 5.7. *If $\varrho > 0$, \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n , and $K \subset \mathcal{M}^n$ is a convex body of constant width $D > 0$ where $D + 2\varrho < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, then $K^{(\varrho)}$ is a convex body of constant width $D + 2\varrho$ satisfying $R(K^{(\varrho)}) = R(K) + \varrho$, $r(K^{(\varrho)}) = r(K) + \varrho$ and*

$$K^{(\varrho)} = \bigcap_{x \in K} B_{\mathcal{M}^n}(x, D + \varrho). \quad (16)$$

Proof. We drop the reference to \mathcal{M}^n in the formulas to simplify notation. First we verify (16). For any $y \in K$, we have $B(y, \varrho) \subset B(x, D + \varrho)$ for any $x \in K$; therefore, $K^{(\varrho)} \subset \bigcap_{x \in K} B(x, D + \varrho)$.

On the other hand, let $z \in \mathcal{M}^n \setminus K^{(\varrho)}$, and let $y \in K$ be closest to z . In particular, $d(z, y) > \varrho$. We deduce from Lemma 5.3 (i) that the unit vector $v \in T_y$ pointing towards z along $[y, z]$ is an exterior unit normal to K at y , and hence there exists $x \in K$ such that $-v \in T_x$ points towards x along $[y, x]$ and $d(x, y) = D$. It follows that $d(x, z) = d(x, y) + d(y, z) > D + \varrho$, completing the proof of (16).

Now (16) and $D + 2\varrho < \frac{\pi}{2}$ yield that $K^{(\varrho)}$ is convex, and Lemma 2.4 implies that $\text{diam } K^{(\varrho)} = D + 2\varrho < \frac{\pi}{2}$.

To verify that $K^{(\varrho)}$ has constant width $D + 2\varrho$, let $z \in \partial K^{(\varrho)}$, and let $v \in T_z$ be an exterior unit normal to $K^{(\varrho)}$ at z . Let $y \in K$ be closest to z , and hence $d(z, y) = \varrho$. Since $B(y, \varrho) \subset K^{(\varrho)}$, $-v \in T_z$ points towards y along $[z, y]$. For the unit vector $w \in T_y$ pointing towards z along $[y, z]$, w is an exterior unit normal to K at y by Lemma 5.3 (i), thus there exists $x_0 \in K$ such that $-v \in T_{x_0}$ points towards x_0 along $[y, x_0]$ and $d(x_0, y) = D$. For the $x \in \partial B(x_0, \varrho) \subset K^{(\varrho)}$ satisfying that $y, x_0 \in [z, x]$, we have $d(x, z) = D + 2\varrho$, verifying that $K^{(\varrho)}$ has constant width $D + 2\varrho$. \square

6. Angles and balls

The main tool to prove Proposition 7.4 is Lemma 6.3. However, we first point out the following two simple observations.

Claim 6.1. *The sum of any two angles of a triangle T is less than π provided that either T is a triangle in \mathbb{R}^n or H^n , or T lies in S^n and each side is less than $\pi/2$.*

Proof. The only non-trivial case is when T lies in S^n . In this case, we may assume that $n = 2$. Let γ be the third angle of T , and let T' be the spherical triangle having an angle

γ enclosed by two sides of lengths $\pi/2$. Since $V(T) < V(T') = \gamma$, and $\pi + V(T)$ is the sum of the angles of T , we conclude Claim 6.1. \square

Claim 6.2. *Given $y_0, y_1, z \in \mathcal{M}^n$ not lying on a geodesic arc and $w \in [z, y_0]_{\mathcal{M}^n} \setminus \{z, y_0\}$ where \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , we have*

$$\angle(y_0, y_1, z) - \angle(y_0, z, y_1) > \angle(y_0, y_1, w) - \angle(y_0, w, y_1).$$

Proof. We may assume that $n = 2$. If $\mathcal{M}^2 = H^2$ or $\mathcal{M}^2 = \mathbb{R}^2$, then applying Claim 6.1 to the triangle $[z, w, y_1]_{\mathcal{M}^2}$ shows that $\angle(y_0, z, y_1) < \angle(y_0, w, y_1)$. Since readily $\angle(y_0, y_1, z) > \angle(y_0, y_1, w)$, we deduce Claim 6.2 in this case.

Therefore let $\mathcal{M}^2 = S^2$. Writing Y, Z, W to denote the angles of the triangle $T = [y_1, z, w]_{\mathcal{M}^2}$ at y_1, z, w , respectively, we deduce that

$$\angle(y_0, y_1, z) - \angle(y_0, z, y_1) - (\angle(y_0, y_1, w) - \angle(y_0, w, y_1)) = Y - Z + \pi - W = 2Y - V(T).$$

However, the “two-gon” M containing T and bounded by the two half-circles connecting y_1 and its antipodal $-y_1$ and passing through z and w , respectively, has area $2Y$, thus

$$2Y - V(T) = V(M) - V(T) > 0,$$

completing the proof of Claim 6.2. \square

Lemma 6.3. *For $B = B_{\mathcal{M}^n}(y_0, R)$ where \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n and $R < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, let $y_1 \in \partial B$ and $p \in \mathcal{M}^n$ such that $0 < \angle(y_1, y_0, p) \leq \frac{\pi}{2}$, let $u \in T_p$ denote the unit vector such that $-u$ points along the segment $[p, y_0]_{\mathcal{M}^n}$ and let the hyperplane H in \mathcal{M}^n be the perpendicular bisector of the geodesic segment $[p, y_1]_{\mathcal{M}^n}$.*

If $d(p, y_0) \geq R + \eta$ where $\eta \in (0, \frac{R}{2})$ provided $\mathcal{M}^n = H^n$ or $\mathcal{M}^n = \mathbb{R}^n$, and $\eta \in (0, \min\{\frac{R}{3}, \frac{\pi}{6} - \frac{R}{3}\})$ provided $\mathcal{M}^n = S^n$, then the angle of $\sigma_H N_B(y_1)$ and u in T_p is at least

$$\begin{aligned} \frac{2 \cdot \eta}{\sqrt{26} \cdot R} &> \frac{\eta}{3R} && \text{if } \mathcal{M}^n = \mathbb{R}^n, \\ \frac{\sinh \eta}{\sinh(\frac{5}{2}R) \cdot \sqrt{2} \cosh R} &> \frac{\eta}{\sinh 5R} && \text{if } \mathcal{M}^n = H^n, \\ \frac{3 \cdot \eta}{8\sqrt{2} \cdot R} &> \frac{\eta}{4R} && \text{if } \mathcal{M}^n = S^n. \end{aligned}$$

Proof. We observe that the angle of $\sigma_H N_B(y_1)$ and u in T_p is $|\angle(y_0, y_1, p) - \angle(y_0, p, y_1)|$. Let $p_0 \in [p, y_0]_{\mathcal{M}^n}$ such that $d(p_0, y_0) = R + \eta$, and let $q = \partial B \cap [p, y_0]_{\mathcal{M}^n}$. Readily

$$\angle(y_0, y_1, q) = \angle(y_0, q, y_1), \tag{17}$$

and it follows from Claim 6.2 and (17) that

$$\angle(y_0, y_1, p) - \angle(y_0, p, y_1) > \angle(y_0, y_1, p_0) - \angle(y_0, p_0, y_1) > \angle(y_0, y_1, q) - \angle(y_0, q, y_1) = 0. \quad (18)$$

Therefore Lemma 6.3 holds in Case 1 if

$$\angle(y_0, y_1, p_0) - \angle(y_0, p_0, y_1) \geq \tilde{\gamma}\eta \quad (19)$$

for the suitable $\tilde{\gamma} > 0$ depending on \mathcal{M}^n and R .

We write P, Q, Y to denote the angles of the triangle $[p_0, q, y_1]_{\mathcal{M}^n}$ at p_0, q, y_1 . Since $\angle(q, y_0, y_1) \leq \frac{\pi}{2}$ in the triangle $[q, y_0, y_1]_{\mathcal{M}^n}$ where the other two angles are $\pi - Q$, we have

$$\tan Q \leq \begin{cases} \frac{-1}{\cosh R} & \text{if } \mathcal{M}^n \text{ is } H^n, \\ -1 & \text{if } \mathcal{M}^n \text{ is } \mathbb{R}^n \text{ or } S^n. \end{cases}$$

In particular, as $\cosh R \geq 1$, we have

$$\sin Q \geq \begin{cases} \frac{1}{\sqrt{2} \cosh R} & \text{if } \mathcal{M}^n \text{ is } H^n, \\ \frac{1}{\sqrt{2}} & \text{if } \mathcal{M}^n \text{ is } \mathbb{R}^n \text{ or } S^n. \end{cases} \quad (20)$$

On the one hand, first applying (17), and after that Claim 6.1 imply that

$$\angle(y_0, y_1, p_0) - \angle(y_0, p_0, y_1) = (Y + \pi - Q) - P > Y. \quad (21)$$

On the other hand, we define

$$f_{\mathcal{M}^n}(t) = \begin{cases} t & \text{if } \mathcal{M}^n = \mathbb{R}^n, \\ \sinh t & \text{if } \mathcal{M}^n = H^n, \\ \sin t & \text{if } \mathcal{M}^n = S^n. \end{cases}$$

We observe that $d_{\mathcal{M}^n}(p_0, y_1) < \frac{5}{2}R$ if $\mathcal{M}^n = H^n$, and $d_{\mathcal{M}^n}(q, y_1) < R$ if $\mathcal{M}^n = S^n$. For $\mathcal{M}^n = \mathbb{R}^n$ since $Q \in (\frac{\pi}{2}, \frac{3\pi}{4}]$, using the Law of Cosines for the triangle $[p_0, q, y_1]_{\mathbb{R}^n}$ we deduce $d_{\mathbb{R}^n}(p_0, y_1) < \frac{\sqrt{13}}{2}R$. For $\mathcal{M}^n = S^n$ using the Law of Cosines in the triangle $[p_0, q, y_1]_{S^n}$ we have

$$\begin{aligned} \cos(d_{S^n}(p_0, y_1)) &= \cos(\eta) \cdot \cos(d_{S^n}(q, y_1)) + \sin(\eta) \cdot \sin(d_{S^n}(q, y_1)) \cdot \cos(Q) \geq \\ &\geq \cos(\eta) \cdot \cos(R) - \frac{1}{\sqrt{2}} \cdot \sin(\eta) \cdot \sin(R) > \cos(R + \eta) \end{aligned}$$

and since by the choice of η we know that $R + \eta < \frac{\pi}{2}$, from this we get

$$\frac{1}{f_{S^n}(d_{S^n}(p_0, y_1))} > \frac{1}{\sin(R + \eta)} > \frac{3}{4 \cdot R}$$

also using the upper bound $\eta < \frac{R}{3}$. Substituting $d_{\mathcal{M}^n}(p_0, q) = \eta$ in the Law of Sines for the triangle $[p, q, y_1]_{\mathcal{M}^n}$ and using (20) and $\frac{t}{2} < \sin t < t$ for $t \in (0, \frac{\pi}{2})$, we deduce that

$$Y > \sin Y = \frac{f_{\mathcal{M}^n}(d_{\mathcal{M}^n}(p_0, q))}{f_{\mathcal{M}^n}(d_{\mathcal{M}^n}(p_0, y_1))} \cdot \sin Q > \begin{cases} \frac{2 \cdot \eta}{\sqrt{26} \cdot R} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \\ \frac{\sinh \eta}{\sinh(\frac{5}{2}R) \cdot \sqrt{2} \cosh R} & \text{if } \mathcal{M}^n = H^n, \\ \frac{3 \cdot \eta}{8\sqrt{2} \cdot R} & \text{if } \mathcal{M}^n = S^n. \end{cases}$$

Therefore if $d(p, y_0) \geq R + \eta$, then Lemma 6.3 follows from (19) and (21). \square

7. The gap between two-point symmetrization and its convex hull

We write $\angle(x, y)$ to denote the angle of $x, y \in \mathbb{R}^n \setminus \{o\}$. In Proposition 7.4, we use the following observation to estimate the extra volume from below.

Claim 7.1. *For $r > 0$ and $w_1, w_2 \in \mathbb{R}^n$ with $0 < \|w_1 - w_2\| < 2r$, let $p \in \partial B(w_1, r) \cap \partial B(w_2, r)$, and for $i = 1, 2$, let $H_i^+ = \{x \in \mathbb{R}^n : \langle x, p - w_i \rangle \geq \langle p, p - w_i \rangle\}$. If $\angle(w_1, p, w_2) = \alpha$ satisfies $0 < \alpha_0 \leq \alpha \leq \alpha_1 < \pi$ and $c_n = \frac{1}{2^{4n} n^n}$, then*

$$V(\text{conv}(B(w_1, r) \cup B(w_2, r)) \cap H_1^+ \cap H_2^+) \geq c_n r^n \alpha_0^{n+1} \cos \frac{\alpha_1}{2}.$$

Proof. Let $\omega_{-1} = 1$ and ω_k be the k dimensional Lebesgue measure of S^k for $k \geq 0$. First we verify that

$$V(\text{conv}(B(w_1, r) \cup B(w_2, r)) \cap H_1^+ \cap H_2^+) \geq \tilde{c}_n \cdot \frac{(1 - \cos \frac{\alpha}{2})^{\frac{n}{2}+1} \cos \frac{\alpha}{2}}{\alpha/2} \cdot r^n \quad (22)$$

where $\tilde{c}_2 = 1$ and $\tilde{c}_n = \frac{2\omega_{n-3}}{n(n-1)(n-2)}$. Let $w = \frac{1}{2}(w_1 + w_2)$, and let $u = \frac{p-w}{\|p-w\|} \in S^{n-1}$. For $q = w + ru$, we observe that $p \in [w, q]$ and $\|q - p\| = r(1 - \cos \frac{\alpha}{2})$. For $i = 1, 2$, we consider $v_i = w_i + ru \in \partial B(w_i, r)$, and the point $\tilde{v}_i \in [v_1, v_2]$ with $\langle \tilde{v}_i, p - w_i \rangle = \langle p, p - w_i \rangle$, and hence

$$\|\tilde{v}_i - q\| = \frac{\|q - p\|}{\tan \frac{\alpha}{2}} = \frac{1 - \cos \frac{\alpha}{2}}{\tan \frac{\alpha}{2}} \cdot r.$$

It follows that the area of the triangle $[p, \tilde{v}_1, \tilde{v}_2]$ is

$$\frac{(1 - \cos \frac{\alpha}{2})^2}{\tan \frac{\alpha}{2}} \cdot r^2 \geq \frac{(1 - \cos \frac{\alpha}{2})^2 \cos \frac{\alpha}{2}}{\alpha/2} \cdot r^2.$$

If $n = 2$, then we have verified (22). If $n \geq 3$, then let L be the affine $(n-2)$ -space passing through p and orthogonal to the linear two-space $\text{lin}\{v_1, v_2\}$ containing all of our points

defined so far. Then L intersects $\text{conv}(B(w_1, r) \cup B(w_2, r))$ in an $(n-2)$ -dimensional ball centered at p and of radius

$$\sqrt{\|q-p\| \cdot (2r - \|q-p\|)} \geq \sqrt{r^2 \left(1 - \cos \frac{\alpha}{2}\right)},$$

completing the proof of (22).

Since $\left(\frac{1-\cos t}{t}\right)' = \frac{t \sin t + \cos t - 1}{t^2} > 0$ for $t \in (0, \frac{\pi}{2})$, we deduce that $\frac{(1-\cos \frac{\alpha}{2})^{\frac{n}{2}+1}}{\alpha/2}$ is monotonically increasing in $\alpha \in (0, \pi)$. However, $1 - \cos t = \frac{(\sin t)^2}{1 + \cos t} \geq \frac{t^2}{8}$ for $t \in (0, \frac{\pi}{2})$, therefore (22) yields Claim 7.1 with $c'_n = \tilde{c}_n/2^{\frac{5n}{2}+4}$ instead of c_n .

If $n = 2$, then $c'_n > c_n$ and Claim 7.1 readily holds. If $n \geq 3$, then we use that $\omega_{k-1} = \frac{k\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$ holds for $k \geq 1$ and that $\Gamma(t+1) < (\frac{t}{e})^t \sqrt{2\pi(t+1)}$ holds for $t \geq \frac{1}{2}$ according to Artin [4] to conclude

$$\begin{aligned} c'_n &= \frac{2\omega_{n-3}}{2^{\frac{5n}{2}+4}n(n-1)(n-2)} = \frac{\pi^{\frac{n}{2}-1}}{2^{\frac{5n}{2}+3}n(n-1)\Gamma(\frac{n}{2})} > \frac{(2e\pi)^{\frac{n}{2}-1}}{2^{\frac{5n}{2}+3}n(n-1)(n-2)^{\frac{n}{2}-1}\sqrt{\pi n}} \\ &> \left(\frac{e\pi}{2^4}\right)^{\frac{n}{2}} \frac{1}{2^4 e \pi^{\frac{3}{2}} n^n} > \frac{1}{2^{\frac{n}{2}+8} n^n} > \frac{1}{2^{4n} n^n} = c_n \end{aligned}$$

where $\frac{e\pi}{2^4} > \frac{1}{2}$ and $e\pi^{\frac{3}{2}} < 2^4$. \square

To compare a bounded portion of the hyperbolic or the spherical geometry to Euclidean geometry, we consider the map $\varphi: \mathbb{R}^{n+1} \setminus e^\perp \rightarrow e + e^\perp$ defined by

$$\varphi(x) = \frac{x}{\langle x, e \rangle}.$$

Writing B^{n+1} to denote the unit Euclidean ball centered at the origin in \mathbb{R}^{n+1} and $S_+^n = \{x \in S^n : \langle x, e \rangle > 0\}$, we consider the diffeomorphisms

$$\begin{aligned} \varphi_{H^n} &: H^n \rightarrow e + (e^\perp \cap \text{int } B^{n+1}) \\ \varphi_{S^n} &: S_+^n \rightarrow e + e^\perp \end{aligned}$$

that are the restrictions of φ to H^n and S^n , respectively. We observe that φ_{H^n} and φ_{S^n} map convex subsets of H^n and S_+^n , respectively into convex subsets of $e + e^\perp$. We observe that if \mathcal{M}^n is either H^n or S^n , then for any ball $B_{\mathcal{M}^n}(z, r)$ in \mathcal{M}^n , there exists a hyperplane Π in \mathbb{R}^{n+1} such that $\partial_{\mathcal{M}^n} B_{\mathcal{M}^n}(z, r) = \Pi \cap \mathcal{M}^n$. Since $\varphi|_\Pi$ maps lines into lines, it is a projective map $\Pi \rightarrow e + e^\perp$, therefore we have

$$\varphi_{H^n} B_{H^n}(z, r) \text{ is an ellipsoid for } z \in H^n \text{ and } r > 0, \quad (23)$$

$$\varphi_{S^n} B_{S^n}(z, r) \text{ is an ellipsoid for } z \in S_+^n \text{ and } r > 0 \text{ with } B_{S^n}(z, r) \subset S_+^n. \quad (24)$$

It follows from (2) that for $z \in H^n$ with $d_{H^n}(e, z) = r$ or $z \in S_+^n$ with $d_{S^n}(e, z) = r < \frac{\pi}{2}$, if $u \in T_z$, then

$$\frac{\sqrt{-\mathcal{B}(u, u)}}{(\cosh r)^2} \leq \|D\varphi_{H^n}(z)u\| \leq \frac{\sqrt{-\mathcal{B}(u, u)}}{\cosh r} \text{ and } |\det D\varphi_{H^n}(z)| = (\cosh r)^{-(n+1)} \quad (25)$$

$$\frac{\|u\|}{(\cos r)^2} \geq \|D\varphi_{S^n}(z)u\| \geq \frac{\|u\|}{\cos r} \text{ and } |\det D\varphi_{S^n}(z)| = (\cos r)^{-(n+1)} \quad (26)$$

where $D\varphi_{H^n}(z)$ and $D\varphi_{S^n}(z)$ stand for the differentials at z , which are linear maps $T_z \rightarrow e^\perp$ (here T_z is equipped with the scalar product $-\mathcal{B}(\cdot, \cdot)$ if $\mathcal{M}^n = H^n$).

If \mathcal{M}^n is either H^n or S^n , then we observe that $\partial_{\mathcal{M}^n} B_{\mathcal{M}^n}(z, r)$ is contained in a hyperplane of \mathbb{R}^{n+1} for $z \in \mathcal{M}^n$ and $r > 0$ (even $r \in (0, \pi)$ provided $\mathcal{M}^n = S^n$); namely

$$\begin{aligned} \partial_{H^n} B_{H^n}(z, r) &= \{x \in H^n : \mathcal{B}(x, z) = \cosh r\} \\ \partial_{S^n} B_{S^n}(z, r) &= \{x \in S^n : \langle x, z \rangle = \cos r\}. \end{aligned} \quad (27)$$

Lemma 7.2. *Let $u \in T_e = e^\perp$ with $\|u\| = 1$ where \mathcal{M}^n is either H^n or S^n (in particular, $\mathcal{B}(u, u) = -1$ provided $\mathcal{M}^n = H^n$).*

- (i): *If $\mathcal{M}^n = H^n$, $r > 0$ and $z = e \cosh r + u \sinh r \in H^n$ (and hence $e \in \partial B_{H^n}(z, r)$), then $\varphi_{H^n} B_{H^n}(z, r) \subset e + T_e$ is an ellipsoid having e on its boundary with exterior normal $-u$ such that one of its principal axes is parallel to u and is of length $a = \frac{\sinh 2r}{\cosh 2r}$, and all other principal axes are of length $b = \frac{2 \sinh r}{\sqrt{\cosh 2r}}$ where $\frac{\sinh 2r}{\cosh 2r} < \frac{2 \sinh r}{\sqrt{\cosh 2r}}$.*
- (ii): *If $\mathcal{M}^n = S^n$, $r \in (0, \frac{\pi}{4})$ and $z_0 = e \cos r + u \sin r \in S^n$ (and hence $e \in \partial B_{S^n}(z, r)$), then $\varphi_{S^n} B_{S^n}(z, r) \subset e + T_e$ is an ellipsoid having e on its boundary with exterior normal $-u$ such that one of its principal axes is parallel to u and is of length $a = \frac{\sin 2r}{\cos 2r}$, and all other principal axes are of length $b = \frac{2 \sin r}{\sqrt{\cos 2r}}$ where $\frac{\sin 2r}{\cos 2r} > \frac{2 \sin r}{\sqrt{\cos 2r}}$.*

Proof. In both cases, we consider an orthonormal basis e_0, \dots, e_n of \mathbb{R}^{n+1} where $e_0 = e$ and $e_1 = u$, and for $x \in \mathbb{R}^{n+1}$, we write its coordinates with respect to e_0, \dots, e_n . In particular, for $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ with $x_0 \neq 0$, we have $\varphi(x) = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$. We observe that the ellipsoid $\varphi_{\mathcal{M}^n} B_{\mathcal{M}^n}(z, r)$ in $e + T_e$ has an axial rotational symmetry around the line $e + \mathbb{R}u$. It follows that $-u$ is an exterior normal to the ellipsoid at the boundary point e , and one of the principal axes of the ellipsoid is parallel to u .

If $\mathcal{M}^n = H^n$, then $e \cosh 2r + u \sinh 2r \in H^n$ is the diametrically opposite point of $B_{H^n}(z, r)$ to e ; therefore, the principal axis of the ellipsoid $\varphi_{H^n} B_{H^n}(z, r)$ parallel to u is of length $a = \frac{\sinh 2r}{\cosh 2r}$.

To determine the common length b of the principal axes of the ellipsoid $\varphi_{H^n} B_{H^n}(z, r)$ orthogonal to u , we note that according to (27), $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ satisfies $x \in \partial_{H^n} B_{H^n}(z, r)$ if and only if

$$\begin{aligned} 1 + x_1^2 + \dots + x_n^2 &= x_0^2 \\ x_0 \cdot \cosh r - x_1 \cdot \sinh r &= \cosh r. \end{aligned} \quad (28)$$

We deduce from (28) and the symmetries of $\varphi_{H^n} B_{H^n}(z, r)$ that

$$\begin{aligned} \left(\frac{b}{2}\right)^2 &= \max \left\{ \frac{x_0^2}{x_0^2} : 1 + x_1^2 + x_2^2 = x_0^2 \text{ and } x_0 \cdot \cosh r - x_1 \cdot \sinh r = \cosh r \right\} \\ &= \max \left\{ \frac{x_0^2 - (x_0 - 1)^2 \frac{(\cosh r)^2}{(\sinh r)^2} - 1}{x_0^2} : x_0 \geq 1 \right\} \\ &= \max \left\{ 1 - \frac{(\cosh r)^2}{(\sinh r)^2} + \frac{2}{x_0} \cdot \frac{(\cosh r)^2}{(\sinh r)^2} - \frac{1}{x_0^2} \cdot \frac{(\cosh r)^2 + (\sinh r)^2}{(\sinh r)^2} : x_0 \geq 1 \right\}. \end{aligned} \quad (29)$$

For $f(t) = \frac{2}{t} \cdot \frac{(\cosh r)^2}{(\sinh r)^2} - \frac{1}{t^2} \cdot \frac{(\cosh r)^2 + (\sinh r)^2}{(\sinh r)^2}$, we have $f'(t) = \frac{2}{t^3} \cdot \frac{1}{(\sinh r)^2} ((\cosh r)^2 + (\sinh r)^2 - t(\cosh r)^2)$, and hence f has its maximum at $t = \frac{(\cosh r)^2 + (\sinh r)^2}{(\cosh r)^2}$. Therefore, (29) and $(\cosh r)^2 - (\sinh r)^2 = 1$ imply

$$\begin{aligned} \left(\frac{b}{2}\right)^2 &= \frac{-1}{(\sinh r)^2} + \frac{2(\cosh r)^2}{(\cosh r)^2 + (\sinh r)^2} \cdot \frac{(\cosh r)^2}{(\sinh r)^2} - \frac{(\cosh r)^4}{((\cosh r)^2 + (\sinh r)^2)(\sinh r)^2} \\ &= \frac{(\cosh r)^2((\cosh r)^2 - 1) - (\sinh r)^2}{((\cosh r)^2 + (\sinh r)^2)(\sinh r)^2} = \frac{(\sinh r)^2}{(\cosh r)^2 + (\sinh r)^2}. \end{aligned}$$

Since $\cosh 2r = (\cosh r)^2 + (\sinh r)^2$, we have $b = \frac{2 \sinh r}{\sqrt{\cosh 2r}}$.

To compare a and b , using $\sinh 2r = 2(\sinh r)(\cosh r)$ yields $\frac{a}{b} = \frac{\frac{\cosh r}{\sqrt{\cosh 2r}}}{\frac{2 \sinh r}{\sqrt{\cosh 2r}}} = \frac{\cosh r}{2 \sinh r} < 1$.

Let us sketch the argument, analogously to the one above, in the spherical case $\mathcal{M}^n = H^n$ and $0 < r < \frac{\pi}{4}$. In particular, $e \cos 2r + u \sin 2r \in S^n$ is the diametrically opposite point of $B_{S^n}(z, r)$ to e , and hence the principal axis of the ellipsoid $\varphi_{S^n} B_{S^n}(z, r)$ parallel to u is of length $a = \frac{\sin 2r}{\cos 2r}$.

To determine the common length b of the principal axes of the ellipsoid $\varphi_{S^n} B_{S^n}(z, r)$ orthogonal to u , we note that according to (27), $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ satisfies $x \in \partial_{S^n} B_{S^n}(z, r)$ if and only if

$$\begin{aligned} x_0^2 + x_1^2 + \dots + x_n^2 &= 1 \\ x_0 \cdot \cos r + x_1 \cdot \sin r &= \cos r. \end{aligned} \quad (30)$$

Using $(\cos r)^2 + (\sin r)^2 = 1$ and similar argument as in the hyperbolic case, we deduce that

$$b = \frac{2 \sin r}{\sqrt{(\cos r)^2 - (\sin r)^2}} = \frac{2 \sin r}{\sqrt{\cos 2r}}.$$

To compare a and b , using $\sin 2r = 2 \sin r \cos r$ yields $\frac{a}{b} = \frac{\frac{\cos r}{\sqrt{\cos 2r}}}{\frac{2 \sin r}{\sqrt{\cos 2r}}} = \frac{\cos r}{2 \sin r} = \frac{\cos r}{\sqrt{(\cos r)^2 - (\sin r)^2}} > 1$. \square

Lemma 7.3. *If $[y_0, y_1, p]$ is a triangle in \mathcal{M}^n where \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , and $\angle(y_0, y_1, p) \leq \frac{\pi}{2}$, $d_{\mathcal{M}^n}(y_0, y_1) = R$ and $d_{\mathcal{M}^n}(y_0, p) \geq R + \eta$ for $R > 0$ and $\eta \geq 0$, then*

$$d_{\mathcal{M}^n}(y_1, p) \geq \begin{cases} \sqrt{2R \cdot \eta} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \\ \sqrt{\tanh R \cdot \eta} & \text{if } \mathcal{M}^n = H^n, \\ \sqrt{\tan R \cdot \eta} & \text{if } \mathcal{M}^n = S^n \text{ and } R, \eta \leq \frac{\pi}{4}. \end{cases}$$

Proof. We use $\angle(y_0, y_1, p) \leq \frac{\pi}{2}$ and the Law of Cosines in \mathcal{M}^n .

If $\mathcal{M}^n = \mathbb{R}^n$, then

$$d_{\mathbb{R}^n}(y_1, p) \geq \sqrt{(R + \eta)^2 - R^2} \geq \sqrt{2R\eta}.$$

If $\mathcal{M}^n = H^n$, then

$$\cosh d_{H^n}(y_1, p) \geq \frac{\cosh(R + \eta)}{\cosh R} = \cosh \eta + \tanh R \cdot \sinh \eta.$$

Therefore, Lemma 7.3 follows in the case $\mathcal{M}^n = H^n$ if

$$\cosh \left(\sqrt{\tanh R} \cdot \sqrt{\eta} \right) \leq \cosh \eta + \tanh R \cdot \sinh \eta. \quad (31)$$

The equation (31) readily holds if $\eta = 0$. Thus using differentiation, it is sufficient to verify

$$\sinh \left(\sqrt{\tanh R} \cdot \sqrt{\eta} \right) \cdot \frac{\sqrt{\tanh R}}{2\sqrt{\eta}} \leq \sinh \eta + \tanh R \cdot \cosh \eta \quad (32)$$

for $\eta > 0$. If $\eta \in (0, \tanh R]$, then we use that $\sinh t/t$ is increasing for $t > 0$, $\tanh R < 1$ and $\sinh 1 < 2$, and hence

$$\begin{aligned} \sinh \left(\sqrt{\tanh R} \cdot \sqrt{\eta} \right) \cdot \frac{\sqrt{\tanh R}}{2\sqrt{\eta}} &= \frac{\sinh \left(\sqrt{\tanh R} \cdot \sqrt{\eta} \right)}{\sqrt{\tanh R} \cdot \sqrt{\eta}} \cdot \frac{\tanh R}{2} \\ &< \tanh R < \tanh R \cdot \cosh \eta. \end{aligned}$$

On the other hand, if $\eta \geq \tanh R$, then

$$\sinh \left(\sqrt{\tanh R} \cdot \sqrt{\eta} \right) \cdot \frac{\sqrt{\tanh R}}{2\sqrt{\eta}} < \sinh \eta,$$

proving (32), and in turn (31).

If $\mathcal{M}^n = S^n$, then we use a similar argument as in the hyperbolic case; in particular, we have

$$\cos d_{S^n}(y_1, p) \leq \frac{\cos(R + \eta)}{\cos R} = \cos \eta - \tan R \cdot \sin \eta.$$

Therefore, Lemma 7.3 follows in the case $\mathcal{M}^n = S^n$ if

$$\cos \left(\sqrt{\tan R} \cdot \sqrt{\eta} \right) \geq \cos \eta - \tan R \cdot \sin \eta. \quad (33)$$

The equation (33) readily holds if $\eta = 0$. Thus using differentiation, it is sufficient to verify

$$\sin \left(\sqrt{\tan R} \cdot \sqrt{\eta} \right) \cdot \frac{\sqrt{\tan R}}{2\sqrt{\eta}} \leq \sin \eta + \tan R \cdot \cos \eta \quad (34)$$

for $\eta > 0$. If $\eta \leq \tan R$ where $\tan R \leq 1$ because of $R \leq \frac{\pi}{4}$, then we use that $\sin t < t$ for $t > 0$ and $\cos 1 > \cos \frac{\pi}{3} = \frac{1}{2}$, and hence

$$\sin \left(\sqrt{\tan R} \cdot \sqrt{\eta} \right) \cdot \frac{\sqrt{\tan R}}{2\sqrt{\eta}} \leq \frac{\tan R}{2} < \tan R \cdot \cos \eta.$$

On the other hand, if $\eta \geq \tan R$, then

$$\sin \left(\sqrt{\tan R} \cdot \sqrt{\eta} \right) \cdot \frac{\sqrt{\tan R}}{2\sqrt{\eta}} < \sin \eta,$$

proving (34), and in turn (33). \square

Proposition 7.4. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $X \subset \mathcal{M}^n$ be a convex body of constant width D for $D > 0$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$. In addition, let $x_1, x_2 \in X$ satisfy $d_{\mathcal{M}^n}(x_1, x_2) = D$, let y_0 be the midpoint of $[x_1, x_2]_{\mathcal{M}^n}$, and let*

$$\varrho = \begin{cases} \frac{D}{2} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = H^n, \\ \min \left\{ \frac{D}{2}, \frac{\pi}{8} - \frac{D}{4} \right\} & \text{if } \mathcal{M}^n = S^n \text{ and } D < \frac{\pi}{2}. \end{cases}$$

If there exists $z \in \partial_{\mathcal{M}^n} X$ such that $d_{\mathcal{M}^n}(z, y_0) \geq \frac{D}{2} + \eta$ for $\eta \in (0, \eta_0)$, then there exists a hyperplane H such that

$$V_{\mathcal{M}^n} \left(\text{conv } \tau_H X^{(\varrho)} \right) - V_{\mathcal{M}^n}(\tau_H X^{(\varrho)}) \geq \tilde{\gamma}_1 \eta^{\frac{3n+2}{2}}$$

where

$$\tilde{\gamma}_1 = \begin{cases} \frac{1}{2^{12n} n^n} \cdot \frac{1}{D^{\frac{n+2}{2}}} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \text{ and } D < \frac{\pi}{2}; \\ \frac{1}{2^{8n} n^n} \cdot \frac{1}{(\sinh 10D)^{\frac{n+2}{2}}} & \text{if } \mathcal{M}^n = H^n; \end{cases}$$

$$\eta_0 = \begin{cases} \frac{D}{2} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \text{ and } D \leq \frac{\pi}{6}; \\ \frac{3}{4\pi} \left(\frac{\pi}{2} - D\right)^2 & \text{if } \mathcal{M}^n = S^n \text{ and } \frac{\pi}{6} \leq D < \frac{\pi}{2}; \\ \min \left\{1, \frac{D}{2}\right\} & \text{if } \mathcal{M}^n = H^n. \end{cases}$$

Proof. We may assume that z is the farthest point of X from y_0 , and hence

$$X \subset B_{\mathcal{M}^n}(y_0, d_{\mathcal{M}^n}(z, y_0)). \quad (35)$$

Let $Y = X^{(\varrho)}$ that is a convex body of diameter $D + 2\varrho$ according to Lemma 5.7. There exist $y_1, y_2 \in \partial_{\mathcal{M}^n} Y$ such that $x_1, x_2 \in [y_1, y_2]_{\mathcal{M}^n}$ and $d_{\mathcal{M}^n}(x_i, y_i) = \varrho$, $i = 1, 2$. It follows from (35) that there exists $p \in \partial_{\mathcal{M}^n} Y$ such that $z \in [p, y_0]_{\mathcal{M}^n}$, and hence $-N_Y(p) \in T_p$ points towards y_0 along $[p, y_0]_{\mathcal{M}^n}$ and $d_{\mathcal{M}^n}(p, y_0) \geq \frac{D}{2} + \varrho + \eta$. We may assume that $\angle(p, y_0, y_1) \leq \frac{\pi}{2}$ after possibly interchanging y_1 and y_2 .

Let H be the perpendicular bisector hyperplane in \mathcal{M}^n of the segment $[p, y_1]_{\mathcal{M}^n}$, let H^+ be the corresponding half-space containing p , and let $\tilde{x} = \sigma_H x_1$. We observe that $p \in B_{\mathcal{M}^n}(\tilde{x}, \varrho) \subset \sigma_H Y$ and $p \in B_{\mathcal{M}^n}(z, \varrho) \subset Y$.

It follows from applying Lemma 6.3 with $R = \frac{D}{2} + \varrho \leq D$ that for $N_Y(p) \in T_p$ and $\sigma_H N_Y(y_1) = N_{\sigma_H Y}(p) \in T_p$, we have

$$\angle(p, y_1, y_0) - \angle(y_1, p, y_0) = \angle(N_{\sigma_H Y}(p), N_Y(p)) \geq \tilde{\gamma} \cdot \eta \quad (36)$$

where

$$\tilde{\gamma} = \begin{cases} \frac{1}{4D} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n; \\ \frac{1}{\sinh 5D} & \text{if } \mathcal{M}^n = H^n. \end{cases} \quad (37)$$

Since $d_{\mathcal{M}^n}(p, y_0) \geq \frac{D}{2} + \varrho + \eta = R + \eta$ and $\angle(y_0, y_1, p) \leq \frac{\pi}{2}$, we deduce from Lemma 7.3 applied with $R = \frac{D}{2} + \varrho = D$ if $\mathcal{M}^n = \mathbb{R}^n$ or $\mathcal{M}^n = H^n$, and with $R = \frac{D}{2} + \varrho > \frac{D}{2}$ if $\mathcal{M}^n = S^n$ that

$$d_{\mathcal{M}^n}(y_1, p) \geq 4\tilde{\gamma}_0 \cdot \sqrt{\eta}$$

where

$$\tilde{\gamma}_0 = \begin{cases} \frac{\sqrt{D}}{4} & \text{if } \mathcal{M}^n = \mathbb{R}^n; \\ \frac{\sqrt{\tanh D}}{4} & \text{if } \mathcal{M}^n = H^n; \\ \frac{\sqrt{\tan \frac{D}{2}}}{4} & \text{if } \mathcal{M}^n = S^n; \end{cases} \quad (38)$$

and hence η satisfies

$$\tilde{\gamma}_0 \sqrt{\eta} < \varrho \quad \text{and} \quad \eta < \varrho.$$

In turn, we deduce that

$$B_{\mathcal{M}^n}(p, 2\tilde{\gamma}_0\sqrt{\eta}) \subset H^+. \quad (39)$$

Let $\tilde{x}_\eta \in [\tilde{x}, p]_{\mathcal{M}^n}$ and $z_\eta \in [z, p]_{\mathcal{M}^n}$ such that $d_{\mathcal{M}^n}(p, \tilde{x}_\eta) = \tilde{\gamma}_0\sqrt{\eta}$ and $d_{\mathcal{M}^n}(p, z_\eta) = \tilde{\gamma}_0\sqrt{\eta}$. It follows that $N_{\sigma_H Y}(p)$ is the exterior unit normal to $B_{\mathcal{M}^n}(\tilde{x}_\eta, \tilde{\gamma}_0\sqrt{\eta})$ at p and $N_Y(p)$ is the exterior unit normal to $B_{\mathcal{M}^n}(z_\eta, \tilde{\gamma}_0\sqrt{\eta})$ at p ; moreover, (39) yields that

$$\begin{aligned} B_{\mathcal{M}^n}(\tilde{x}_\eta, \tilde{\gamma}_0\sqrt{\eta}) \cup B_{\mathcal{M}^n}(z_\eta, \tilde{\gamma}_0\sqrt{\eta}) &\subset (B_{\mathcal{M}^n}(\tilde{x}, \varrho) \cap B_{\mathcal{M}^n}(p, 2\tilde{\gamma}_0\sqrt{\eta})) \\ &\cup (B_{\mathcal{M}^n}(z, \varrho) \cap B_{\mathcal{M}^n}(p, 2\tilde{\gamma}_0\sqrt{\eta})) \\ &\subset (B_{\mathcal{M}^n}(\tilde{x}, \varrho) \cap H^+) \cup (B_{\mathcal{M}^n}(z, \varrho) \cap H^+) \\ &\subset (\sigma_H Y \cap H^+) \cup (Y \cap H^+) \subset \tau_H Y. \end{aligned} \quad (40)$$

Let H_x^+ be the half-space of \mathcal{M}^n touching $B_{\mathcal{M}^n}(\tilde{x}_\eta, \tilde{\gamma}_0\sqrt{\eta})$ at p with $H_x^+ \cap B_{\mathcal{M}^n}(\tilde{x}_\eta, \tilde{\gamma}_0\sqrt{\eta}) = \{p\}$, and hence $\text{int } H_x^+ \cap \sigma_H Y = \emptyset$. Similarly, let H_z^+ be the half-space of \mathcal{M}^n touching $B_{\mathcal{M}^n}(z_\eta, \tilde{\gamma}_0\sqrt{\eta})$ at p with $H_z^+ \cap B_{\mathcal{M}^n}(z_\eta, \tilde{\gamma}_0\sqrt{\eta}) = \{p\}$, and hence $\text{int } H_z^+ \cap Y = \emptyset$. Therefore if either $\mathcal{M}^n = S^n$ and $D < \frac{\pi}{2}$, or $\mathcal{M}^n = \mathbb{R}^n$ or $\mathcal{M}^n = H^n$, then

$$\text{conv}\{B_{\mathcal{M}^n}(\tilde{x}_\eta, \tilde{\gamma}_0\sqrt{\eta}), B_{\mathcal{M}^n}(z_\eta, \tilde{\gamma}_0\sqrt{\eta})\} \cap (\text{int } H_x^+ \cap \text{int } H_z^+) \subset \left(\text{conv } \tau_H X^{(\varrho)}\right) \setminus \tau_H X^{(\varrho)}. \quad (41)$$

In the triangle $[y_1, p, y_0]_{\mathcal{M}^n}$, we have $d_{\mathcal{M}^n}(p, y_0) \geq D + \varrho = d_{\mathcal{M}^n}(y_1, y_0)$, thus $\angle(y_1, p, y_0) < \angle(p, y_1, y_0)$, and hence $\angle(y_1, p, y_0) < \frac{\pi}{2}$ because either $\mathcal{M}^n = S^n$ and all sides of $[y_1, p, y_0]_{S^n}$ are acute, or $\mathcal{M}^n = \mathbb{R}^n$ or $\mathcal{M}^n = H^n$. We deduce from (36) and $\angle(y_1, p, y_0) < \frac{\pi}{2}$ that

$$\angle(N_{\sigma_H Y}(p), N_Y(p)) < \frac{\pi}{2}. \quad (42)$$

The rest of the argument is divided into three cases depending on \mathcal{M}^n .

Case 1. $\mathcal{M}^n = \mathbb{R}^n$

In this case, (36) reads as

$$\angle(\tilde{x}_\eta, p, z_\eta) = \angle(N_{\sigma_H Y}(p), N_Y(p)) \geq \frac{\eta}{4D}.$$

Since $\angle(N_{\sigma_H Y}(p), N_Y(p)) < \frac{\pi}{2}$ according to (42), we deduce from (38), (41) and Claim 7.1 (where $c_n = \frac{1}{2^{4n}n^n}$) that

$$V\left(\left(\text{conv } \tau_H X^{(\varrho)}\right) \setminus \tau_H X^{(\varrho)}\right) \geq \frac{c_n}{\sqrt{2}} \cdot \left(\frac{\sqrt{D} \cdot \sqrt{\eta}}{4}\right)^n \cdot \left(\frac{\eta}{4D}\right)^{n+1}$$

$$= \frac{c_n}{4^{2n+1}\sqrt{2}} \cdot \frac{\eta^{\frac{3n}{2}+1}}{D^{\frac{n}{2}+1}} > \frac{1}{2^{9n}n^n} \cdot \frac{\eta^{\frac{3n}{2}+1}}{D^{\frac{n}{2}+1}}.$$

Case 2. $\mathcal{M}^n = H^n$

In this case, we may assume that

$$p = e.$$

We consider the ellipsoids $E_x = \varphi_{H^n} B_{H^n}(\tilde{x}_\eta, \tilde{\gamma}_0 \sqrt{\eta}) \subset e + T_e$ and $E_z = \varphi_{H^n} B_{H^n}(z_\eta, \tilde{\gamma}_0 \sqrt{\eta}) \subset e + T_e$ satisfying that $N_{\sigma_H Y}(p)$ is an exterior normal to E_x at $p = e$, and $N_Y(p)$ is an exterior normal to E_z at $p = e$. In addition, $N_{\sigma_H Y}(p)$ and $N_Y(p)$ are interior normals for the half-spaces

$$\tilde{H}_x^+ = \varphi_{H^n} H_x^+ \quad \text{and} \quad \tilde{H}_z^+ = \varphi_{H^n} H_z^+$$

in $e + T_e$, respectively. We deduce from (41) that

$$\text{conv}_{e+T_e}(E_x \cup E_z) \cap (\text{int } \tilde{H}_x^+) \cap (\text{int } \tilde{H}_z^+) \subset \varphi_{H^n} \left(\left(\text{conv}_{H^n} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right). \quad (43)$$

According to Lemma 7.2, E_z has axial rotational symmetry around $p + \mathbb{R} N_Y(p)$, and the axis of E_z contained in $p + \mathbb{R} N_Y(p)$ is of length $\tanh 2\tilde{\gamma}_0 \sqrt{\eta} = \frac{\sinh 2\tilde{\gamma}_0 \sqrt{\eta}}{\cosh 2\tilde{\gamma}_0 \sqrt{\eta}} < \frac{2 \sinh \tilde{\gamma}_0 \sqrt{\eta}}{\sqrt{\cosh 2\tilde{\gamma}_0 \sqrt{\eta}}}$ where $\frac{2 \sinh \tilde{\gamma}_0 \sqrt{\eta}}{\sqrt{\cosh 2\tilde{\gamma}_0 \sqrt{\eta}}}$ is the length of the orthogonal axes. Therefore setting

$$z^* = p - N_Y(p) \cdot \frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2},$$

using $\tilde{B}(\cdot, \cdot)$ to denote n -dimensional balls in $e + T_e$, we have

$$\tilde{B}\left(z^*, \frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2}\right) \subset E_z,$$

and has $N_Y(p)$ as exterior unit normal at p . Similarly,

$$x^* = p - N_{\sigma_H Y}(p) \cdot \frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2},$$

satisfies that

$$\tilde{B}\left(x^*, \frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2}\right) \subset E_x,$$

and has $N_{\sigma_H Y}(p)$ as exterior unit normal at p . We deduce from (43) that

$$\text{conv}_{e+T_e} \left\{ \tilde{B}\left(x^*, \frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2}\right), \tilde{B}\left(z^*, \frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2}\right) \right\} \cap (\text{int } \tilde{H}_x^+) \cap (\text{int } \tilde{H}_z^+) \subset \quad (44)$$

$$\subset \varphi_{H^n} \left(\left(\operatorname{conv}_{H^n} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right).$$

It follows from (36) and Lemma 6.3 that for $R = \frac{D}{2} + \varrho \leq D$, we have

$$\angle(\tilde{x}^*, p, z^*) = \angle(N_{\sigma_H Y}(p), N_Y(p)) > \frac{\eta}{\sinh 5D}. \quad (45)$$

Since $\angle(N_{\sigma_H Y}(p), N_Y(p)) < \frac{\pi}{2}$ according to (42), and (25) yields that

$$V_{H^n}(X) \geq V_{\mathbb{R}^n}(\varphi_{H^n} X)$$

for any bounded Borel set $X \subset H^n$, we deduce from (44), (45) and Claim 7.1 that

$$V_{H^n} \left(\left(\operatorname{conv}_{H^n} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right) \geq \frac{c_n}{\sqrt{2}} \cdot \left(\frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2} \right)^n \cdot \left(\frac{\eta}{\sinh 5D} \right)^{n+1}. \quad (46)$$

Here $\tilde{\gamma}_0 = \frac{\sqrt{\tanh D}}{4} < \frac{1}{4}$ and $\eta \in (0, 1)$ yield that

$$\frac{\tanh 2\tilde{\gamma}_0 \sqrt{\eta}}{2} > 0.9 \cdot \frac{\sqrt{\eta \tanh D}}{4}.$$

On the other hand, $\tanh D > \frac{1}{5} \tanh 5D$ and $\sinh 10D = 2 \sinh 5D \cdot \cosh 5D$ imply

$$\begin{aligned} \frac{(\tanh D)^{\frac{n}{2}}}{(\sinh 5D)^{n+1}} &> \frac{\left(\frac{1}{5} \tanh 5D\right)^{n/2}}{(\sinh 5D)^{n+1}} = \left(\frac{2}{5}\right)^{\frac{n}{2}} \frac{1}{(\sinh 10D)^{\frac{n}{2}} \sinh 5D} \\ &> \left(\frac{2}{5}\right)^{\frac{n}{2}} \frac{1}{(\sinh 10D)^{\frac{n}{2}+1}}. \end{aligned}$$

Using these last two estimates and $0.9\sqrt{\frac{2}{5}} > \frac{1}{2}$, we deduce from (46) that

$$\begin{aligned} V \left(\left(\operatorname{conv} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right) &\geq \frac{c_n}{4^n \sqrt{2}} \left(0.9 \sqrt{\frac{2}{5}} \right)^n \cdot \frac{\eta^{\frac{3n}{2}+1}}{(\sinh 10D)^{\frac{n}{2}+1}} \\ &> \frac{1}{2^{8n} n^n} \cdot \frac{\eta^{\frac{3n}{2}+1}}{(\sinh 10D)^{\frac{n}{2}+1}}. \end{aligned}$$

Case 3. $\mathcal{M}^n = S^n$

We may assume once more that $p = e$. Considering the ellipsoids $E_x = \varphi_{S^n} B_{S^n}(\tilde{x}_\eta, \tilde{\gamma}_0 \sqrt{\eta}) \subset e + T_e$ and $E_z = \varphi_{S^n} B_{S^n}(z_\eta, \tilde{\gamma}_0 \sqrt{\eta}) \subset e + T_e$, we have $N_{\sigma_H Y}(p)$ as an exterior normal to E_x at p and $N_Y(p)$ an exterior normal to E_z at p . Then again $N_{\sigma_H Y}(p)$ and $N_Y(p)$ are interior normals for the half-spaces

$$\tilde{H}_x^+ = \varphi_{S^n} H_x^+ \quad \text{and} \quad \tilde{H}_z^+ = \varphi_{S^n} H_z^+$$

in $e + T_e$, respectively. We deduce from (41) that

$$\text{conv}_{e+T_e}(E_x \cup E_z) \cap (\text{int } \tilde{H}_x^+) \cap (\text{int } \tilde{H}_z^+) \subset \varphi_{S^n} \left(\left(\text{conv}_{H^n} \tau_H X^{(e)} \right) \setminus \tau_H X^{(e)} \right). \quad (47)$$

Lemma 7.2 implies that E_z has axial rotational symmetry around $p + \mathbb{R} N_Y(p)$, and the axis of E_z contained in $p + \mathbb{R} N_Y(p)$ is of length $\tan(2\tilde{\gamma}_0\sqrt{\eta}) > \frac{2\sin(\tilde{\gamma}_0\sqrt{\eta})}{\sqrt{\cos(2\tilde{\gamma}_0\sqrt{\eta})}}$ where $\frac{2\sin(\tilde{\gamma}_0\sqrt{\eta})}{\sqrt{\cos(2\tilde{\gamma}_0\sqrt{\eta})}}$ is the length of the orthogonal axes. Since if $a = \tan(2\tilde{\gamma}_0\sqrt{\eta})$ is the major axis and $b = \frac{2\sin(\tilde{\gamma}_0\sqrt{\eta})}{\sqrt{\cos(2\tilde{\gamma}_0\sqrt{\eta})}}$ is the minor axis of an ellipse, then its minimal radius of curvature is $\frac{b^2}{2a} = \tan \tilde{\gamma}_0\sqrt{\eta}$, we deduce for

$$\begin{aligned} z^* &= p - N_Y(p) \cdot \tan \tilde{\gamma}_0\sqrt{\eta}, \\ x^* &= p - N_{\sigma_H Y}(p) \cdot \tan \tilde{\gamma}_0\sqrt{\eta}, \end{aligned}$$

that

$$\tilde{B}(z^*, \tan \tilde{\gamma}_0\sqrt{\eta}) \subset E_z,$$

and has $N_Y(p)$ as exterior unit normal at p . Similarly,

$$\tilde{B}(x^*, \tan \tilde{\gamma}_0\sqrt{\eta}) \subset E_x,$$

and has $N_{\sigma_H Y}(p)$ as exterior unit normal at p . We deduce from (47) that

$$\begin{aligned} \text{conv}_{e+T_e} \left\{ \tilde{B}(x^*, \tan \tilde{\gamma}_0\sqrt{\eta}), \tilde{B}(z^*, \tan \tilde{\gamma}_0\sqrt{\eta}) \right\} \cap (\text{int } \tilde{H}_x^+) \cap (\text{int } \tilde{H}_z^+) &\subset \\ &\subset \varphi_{S^n} \left(\left(\text{conv}_{H^n} \tau_H X^{(e)} \right) \setminus \tau_H X^{(e)} \right). \end{aligned} \quad (48)$$

From (36) and Lemma 6.3 setting $R = \frac{D}{2} + \varrho \leq D$, we have

$$\angle(\tilde{x}^*, p, z^*) = \angle(N_{\sigma_H Y}(p), N_Y(p)) > \frac{\eta}{4D}. \quad (49)$$

Since (26) yields that

$$V_{S^n}(X) \geq (\cos r)^{n+1} \cdot V_{\mathbb{R}^n}(\varphi_{S^n} X)$$

for any bounded Borel set $X \subset B_{S^n}(e, r)$, and

$$\text{conv} \left\{ B_{S^n}(\tilde{x}_\eta, \tilde{\gamma}_0\sqrt{\eta}), B_{S^n}(z_\eta, \tilde{\gamma}_0\sqrt{\eta}) \right\} \subset B_{S^n}(p, 2\varrho) \subset B_{S^n}\left(e, \frac{\pi}{4}\right), \quad (50)$$

we deduce from (48), (49) and Claim 7.1 that

$$V_{\mathbb{R}^n} \left(\varphi_{S^n} \left(\left(\operatorname{conv}_{S^n} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right) \right) \geq \frac{1}{2^{\frac{n+1}{2}}} \cdot \frac{c_n}{\sqrt{2}} \cdot (\tan(\tilde{\gamma}_0 \sqrt{\eta}))^n \cdot \left(\frac{\eta}{4D} \right)^{n+1}.$$

Since $\eta < \frac{D}{2} < \frac{\pi}{4}$, we have

$$0 < \tilde{\gamma}_0 \sqrt{\eta} < \frac{\sqrt{\eta}}{4} < \frac{\pi}{2}$$

implying

$$\tan(\tilde{\gamma}_0 \sqrt{\eta}) \geq \frac{\sqrt{\tan\left(\frac{D}{2}\right)\eta}}{4} \geq \frac{\sqrt{\frac{D}{2} \cdot \eta}}{4}.$$

This gives us

$$V_{\mathbb{R}^n} \left(\varphi_{S^n} \left(\left(\operatorname{conv}_{S^n} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right) \right) \geq \frac{\eta^{\frac{3n}{2}+1}}{D^{\frac{n+2}{2}} \cdot 2^{9n+3} \cdot n^n},$$

concluding via (50) that

$$\begin{aligned} V_{S^n} \left(\left(\operatorname{conv}_{S^n} \tau_H X^{(\varrho)} \right) \setminus \tau_H X^{(\varrho)} \right) &\geq \frac{\eta^{\frac{3n}{2}+1} \cdot \left(\cos\left(\frac{\pi}{4}\right) \right)^{n+1}}{D^{\frac{n+2}{2}} \cdot 2^{9n+3} \cdot n^n} \\ &= \frac{\eta^{\frac{3n}{2}+1}}{D^{\frac{n+2}{2}} \cdot 2^{\frac{19n+7}{2}} \cdot n^n} > \frac{\eta^{\frac{3n}{2}+1}}{D^{\frac{n+2}{2}} \cdot 2^{12n} \cdot n^n}. \quad \square \end{aligned}$$

8. Estimates about volumes of balls and spherical caps

We prepare the proof of Theorem 1.3 with a series of lemmas about balls.

Lemma 8.1. *Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n . Let r be a positive number, for $\mathcal{M}^n = S^n$ we also assume $r \leq \frac{\pi}{2}$. For $0 < s < r$ we can give the following lower bound for the volume of a ball of radius $r - s$:*

$$V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r - s)) \geq V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r)) - s \cdot (f_{\mathcal{M}^n}(r))^{n-1} \cdot n \cdot \kappa_n$$

where

$$f_{\mathcal{M}^n}(t) = \begin{cases} t & \text{if } \mathcal{M}^n = \mathbb{R}^n \\ \sinh t & \text{if } \mathcal{M}^n = H^n \\ \sin t & \text{if } \mathcal{M}^n = S^n \end{cases}.$$

Proof. For $\varrho > 0$ the Lebesgue measure of the ball of radius ϱ in \mathcal{M}^n is

$$V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, \varrho)) = \int_0^{\varrho} (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt. \quad (51)$$

We observe that $f_{\mathcal{M}^n}$ is monotonically increasing (for $\mathcal{M}^n = S^n$ we assume $\varrho \leq \frac{\pi}{2}$), so

$$\begin{aligned} V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r)) &= \int_0^{r-s} (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt + \int_{r-s}^r (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt \leq \\ &\leq \int_0^{r-s} (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt + s \cdot (f_{\mathcal{M}^n}(r))^{n-1} \cdot n \cdot \kappa_n = \\ &= V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r-s)) + s \cdot (f_{\mathcal{M}^n}(r))^{n-1} \cdot n \cdot \kappa_n. \quad \square \end{aligned}$$

Lemma 8.2. Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , and let $r > 0$ where we also assume $r \leq \frac{\pi}{3}$ if $\mathcal{M}^n = S^n$. For $0 < s < \frac{r}{2}$ we can give the following upper bounds for the volume of a ball of radii $r-s$ and $r+s$:

$$V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r-s)) \leq V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r)) - s \cdot \left(f_{\mathcal{M}^n}\left(\frac{r}{2}\right)\right)^{n-1} \cdot n \cdot \kappa_n$$

and

$$V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r+s)) \leq V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r)) + s \cdot \left(f_{\mathcal{M}^n}\left(\frac{3r}{2}\right)\right)^{n-1} \cdot n \cdot \kappa_n$$

where

$$f_{\mathcal{M}^n}(t) = \begin{cases} t & \text{if } \mathcal{M}^n = \mathbb{R}^n \\ \sinh t & \text{if } \mathcal{M}^n = H^n \\ \sin t & \text{if } \mathcal{M}^n = S^n \end{cases}.$$

Proof. We use a similar argument as in Lemma 8.1. Using (51) for $\varrho = r$ we have

$$\begin{aligned} V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r)) &= \int_0^{r-s} (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt + \int_{r-s}^r (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt \geq \\ &\geq \int_0^{r-s} (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt + s \cdot \left(f_{\mathcal{M}^n}\left(\frac{r}{2}\right)\right)^{n-1} \cdot n \cdot \kappa_n = \\ &= V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r-s)) + s \cdot \left(f_{\mathcal{M}^n}\left(\frac{r}{2}\right)\right)^{n-1} \cdot n \cdot \kappa_n \end{aligned}$$

by the choice of s and the monotonicity of $f_{\mathcal{M}^n}$.

By the choice $\varrho = r+s$ we can obtain the other inequality:

$$V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r+s)) = \int_0^r (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt + \int_r^{r+s} (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n dt \leq$$

$$\begin{aligned} &\leq \int_0^r (f_{\mathcal{M}^n}(t))^{n-1} \cdot n \cdot \kappa_n \, dt + s \cdot \left(f_{\mathcal{M}^n} \left(\frac{3r}{2} \right) \right)^{n-1} \cdot n \cdot \kappa_n = \\ &= V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, r)) + s \cdot \left(f_{\mathcal{M}^n} \left(\frac{3r}{2} \right) \right)^{n-1} \cdot n \cdot \kappa_n. \quad \square \end{aligned}$$

Remark. Note that we only used $r \leq \frac{\pi}{3}$ for the second inequality, the first upper estimate holds for $r \leq \frac{\pi}{2}$.

Proposition 8.3. *If $X \subset \mathcal{M}^n$ is compact with diameter at most $D > 0$ (also $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$), $0 < \varepsilon < V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, \frac{D}{2}))$ and $V_{\mathcal{M}^n}(X) \geq V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(x_0, \frac{D}{2})) - \varepsilon$, then for $0 < \varrho \leq \frac{D}{2}$ satisfying also that $D + 2\varrho < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, we have*

$$V_{\mathcal{M}^n}(X^{(\varrho)}) \geq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(x_0, \frac{D}{2} + \varrho\right)\right) - E_{\mathcal{M}^n}(\varepsilon, D)$$

where

$$E_{\mathcal{M}^n}(\varepsilon, D) = \begin{cases} \varepsilon \cdot 4^{n-1} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \\ \varepsilon \cdot \left(4 \cosh \frac{3D}{4}\right)^{n-1} & \text{if } \mathcal{M}^n = H^n \end{cases}.$$

Proof. We choose $\tilde{E}_{\mathcal{M}^n}(\varepsilon, D)$ so that it satisfies

$$V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(x_0, \frac{D}{2} - \tilde{E}_{\mathcal{M}^n}(\varepsilon, D)\right)\right) \leq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(x_0, \frac{D}{2}\right)\right) - \varepsilon.$$

Let $R = \frac{D}{2} + \varrho$. Applying Lemma 8.2 we can set

$$\tilde{E}_{\mathcal{M}^n}(\varepsilon, D) = \frac{\varepsilon}{\left(f_{\mathcal{M}^n}\left(\frac{D}{4}\right)\right)^{n-1} \cdot n \cdot \kappa_n}$$

where

$$f_{\mathcal{M}^n}(t) = \begin{cases} t & \text{if } \mathcal{M}^n = \mathbb{R}^n \\ \sinh t & \text{if } \mathcal{M}^n = H^n \\ \sin t & \text{if } \mathcal{M}^n = S^n \end{cases}.$$

Hence the Isoperimetric Inequality Theorem 1.1 yields

$$V_{\mathcal{M}^n}(X^{(\varrho)}) \geq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(x_0, R - \tilde{E}_{\mathcal{M}^n}(\varepsilon, D)\right)\right).$$

Now using Lemma 8.1 we have

$$\begin{aligned}
V_{\mathcal{M}^n} \left(B_{\mathcal{M}^n} \left(x_0, R - \tilde{E}_{\mathcal{M}^n}(\varepsilon, D) \right) \right) &\geq V_{\mathcal{M}^n} (B_{\mathcal{M}^n} (x_0, R)) \\
&\quad - \tilde{E}_{\mathcal{M}^n}(\varepsilon, D) \cdot f_{\mathcal{M}^n}(R)^{n-1} \cdot n \cdot \kappa_n \\
&= V_{\mathcal{M}^n} (B_{\mathcal{M}^n} (x_0, R)) - \varepsilon \cdot \left(\frac{f_{\mathcal{M}^n}(R)}{f_{\mathcal{M}^n}(\frac{R}{2})} \right)^{n-1}.
\end{aligned}$$

For $\mathcal{M}^n = \mathbb{R}^n$ we are already done. It is trivial that

$$\frac{f_{S^n}(R)}{f_{S^n}(\frac{R}{4})} \leq \frac{f_{S^n}(R)}{f_{S^n}(\frac{R}{4})} \leq 4.$$

Finally, in the hyperbolic case

$$\frac{f_{H^n}(R)}{f_{H^n}(\frac{R}{4})} \leq \frac{f_{H^n}(R)}{f_{H^n}(\frac{R}{4})} = 4 \cosh\left(\frac{R}{2}\right) \cosh\left(\frac{R}{4}\right) \leq 4 \cosh\left(\frac{3R}{4}\right),$$

which finishes the proof. \square

If \mathcal{M}^n is either \mathbb{R}^n , H^n or S^n , then we need some lower bound on the volume of spherical caps. If H^+ is a half-space in \mathcal{M}^n such that $\partial_{\mathcal{M}^n} H^+ \cap \text{int } B_{\mathcal{M}^n}(x_0, t) \neq \emptyset$ for $x_0 \in \mathcal{M}^n$ and $t > 0$ where $t < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, then $C = H^+ \cap B_{\mathcal{M}^n}(x_0, t)$ is called a spherical cap, and the depth δ of C is the maximal distance of points of C from $\partial_{\mathcal{M}^n} H^+$. We observe that if $\delta < t$, then the distance of x_0 from H^+ is $t - \delta$.

We need some estimate about the $(n-1)$ -volume κ_{n-1} of an $(n-1)$ -dimensional Euclidean unit ball. It follows from $\frac{\kappa_{n-1}}{\kappa_n} > \sqrt{\frac{n}{2\pi}}$ and $\Gamma(x+1) < \left(\frac{x}{e}\right)^x \cdot \sqrt{2\pi(x+1)}$ that

$$\frac{\kappa_{n-1}}{n} > \frac{\kappa_n}{\sqrt{2n\pi}} = \frac{\pi^{\frac{n}{2}}}{\sqrt{2n\pi} \cdot \Gamma(\frac{n}{2} + 1)} > \frac{\pi^{\frac{n}{2}}}{\sqrt{2n\pi} \cdot \left(\frac{n}{2e}\right)^{\frac{n}{2}} \cdot \sqrt{\pi(n+2)}} > \frac{1}{n^{\frac{n}{2}}}. \quad (52)$$

In the Euclidean case, if $H \cap \text{int } B_{\mathbb{R}^n}(x_0, t) \neq \emptyset$ for a hyperplane H of \mathbb{R}^n where the distance of x_0 from H is $t - \delta$ for $\delta \in (0, t]$ (and hence the small cap cut off by H is of depth δ), then $H \cap B_{\mathbb{R}^n}(x_0, t)$ is an $(n-1)$ -dimensional Euclidean ball of radius a where

$$a = \sqrt{t^2 - (t - \delta)^2} \geq \sqrt{t\delta}. \quad (53)$$

Lemma 8.4. For $x_0 \in \mathbb{R}^n$, $t > 0$ and $0 < \delta \leq t$, if H^+ is a half-space in \mathbb{R}^n such that $\partial_{\mathbb{R}^n} H^+ \cap \text{int } B_{\mathbb{R}^n}(x_0, t) \neq \emptyset$, and $H^+ \cap B_{\mathbb{R}^n}(x_0, t)$ is a spherical cap of depth at least δ , then

$$V_{\mathbb{R}^n} (H^+ \cap B_{\mathbb{R}^n}(x_0, t)) \geq \frac{2\kappa_{n-1}}{n+1} \cdot t^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}} \geq \frac{1}{n^{\frac{n}{2}}} \cdot t^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}}.$$

Proof. It follows from applying first (53) and then (52) that

$$V_{\mathbb{R}^n}(H^+ \cap B_{S^n}(x_0, t)) \geq \int_0^\delta \kappa_{n-1} t^{\frac{n-1}{2}} \cdot s^{\frac{n-1}{2}} ds = \frac{2\kappa_{n-1}}{n+1} \cdot t^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}} \geq \frac{1}{n^{\frac{n}{2}}} \cdot t^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}}. \quad \square$$

In Lemma 8.5, we use two estimates for hyperbolic volume. Concerning balls, if $r > 0$ and $x_0 \in H^n$, then

$$V_{H^n}(B_{H^n}(x_0, r)) = \int_0^r n\kappa_n (\sinh s)^{n-1} ds \geq \int_0^r n\kappa_n s^{n-1} ds = \kappa_n r^n. \quad (54)$$

It is a reasonable lower bound if $r \leq 2$.

Next let ℓ be any line in H^n , let $\delta > 0$, and let $x_s \in \ell$ for $s \in [0, \delta]$ be a parametrization of the segment $[x_0, x_\delta]_{H^n} \subset \ell$ where $d_{H^n}(x_0, x_s) = s$. In addition, let H_s be the hyperplane in H^n passing through x_s and orthogonal to ℓ . Now if X is any compact set lying between H_0 and H_δ , then the fact that the distance between any point of H_s and any point of H_t for $0 \leq s < t \leq \delta$ is at least $t - s$ yields that

$$V_{H^n}(X) \geq \int_0^\delta V_{H^{n-1}}(H_s \cap X) ds. \quad (55)$$

Lemma 8.5. For $x_0 \in H^n$, $t > 0$ and $0 < \delta \leq \min\{\frac{t}{2}, 1\}$, if H^+ is a half-space in H^n such that $H^+ \cap B_{H^n}(x_0, t)$ is a spherical cap of depth at least δ , then

$$V_{H^n}(H^+ \cap B_{H^n}(x_0, t)) \geq n^{-\frac{(n-1)}{2}} \cdot (\tanh(t - \delta))^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}}.$$

Proof. We may assume that $H^+ = H_\delta^+$ is a half-space in H^n such that $H_\delta^+ \cap B_{H^n}$ is a spherical cap of depth δ .

First, let $w_0 \in \partial_{H^n} B_{H^n}(x_0, t)$ be the point such that the segment $[x_0, w_0]_{H^n}$ intersects $\partial_{H^n} H_\delta^+$ in some w_δ . For any $s \in [0, \delta]$, let w_s be the point of $[x_0, w_0]_{H^n}$ with $d_{H^n}(x_0, w_s) = s$, and let H_s be the hyperplane of H^n passing through w_s and orthogonal to $[x_0, w_0]_{H^n}$. The Law of Cosines yields that H_s intersects $B_{H^n}(x_0, t)$ in an $(n-1)$ -ball of radius a_s where

$$\begin{aligned} \cosh a_s &= \frac{\cosh t}{\cosh(t-s)} = \frac{\cosh(t-s) \cosh s + \sinh(t-s) \cdot \sinh s}{\cosh(t-s)} \\ &= \cosh s + \tanh(t-s) \cdot \sinh s \geq 1 + s \cdot \tanh(t-\delta). \end{aligned}$$

Since $\cosh 1 + \sinh 1 = e \leq \cosh 2$ and $\cosh z \leq 1 + z^2$ if $|z| \leq 2$, we deduce that

$$a_s \geq \sqrt{s \cdot \tanh(t-\delta)}$$

It follows from (54) that if $s \in [0, \delta]$, then

$$V_{H^{n-1}}(H_s \cap B_{H^n}(x_0, t)) \geq \kappa_{n-1} a_s^{n-1} \geq \kappa_{n-1} \cdot s^{\frac{n-1}{2}} \cdot (\tanh(t - \delta))^{\frac{n-1}{2}};$$

therefore, (55) yields that

$$\begin{aligned} V_{H^n}(H_\delta^+ \cap B_{H^n}(x_0, t)) &\geq \int_0^\delta \kappa_{n-1} \cdot s^{\frac{n-1}{2}} \cdot (\tanh(t - \delta))^{\frac{n-1}{2}} ds \\ &= \kappa_{n-1} \cdot (\tanh(t - \delta))^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}}. \end{aligned}$$

Since (52) yields $\kappa_{n-1} \geq n^{\frac{-(n-1)}{2}}$, we conclude Lemma 8.5. \square

In the case of spherical caps C , the analogue of (55) does not hold in the spherical space; therefore, we estimate the volume of C by projecting C into a Euclidean space of the same dimension.

Lemma 8.6. For $x_0 \in S^n$, $0 < t < \frac{\pi}{2}$ and $0 < \delta \leq \frac{t}{2}$, if H^+ is a hemisphere in S^n whose distance from x_0 is $t - \delta$, then

$$V_{S^n}(H^+ \cap B_{S^n}(x_0, t)) \geq \frac{1}{2^n \cdot n^{\frac{n}{2}}} \cdot t^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}}.$$

Proof. The intersection $\partial_{S^n} H^+ \cap B_{S^n}(x_0, t)$ is an $(n-1)$ -ball in S^n of some radius a . Let $s = t - \delta$, then

$$\cos(s + \delta) = \cos(t) = \cos(s) \cdot \cos(a).$$

We claim that $a \geq \sqrt{s\delta}$, which is equivalent to

$$\cos(s) \cdot \cos(\sqrt{s\delta}) \geq \cos(s + \delta). \quad (56)$$

This holds with equality if $\delta = 0$, so it is sufficient to see

$$\cos(s) \cdot \frac{\sin(\sqrt{s\delta})}{\sqrt{s\delta}} \cdot \frac{s}{2} \leq \sin(s + \delta),$$

after differentiation. Since $0 < s < \frac{\pi}{2}$, we have $s < \tan(s)$, so

$$\cos(s) \cdot \frac{\sin(\sqrt{s\delta})}{\sqrt{s\delta}} \cdot \frac{s}{2} < \cos(s) \cdot \frac{s}{2} < \frac{\sin(s)}{2}.$$

Here $\delta < \frac{\pi}{3}$ so $\frac{1}{2} < \cos(\delta)$, therefore

$$\cos(s) \cdot \frac{\sin(\sqrt{s\delta})}{\sqrt{s\delta}} \cdot \frac{s}{2} < \sin(s) \cdot \cos(\delta) < \sin(s + \delta),$$

proving (56).

Let w be the closest point of H^+ to x_0 , and let $\Pi: S^n \rightarrow w^\perp$ be the orthogonal projection. Then $\Pi(\partial_{S^n} H^+ \cap B_{S^n}(x_0, t))$ is a Euclidean $(n-1)$ -ball of radius $\sin(a)$. By the choice of H^+ and from $d_{S^n}(x_0, w) = t - \delta$, there is a point $\tilde{w} \in H^+ \cap \partial_{S^n} B_{S^n}(x_0, t)$ such that $d_{S^n}(w, \tilde{w}) = \delta$ and $w \in [\tilde{w}, x_0]_{S^n}$. Hence, there is a cone $C \subset \Pi(H^+ \cap B_{S^n}(x_0, t))$ with base $\Pi(\partial_{S^n} H^+ \cap B_{S^n}(x_0, t))$ and height $\sin(\delta)$. Thus,

$$\begin{aligned} V_{S^n}(H^+ \cap B_{S^n}(x_0, t)) &\geq V_{w^\perp}(\Pi(H^+ \cap B_{S^n}(x_0, t))) \\ &\geq V_{w^\perp}(C) = \frac{\kappa_{n-1}}{n} \cdot (\sin(a))^{n-1} \cdot \sin(\delta). \end{aligned}$$

From (56) we can imply

$$\sin(a) \geq \sin\left(\sqrt{\frac{t\delta}{2}}\right),$$

so from $\sqrt{\frac{t\delta}{2}} < \frac{\pi}{4}$ and $\delta < \frac{\pi}{3}$ we have

$$V_{S^n}(H^+ \cap B_{S^n}(x_0, t)) \geq \frac{\kappa_{n-1}}{n} \cdot \left(\frac{\sqrt{t\delta}}{2}\right)^{n-1} \cdot \frac{\delta}{2} = \frac{1}{2^n} \cdot t^{\frac{n-1}{2}} \cdot \delta^{\frac{n+1}{2}}.$$

Finally, (52) completes the proof. \square

9. The proof of Theorem 1.3

We need some upper bound on the volume of balls in \mathcal{M}^n where \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n .

Lemma 9.1. *If \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n , and $r > 0$ where $r < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, then*

$$V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(z_0, r)) \leq \begin{cases} \frac{2^{\frac{3n}{2}}}{n^{\frac{n+1}{2}}} \cdot r^n & \text{if either } \mathcal{M}^n = \mathbb{R}^n, \text{ or} \\ & \mathcal{M}^n = S^n, \text{ or } \mathcal{M}^n = H^n \text{ and } r \leq 1 \\ \frac{2^{\frac{3n+2}{2}}}{n^{\frac{n+1}{2}}} e^{\frac{(n-1)r}{2}} & \text{if } \mathcal{M}^n = H^n \text{ and } r > 0. \end{cases}$$

Remark. The second bound in the hyperbolic case is worse than the first bound if $r \leq 1$.

Proof. We set $r = D/2$. In the Euclidean case, it follows from $\Gamma(x+1) > \left(\frac{x}{e}\right)^x \cdot \sqrt{2\pi x}$ for $x \geq 1$ and from $n \geq 2$ and $\frac{1}{\sqrt{\pi}} \cdot \left(\frac{e\pi}{2^4}\right)^{\frac{n}{2}} \leq \frac{1}{2}$ that

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} < \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2e}\right)^{\frac{n}{2}} \cdot \sqrt{\pi n}} = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{e\pi}{2^4}\right)^{\frac{n}{2}} \cdot \frac{2^{\frac{5n}{2}}}{n^{\frac{n+1}{2}}} < \frac{1}{2} \cdot \frac{2^{\frac{5n}{2}}}{n^{\frac{n+1}{2}}}. \quad (57)$$

In the spherical case, if $r \in (0, \frac{\pi}{2})$, then

$$V_{S^n}(B_{S^n}(z_0, r)) = n\kappa_n \int_0^r (\sin s)^{n-1} ds \leq n\kappa_n \int_0^r s^{n-1} ds = r^n \kappa_n < \frac{2^{\frac{5n}{2}}}{n^{\frac{n+1}{2}}} \cdot r^n. \quad (58)$$

Finally, in the hyperbolic case, if $r \in (0, 1]$, then $\sinh s < \sqrt{2}s$ for $s \in (0, 1]$ yields

$$V_{H^n}(B_{H^n}(z_0, r)) = n\kappa_n \int_0^r (\sinh s)^{n-1} ds \leq n\kappa_n \int_0^r \sqrt{2}^{n-1} s^{n-1} ds < \frac{2^{\frac{3n}{2}}}{n^{\frac{n+1}{2}}} \cdot r^n. \quad (59)$$

Therefore, (57), (58) and (59) yield Lemma 9.1 if either $\mathcal{M}^n = \mathbb{R}^n$, or $\mathcal{M}^n = S^n$, or $\mathcal{M}^n = H^n$ and $r \leq 1$.

Finally, we consider the case $\mathcal{M}^n = H^n$ and $r \geq 1$. As a first step, we observe that $\sinh s \leq e^s/2$ for $s \geq 0$; therefore, we deduce using (57) and $\frac{n}{n-1} \leq 2$ that

$$\begin{aligned} V_{H^n}(B_{H^n}(z_0, r)) &= n\kappa_n \int_0^r (\sinh s)^{n-1} ds \leq n\kappa_n \int_0^r \frac{e^{(n-1)s}}{2^{n-1}} ds \\ &\leq n\kappa_n \cdot \frac{e^{(n-1)r}}{(n-1)2^{n-1}} \leq \frac{2 \cdot 2^{\frac{3n}{2}} e^{(n-1)r}}{n^{\frac{n+1}{2}}} \leq \frac{2^{\frac{3n+2}{2}} e^{(n-1)r}}{n^{\frac{n+1}{2}}}, \end{aligned}$$

proving Lemma 9.1 also if $\mathcal{M}^n = H^n$ and $r \geq 1$. \square

Now we are ready to prove Theorem 1.3, which we are restating including the exact values of $\varepsilon_{\mathcal{M}^n}(D)$.

Theorem 9.2. For $n \geq 2$ if \mathcal{M}^n is either \mathbb{R}^n , S^n or H^n , $D > 0$ (where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$) and $X \subset \mathcal{M}^n$ is measurable with $\text{diam} X \leq D$ and

$$V_{\mathcal{M}^n}(X) \geq (1 - \varepsilon)V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(z_0, \frac{D}{2}\right)\right),$$

for $\varepsilon \in [0, \varepsilon_{\mathcal{M}^n}(D))$, then there exists a $c \in \mathcal{M}^n$ such that

$$B\left(c, \frac{D}{2} - \gamma_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}}\right) \subset \text{conv}_{\mathcal{M}^n} X \subset B\left(c, \frac{D}{2} + \gamma_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}}\right)$$

where

$$\gamma_{\mathcal{M}^n}(D) = \begin{cases} e^{21n} \cdot D & \text{if } \mathcal{M}^n = H^n \text{ and } D \leq 2, \text{ or } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n; \\ n \cdot e^{7D+8} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 1, \end{cases}$$

$$\varepsilon_{\mathcal{M}^n}(D) = \begin{cases} e^{-28n} n^{-\frac{n}{2}} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n \text{ and } D \leq \frac{\pi}{6}, \\ & \text{or } \mathcal{M}^n = H^n \text{ and } D \leq 2; \\ e^{-30n} n^{-\frac{n}{2}} \left(\frac{\pi}{2} - D\right)^{3n+2} & \text{if } \mathcal{M}^n = S^n \text{ and } \frac{\pi}{6} \leq D < \frac{\pi}{2}; \\ e^{-18D} n^{-\frac{n}{2}} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 2 \end{cases}$$

In addition, $V_{\mathcal{M}^n}((\text{conv}_{\mathcal{M}^n} X) \setminus X) \leq \varepsilon \cdot V_{\mathcal{M}^n}(B_{\mathcal{M}^n}(z_0, \frac{D}{2}))$.

Proof. Let \mathcal{M}^n be either \mathbb{R}^n , H^n or S^n , let $D > 0$ where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$, and let $X \subset \mathcal{M}^n$ be measurable satisfying $\text{diam } X \leq D$ and $V(X) \geq (1 - \varepsilon)V(B(z_0, \frac{D}{2}))$. We set

$$\tilde{\varepsilon} = \varepsilon \cdot V\left(B\left(z_0, \frac{D}{2}\right)\right) \quad \text{and} \quad \tilde{\varepsilon}_{\mathcal{M}^n}(D) = \varepsilon_{\mathcal{M}^n}(D) \cdot V\left(B\left(z_0, \frac{D}{2}\right)\right),$$

and hence $V(X) \geq V(B(z_0, \frac{D}{2})) - \tilde{\varepsilon}$ and $\tilde{\varepsilon} < \tilde{\varepsilon}_{\mathcal{M}^n}(D)$.

We consider a $\tilde{X} \subset \mathcal{M}^n$ that has maximal volume under the conditions $X \subset \tilde{X}$ and $\text{diam } \tilde{X} \leq D$. In particular, \tilde{X} is a convex body of constant width D according to Corollary 5.5. We fix some $x_1, x_2 \in \tilde{X}$ satisfying that $d_{\mathcal{M}^n}(x_1, x_2) = D$, let y_0 be the midpoint of $[x_1, x_2]_{\mathcal{M}^n}$, and let

$$\varrho = \begin{cases} \frac{D}{2} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = H^n, \\ \min\left\{\frac{D}{2}, \frac{\pi}{8} - \frac{D}{4}\right\} & \text{if } \mathcal{M}^n = S^n \text{ and } D < \frac{\pi}{2}, \end{cases}$$

be the ϱ in Proposition 7.4 that satisfies $D + 2\varrho < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$.

We consider the parallel domain $\tilde{X}^{(\varrho)}$, which is a convex body of constant width $D + 2\varrho$ by Lemma 5.7. In addition, Proposition 8.3 yields that

$$V_{\mathcal{M}^n}(\tilde{X}^{(\varrho)}) \geq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(x_0, \frac{D}{2} + \varrho\right)\right) - \tilde{E}_{\mathcal{M}^n}(\tilde{\varepsilon}, D) \quad (60)$$

where

$$\tilde{E}_{\mathcal{M}^n}(\tilde{\varepsilon}, D) = \begin{cases} \tilde{\varepsilon} \cdot 4^n & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \\ \tilde{\varepsilon} \cdot 4^n (\cosh D)^n & \text{if } \mathcal{M}^n = H^n \end{cases}.$$

As we verify it at the end of the proof of Theorem 9.2, we have

$$\tilde{E}_{\mathcal{M}^n}(\tilde{\varepsilon}_{\mathcal{M}^n}(D), D) < \tilde{\gamma}_1 \eta_0^{\frac{3n+2}{2}} \quad (61)$$

where the constants $\tilde{\gamma}_1$ and η_0 come from Proposition 7.4; namely,

$$\tilde{\gamma}_1 = \begin{cases} \frac{1}{2^{12n} n^n} \cdot \frac{1}{D^{\frac{n+2}{2}}} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \text{ and } D < \frac{\pi}{2}; \\ \frac{1}{2^{8n} n^n} \cdot \frac{1}{(\sinh 10D)^{\frac{n+2}{2}}} & \text{if } \mathcal{M}^n = H^n; \end{cases}$$

We recall that two-point symmetrization preserves Lebesgue measure and does not increase diameter, and hence

$$V_{\mathcal{M}^n}(\tilde{X}^{(\varrho)}) = V_{\mathcal{M}^n}(\tau_H \tilde{X}^{(\varrho)}) \leq V_{\mathcal{M}^n}(\operatorname{conv}_{\mathcal{M}^n} \tau_H \tilde{X}^{(\varrho)}) \leq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(z_0, \frac{D}{2} + \varrho\right)\right)$$

by the Isodiametric Inequality 1.2. Let

$$\frac{D}{2} + \eta = \min \left\{ \frac{D}{2} + \eta_0, \max_{z \in \tilde{X}} d_{\mathcal{M}^n}(z, y_0) \right\},$$

and hence Proposition 7.4 and (60) yield the existence of a hyperplane H such that

$$\begin{aligned} \tilde{\gamma}_1 \eta^{\frac{3n+2}{2}} &\leq V_{\mathcal{M}^n}(\operatorname{conv}_{\mathcal{M}^n} \tau_H \tilde{X}^{(\varrho)}) - V_{\mathcal{M}^n}(\tau_H \tilde{X}^{(\varrho)}) \\ &= V_{\mathcal{M}^n}(\operatorname{conv}_{\mathcal{M}^n} \tau_H \tilde{X}^{(\varrho)}) - V_{\mathcal{M}^n}(\tilde{X}^{(\varrho)}) \\ &\leq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(z_0, \frac{D}{2} + \varrho\right)\right) - V_{\mathcal{M}^n}(\tilde{X}^{(\varrho)}) \leq \tilde{E}_{\mathcal{M}^n}(\tilde{\varepsilon}, D). \end{aligned}$$

It follows, using the condition (61) and $\sinh 10D \cdot (\cosh D)^2 \leq \sinh 11D \cdot \cosh D \leq \sinh 12D$, that

$$\frac{\eta}{\tilde{\varepsilon}^{\frac{2}{3n+2}}} \leq \begin{cases} \left(2^{12n} n^n 4^n \cdot D^{\frac{n+2}{2}}\right)^{\frac{2}{3n+2}} \leq 2^{10} n D^{\frac{n+2}{3n+2}} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n; \\ \left(2^{8n} n^n (\sinh 10D)^{\frac{n+2}{2}} \cdot 4^n (\cosh D)^n\right)^{\frac{2}{3n+2}} \\ \leq 2^7 n (\sinh 12D)^{\frac{n+2}{3n+2}} & \text{if } \mathcal{M}^n = H^n. \end{cases}.$$

We observe that $\frac{n+2}{3n+2} \leq \frac{1}{2}$, and $\sinh t \leq e^{22t}$ for $t \in [0, 24]$; thus if $D \leq 2$, then

$$(\sinh 12D)^{\frac{n+2}{3n+2}} \leq \begin{cases} (e^{22} \cdot 12 \cdot D)^{\frac{n+2}{3n+2}} \leq e^{13} \cdot D^{\frac{n+2}{3n+2}} & \text{if } D \leq 2; \\ e^{12D \cdot \frac{n+2}{3n+2}} \leq e^{6D} & \text{if } D \geq 1. \end{cases}$$

Next, Lemma 9.1 yields

$$\begin{aligned} \frac{\tilde{\varepsilon}^{\frac{2}{3n+2}}}{\varepsilon^{\frac{2}{3n+2}}} &= V\left(B\left(z_0, \frac{D}{2}\right)\right)^{\frac{2}{3n+2}} \\ &\leq \begin{cases} \left(\frac{2^{\frac{3n}{2}}}{n^{\frac{n+1}{2}}} \cdot \frac{D^n}{2^n}\right)^{\frac{2}{3n+2}} \leq \frac{e \cdot D^{\frac{2n}{3n+2}}}{n^{\frac{1}{3}}} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n, \\ & \text{or } \mathcal{M}^n = H^n \text{ and } D \leq 2; \\ \left(\frac{2^{\frac{3n+2}{2}} e^{\frac{(n-1)D}{2}}}{n^{\frac{n+1}{2}}}\right)^{\frac{2}{3n+2}} \leq \frac{e^{\frac{D}{3}+1}}{n^{\frac{1}{3}}} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 1. \end{cases} \end{aligned} \quad (62)$$

Since $2^7 < e^6$ and $2^{10} < e^7$, we deduce from the estimates above that

$$R(\tilde{X}) \leq \max_{z \in \tilde{X}} d_{\mathcal{M}^n}(z, y_0) \leq \frac{D}{2} + \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}}$$

where

$$\tilde{\gamma}_2 = \begin{cases} e^{20n} \cdot D & \text{if } \mathcal{M}^n = H^n \text{ and } D \leq 2, \text{ or } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n; \\ e^{7D+7} n & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 1. \end{cases}$$

We deduce from $\varepsilon < \varepsilon_{\mathcal{M}^n}(D)$ (see the argument at the end of the proof of Theorem 9.2) that

$$\tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}} \leq \frac{D}{8}. \quad (63)$$

In turn, Lemma 5.6 yields that

$$r(\tilde{X}) \geq \frac{D}{2} - \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}} \geq \frac{3D}{8}.$$

In particular, writing c to denote the circumcenter of \tilde{X} , we have

$$B_{\mathcal{M}^n}\left(c, \frac{D}{2} - \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}}\right) \subset \tilde{X} \subset B_{\mathcal{M}^n}\left(c, \frac{D}{2} + \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}}\right). \quad (64)$$

Next, we write $\overline{X} = \text{conv } X$, and hence

$$V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(c, \frac{D}{2}\right)\right) - \tilde{\varepsilon} \leq V_{\mathcal{M}^n}(X) \leq V_{\mathcal{M}^n}(\overline{X}) \leq V_{\mathcal{M}^n}(\tilde{X}) \leq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(c, \frac{D}{2}\right)\right). \quad (65)$$

For any $x \in \partial_{\mathcal{M}^n} \overline{X}$, writing H_x^+ to denote the closed “supporting” half-space of \mathcal{M}^n such that $x \in H_x^+$ and $\overline{X} \cap \text{int } H_x^+ = \emptyset$, we deduce that

$$V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}\left(c, \frac{D}{2} - \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}}\right) \cap H_x^+\right) \leq \tilde{\varepsilon}. \quad (66)$$

We write δ_x to denote the depth of $B_{\mathcal{M}^n}\left(c, \frac{D}{2} - \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}}\right) \cap H_x^+$ for $x \in \partial_{\mathcal{M}^n} \overline{X}$, and hence combining (66) with Lemmas 8.4, 8.5 and 8.6 yield that

$$\tilde{\varepsilon} \geq \begin{cases} \frac{(\frac{D}{4})^{\frac{n-1}{2}}}{n^{\frac{n}{2}}} (\min\{\delta_x, \frac{D}{8}\})^{\frac{n+1}{2}} \geq \frac{D^{\frac{n-1}{2}}}{2^n \cdot n^{\frac{n}{2}}} (\min\{\delta_x, \frac{D}{8}\})^{\frac{n+1}{2}} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \\ \frac{(\frac{D}{4})^{\frac{n-1}{2}}}{2^n \cdot n^{\frac{n}{2}}} (\min\{\delta_x, \frac{D}{8}\})^{\frac{n+1}{2}} \geq \frac{D^{\frac{n-1}{2}}}{2^{2n} \cdot n^{\frac{n}{2}}} (\min\{\delta_x, \frac{D}{8}\})^{\frac{n+1}{2}} & \text{if } \mathcal{M}^n = S^n \\ \frac{(\tanh \frac{D}{4})^{\frac{n-1}{2}}}{n^{\frac{n}{2}}} (\min\{\delta_x, 1, \frac{D}{8}\})^{\frac{n+1}{2}} & \text{if } \mathcal{M}^n = H^n. \end{cases}$$

Since $\tilde{\varepsilon} \leq \tilde{\varepsilon}_{\mathcal{M}}(D)$, we deduce that if $x \in \partial_{\mathcal{M}^n} \overline{X}$, then $\delta_x < \frac{D}{8}$ if $\mathcal{M}^n = \mathbb{R}^n$ or $\mathcal{M}^n = S^n$, and $\delta_x < \min \left\{ 1, \frac{D}{8} \right\}$ if $\mathcal{M}^n = H^n$. Therefore, if $x \in \partial_{\mathcal{M}^n} \overline{X}$, then

$$\delta_x \leq \tilde{\gamma}_3 \tilde{\varepsilon}^{\frac{2}{n+1}}$$

where

$$\tilde{\gamma}_3 = \begin{cases} \frac{4n}{D^{\frac{n-1}{n+1}}} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \\ \frac{8n}{D^{\frac{n-1}{n+1}}} & \text{if } \mathcal{M}^n = S^n \\ \frac{n}{(\tanh \frac{D}{4})^{\frac{n-1}{n+1}}} & \text{if } \mathcal{M}^n = H^n. \end{cases}$$

It follows from Lemma 9.1 that

$$V \left(B \left(z_0, \frac{D}{2} \right) \right)^{\frac{2}{n+1}} \leq \begin{cases} \frac{2^4}{n} \cdot D^{\frac{2n}{n+1}} & \text{if either } \mathcal{M}^n = \mathbb{R}^n, \text{ or} \\ & \mathcal{M}^n = S^n, \text{ or } \mathcal{M}^n = H^n \text{ and } D \leq 2 \\ \frac{e^{\frac{2D}{3}+2}}{n} & \text{if } \mathcal{M}^n = H^n \text{ and } r > 0. \end{cases}$$

Now $\tilde{\varepsilon} = \varepsilon \cdot V \left(B \left(z_0, \frac{D}{2} \right) \right)$, $(\tanh \frac{D}{4})^{-\frac{n-1}{n+1}} < e$ for $D \geq 1$ and $\tanh t \geq \frac{t}{2}$ for $t \in [0, 1]$ imply

$$\delta_x \leq \tilde{\gamma}_4 \varepsilon^{\frac{2}{n+1}}$$

for any $x \in \partial_{\mathcal{M}^n} \overline{X}$ where

$$\tilde{\gamma}_4 = \begin{cases} 2^7 D & \text{if either } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n, \text{ or } \mathcal{M}^n = H^n \text{ and } D \leq 2 \\ e^{D+3} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 1. \end{cases}$$

In turn, we conclude from (64) that

$$B_{\mathcal{M}^n} \left(c, \frac{D}{2} - e \cdot \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}} \right) \subset \overline{X} \subset B_{\mathcal{M}^n} \left(c, \frac{D}{2} + \tilde{\gamma}_2 \varepsilon^{\frac{2}{3n+2}} \right).$$

Finally, (65) yields that $V_{\mathcal{M}^n} (\overline{X} \setminus X) \leq \varepsilon V_{\mathcal{M}^n} (B_{\mathcal{M}^n} (z_0, \frac{D}{2}))$.

We still need to verify that our choice of $\varepsilon_{\mathcal{M}^n}(D)$ works; in other words, it satisfies (61) and (63). We prove (61) on a case-by-case basis using (62) and the values of $\tilde{\gamma}_1$ and η_0 in Proposition 7.4. It follows from $\tilde{\varepsilon} = \varepsilon \cdot V \left(B \left(z_0, \frac{D}{2} \right) \right)$, $\cosh D < 4$ if $0 < D \leq 2$, $\cosh D < e^D$ if $D \geq 2$ and Lemma 9.1 that

$$\tilde{E}_{\mathcal{M}^n} (\tilde{\varepsilon}, D)^{\frac{2}{3n+2}}$$

$$\begin{aligned}
&= \begin{cases} \varepsilon^{\frac{2}{3n+2}} \cdot V\left(B\left(z_0, \frac{D}{2}\right)\right)^{\frac{2}{3n+2}} \cdot 4^{\frac{2n}{3n+2}} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \\ \varepsilon^{\frac{2}{3n+2}} \cdot V\left(B\left(z_0, \frac{D}{2}\right)\right)^{\frac{2}{3n+2}} \cdot 4^{\frac{2n}{3n+2}} (\cosh D)^{\frac{2n}{3n+2}} & \text{if } \mathcal{M}^n = H^n \end{cases} \\
&\leq \begin{cases} \varepsilon^{\frac{2}{3n+2}} \cdot D^{\frac{2n}{3n+2}} \cdot \frac{2^4}{n^{\frac{n+1}{3n+2}}} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \text{ or } \mathcal{M}^n = H^n \text{ and } D \leq 2 \\ \varepsilon^{\frac{2}{3n+2}} \cdot e^D \cdot \frac{2^{\frac{7}{3}}}{n^{\frac{n+1}{3n+2}}} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 2 \end{cases}
\end{aligned}$$

We deduce from $2^4 < e^3$, $2^{\frac{7}{3}} < e^2$, $2^{12n \cdot \frac{2}{3n+2}} < 2^8 < e^6$, $2^{8n \cdot \frac{2}{3n+2}} < 2^{\frac{16}{3}} < e^4$ and

$$(\sinh 10D)^{\frac{n+2}{2} \cdot \frac{2}{3n+2}} < \begin{cases} (e^{20} D)^{\frac{n+2}{3n+2}} < e^{10} D^{\frac{n+2}{3n+2}} & \text{if } 0 < D \leq 2 \\ e^{5D} & \text{if } D \geq 2 \end{cases}$$

the estimate

$$\frac{\tilde{E}_{\mathcal{M}^n}(\tilde{\varepsilon}, D)^{\frac{2}{3n+2}}}{\tilde{\gamma}_1^{\frac{2}{3n+2}}} \leq \begin{cases} \varepsilon^{\frac{2}{3n+2}} \cdot D \cdot e^{13} n^{\frac{n}{3n+2}} & \text{if } \mathcal{M}^n = \mathbb{R}^n \text{ or } \mathcal{M}^n = S^n \text{ or } \\ & \mathcal{M}^n = H^n \text{ and } D \leq 2 \\ \varepsilon^{\frac{2}{3n+2}} \cdot e^{6D+6} n^{\frac{n}{3n+2}} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 2. \end{cases}$$

Therefore, using $3n+2 \leq 4n$, we may choose

$$\varepsilon_{\mathcal{M}^n}(D) = \begin{cases} e^{-28n} n^{-\frac{n}{2}} & \text{if } \mathcal{M}^n = \mathbb{R}^n, \text{ or } \mathcal{M}^n = S^n \text{ and } D \leq \frac{\pi}{6}, \\ & \text{or } \mathcal{M}^n = H^n \text{ and } D \leq 2; \\ e^{-30n} n^{-\frac{n}{2}} \left(\frac{\pi}{2} - D\right)^{3n+2} & \text{if } \mathcal{M}^n = S^n \text{ and } \frac{\pi}{6} \leq D < \frac{\pi}{2}; \\ e^{-18D} n^{-\frac{n}{2}} & \text{if } \mathcal{M}^n = H^n \text{ and } D \geq 2 \end{cases}$$

in order to ensure that (61) holds.

Luckily, this choice of $\varepsilon_{\mathcal{M}^n}(D)$ also satisfies (63). \square

10. The Euclidean case revisited

We write B^n to denote the Euclidean unit ball centered at the origin, and $X \Delta Y$ to denote the symmetric difference of the sets X and Y . The goal of this section is to prove the following theorem.

Theorem 10.1. *For $n \geq 2$, if $X \subset \mathbb{R}^n$ is measurable with $\text{diam} X \leq D$ for $D > 0$ and*

$$V(X) \geq (1 - \varepsilon) V\left(\frac{D}{2} B^n\right)$$

for $\varepsilon \in [0, \frac{1}{2}]$, then there exists a $z \in \mathbb{R}^n$ such that

$$V\left(X \Delta \left(z + \frac{D}{2} B^n\right)\right) \leq \gamma_0 \sqrt{\varepsilon} \cdot V\left(\frac{D}{2} B^n\right) \quad (67)$$

for $\gamma_0 = cn^{\frac{5}{2}}(\log n)^5$, and assuming $\varepsilon < c^{-n}$, we have

$$z + \left(1 - c\varepsilon^{\frac{1}{n+1}}\right) \frac{D}{2} B^n \subset \text{conv} X \subset z + \left(1 - c\varepsilon^{\frac{1}{n+1}}\right) \frac{D}{2} B^n \quad (68)$$

where $V((\text{conv} X) \setminus X) \leq \varepsilon \cdot V(B_{\mathcal{M}^n}(z_0, \frac{D}{2}))$ and $c > 1$ is an absolute constant.

The first stability forms of the Brunn-Minkowski inequality were due to Minkowski himself (see Groemer [29]). If the distance of convex bodies K and C in \mathbb{R}^n is measured in terms of the Hausdorff distance, then Diskant [18] and Groemer [28] provided close to be optimal stability versions (see Groemer [29]). However, the natural distance is in terms volume of the symmetric difference, and the optimal result is due to Figalli, Maggi, Pratelli [21,22] (see Kolesnikov, Milman [34] Section 12 and Klartag, Lehec [33] for an improvement of the factor γ^* involved in Theorem 10.2).

For convex bodies K and C in \mathbb{R}^n , we define the “homothetic distance” $A(K, C)$ of convex bodies K and C to be

$$A(K, C) = \min \{V(\alpha K \Delta (x + \beta C)) : x \in \mathbb{R}^n\}$$

where $\alpha = V(K)^{\frac{-1}{n}}$ and $\beta = V(C)^{\frac{-1}{n}}$, and $K \Delta Q$ stands for the symmetric difference of K and Q . In addition, let $\sigma(K, C) = \max \left\{ \frac{V(C)}{V(K)}, \frac{V(K)}{V(C)} \right\}$. Now Figalli, Maggi, Pratelli [21,22] proved Theorem 10.2 up to the factor γ^* depending on n . The original factor γ^* of [22] was improved by Kolesnikov, Milman [34], and Section 12.2 of [34] used the upper bound $c_1 \sqrt[n]{n}$ on the Cheeger constant that was available that time for an absolute constant $c_1 > 0$. However, since then Klartag, Lehec [33] proved the upper bound $c_2(\log n)^5$ on the Cheeger constant for an absolute constant $c_2 > 0$ (note that the Cheeger constant in \mathbb{R}^n is upper bounded by an absolute constant according to the celebrated Kannan-Lovász-Simonovits conjecture).

Theorem 10.2 (Figalli-Maggi-Pratelli, Kolesnikov-Milman, Klartag-Lehec). *For convex bodies K and C in \mathbb{R}^n , and $\gamma^* = c_0 n^{-5}(\log n)^{-10}$ where $c \in (0, 1]$ is an absolute constant, we have*

$$V(K + C)^{\frac{1}{n}} \geq (V(K)^{\frac{1}{n}} + V(C)^{\frac{1}{n}}) \left[1 + \frac{\gamma^*}{\sigma(K, C)^{\frac{1}{n}}} \cdot A(K, C)^2 \right].$$

Here the exponent 2 of $A(K, C)^2$ is optimal (cf. Figalli, Maggi, Pratelli [22]). We note that Diskant [18] and Groemer [28] verified stability versions of the Brunn-Minkowski inequality in terms of the Hausdorff distance (see also Groemer [29]).

If C is a convex body in \mathbb{R}^n , then its support function is

$$h_C(u) = \max_{x \in C} \langle x, u \rangle \quad \text{for } u \in \mathbb{R}^n,$$

and hence $\text{diam } C = \max_{u \in S^{n-1}} (h_C(u) + h_C(-u))$. Since $h_{C+M} = h_C + h_M$ for another convex body M , it follows that the origin symmetric convex body $\frac{1}{2}(C - C)$ satisfies

$$\frac{1}{2}(C - C) \subset \frac{\text{diam } C}{2} B^n. \quad (69)$$

Proof of Theorem 10.1. We may assume that $D = 2$. We consider the convex body $K = \text{cl conv } X$, and hence $\text{diam } K \leq 2$ and $V(K) \geq (1 - \varepsilon)\kappa_n$ for $V(B^n) = \kappa_n$. It follows from the Isodiametric Inequality Theorem 1.2 that $V(K) \leq \kappa_n$. We may translate K in a way such that

$$A(K, -K) = V(K)^{-1} \cdot V(K \Delta (-K)) \geq \kappa_n^{-1} \cdot V(K \Delta (-K)).$$

As $1 - \varepsilon \geq (1 + 2\varepsilon)^{-1}$ follows from $\varepsilon \leq \frac{1}{2}$, we deduce from (69), $\sigma(K, -K) = 1$ and Theorem 10.2 that

$$\begin{aligned} \kappa_n &\geq V\left(\frac{1}{2}(K - K)\right) \geq V(K) \left(1 + \gamma^* \cdot \frac{V(K \Delta (-K))}{\kappa_n^2}\right)^2 \\ &\geq \frac{\kappa_n}{1 + 2\varepsilon} \left(1 + \gamma^* \cdot \frac{V(K \Delta (-K))}{\kappa_n^2}\right)^2; \end{aligned}$$

therefore,

$$V(K \Delta (-K)) \leq \sqrt{\frac{2\varepsilon}{\gamma^*}} \cdot \kappa_n. \quad (70)$$

In particular, (69) and (70) yield that $K_0 = K \cap (-K) \subset \frac{1}{2}(K - K)$ satisfies

$$\left(1 - \sqrt{\frac{\varepsilon}{\gamma^*}}\right) \cdot \kappa_n \leq V(K) - \frac{1}{2} V(K \Delta (-K)) = V(K_0) \leq \kappa_n, \quad (71)$$

and hence

$$V(K \Delta B^n) \leq V(K \Delta K_0) + V(K_0 \Delta B^n) \leq 2\sqrt{\frac{\varepsilon}{\gamma^*}},$$

proving (67).

Concerning the estimate for the Hausdorff distance of K from B^n , let δ be the maximal depth of a cap of B^n not overlapping with $K_0 \subset B^n$. It follows from (71) and Lemma 8.4 that

$$\sqrt{\frac{\varepsilon}{\gamma^*}} \cdot \kappa_n \geq \frac{2\kappa_{n-1}}{n+1} \cdot \delta^{\frac{n+1}{2}},$$

and hence

$$\delta \leq \left(\frac{\varepsilon}{\gamma^*} \right)^{\frac{1}{n+1}} \cdot \left(\frac{(n+1)\kappa_n}{2\kappa_{n-1}} \right)^{\frac{2}{n+1}},$$

proving (68). \square

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