

# Ultraproducts and Related Constructions

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**Abstract:** In this work, we survey some research directions in which the ultraproduct construction and methods based on ultrafilters play significant roles. Rather different areas of mathematics have been considered: topics we are reviewing here include some aspects of the model theory of first-order and second-order existential logics, finite Ramsey theory and general topology. Special emphasis has been made for producing a uniform treatment and for highlighting interconnections between these different subjects.

**Keywords:** ultrafilter; ultraproduct; first order logic; second order logic; finite Ramsey theory; ultraproducts in general topology

**MSC:** 03C20; 05C55; 54D99

## 1. Introduction

The ultraproduct construction was originally developed as a tool of the model theory of first-order logic. With the aid of this construction, profound results of first-order logic can be proven, such as the compactness theorem, theorems about axiomatizability, and characterizations of elementary equivalence. Further, ultrafilters and ultraproducts naturally arise in other areas of mathematics, such as finite and infinite combinatorics, measure theory, functional analysis, topology and the theory of social decisions (voting systems). To illustrate the wide and diverse applications of related ideas and constructions, we refer to [1].

In this work we survey research directions from first- and higher-order logic, from finite Ramsey theory and from general topology where ultrafilters and ultraproducts play a significant role. Our selection is inevitably subjective, and the topics and results covered here are also obviously bounded by our actual knowledge (however, we try to find the balance between reviewing the classical and the most important results, as well as less widely known but interesting results).

In more detail, here is an outline of the current research situation we are intending to review (all relevant definitions will be recalled at the appropriate points of this work):

- Ultrapowers modulo  $\kappa$ -regular ultrafilters are consistently  $\kappa^{++}$ -universal;
- Recent results about Keisler's order;
- Generalizations and density versions of Hindman's theorem;
- Ultraproducts of dynamical systems.

The theories of infinitesimal numbers in particular or nonstandard analysis in general are other topics where the ultraproduct construction (or, more generally, saturated structures) is of considerable importance. We decided not to survey these topics in the present work, because on the one hand, the subject is rather large, so to survey it we would need much more space. On the other hand, excellent texts on these topics are available (we refer to [2–4], and to the more recent [5,6]; see also [7]).

During the preparation of this manuscript, it became clear that it is impossible to



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produce a completely self-contained text that discusses the recent (sometimes rather special) developments of a particular subject. Hence, the present survey is intended for readers who are familiar with some aspects of ultrafilters and ultraproducts and interested in another related subject (either for learning the basics or to review recent developments of some other closely related subjects). Consequently, the depth and detail of the text is constantly changing. For example, in Section 2, we provide a detailed definition of ultraproducts, but in Section 3.2 we assume that reader is familiar with the notion of types. Each section begins at the level of an introductory graduate course, but our intentions are not to provide a systematic introduction. Instead, at the beginning of each section, we intend to fix some notations and quickly remind the reader of some facts they already know. For the reader's convenience, we include most of the relevant definitions—again, not for the reader to learn the subject, but for helping the reader so they do not need to search external sources.

We include proofs that, in complexity, are at the level of a related introductory graduate course (sometimes, these results are part of the folklore or may be exercises in an introductory course). We made one exception in Section 6, where we sketch Glazer's proof for Hindman's theorem (see Theorem 20 below). This proof is highly nontrivial, but so elegant and flexible that it deserves more detail. We made an effort to refer to the original sources where the results first appeared (but surely, our historical notes may contain inaccuracies; we apologize for these). After the introductory parts, we recall the questions that were the driving forces for further investigations, and sometimes (after a big change in the presentation of the details), we are able to review some very recent results as well. In some other cases, the recent results seem to be too technical, so we decided not to recall them in detail, because to do so, one would need a disproportionate amount of further technical preparation.

The structure of the remaining part of the present work is as follows. At the end of this section, we sum up our system of notation. In Section 2, we recall the definition and basic properties of ultrafilters and ultraproducts. We emphasize again that we assume our readers are more or less familiar with the content of this section; our main intention is only to fix notation, to exclude ambiguities, and to help the reader to find the precise version of a definition they essentially already know. In Section 3, we concentrate on the role of ultrafilters in first-order logic. This section is divided into three subsections: Section 3.1 is devoted to regular ultrafilters, Section 3.2 is devoted to good ultrafilters, and Section 3.3 is about Keisler's order (relevant definitions will be recalled at the beginning of each subsection). The main theme of Section 4 is the variants of the Isomorphic Ultrapowers Theorem: elementary equivalent structures have isomorphic ultrapowers. In Section 5, we survey some results that generalize the corresponding results of first-order model theory to the context of second-order existential logic (as mentioned already, the first-order case is discussed in the earlier sections of the present work). Section 6 is devoted to ultrafilter methods in finite Ramsey theory. As we mentioned, in this section, we discuss Glazer's proof for Hindman's theorem as well. In Section 7, we review the ultraproduct construction in general topology. Finally, in Section 8, we mention some open problems and research directions that remain open at the time, when this work had been prepared.

### Notation

Our notation is mostly standard, but the following list may be helpful.

1. Throughout,  $\omega$  denotes the set of natural numbers, and for every  $n \in \omega$  we have  $n = \{0, 1, \dots, n-1\}$ . Let  $A$  and  $B$  be sets. Then,  ${}^A B$  denotes the set of functions whose domain is  $A$  and whose range is a subset of  $B$ .
2. In addition,  $|A|$  denotes the cardinality of  $A$ , and  $\mathcal{P}(A)$  denotes the power set of  $A$ , that is,  $\mathcal{P}(A)$  consists of all subsets of  $A$ . For a cardinal  $\kappa$ ,  $[A]^\kappa$  (respectively,  $[A]^{<\kappa}$ ) denotes the set of subsets of  $A$  of cardinality  $\kappa$  (respectively, of cardinality smaller than  $\kappa$ ).
3. If  $\mathcal{A}$  and  $\mathcal{B}$  are first-order structures, then  $\mathcal{A} \leq \mathcal{B}$  denotes the fact that  $\mathcal{A}$  is a substructure of  $\mathcal{B}$ . Structures will be denoted by calligraphic letters, and their underlying

sets will be denoted by the corresponding Latin letter. If  $\langle R_i : i \in I \rangle$  is a sequence of functions or relations (possibly with higher arities) on  $A$ , then  $\langle \mathcal{A}, R_i \rangle_{i \in I}$  denotes the structure obtained from  $\mathcal{A}$  by adjoining a new symbol  $r_i$  to the language of  $\mathcal{A}$  for all  $i \in I$  and interpreting these new symbols in the natural way.

## 2. Technical Introduction

For completeness, we start by recalling the basic definitions and some classical results related to filters, ultrafilters, and ultraproducts. We assume our readers are familiar with the content of this section—we included it just to fix notation and to provide internal, quick references for some well-known details. Experienced readers may completely skip this section. For notation, concepts, and proofs not recalled here, and for further results, we refer to [2].

**Definition 1.** Let  $I$  be any nonempty set.  $\mathcal{F} \subseteq \mathcal{P}(I)$  is defined to be a filter over  $I$  iff

- (1)  $\emptyset \notin \mathcal{F}$  and  $\mathcal{F} \neq \emptyset$ ;
- (2) if  $x \subseteq y \subseteq I$  and  $x \in \mathcal{F}$ , then  $y \in \mathcal{F}$ ;
- (3) if  $x, y \in \mathcal{F}$  then  $x \cap y \in \mathcal{F}$ .

$\mathcal{F}$  is defined to be an ultrafilter iff, in addition,

- (4) for all  $x \subseteq I$  either  $x \in \mathcal{F}$ , or  $I - x \in \mathcal{F}$  holds as well.

Let  $I$  be a nonempty set. For  $a \in I$ , define  $\mathcal{F}_a \subseteq \mathcal{P}(I)$  as

$$\mathcal{F}_a = \{x \subseteq I : a \in x\}.$$

It is routine to check that  $\mathcal{F}_a$  is an ultrafilter over  $I$ .  $\mathcal{F}_a$  is called the principal ultrafilter (over  $I$ ) generated by  $a$ . In particular, (principal) ultrafilters exist over each nonempty set. Further, if  $I$  is finite, then all ultrafilters over  $I$  are principal because of the following. Suppose, seeking a contradiction, that  $I$  is finite and  $\mathcal{F} \subseteq \mathcal{P}(I)$  is a nonprincipal ultrafilter over  $I$ . Then, for all  $a \in I$ , we have  $\{a\} \notin \mathcal{F}$  (because otherwise  $\mathcal{F}$  would be the principal ultrafilter generated by  $a$ ). Hence,  $I - \{a\} \in \mathcal{F}$  holds for all  $a \in I$ . However, then, since  $I$  is finite,

$$\mathcal{F} \ni \bigcap_{a \in I} I - \{a\} = \emptyset,$$

which contradicts Definition 1 (4).

An ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$  is defined to be *nonprincipal* iff, for all  $a \in I$ , we have  $\mathcal{F} \neq \mathcal{F}_a$ . Nonprincipal ultrafilters exist over each infinite set because of the following. For an infinite  $I$ , define

$$\mathcal{F} = \{x \subseteq I : I - x \text{ is finite}\}.$$

It is easy to check that  $\mathcal{F}$  is a filter (the so-called Fréchet filter over  $I$ ). By Zorn's Lemma, it can be extended to an ultrafilter, and clearly, each ultrafilter extending the Fréchet filter is nonprincipal.

Now, in order to describe the ultraproduct construction, first, we recall direct products of first-order structures.

Throughout this section, we fix a first-order language  $L$ ; constant symbols in  $L$  are treated as 0-ary function symbols.

**Definition 2.** Let  $I$  be a nonempty set, and for each  $i \in I$ , let  $\mathcal{A}_i$  be an  $L$ -structure. Then, the direct product

$$\mathcal{A} := \prod_{i \in I} \mathcal{A}_i$$

is the  $L$ -structure defined as follows.

- The underlying set  $A$  of  $\mathcal{A}$  is  $A = \prod_{i \in I} A_i$ , that is,  $A$  is the (set-theoretical) direct product of the underlying sets of the  $\mathcal{A}_i$ .
- If  $f$  is an  $n$ -ary function symbol in  $L$ , then its interpretation  $f^{\mathcal{A}}$  in  $\mathcal{A}$  is defined coordinatewise; that is, for  $s_0, \dots, s_{n-1} \in A$ , we define

$$f^{\mathcal{A}}(s_0, \dots, s_{n-1}) = \langle f^{A_i}(s_0(i), \dots, s_{n-1}(i)) : i \in I \rangle.$$

- If  $R$  is an  $n$ -ary relation symbol in  $L$ , then its interpretation  $R^{\mathcal{A}}$  in  $\mathcal{A}$  is defined coordinatewise as well; that is, for any  $s_0, \dots, s_{n-1} \in A$

$$\langle s_0, \dots, s_{n-1} \rangle \in R^{\mathcal{A}} \text{ iff } (\forall i \in I) (\langle s_0(i), \dots, s_{n-1}(i) \rangle \in R^{A_i}).$$

Keeping the notation of Definition 2, suppose  $\mathcal{F} \subseteq \mathcal{P}(I)$  is a filter. We define a binary relation  $\sim_{\mathcal{F}}$  on  $A$  as follows. For any  $s, z \in A$

$$s \sim_{\mathcal{F}} z \text{ iff } \{i \in I : s(i) = z(i)\} \in \mathcal{F}.$$

It is easy to see that  $\sim_{\mathcal{F}}$  is an equivalence relation, and further, it is compatible with all coordinatewise defined functions and relations; that is, if  $f$  is an  $n$ -ary function symbol and  $R$  is an  $n$ -ary relation symbol in  $L$  and  $s_0, \dots, s_{n-1}, z_0, \dots, z_{n-1} \in A$  are such that  $s_0 \sim_{\mathcal{F}} z_0, \dots, s_{n-1} \sim_{\mathcal{F}} z_{n-1}$ , then

$$f^{\mathcal{A}}(s_0, \dots, s_{n-1}) \sim_{\mathcal{F}} f^{\mathcal{A}}(z_0, \dots, z_{n-1})$$

and

$$\{i \in I : \langle s_0(i), \dots, s_{n-1}(i) \rangle \in R^{A_i}\} \in \mathcal{F} \text{ iff } \{i \in I : \langle z_0(i), \dots, z_{n-1}(i) \rangle \in R^{A_i}\} \in \mathcal{F}.$$

Now we are ready to recall the definitions of reduced products and ultraproducts.

**Definition 3.** Keeping the notation of Definition 2, assume further that  $\mathcal{F} \subseteq \mathcal{P}(I)$  is a filter. Then the reduced product

$$\mathcal{A} := \prod_{i \in I} \mathcal{A}_i / \mathcal{F}$$

is the  $L$ -structure defined as follows.

- The underlying set  $A$  of  $\mathcal{A}$  is  $A = \prod_{i \in I} A_i / \sim_{\mathcal{F}}$ ; that is,  $A$  is the set of equivalence classes of  $\sim_{\mathcal{F}}$ .
- If  $f$  is an  $n$ -ary function symbol in  $L$ , then its interpretation  $f^{\mathcal{A}}$  in  $\mathcal{A}$  is defined as follows: for  $s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}} \in A$ , we define

$$f^{\mathcal{A}}(s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}}) = f^{\prod_{i \in I} \mathcal{A}_i}(s_0, \dots, s_{n-1}) / \sim_{\mathcal{F}}.$$

- If  $R$  is an  $n$ -ary relation symbol in  $L$ , then its interpretation  $R^{\mathcal{A}}$  in  $\mathcal{A}$  is defined as follows. For any  $s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}} \in A$

$$\langle s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}} \rangle \in R^{\mathcal{A}} \text{ iff } \{i \in I : \langle s_0(i), \dots, s_{n-1}(i) \rangle \in R^{A_i}\} \in \mathcal{F}.$$

If  $\mathcal{F}$  is an ultrafilter, then the corresponding reduced product is called the ultraproduct modulo  $\mathcal{F}$ .

If each  $\mathcal{A}_i$  is the same structure  $\mathcal{B}$ , then instead of reduced products and ultraproducts, we say reduced powers and ultrapowers, respectively, and denote them by  ${}^I \mathcal{B} / \mathcal{F}$ .

Note that by the remark just before Definition 3, reduced products are well defined; that is, if  $f$  is an  $n$ -ary function symbol in  $L$ , then  $f^{\mathcal{A}}(s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}})$  does not depend on the particular choice of the representatives  $s_0, \dots, s_{n-1}$  of the equivalence classes  $s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}}$  and similarly, if  $R$  is an  $n$ -ary relation symbol of  $L$ , then

$$\{i \in I : \langle s_0(i), \dots, s_{n-1}(i) \rangle \in R^{A_i}\} \in \mathcal{F}$$

also does not depend on the representatives  $s_0, \dots, s_{n-1}$  of  $s_0 / \sim_{\mathcal{F}}, \dots, s_{n-1} / \sim_{\mathcal{F}}$ .

For generalizations of the ultraproduct construction, we refer to Chapters 6.4 and 6.5 of [2,8].

One of the fundamental properties of ultraproducts is that they preserve the validity of the first-order formulas: if a first order formula  $\varphi$  is true in each  $\mathcal{A}_i$ , then  $\varphi$  remains true in their ultraproduct. In fact, the following stronger theorem is true.

**Theorem 1.** (Łoś.) *Let  $I$  be a nonempty set; for each  $i \in I$  let  $\mathcal{A}_i$  be an  $L$ -structure,  $\mathcal{F} \subseteq \mathcal{P}(I)$  is an ultrafilter and  $\varphi$  is a first-order formula of  $L$ . Then, the following are equivalent:*

- (1)  $\prod_{i \in I} \mathcal{A}_i / \mathcal{F} \models \varphi$ ;
- (2)  $\{i \in I : \mathcal{A}_i \models \varphi\} \in \mathcal{F}$ .

The proof is based on an induction on the complexity of  $\varphi$  and can be found in almost all textbooks on model theory.

It is also natural to ask that what kind of formulas are preserved under reduced products. The answer is the following: a sentence  $\varphi$  is preserved under all reduced products iff  $\varphi$  is logically equivalent with a Horn sentence. For the proof, further details, and historical remarks, we refer to Section 6.2 (in particular, to Theorem 6.2.5) of [2].

An important consequence of Theorem 1 is the compactness theorem.

**Theorem 2.** (Compactness Theorem.) *Let  $\Sigma$  be a set of first-order formulas. Then, the following are equivalent.*

- (1) *Each finite subset of  $\Sigma$  has a model.*
- (2)  *$\Sigma$  has a model.*

**Proof.** Clearly, (2) implies (1). For the converse direction, assume (1). We proceed by a case distinction. If  $\Sigma$  is finite, then (2) holds obviously. If  $\Sigma$  is infinite, then we will see in Section 3.1 that there is an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(\Sigma)$  and an injective function  $f : \Sigma \rightarrow \mathcal{F}$  such that

$$\nu(\varphi) := \{\psi \in \Sigma : \varphi \in f(\psi)\}$$

is finite for all  $\varphi \in \Sigma$  (in the terminology of Section 3.1,  $\mathcal{F}$  can be chosen to be any regular ultrafilter on  $\Sigma$ ). Then, by (1), for each  $\varphi \in \Sigma$ , there exists a structure  $\mathcal{A}_\varphi$  such that  $\mathcal{A}_\varphi \models \nu(\varphi)$ . Let

$$\mathcal{A} := \prod_{\varphi \in \Sigma} \mathcal{A}_\varphi / \mathcal{F}.$$

We claim that  $\mathcal{A} \models \Sigma$ . To verify this, assume  $\psi \in \Sigma$ . As  $f(\psi) \in \mathcal{F}$ , by Theorem 1, it is enough to show that  $\mathcal{A}_\varphi \models \psi$  for all  $\varphi \in f(\psi)$ . However, if  $\varphi \in f(\psi)$ , then  $\psi \in \nu(\varphi)$ . Hence, by construction,  $\mathcal{A}_\varphi \models \psi$ , as desired.  $\square$

We close this section with some well-known applications. Recall that for a class  $K$  of  $L$ -structures, the theory  $TH(K)$  of  $K$  is defined to be

$$TH(K) = \{\varphi : K \models \varphi\}$$

, and conversely, for a set  $\Sigma$  of  $L$ -formulas, the class  $Mod(\Sigma)$  of models of  $\Sigma$  is defined to be

$$Mod(\Sigma) = \{\mathcal{A} : \mathcal{A} \models \Sigma\}.$$

Two  $L$ -structures  $\mathcal{A}$  and  $\mathcal{B}$  are defined to be *elementarily equivalent* (in symbols  $\mathcal{A} \equiv_e \mathcal{B}$ ) iff  $TH(\mathcal{A}) = TH(\mathcal{B})$ .

$K$  is closed under elementary equivalence iff  $\mathcal{A} \in K, \mathcal{A} \equiv_e \mathcal{B}$  imply  $\mathcal{B} \in K$ .

A class  $K$  of  $L$ -structures is defined to be *axiomatizable* iff there exists a set  $\Sigma$  of formulas such that  $K = \text{Mod}(\Sigma)$ . Further,  $K$  is *finitely axiomatizable* iff  $\Sigma$  can be chosen to be finite.

It is natural and important to describe axiomatizable classes with their closure properties. The next classical result settles this question.

**Theorem 3.** *Let  $K$  be a class of  $L$ -structures. Then, the following are equivalent.*

- (1)  $K$  is closed under ultraproducts and elementary equivalence.
- (2)  $K$  is axiomatizable.

**Theorem 4.** *Let  $K$  be a class of  $L$ -structures. Then, the following are equivalent.*

- (1)  $K$  and the complement of  $K$  are closed under ultraproducts.
- (2)  $K$  is finitely axiomatizable.

For the proofs, we refer to Theorem 4.1.12 of [2]. For a further related result we refer to [9] as well.

### 3. Ultrafilters and First-Order Logic

In this section, we will investigate special kinds of ultrafilters and relate their extra features to model-theoretic properties of ultraproducts modulo them. More specifically, we will recall some classical results on regular and good ultrafilters. At the end of this section, we will focus on Keisler's order.

#### 3.1. Regular Filters

**Definition 4.** *Let  $I$  be any nonempty set.  $\mathcal{E} \subseteq \mathcal{P}(I)$  is defined to be point finite iff*

$$\{x \in \mathcal{E} : i \in x\}$$

*is finite for all  $i \in I$ .*

**Definition 5.** *Let  $I$  be a nonempty set, and let  $\kappa$  be a cardinal. A filter  $\mathcal{F} \subseteq \mathcal{P}(I)$  is defined to be  $\kappa$ -regular iff there exists a point finite  $\mathcal{E} \subseteq \mathcal{F}$  such that  $|\mathcal{E}| = \kappa$ .*

*Further,  $\mathcal{F}$  is regular iff it is  $|I|$ -regular.*

Regular ultrafilters over infinite sets always exist (see, e.g., Proposition 4.3.5 of [2]). In fact, in our usual set theory  $ZFC$ , it is impossible to construct “non-trivial” ultrafilters that are not regular. More specifically, an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$  is defined to be *countably incomplete* iff it is not closed under countably infinite intersections, and it is defined to be *uniform* iff for all  $x \in \mathcal{F}$  we have  $|x| = |I|$ . Let  $R$  denote the statement

$R =$  “uniform ultrafilters over infinite sets are regular”.

Donder in [10] proved that if  $ZFC$  is consistent, then so is  $ZFC + R$ . Therefore, if  $ZFC$  is consistent, then  $R$  cannot be refuted from  $ZFC$ . On the other hand, by some results of Foreman, Magidor, and Shelah,  $R$  cannot be proved from  $ZFC$ ; for details, we refer to [11].

Further, if  $I$  is infinite, then there does not exist an  $|I|^+$ -regular filter over  $I$ . In fact, there does not exist a point finite  $\mathcal{E} \subseteq \mathcal{P}(I)$  with  $|\mathcal{E}| \geq |I|^+$  because of the following. Suppose  $\mathcal{E} \subseteq \mathcal{P}(I)$  is point-finite. By the axiom of choice, there exists a function  $f : \mathcal{E} \rightarrow I$  such that for all  $x \in \mathcal{E}$ , we have  $f(x) \in x$ . Since  $\mathcal{E}$  is point-finite and  $I$  is infinite,

$$|\mathcal{E}| = |\text{dom}(f)| \leq |\text{ran}(f)| \cdot \aleph_0 \leq |I| \cdot \aleph_0 = |I|,$$

as desired.

It is natural to investigate the possible cardinalities of ultrapowers in general and the cardinalities of ultrapowers modulo regular ultrafilters in particular. In order to study this question, suppose that  $A$  and  $I$  are sets and  $\mathcal{F}$  is a filter over  $I$ . As the reduced power



${}^I A/\mathcal{F}$  is the set of equivalence classes of an equivalence relation on the direct power  ${}^I A$ , clearly,  $|{}^I A/\mathcal{F}| \leq |A|^{|I|}$ .

Reduced powers modulo a regular filter have large (in fact, in view of the previous paragraph, the “largest possible”) cardinalities. More precisely, the following is true.

**Theorem 5.** *If  $A$  is an infinite set and  $\mathcal{F}$  is a regular ultrafilter over  $I$ , then  $|{}^I A/\mathcal{F}| = |A|^{|I|}$ .*

For the proof, see, e.g., Proposition 4.3.7 of [2].

Regular ultrapowers are large in another sense as well: for a cardinal  $\kappa$ , a structure  $\mathcal{A}$  is defined to be  $\kappa$ -universal iff  $\mathcal{B} \equiv_e \mathcal{A}$ ,  $|B| < \kappa$  imply that  $\mathcal{B}$  can be elementarily embedded into  $\mathcal{A}$ . Further,  $\mathcal{A}$  is defined to be universal iff it is  $|A|$ -universal. Clearly, in an intuitive sense, universal structures are large. Ultrapowers modulo a regular ultrafilter are still large in this sense.

**Theorem 6.** *Suppose we have  $|L| \leq \kappa$  for our language  $L$ . If  $\mathcal{A}$  is an infinite  $L$ -structure and  $\mathcal{F}$  is a  $\kappa$ -regular ultrafilter over a set  $I$ , then the ultrapower  ${}^I \mathcal{A}/\mathcal{F}$  is  $\kappa^+$ -universal.*

For the proof, see [12,13] and Theorem 4.3.12 of [2].

If  $\mathcal{A}$  is an infinite  $L$ -structure (with  $|L| \leq \kappa$ ) and  $\mathcal{F} \subseteq \mathcal{P}(\kappa)$  is a  $\kappa$ -regular ultrafilter, then by Theorem 5,

$$|{}^\kappa \mathcal{A}/\mathcal{F}| = |A|^\kappa \geq 2^\kappa \geq \kappa^+.$$

Hence, the cardinality of  ${}^I \mathcal{A}/\mathcal{F}$  does not exclude the possibility that  ${}^I \mathcal{A}/\mathcal{F}$  is  $\kappa^{++}$ -universal. Indeed, there is no obvious reason for why  ${}^I \mathcal{A}/\mathcal{F}$  cannot be  $\kappa^{++}$ -universal. In fact, in Conjecture 19 of [2], Chang and Keisler conjectured that

if  $\mathcal{F}$  is a regular ultrafilter over  $\kappa$  and  $\mathcal{A}$  is infinite, then  ${}^\kappa \mathcal{A}/\mathcal{F}$  is  $\kappa^{++}$ -universal.

It turned out that this statement is independent of ZFC. In fact, in [14], there is a combinatorial principle (beyond ZFC) implying that the conjecture is true. Further, among others, in [15], it was shown that, relative to the consistency of the existence of supercompact cardinals, it is consistent that the conjecture fails on  $\kappa = \aleph_\omega$ . For further related results we also refer to [16,17].

### 3.2. Good Ultrafilters

Let  $\mathcal{A}$  be an  $L$ -structure and let  $X \subseteq A$ . Recall that  $\mathcal{A}$  is defined as *saturated* over  $X$  iff each type over  $X$  can be realized in  $\mathcal{A}$ . Further, for a cardinal  $\kappa$ ,  $\mathcal{A}$  is defined as  $\kappa$ -saturated iff it is saturated over all  $X \in [A]^{<\kappa}$  and  $\mathcal{A}$  is defined as saturated iff it is  $|A|$ -saturated.

Let  $T$  be a first-order theory. Saturated models of  $T$  do not necessarily exist (in general,  $\kappa^+$ -saturated models of  $T$  have cardinality of at least  $2^\kappa$  because  $2^\kappa$  many types might exist over a set of cardinality  $\kappa$ ). However, saturated models of  $T$  are unique up to their cardinalities (this can be shown by a straightforward back-and-forth argument, the details of which can be found in Chapter 5 of [2] in general and Lemma 5.1.11 of [2] in particular).

It is natural to investigate saturation properties of ultraproducts and ultrapowers. We start with two simple observations.

Let  $I$  be a set and let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be an ultrafilter. If our language  $L$  is countable and  $\mathcal{F}$  is countably incomplete, then each ultraproduct modulo  $\mathcal{F}$  is  $\aleph_1$ -saturated (for a simple proof, we refer to Theorem 6.1.1. of [2]).

Let  $\mathcal{A}$  be an  $L$ -structure. Then, the diagonal embedding  $\delta : A \rightarrow {}^I \mathcal{A}/\mathcal{F}$  is defined as

$$\delta(a) = \langle a, a, \dots \rangle / \mathcal{F}$$

for all  $a \in A$ . It is easy to see that  $\delta$  is an elementary embedding. If  $\mathcal{F}$  is  $\max\{|A|, |L|\}$ -regular, then  ${}^I \mathcal{A}/\mathcal{F}$  will be saturated over the range of  $\delta$  because of the following. Let  $p$  be any type in  ${}^I \mathcal{A}/\mathcal{F}$  over the range of  $\delta$ , and let  $\mathcal{E} \subseteq \mathcal{F}$  be point-finite with  $|\mathcal{E}| = \max\{|A|, |L|\}$ . Then, there exists a bijection  $f : p \rightarrow \mathcal{E}$ . For each  $i \in I$  let

$$\nu(i) = \{\varphi(v, \delta(\bar{c})) : i \in f(\varphi(v, \delta(\bar{c})))\}.$$

Observe that  $\nu(i)$  is finite because  $\mathcal{E}$  is point-finite. Hence, for each  $i \in I$ , there exists  $a_i \in A$  such that

$$\mathcal{A} \models \bigwedge_{\varphi(v, \delta(\bar{c})) \in \nu(i)} \varphi(a_i, \bar{c}).$$

Let  $a = \langle a_i : i \in I \rangle / \mathcal{F}$ . Clearly, if  $\varphi(v, \delta(\bar{c})) \in p$  and  $i \in f(\varphi(v, \delta(\bar{c})))$ , then  $\varphi(v, \delta(\bar{c})) \in \nu(i)$ . Hence  $\mathcal{A} \models \varphi(a_i, \bar{c})$ . Therefore, by Łoś Lemma (see Theorem 1 above),

$${}^I \mathcal{A} / \mathcal{F} \models \varphi(a, \delta(\bar{c})).$$

It follows that  $a$  realizes  $p$ , as desired.

Can an ultrapower or ultrapower be more saturated? Counterexamples show that using regular ultrafilters, one cannot achieve stronger saturation properties of ultrapowers; for this, we need ultrafilters with more special features. More specifically, in Definition 6 we recall the definition of good ultrafilters and then illustrate how they can be used to obtain some control on saturation properties of ultrapowers modulo them.

**Definition 6.** Let  $I$  be a nonempty set,  $\kappa$  a cardinal and let  $f, g : [\kappa]^{<\aleph_0} \rightarrow \mathcal{P}(I)$  be functions. Then

- $f$  is defined as being monotone iff  $x \subseteq y \in [\kappa]^{<\aleph_0}$  implies  $f(x) \supseteq f(y)$ ;
- $g$  is defined as being additive iff for all  $x, y \in [\kappa]^{<\aleph_0}$  we have  $g(x \cup y) = g(x) \cap g(y)$ ;
- $g$  is a refinement of  $f$  iff for all  $x \in [\kappa]^{<\aleph_0}$  we have  $g(x) \subseteq f(x)$ .

An ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$  is defined as being  $\lambda$ -good iff for all  $\kappa < \lambda$ , each monotone  $f : [\kappa]^{<\aleph_0} \rightarrow \mathcal{F}$  has an additive refinement  $g : [\kappa]^{<\aleph_0} \rightarrow \mathcal{F}$ .

Finally,  $\mathcal{F}$  is a good ultrafilter iff it is  $|I|^+$ -good.

Keeping the notation of Definition 6, an arbitrary function  $h : [\kappa]^{<\aleph_0} \rightarrow \mathcal{F}$  has a monotone refinement  $f$ ; just define

$$f(x) = \bigcap_{y \subseteq x} h(y)$$

for all  $x \in [\kappa]^{<\aleph_0}$ . Hence,  $\mathcal{F}$  is a  $\lambda$ -good ultrafilter iff for all  $\kappa < \lambda$ , each  $h : [\kappa]^{<\aleph_0} \rightarrow \mathcal{F}$  has an additive refinement  $g : [\kappa]^{<\aleph_0} \rightarrow \mathcal{F}$ .

Observe further that each ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$  is  $\aleph_1$ -good, because of the following. If  $f : [\aleph_0]^{<\aleph_0} \rightarrow \mathcal{F}$  is any monotone function, then define  $g$  as being

$$g(x) = f(\{0, 1, \dots, \max(x)\})$$

for all  $x \in [\aleph_0]^{<\aleph_0}$ . Clearly,  $g$  is an additive refinement of  $f$ .

Now, we recall the existence theorem for good ultrafilters.

**Theorem 7.** For each infinite cardinal  $\kappa$ , there exists a countably incomplete, good ultrafilter on  $\kappa$ .

For the proof, we refer to Theorem 6.1.4 of [2] and Theorem VI.3.1. of [18]. Originally, Keisler proved Theorem 7 in [13] by assuming the generalized continuum hypothesis. Later, in [19], Kunen proved it in ZFC.

As the next theorem shows, the significance of  $\kappa$ -good ultrafilters is that ultrapowers modulo them are  $\kappa$ -saturated.

**Theorem 8.** Suppose  $\kappa$  is an infinite cardinal; our language  $L$  satisfies  $|L| < \kappa$ , and  $\mathcal{F} \subseteq \mathcal{P}(I)$  is a countably incomplete,  $\kappa$ -good ultrafilter. Then, each ultrapower modulo  $\mathcal{F}$  is  $\kappa$ -saturated.

For the proof, we refer to Theorem 6.1.8 of [2].



### 3.3. Keisler's Order

Let  $\mathcal{A}$  be an  $L$ -structure and let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be a regular ultrafilter. However,  ${}^I\mathcal{A}/\mathcal{F}$  is not saturated for all  $\mathcal{A}$ , it still may be saturated, or at least  $\lambda$ -saturated for some special  $\mathcal{A}$  and  $\lambda$ . For example, if  $L$  is empty (does not contain any function or relation symbols), then  ${}^I\mathcal{A}/\mathcal{F}$  will be saturated for each  $L$ -structure  $\mathcal{A}$ . This motivates the idea to “measure the complexity” of a structure  $\mathcal{A}$  with the class of regular ultrafilters for which the corresponding ultrapower of  $\mathcal{A}$  is saturated: the complexity of  $\mathcal{A}$  is smaller if it is easier to find saturated ultrapowers of it. The precise definition is as follows.

**Definition 7.** Let  $L_1$  and  $L_2$  be countable languages, and let  $T_1, T_2$  be complete theories in  $L_1$ , respectively, in  $L_2$ . Then  $T_1 \preceq T_2$  iff for any cardinal  $\kappa$ , any pair of structures  $\mathcal{A} \models T_1$ ,  $\mathcal{B} \models T_2$ , and any regular ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ ,  $\kappa^+$ -saturatedness of  ${}^\kappa\mathcal{B}/\mathcal{F}$  implies  $\kappa^+$ -saturatedness of  ${}^\kappa\mathcal{A}/\mathcal{F}$ .

It is easy to see that  $\preceq$  is a preorder (reflexive and transitive). Two theories  $T_1, T_2$  are defined as being equivalent iff  $T_1 \preceq T_2$  and  $T_2 \preceq T_1$ . This relation is an equivalence relation and  $\preceq$  induces a partial ordering on the equivalence classes. This partial ordering is called *Keisler's order*, and it was introduced in [20]. Since then, Keisler's order has been thoroughly investigated. For example, by Theorem VI.4.3 of [2], any theory having the strict order property is  $\preceq$ -minimal.

In [21], a combinatorial tool was developed to study the preorder structure of  $\preceq$ . With the aid of this combinatorial tool, the existence of a minimal unstable theory has been established. Sufficient conditions for maximality of a theory have also been presented. In [22], the authors mainly study Keisler's order on unstable and on simple, unstable theories.

For a more recent related result, we refer to [23] where the authors show that there exists a continuum-sized family of first-order theories (in countable languages) such that Keisler's order on them is order isomorphic with  $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ , where, for  $x, y \in [\omega]^{\aleph_0}$ ,  $x \subseteq^* y$  iff  $x - y$  is finite (that is,  $y$  contains  $x$  modulo a finite set, or  $y$  “almost contains  $x$ ”).

### 4. Ultraproducts and First-Order Logic

Suppose  $\mathcal{A}$  is an infinite  $L$ -structure. Then there exists another  $L$ -structure  $\mathcal{B}$  such that  $\mathcal{A} \equiv_e \mathcal{B}$ , but  $\mathcal{A}$  and  $\mathcal{B}$  are not isomorphic because of the following. Let  $\mathcal{F}$  be any  $|A|$ -regular ultrafilter and choose  $\mathcal{B}$  as the ultrapower of  $\mathcal{A}$  modulo  $\mathcal{F}$ . Then, on the one hand, by the Łoś Lemma (Theorem 1 above),  $\mathcal{A}$  and  $\mathcal{B}$  are elementarily equivalent. On the other hand, by Theorem 5,

$$|B| \geq |A|^{|A|} \geq 2^{|A|} > |A|;$$

hence  $\mathcal{A}$  and  $\mathcal{B}$  cannot be isomorphic. Therefore, ultrapowers (modulo regular enough ultrafilters) produce elementarily equivalent but nonisomorphic pairs of models.

Surprisingly, a kind of converse of this phenomena is true as well: if  $\mathcal{A}$  and  $\mathcal{B}$  are elementarily equivalent, then they have isomorphic ultrapowers. This result and similar results are called *Isomorphic Ultrapowers Theorems*. These theorems are of central importance because they provide model-theoretic characterizations for elementary equivalence. The present section is devoted to the Isomorphic Ultrapowers Theorems and their short history. We start with a very simple observation.

**Theorem 9.** Assume the Continuum Hypothesis:  $2^{\aleph_0} = \aleph_1$ . Assume that  $\mathcal{A}, \mathcal{B}$  are countable  $L$ -structures and  $L$  is also countable. Let  $\mathcal{F} \subseteq \mathcal{P}(\omega)$  be a regular ultrafilter. Then the following are equivalent:

- (1)  $\mathcal{A}$  and  $\mathcal{B}$  are elementarily equivalent.
- (2)  ${}^\omega\mathcal{A}/\mathcal{F}$  and  ${}^\omega\mathcal{B}/\mathcal{F}$  are isomorphic.

**Proof.** First, we show that (1) implies (2). By the Łoś Lemma (Theorem 1 above) we have

$${}^\omega\mathcal{A}/\mathcal{F} \equiv_e \mathcal{A} \stackrel{(1)}{\equiv_e} \mathcal{B} \equiv_e {}^\omega\mathcal{B}/\mathcal{F},$$

so  ${}^\omega \mathcal{A}/\mathcal{F} \equiv_e {}^\omega \mathcal{B}/\mathcal{F}$ . By the Continuum Hypothesis and Theorem 5,

$$|{}^\omega \mathcal{A}/\mathcal{F}| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph_1$$

and by the fourth paragraph of Section 3.2,  ${}^\omega \mathcal{A}/\mathcal{F}$  is  $\aleph_1$ -saturated and therefore saturated. Similarly,  $|{}^\omega \mathcal{B}/\mathcal{F}| = \aleph_1$ , and it is also saturated. Now, (2) follows from the uniqueness of saturated models (see the second paragraph of Section 3.2).

To show that (2) implies (1), observe that

$$\mathcal{A} \equiv_e {}^\omega \mathcal{A}/\mathcal{F} \stackrel{(2)}{\equiv_e} {}^\omega \mathcal{B}/\mathcal{F} \equiv_e \mathcal{B}.$$

□

After this easy observation, it is natural to ask if the Continuum Hypothesis and/or the cardinality assumptions on  $\mathcal{A}, \mathcal{B}$  and  $L$  can be eliminated from the conditions. In this direction, Keisler proved the following.

**Theorem 10.** *Let  $\mathcal{A}, \mathcal{B}$  be  $L$ -structures, assume  $|L| \leq \kappa$ ,  $\mathcal{A}, \mathcal{B} \leq \kappa^+$ , and assume  $2^\kappa = \kappa^+$ . Let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be a countably incomplete,  $\kappa^+$ -good ultrafilter. Then, the following are equivalent.*

- (1)  $\mathcal{A}$  and  $\mathcal{B}$  are elementarily equivalent.
- (2)  ${}^I \mathcal{A}/\mathcal{F}$  and  ${}^I \mathcal{B}/\mathcal{F}$  are isomorphic.

For the proof, we refer to 6.1.9 of [2]. The proof of (1)  $\Rightarrow$  (2) is the same as in Theorem 9. The proof of (1)  $\Rightarrow$  (2) (which is in the harder direction) is still similar to the appropriate part of the proof of Theorem 9: Theorem 8, together with the assumption  $2^\kappa = \kappa^+$ , implies that  ${}^\omega \mathcal{A}/\mathcal{F}$  and  ${}^\omega \mathcal{B}/\mathcal{F}$  are elementarily equivalent, equinumerous, saturated structures, so the uniqueness theorem of saturated models implies (2).

Finally Shelah proved a version of the Isomorphic Ultrapowers Theorem without assuming any instance of the Generalized Continuum Hypothesis:

**Theorem 11.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $L$ -structures. Then, the following are equivalent.*

- (1)  $\mathcal{A}$  and  $\mathcal{B}$  are elementarily equivalent.
- (2) There is an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$  such that  ${}^I \mathcal{A}/\mathcal{F}$  and  ${}^I \mathcal{B}/\mathcal{F}$  are isomorphic.

For the proof, we refer to [24]. In the proof of Theorem 11, a particular ultrafilter  $\mathcal{F}$  has been constructed on the set  $I = 2^{\max\{|A|, |B|\}}$  that depends on  $\mathcal{A}$  and  $\mathcal{B}$  (but in (2) of Theorem 11, both of the ultrapowers of  $\mathcal{A}$  and  $\mathcal{B}$  are taken modulo the same  $\mathcal{F}$ ). At that time, it remained open whether ultrafilters over smaller sets may be used in (2). Roughly speaking, it turned out that they consistently could not. More precisely, Shelah proved the following theorem.

**Theorem 12.** *It is consistent with ZFC that there exist two elementarily equivalent countable structures such that they do not have isomorphic ultrapowers modulo any ultrafilter over a countable set.*

For the proof and further details, we refer to [25]. After this result, it was natural to ask how many pairwise non-isomorphic ultrapowers a countable structure may have modulo an ultrafilter over a countable set. Independently, this question has become significant in functional analysis and in the theory of operator algebras (see [26]). The answer is a dichotomy theorem due to Farah and Shelah, and it is as follows.

**Theorem 13.** *Let  $\mathcal{A}$  be a countable structure with a countable language (in fact, it would be enough to assume  $|A| \leq 2^{\aleph_0}$  only). Let*

$$U = \{{}^\omega \mathcal{A}/\mathcal{F} : \mathcal{F} \subseteq \mathcal{P}(\omega) \text{ is a nonprincipal ultrafilter}\}.$$

Then, counting up to isomorphisms, there are only two possibilities: either  $|U| = 1$  or  $|U| = 2^{2^{\aleph_0}}$ .

For the proof, we refer to [27]. Keeping the notation of Theorem 13, we note the following. Each nonprincipal ultrafilter on  $\omega$  is countably incomplete. Hence, according to the fourth paragraph of Section 3.2, each element of  $U$  is  $\aleph_1$ -saturated (and clearly, pairwise elementarily equivalent). Therefore, if  $2^{\aleph_0} = \aleph_1$ , then  $|U| = 1$  should hold because of the uniqueness theorem of saturated structures.

Suppose now that  $2^{\aleph_0} \neq \aleph_1$ . By [26], if  $\mathcal{A}$  is stable, then all of its ultrapowers (modulo a nonprincipal ultrafilter over  $\omega$ ) are still saturated, so in this case,  $|U| = 1$  follows, similarly to the previous paragraph. The remaining case is when the Continuum Hypothesis fails and  $\mathcal{A}$  is unstable. Then  $|U| = 2^{2^{\aleph_0}}$  by a beautiful tour de force presented in [27].

## 5. Ultraproducts and Higher-Order Logic

Can the Łoś Lemma generalized to higher-order logic, or more specifically, what kind of ultraproducts (if any) preserve the validity of certain higher-order formulas? In this section, we discuss related questions.

Let  $L$  be a first-order language. If  $R_0, \dots, R_{n-1}$  are symbols denoting relations, then  $L(R_0, \dots, R_{n-1})$  is the first-order language obtained from  $L$  by adjoining  $R_0, \dots, R_{n-1}$  to  $L$ .

The simplest higher-order formulas are the second-order existential formulas (also called  $\Sigma_1^1$ -formulas). They have the following form:

$$\exists R_0, \dots, \exists R_{n-1} \varphi,$$

where  $R_0, \dots, R_{n-1}$  are variables denoting relations (possibly with higher arities) and  $\varphi$  is a first order formula of the language  $L(R_0, \dots, R_{n-1})$ . Ultraproducts still preserve the validity of  $\Sigma_1^1$ -formulas in one direction.

**Theorem 14.** Suppose  $\mathcal{F} \subseteq \mathcal{P}(I)$  is an ultrafilter,  $\mathcal{A}_i$  is an  $L$ -structure for all  $i \in I$ , and  $\psi$  is a  $\Sigma_1^1$ -formula such that

$$\{i \in I : \mathcal{A}_i \models \psi\} \in \mathcal{F}.$$

$$\text{Then } \prod_{i \in I} \mathcal{A}_i / \mathcal{F} \models \psi.$$

**Proof.** Suppose  $\psi = \exists R_0, \dots, \exists R_{n-1} \varphi$ , where  $\varphi$  is a first-order formula in  $L$ . Let  $J = \{i \in I : \mathcal{A}_i \models \psi\}$ . For each  $j \in J$ , there exist relations  $R_0^j, \dots, R_{n-1}^j$  on  $\mathcal{A}_j$  such that

$$\langle \mathcal{A}_j, R_0^j, \dots, R_{n-1}^j \rangle \models \varphi.$$

For  $j \in J$  let  $\mathcal{B}_j = \langle \mathcal{A}_j, R_0^j, \dots, R_{n-1}^j \rangle$ , and for  $i \in I - J$ , let  $\mathcal{B}_i = \mathcal{A}_i$ . By the Łoś Lemma (Theorem 1)

$$\prod_{i \in I} \mathcal{B}_i / \mathcal{F} \models \varphi;$$

hence  $\prod_{i \in I} \mathcal{A}_i / \mathcal{F} \models \psi$ , as desired.  $\square$

Keeping the notation of Theorem 14, it is easy to see that, in general, ultraproducts do not preserve the validity of  $\Sigma_1^1$ -formulas in the other direction:

- For each  $n \in \omega$ , let  $A_n$  be a set with  $|A_n| = n$ ;
- Let  $\mathcal{F} \subseteq \mathcal{P}(\omega)$  be any nonprincipal ultrafilter; and
- Let  $\psi$  be the  $\Sigma_1^1$ -formula expressing the existence of a (unary) function, which is injective, but not surjective.

For each  $k \in \omega$ , there exists a first-order formula expressing that the universe has at least  $k$  distinct elements. Hence, by the Łoś Lemma,  $\prod_{n \in \omega} A_n / \mathcal{F}$  is infinite; therefore,

$$\prod_{n \in \omega} A_n / \mathcal{F} \models \psi.$$

On the other hand, as each  $A_n$  is finite, we have

$$\{n \in \omega : A_n \models \psi\} = \emptyset \notin \mathcal{F}.$$

However, it is natural to ask which ultraproducts preserve the validity of  $\Sigma_1^1$  formulas in both directions. To answer this, we recall some notions and results from [28]. At the technical level, decomposable (sometimes called internal) relations and their characterization will be essential, so we start by recalling their definition.

**Definition 8.** Let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be an ultrafilter; for each  $i \in I$ , let  $A_i$  be a set and let  $k \in \omega$ . A  $k$ -ary relation  $R \subseteq {}^k(\prod_{i \in I} A_i / \mathcal{F})$  is defined as decomposable iff for each  $i \in I$  there exists a  $k$ -ary relation  $R^i \subseteq {}^k A_i$  such that

$$R = \prod_{i \in I} R^i / \mathcal{F}.$$

Easy cardinality considerations show that an infinite ultraproduct always has relations (of arbitrary arity) that are not decomposable.

Next, we turn to characterize decomposable relations. Again, let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be an ultrafilter, and for each  $i \in I$  let  $A_i$  be a set and let  $k \in \omega$ . By a  $k$ -dimensional choice function, we mean a function

$$\hat{\cdot} : {}^k(\prod_{i \in I} A_i / \mathcal{F}) \rightarrow {}^k(\prod_{i \in I} A_i)$$

such that for all  $\bar{a} \in {}^k(\prod_{i \in I} A_i / \mathcal{F})$  we have  $\bar{a} = (\bar{a})^{\hat{\cdot}} / \mathcal{F}$ . Fix a  $k$ -dimensional choice function  $\hat{\cdot}$ .

For  $\bar{a} \in {}^k(\prod_{i \in I} A_i / \mathcal{F})$  and  $R \subseteq {}^k(\prod_{i \in I} A_i / \mathcal{F})$ , we define

$$T^*(\bar{a}, R) = \{i \in I : (\exists \bar{b} \in R)((\bar{a})^{\hat{\cdot}}(i) = (\bar{b})^{\hat{\cdot}}(i))\}.$$

Further,  $\bar{a}$  is defined to be *close to*  $R$  iff  $T^*(\bar{a}, R) \in \mathcal{F}$  and  $R$  is defined to be *closed for*  $\hat{\cdot}$  iff

$$\text{for all } \bar{a} \in {}^k(\prod_{i \in I} A_i / \mathcal{F}), \text{ if } \bar{a} \text{ is close to } R, \text{ then } \bar{a} \in R.$$

By Lemma 3.4 of [28], the family of  $k$ -ary relations closed for  $\hat{\cdot}$  is the set of all closed sets of an appropriate topological space. The topological space depends on the choice function  $\hat{\cdot}$  and the so the obtained topological spaces are called *ultratopologies*. Now the main results of [28] are as follows.

**Theorem 15.** Let  $k \in \omega$  and let  $R \subseteq {}^k(\prod_{i \in I} A_i / \mathcal{F})$  be a  $k$ -ary relation on an ultraproduct. Then the following are equivalent.

- (1)  $R$  is decomposable;
- (2) There exists an ultratopology in which  $R$  is closed;
- (3) There exists an ultratopology in which  $R$  is open;
- (4) There exists an ultratopology in which  $R$  is clopen.

For the proofs, we refer to Theorem 3.8 of [28] and Corollary 2.2 of [29].

Now we can describe the situations when an ultraproduct completely preserves the validity of a  $\Sigma_1^1$ -formula.

**Theorem 16.** Let  $\psi = \exists R_0 \dots \exists R_{n-1} \varphi$  be a  $\Sigma_1^1$ -formula (where  $\varphi$  is a first order formula), let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be an ultrafilter and for each  $i \in I$ , let  $\mathcal{A}_i$  be a first order structure. The following conditions are equivalent.

- (1)  $\{i \in I : \mathcal{A}_i \models \psi\} \in \mathcal{F}$ .
- (2) There is an ultratopology  $\mathcal{C}$  on the ultraproduct  $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i / \mathcal{F}$ , and there are clopen relations  $R_0^{\mathcal{A}}, \dots, R_{n-1}^{\mathcal{A}}$  in  $\langle \mathcal{A}, \mathcal{C} \rangle$  such that

$$\langle \mathcal{A}, R_0^{\mathcal{A}}, \dots, R_{n-1}^{\mathcal{A}} \rangle \models \varphi.$$

For the proof, we refer to Theorem 3.11 of [28].

Decomposable relations play a central role in [30], where a finitary analogue of Morley's categoricity theorem was examined.

Next, we turn to the second order existential generalization of the Keisler–Shelah theorem.

**Definition 9.** Let  $\mathcal{A}_0$  and  $\mathcal{B}_0$  be two  $L$ -structures. Then  $\mathcal{B}_0$  is defined to be inseparable from  $\mathcal{A}_0$  iff for every  $k \in \omega$  and for every finite system of  $k$ -ary relations  $R_0, \dots, R_{n-1} \subseteq {}^k \mathcal{A}_0$ , there exist

- ultrapowers  $\mathcal{A} = {}^I \mathcal{A}_0 / \mathcal{F}$ ,  $\mathcal{B} = {}^I \mathcal{B}_0 / \mathcal{G}$ ;
- a  $k$ -dimensional ultratopology on  $\mathcal{B}$  and
- an isomorphism  $f : \mathcal{A} \rightarrow \mathcal{B}$

such that  $f$  maps each  $R_m^{\mathcal{A}}$  onto a clopen set of  $\mathcal{B}$ , where

$$\langle \mathcal{A}, R_0^{\mathcal{A}}, \dots, R_{n-1}^{\mathcal{A}} \rangle = {}^I \langle \mathcal{A}_0, R_0, \dots, R_{n-1} \rangle / \mathcal{F}.$$

Further,  $\mathcal{A}_0$  and  $\mathcal{B}_0$  are defined to be inseparable from each other iff  $\mathcal{A}_0$  is inseparable from  $\mathcal{B}_0$  and  $\mathcal{B}_0$  is inseparable from  $\mathcal{A}_0$ .

We note that in Definition 9, inseparability is defined in purely model-theoretic terms. Further, inseparability is not a symmetric notion, because we require that the isomorphisms preserve clopen sets only in one direction.

**Theorem 17.** Let  $\mathcal{A}_0$  and  $\mathcal{B}_0$  be two  $L$ -structures. Then, the following are equivalent.

- (1)  $\mathcal{A}_0$  and  $\mathcal{B}_0$  satisfy the same  $\Sigma_1^1$ -formulas (that is, they are  $\Sigma_1^1$ -elementarily equivalent);
- (2)  $\mathcal{A}_0$  and  $\mathcal{B}_0$  are inseparable from each other.

For the proof, we refer to Theorem 4.4 and Corollary 4.5 of [28].

Let us compare Shelah's isomorphism theorem (Theorem 11 above) with Theorem 17. Shelah's isomorphism theorem states that two structures are elementarily equivalent iff they have isomorphic ultrapowers; Theorem 17 states that two structures are  $\Sigma_1^1$ -elementarily equivalent iff they have isomorphic ultrapowers with some additional properties. Hence, Theorem 17 can be regarded as a higher-order generalization of Theorem 11.

Ultratopologies seem to be interesting from a purely topological point of view. They are  $T_1$ , but not Hausdorff spaces. According to Corollary 4.3 of [29], there are infinitely many pairwise non-homeomorphic ultratopologies on each infinite ultraproduct (modulo countably incomplete ultrafilters). Further, by Theorem 4.2 of [29], each ultratopology on an infinite ultraproduct (modulo a countably incomplete ultrafilter) is hereditarily compact, that is, all subspaces of them are compact. Section 3 of [29] and [31] contains further results on cardinalities of dense sets in ultratopologies on ultraproducts of finite sets.

## 6. Ultrafilters and Ramsey Theory

Ramsey theory today is an independent research direction in finite (and infinite) combinatorics. As was demonstrated by the subject-founder [32] as well as in the more recent [33,34], ultrafilters may be used to obtain interesting and deep theorems in this area.

We start by recalling Ramsey's original theorem, which initially was motivated by logic. It appeared in [35] and today is considered the starting point of related investigations. Since then, many essentially different proofs have been found for Ramsey's theorem. We present a proof that is based on ultrafilters (and part of the folklore).

**Theorem 18.** *If  $[\omega]^2$  is colored with finitely many colors, then there exists an infinite  $X \subseteq \omega$  such that  $[X]^2$  is monochromatic.*

**Proof.** Fix a coloration  $f : [\omega]^2 \rightarrow k$  with  $k$  colors and let  $\mathcal{F} \subseteq \mathcal{P}(\omega)$  be a nonprincipal ultrafilter. For each  $i \in \omega$  and  $c \in k$  define  $A_{i,c}$  to be

$$A_{i,c} = \{j \in \omega : f(\langle i, j \rangle) = c\}.$$

Since  $\mathcal{F}$  is an ultrafilter, for each  $i \in \omega$ , there exists a unique  $c(i) \in k$  such that  $A_{i,c(i)} \in \mathcal{F}$ . Further, there exists a unique  $c^* \in k$  such that

$$J := \{i \in \omega : c(i) = c^*\} \in \mathcal{F}.$$

Observe that for any  $i \in J$ , we have  $c(i) = c^*$ , and hence  $A_{i,c^*} \in \mathcal{F}$ .

Now we define an infinite set  $\{a_n : n \in \omega\} \subseteq \omega$  by recursion such that the following stipulations are satisfied for all  $n < m \in \omega$ :

- (a)  $a_n \in J$ ;
- (b)  $a_{n-1} < a_n$ ;
- (c)  $f(\langle a_n, a_m \rangle) = c^*$ .

Observe that  $J \neq \emptyset$  because  $J \in \mathcal{F}$ . Let  $a_0 \in J$  be arbitrary. Suppose  $a_0 < a_1 < \dots < a_{n-1}$  has already been defined. Then

$$J_n := J \cap A_{a_0,c^*} \cap \dots \cap A_{a_{n-1},c^*} \in \mathcal{F}$$

and hence  $J_n \neq \emptyset$  (in fact,  $J_n$  is infinite because  $\mathcal{F}$  is nonprincipal). Let

$$a_n \in J_n \cap \{m \in \omega : m > a_0, \dots, a_{n-1}\}$$

be arbitrary; clearly, (a)–(c) remain true. This completes the recursive construction.

Finally, it is easy to see that, by (c),  $\{a_n : n \in \omega\}$  satisfies the conclusion of the theorem.  $\square$

N. Hindman established the following connection between Ramsey theory and ultrafilters.

**Theorem 19.** *Let  $A$  be any nonempty set and let  $T \subseteq \mathcal{P}(A)$ . Then, the following are equivalent.*

- (1) *If  $A$  is colored with finitely many colors, then there exists  $x \in T$ , which is monochromatic;*
- (2) *There exists an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(A)$  such that for all  $I \in \mathcal{F}$  there exists  $x \in T$  with  $x \subseteq I$ .*

For the proof, we refer to Theorem 3 of Chapter 6.2 of [36].

Now, we turn to discuss a celebrated theorem of Hindman that settles a conjecture of Graham and Rotschild in the affirmative and can be regarded as a common generalization of some classical theorems of Hilbert and Schur. As we will mention below, this result is also a starting point of a variety of more recent investigations.

**Theorem 20.** *If  $\omega$  is colored with finitely many colors, then there exists an infinite  $I \subseteq \omega$  such that the set of (nonempty) finite sums of elements of  $I$  is monochromatic; that is,*

$$\{a_0 + \dots + a_{n-1} : n \in \omega - \{0\}, a_0, \dots, a_{n-1} \in I, a_0, \dots, a_{n-1} \text{ are pairwise distinct}\}$$

*is monochromatic.*



For the proof, we refer to Theorem 15 of Chapter 3.5 of [36], and we sketch another proof below (which also can be found in Chapter 6.2 of [36]). This proof is due to Glazer, but Glazer never published it. Glazer's proof originally appeared in [37]. For an elementary proof for a very closely related variant of this theorem, we refer to [38]. Hindman's original proof appeared in [39].

**Sketch of Glazer's proof of Theorem 20.** For  $x \subseteq \omega$  and  $n \in \omega$ , we define

$$(x) - n = \{m \in \omega : m + n \in x\}.$$

Let  $\mathcal{U}$  denote the set of all ultrafilters over  $\omega$ . We define a binary operation  $\oplus$  on  $\mathcal{U}$  as follows: for  $\mathcal{F}, \mathcal{G} \in \mathcal{U}$

$$\mathcal{F} \oplus \mathcal{G} = \{x \subseteq \omega : \{n \in \omega : (x) - n \in \mathcal{G}\} \in \mathcal{F}\}.$$

It is straightforward to verify that  $\oplus$  is associative, and hence  $\langle \mathcal{U}, \oplus \rangle$  is a semigroup. Further, for each  $x \subseteq \omega$ , let  $N_x = \{\mathcal{F} \in \mathcal{U} : x \in \mathcal{F}\}$ . Then,  $\{N_x : x \subseteq \omega\}$  is a basis of a topology (called Stone topology) and  $\mathcal{U}$  is compact with respect to this topology. Further,  $\oplus$  is continuous in its first variable. It is known that compact semigroups (with a right continuous semigroup operation) contain idempotent elements. It follows that  $\mathcal{U}$  contains an idempotent element  $\mathcal{F} \in \mathcal{U}$  as well; that is, there exists  $\mathcal{F} \in \mathcal{U}$  such that  $\mathcal{F} \oplus \mathcal{F} = \mathcal{F}$ .

Let  $\mathcal{F} \in \mathcal{U}$  be idempotent. For any  $J \subseteq \omega$  let

$$J' = \{n \in \omega : (J) - n \in \mathcal{F}\}.$$

Since  $\mathcal{F}$  is idempotent,  $J \in \mathcal{F}$  implies  $J' \in \mathcal{F}$ . Now, let  $f : \omega \rightarrow k$  be any coloration of  $\omega$  with  $k$  colors. Now, essentially similarly to the second part of the proof of Theorem 18, one can define  $I$  as an infinite recursive sequence:

- There exists  $J \in \mathcal{F}$ , which is monochromatic under  $f$ ; let  $a_0 \in J \cap J'$  be arbitrary;
- If  $a_0 < \dots < a_{n-1}$  has already been defined, then  $a_n$  can be chosen to be a large enough element of

$$\bigcap_{\emptyset \neq b \subseteq \{a_0, \dots, a_{n-1}\}} ((J) - \sum b) \cap ((J - \sum b)') \in \mathcal{F}.$$

Then  $I = \{a_n : n \in \omega\}$  satisfies the conclusion of the theorem.

□

For generalizations and more recent developments, we refer to [40–44].

## 7. Ultraproducts and Topology

We have seen in Section 5 that topological notions naturally arise if one studies the fine structure of ultraproducts. It is also natural to adapt the ultraproduct construction for topological spaces. One natural approach would be to generalize the ultraproduct construction (as described in Section 2) to arbitrary categories, and then to apply the special case of this abstract notion to the category of topological spaces (the morphisms are the continuous functions). We describe this construction in a more concrete way.

As a standard reference for topological notions not recalled in this section, we refer to [45].

**Definition 10.** Let  $I$  be any nonempty set, let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be an ultrafilter, and for each  $i \in I$ , let  $\langle X_i, \tau_i \rangle$  be a topological space. Then, the ultraproduct

$$\langle X, \tau \rangle := \prod_{i \in I} \langle X_i, \tau_i \rangle / \mathcal{F}$$

is a topological space defined as follows.

- The underlining set  $X$  of the ultraproduct is the (usual) ultraproduct of the underlying sets of the factors:  $X = \prod_{i \in I} X_i / \mathcal{F}$ ;
- $\tau$  is the topology on  $X$  generated by the family  $\tau_B$  of “ultraboxes”:

$$\tau_B = \left\{ \prod_{i \in I} G_i / \mathcal{F} : (\forall i \in I)(G_i \in \tau_i) \right\}.$$

Keeping the notation of Definition 10, we note that it is routine to check that  $\tau_B$  is closed under finite intersections; that is,  $\tau_B$  is a basis of a topology on  $X$ .

Recall that Tychonoff’s theorem states that topological products of compact spaces are compact. As a warm up, we show that the analogue of Tychonoff’s theorem is not true for ultraproducts of topological spaces. To see this, for each  $n \in \omega$  let  $X_n$  be an  $n$ -element set and let  $\tau_n$  be the discrete topology on  $X_n$ . Let  $\mathcal{F} \subseteq \mathcal{P}(\omega)$  be any nonprincipal ultrafilter on  $\omega$ . On the one hand, each  $\langle X_n, \tau_n \rangle$  is compact (because  $X_n$  is finite). On the other hand, the ultraproduct

$$\langle X, \tau \rangle = \prod_{n \in \omega} \langle X_n, \tau_n \rangle / \mathcal{F}$$

is infinite (by the Łoś lemma, because  $\mathcal{F}$  is nonprincipal). Further, each one-element subset of  $X$  is open in the ultraproduct topology (because each  $X_n$  is discrete, and hence all one-element ultraboxes belong to  $\tau_B$ ). Hence  $\langle X, \tau \rangle$  is an infinite discrete space; in particular, it cannot be compact. For further properties of ultraproducts of compact Hausdorff spaces we refer to [46].

The argument in the previous paragraph can be rephrased to show that the ultraproducts of discrete spaces remain discrete. In general, it is natural to ask that what kind of properties of topological spaces are preserved by forming ultraproducts of topological spaces. For a comprehensive survey on this topic, we refer to [47]; below, we sum up the related results we consider fundamental.

It is relatively easy to see that ultraproducts preserve the separation axioms  $T_0 - T_3$ . It is harder to show, but it still true that ultraproducts of  $T_{3.5}$ -spaces remain  $T_{3.5}$ . For further information we refer to [47]. However, the separation axiom  $T_4$  is not preserved under ultraproducts: there exist a set  $I$ , an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$ , and an indexed family  $\langle \langle X_i, \tau_i \rangle : i \in I \rangle$  of normal spaces such that

$$\prod_{i \in I} \langle X_i, \tau_i \rangle / \mathcal{F}$$

is not normal. For the proofs we refer to [48].

It is also natural to ask what kind of extra features a topological ultraproduct may have, if we assume some extra properties of the ultrafilter we are working with. For example, as we saw in Section 3.1, it is natural to investigate regular ultrafilters. In particular, according to Theorem 6, the ultrapowers of first-order structures modulo  $\kappa$ -regular ultrafilters are  $\kappa^+$ -universal. For the topological analogue of this result, we recall that a topological space is a  $P_\kappa$ -space iff fewer than  $\kappa$ -many open sets of it have open intersection. Bankston proved the following analogue of Theorem 6.

**Theorem 21.** *Let  $\mathcal{F} \subseteq \mathcal{P}(I)$  be an ultrafilter. Then, the following are equivalent.*

- (1)  $\mathcal{F}$  is  $\kappa$ -regular;
- (2) Each ultraproduct of topological spaces modulo  $\mathcal{F}$  is a  $P_{\kappa^+}$ -space.

For the proof, we refer to Theorem 4.1 of [48].

In order to state a similar result for good ultrafilters we recall that a topological space is  $\kappa$ -Baire iff fewer than  $\kappa$ -many dense open sets have a dense intersection. The following theorem is also due to Bankston.

**Theorem 22.** *Suppose  $\kappa$  is an infinite cardinal and  $\mathcal{F} \subseteq \mathcal{P}(I)$  is a  $\kappa$ -good ultrafilter. Then each ultraproduct of topological spaces modulo  $\mathcal{F}$  is  $\kappa$ -Baire.*

For the proof, we refer to Theorem 2.2. of [49].

Next we recall a theorem of Bankston that can be regarded as a topological analogue of the Isomorphic Ultrapowers Theorems discussed in Section 4 above.

**Theorem 23.** *Each two dense-in-itself  $T_3$  space has homeomorphic ultrapowers.*

For the proof, we refer to Theorem A2.6 of [48].

There is another interesting application of ultraproducts of topological and metric structures that received renewed impetus in the last decade. In [50] and in [51], the ultraproduct construction has been adapted to measure preserving actions of certain infinite (usually finitely generated) groups. With the aid of the ultraproduct construction, deep results have been obtained on approximations of these group actions (see, for example, Theorems 4.7 and 5.2 of [50]; some versions of these theorems were independently obtained in [51]). To present more details would require a considerable amount of further technical preparation, so we choose to stop here.

## 8. Concluding Remarks

In this section, we propose some open problems and possible research directions that may stimulate further investigations. We follow the structure of the previous sections.

We have seen in Theorem 11 that elementarily equivalent structures have isomorphic ultrapowers. The appropriate ultrafilter is constructed in a rather subtle way. This motivates the following problem.

**Problem 1.** *Is it true that if  $\mathcal{A}$  and  $\mathcal{B}$  are elementarily equivalent structures, then for all large enough  $I$  and all regular ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$ , the ultraproducts  ${}^I\mathcal{A}/\mathcal{F}$  and  ${}^I\mathcal{B}/\mathcal{F}$  are isomorphic?*

In Section 5, we investigated preservation properties of certain ultraproducts for second-order existential formulas. It would be natural to go further:

**Problem 2.** *Find generalizations of Theorem 16 for second-order (or even higher-order) formulas.*

**Problem 3.** *Find generalizations of Theorem 17 for second-order (or even higher-order) elementary equivalence.*

As described in Section 5, each ultraproduct can be endowed by a topology in a natural way (the so-obtained topologies are called ultratopologies). According to Theorem 4.2 of [29] (see also the last paragraph of Section 5), ultratopologies in an infinite ultraproduct modulo a countably incomplete ultrafilter are  $T_1$  but not Hausdorff. This motivates the following problem.

**Problem 4.** *Is there a Hausdorff ultratopology on each ultraproduct modulo a countably complete ultrafilter?*

In Section 6, we recalled some results in finite Ramsey Theory that can be established with the aid of certain ultrafilters. There are a variety of related investigations. Hence, instead of formulating a particular problem, we propose the following somewhat vague program.

**Problem 5.** *Devise a systematic and general method with which ultrafilters can be utilized to treat problems in finite Ramsey theory in an uniform way.*

According to Section 7, the ultraproduct construction can be adapted to the context of topological spaces. Related investigations have been focused mainly to general topology. Hence, it would be natural to study the ultraproducts of topological spaces from the point of view of algebraic topology as well:

**Problem 6.** Investigate ultraproducts of topological spaces from the point of view of algebraic topology.

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