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Classification of maps sending lines into translates of a curve

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We list four types of planar curves such that arrangements of their translates are (locally) combinatorially equivalent to an arrangement of lines. We find them by characterising diffeomorphisms $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and continuous curves $C \subset \mathbb{R}^2$ such that $\phi\big(t + C\big)$ is a line for all $t \in \mathbb{R}^2$. There are exactly five maps satisfying (at least locally) this condition. Two of them define the same curve, so we have four different curves. These can be used to define norms giving constructions with $\Omega(n^{4/3})$ unit distances among n points in the plane.

1. Introduction

One of the oldest and best-known problems in combinatorial geometry is Paul Erdős' unit distances problem [5]. What is the maximum number of unit distances among n points on the plane? Erdős conjectured that the maximum number of unit distances

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is $n^{1+o(1)}$. (Through the paper we are going to use the Big-O, Little-o, and Omega notations. A real function $f(x)$ is $o(T(x))$ if $f(x)/T(x) \to 0$ as $x \to \infty$. It is $O(T(x))$ if there is a $c > 0$ such that $f(x)/T(x) \le c$ as $x \to \infty$, and it is $\Omega(T(x))$ if there is a $c > 0$ such that $f(x)/T(x) \ge c$ as $x \to \infty$.)

The conjecture is still open, the best-known upper bound is $O(n^{4/3})$. This bound was proved by Spencer, Szemerédi, and Trotter [13]. It seems that the exponent 4/3 is the limit of the known combinatorial methods, even to prove $o(n^{4/3})$ is out of range of the known techniques.

One reason behind this barrier is that there are norms where one can find n-element point-sets with $\Omega(n^{4/3})$ unit distances. The oldest construction providing such a norm can probably be derived from Jarník's construction [6]. Jarník defined a sequence of centrally symmetric smooth closed convex curves, U_m containing $\Omega(m^{2/3})$ lattice points of the $m \times m$ integer grid. Setting such a convex curve as the unit disk, there are $\Omega(m^{8/3})$ unit distances among the $(2m)^2$ points of the $2m \times 2m$ integer lattice.

For completeness, we include the simple argument. Let's denote the centre of U_m by o. U_m is centrally symmetric, so (by possibly losing a multiplier of 2) we can assume that $o \in [m] \times [m]$. The set,

$$\{(U_m + o) + (i, j) | i, j \in [m]\}$$

has at most $(2m)^2$ points and every translate of o has at least $\Omega(m^{2/3})$ points (the corresponding translate of U_m) at unit distance. If we set $n = m^2$, there are $\Omega(n^{4/3})$ unit distances with this norm.

In this example, the norm changes with n. A nice construction, with a uniform norm, was given by Valtr [16] using translates of a parabola and the $n \times n^2$ integer grid. (see the description of the construction on page 194 in [2]) On the other hand, it was proved by Matousek that most norms, in the sense of Baire category, determine $O(n \log n \log \log n)$ unit distances among n points [8]. This bound was improved recently to $O(n \log n)$ by Alon et al. in [1].

For any strictly convex norm, among n points in the plane, there are $O(n^{4/3})$ unit distances. This claim can be proved using the crossing number inequality from [15] in the same way as proving the Szemerédi-Trotter theorem. This theorem gives a sharp upper bound on the number of incidences, I, between N points and M lines on the real plane, $\mathbb{R}^2$.

Theorem 1.1 *(Szemerédi-Trotter Theorem [12]).*

$$I(N, M) = O\left(N^{\frac{2}{3}} M^{\frac{2}{3}} + N + M\right).$$

An elegant proof of the above theorem was given by Székely, who also showed how to use his proof method to give the $O(n^{4/3})$ bound for unit distances [15].

There are known arrangements of n lines and n points such that

$$I(n,n) = \Omega\left(n^{\frac{4}{3}}\right). \tag{1}$$

Such arrangements were found by Erdős, Elekes [4], Sheffer and Silier [14] using lattice points, i.e. a Cartesian product structure. Recently Guth and Silier gave sharp examples not based on the integer lattice [11].

If there were maps where the images of lines are translates of a single curve, $\mathcal{C}$, then one can map such point-line arrangements to point-curve arrangements. Using part of the curve as (part of) the unit circle, with this norm we have $\Omega(n^{4/3})$ unit distances. Such a map exists: the map

$$(x,y) \mapsto (x, y+x^2) \tag{2}$$

sends the line $(t, at+b)$ to the $(t, t^2+at+b) = \big(t, (t+a/2)^2 - a^2/4 + b\big)$ curve, which is a translate of the parabola $y = x^2$. This map was used over finite fields by Pudlák in [9]. Pudlák noticed that this map gives a one-to-one correspondence between point-parabola incidences and point-line incidences. He used it to define the colouring of the edges of a complete bipartite graph with three colours without creating large monochromatic complete subgraphs. If we apply the map in (2) to Elekes' point-line arrangement in [4] we get a construction very similar to Valtr's [16].

Based on the above observations, it is a natural problem characterizing maps of the plane sending lines into translates of a single curve. As we will see there are exactly four more such maps in addition to Pudlák's map in (2). The first and the third maps in the list below result in the same curve, these are translates of the log (exp) curve as the images of lines. The last curve, which is not listed here, is given by the real and imaginary parts of the complex logarithm function.

1. $$M : (x,y) \mapsto (x, \ln(y)).$$

   For every line $y = ax+b$ with $a > 0$, the $x > -\frac{b}{a}$ part maps to $\big(x, \ln\left(x + \frac{b}{a}\right) + \ln(a)\big)$. This is a translate of the curve $y = \ln(x)$.

2. $$M : (x,y) \mapsto \left(\ln(x), \ln\left(\frac{y}{x}\right)\right).$$

   If we use the notation $\ln(x) = X$, then the image of the line $y = ax + b$ is the

   $$\left(X, \ln\left(1 + \frac{b/a}{e^X}\right) + \ln(a)\right) = \left(X, \ln\left(1 + e^{-(X - \ln(b/a))}\right) + \ln(a)\right)$$

   curve if $a > 0$ and $b > 0$. This is the translate of the curve $y = \ln(1 + e^{-x})$.

3.

$$M : (x, y) \mapsto \left(\ln(x), \frac{y}{x}\right).$$

As in the previous case, if we use the notation $\ln(x) = X$, then the image of the line $y = ax + b$ is the

$$\left(X, e^{-(X-\ln(b))} + a\right)$$

curve if $a > 0$ and $b > 0$. This is the translate of the curve $y = e^{-x}$, so it is similar to the first, $y = \ln(x)$, case.

4. This is probably the most surprising map

$$M : (x, y) \mapsto (\Re(\ln(1 - x + iy)), \Im(\ln(1 - x + iy))).$$

If a generic line is given by the equations $x = t, y = at + b$, then its image after the map is

$$x(t) = \Re\left(\ln\left(t - \frac{1+ib}{1-ia}\right) + \ln(ia - 1)\right),$$
$$y(t) = \Im\left(\ln\left(t - \frac{1+ib}{1-ia}\right) + \ln(ia - 1)\right),$$

the real and imaginary parts of a translate of the complex logarithm function.

2. Preliminaries

Notation 2.1. The group of invertible 3×3 matrices and the Lie algebra of all 3×3 matrices will be denoted by $\mathrm{GL}(3, \mathbb{R})$ and $\mathfrak{gl}(3, \mathbb{R})$ resp. The quotient group of $\mathrm{GL}(3, \mathbb{R})$ by the normal subgroup of scalar matrices is denoted by $\mathrm{PGL}(3, \mathbb{R})$, it is the group of projective linear transformations of the projective plane. The Lie algebra of $\mathrm{PGL}(3, \mathbb{R})$ is the quotient of $\mathfrak{gl}(3, \mathbb{R})$ by the ideal of scalar matrices, we denote it with $\mathfrak{pgl}(3, \mathbb{R})$. It is naturally isomorphic to $\mathfrak{sl}(3, \mathbb{R})$, the Lie algebra of 3×3 matrices of trace 0.

Notation 2.2. We denote by $\mathrm{Aff}(2, \mathbb{R}) \leq \mathrm{GL}(3, \mathbb{R})$ the subgroup of all 3×3 matrices of the form

$$\begin{pmatrix} L & v \\ 0 & 1 \end{pmatrix}$$

where L is a 2×2 invertible matrix (linear transformation), and v is a 2-dimensional column vector (translation). We denote by $\mathfrak{aff}(2, \mathbb{R}) \leq \mathfrak{gl}(3, \mathbb{R})$ the Lie subalgebra of matrices of the form

$$\begin{pmatrix} \Lambda & v \\ 0 & 0 \end{pmatrix}$$

where Λ is a 2×2 matrix, and v is a 2-dimensional column vector. The quotient homomorphism $\mathrm{GL}(3, \mathbb{R}) \to \mathrm{PGL}(3, \mathbb{R})$ maps $\mathrm{Aff}(2, \mathbb{R})$ isomorphically onto its image, and similarly, the quotient homomorphism $\mathfrak{gl}(3, \mathbb{R}) \to \mathfrak{pgl}(3, \mathbb{R})$ maps $\mathfrak{aff}(2, \mathbb{R})$ isomorphically onto its image. We shall often identify $\mathrm{Aff}(2, \mathbb{R})$ and $\mathfrak{aff}(2, \mathbb{R})$ with these images. With this identification $\mathrm{Aff}(2, \mathbb{R})$ becomes the group of affine transformations of $\mathbb{R}^2$, and $\mathfrak{aff}(2, \mathbb{R})$ becomes its Lie algebra.

The following is well-known.

Fact 2.3. Let $\psi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism which maps all lines into lines. Then $\psi_0 \in \mathrm{Aff}(2, \mathbb{R})$.

We need the following local version of this.

Lemma 2.4. *Let $\psi : U \to V$ be a diffeomorphism between connected open subsets $U, V \subseteq \mathbb{R}^2$ which maps all line segments in U into line segments in V. Then ψ can be uniquely extended into a projective linear map $\tilde{\psi} \in \mathrm{PGL}(3, \mathbb{R})$.*

Proof. The standard proof of Fact 2.3 works here if one is careful enough. We recall it for the sake of completeness.

Let $S \subset U$ be a line segment. We define the following set of real numbers.

$$\mathcal{C}_S = \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{If } A,B,C,D \in S \text{ with cross-ratio } (A,B{:}C,D)=\lambda \\ \text{then } \big(\psi(A),\psi(B);\psi(C),\psi(D)\big)=\lambda. \end{array} \right\}$$

We make several observations.

1. If $\lambda \in \mathcal{C}_S$ then $1 - \lambda \in \mathcal{C}_S$.
   Indeed, $(A, B : C, D) = 1 - (A, C : B, D)$.
2. If $\lambda < 0 < \mu$ and $\lambda, \mu \in \mathcal{C}_S$ then $\frac{\mu}{\lambda} \in \mathcal{C}_S$.
   Indeed, let $(A, B : C, D) = \frac{\mu}{\lambda}$. Since this is negative, one of C and D lies inside $\overline{AB}$, the other lies outside. If $C \in \overline{AB}$ then we relabel A, B, C, D to B, A, D, C, this does not change their cross-ratio. So we can assume that $C \notin \overline{AB}$. Then there is a unique $E \in \overline{AB}$ with $(A, B : E, C)) = \lambda$, and an easy calculation shows that $(A, B : E, D) = \mu$. This implies that $\big(\psi(A), \psi(B); \psi(E), \psi(C)\big) = \lambda$ and $\big(\psi(A), \psi(B); \psi(E), \psi(D)\big) = \mu$, hence $\big(\psi(A), \psi(B); \psi(C), \psi(D)\big) = \frac{\mu}{\lambda}$.
3. $0, 1 \in \mathcal{C}_S$.
   Indeed, if $(A, B : C, D) = 0$ then either $A = C$ or $D = B$. This implies that either $\psi(A) = \psi(C)$ or $\psi(D) = \psi(B)$, hence $\big(\psi(A), \psi(B); \psi(C), \psi(D)\big) = 0$. Therefore $0 \in \mathcal{C}_S$, and then (1) implies $1 \in \mathcal{C}_S$.

4. $-1 \in \mathcal{C}_S$.
   Indeed, let $W \subseteq U$ be a convex neighbourhood of S. If $(A, B : C, D) = -1$ then there is a complete quadrangle in W which justifies this, i.e. A and C are the intersection points of the opposite sides, and the diagonals intersect the AB line at C and D. Then ψ maps this quadrangle to a quadrangle in V justifying that $\big(\psi(A), \psi(B); \psi(C), \psi(D)\big) = \lambda$.
5. $\mathbb{Z} \subseteq \mathcal{C}_S$.
   Indeed, starting with $-1 \in \mathcal{C}_S$, and applying (1) and (2) alternately, we obtain that $2, -2, 3, -3, 4, -4, \dots \in \mathcal{C}_S$.
6. $\mathbb{Q} \subseteq \mathcal{C}_S$.
   Indeed, negative rational numbers are in $\mathcal{C}_S$ by (5) and (2). Then by (1), the non-negative rational numbers also belong to $\mathcal{C}_S$.
7. $\mathcal{C}_S = \mathbb{R}$.
   Indeed, by the continuity of the cross-ratio, $\mathcal{C}_S$ is a closed set.

Now let $W \subseteq U$ be a convex open subset. The above observations imply that ψ preserves all cross-ratios in W, hence there is a unique projective linear map $\overline{\psi}_W \in \mathrm{PGL}(3, \mathbb{R})$ which agrees with ψ on W. For overlapping convex open sets, the corresponding projective linear maps must be equal. Since U is connected, all these $\overline{\psi}_W$ must be equal. This proves the lemma. □

Lemma 2.5. *Let $A, B \in \mathfrak{gl}(3, \mathbb{R})$ be matrices whose images $\overline{A}, \overline{B} \in \mathfrak{pgl}(3, \mathbb{R})$ commute. Then, after a suitable base change, $\overline{A}, \overline{B} \in \mathfrak{aff}(2, \mathbb{R})$.*

Proof. Commutators have trace zero. So $[A, B]$ is a scalar matrix with trace 0, hence A and B commute in $\mathfrak{gl}(3, \mathbb{R})$. We distinguish three cases.

- If A has one real and two conjugate complex eigenvalues, then let V be the real part of the linear span of the complex eigenspaces corresponding to the non-real eigenvalues of A.
- If A has three different real eigenvalues, then let V be the linear span of any two of the corresponding eigenspaces.
- If A has only two different real eigenvalues, then let V be the eigenspace corresponding to the eigenvalue with multiplicity two.
- Otherwise, A has a single real eigenvalue of multiplicity three, hence A is a scalar matrix. In this case, we switch the role of A and B, and go through this list again. If B is not a scalar matrix then we obtain our V.
- Finally, if both A and B are scalar matrices, then let V be an arbitrary plane in $\mathbb{R}^3$.

In all cases, V is a plane in $\mathbb{R}^3$ invariant under both A and B. After an appropriate base change, V will be the hyperplane of vectors whose last coordinate is zero. Matrices that map this V into itself are of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

where elements marked with $*$ are arbitrary, and the quotient homomorphism $\mathfrak{gl}(3,\mathbb{R}) \to \mathfrak{pgl}(3,\mathbb{R})$ maps such matrices into $\mathfrak{aff}(2,\mathbb{R})$. Hence, after this base change, $\overline{A}, \overline{B} \in \mathfrak{aff}(2,\mathbb{R})$. □

3. Affine structures

Notation 3.1. For a plane curve $C \subset \mathbb{R}^2$ and a vector $t \in \mathbb{R}^2$ we denote by $t + C$ the translate of C with t.

Problem 3.2. Characterise diffeomorphisms $\phi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and continuous curves $C \subset \mathbb{R}^2$ such that $\phi_0\big(t + C\big)$ is a line for all $t \in \mathbb{R}^2$.

Problem 3.3. Characterize continuous curves $C \subset \mathbb{R}^2$ and diffeomorphisms $\phi : U \to V$ between connected, open subsets $U, V \subseteq \mathbb{R}^2$ such that $\phi\big((t + C) \cap U\big)$ is contained in a line for all $t \in \mathbb{R}^2$.

By composing ϕ_0 on the left with an appropriate translation, one can reduce Problem 3.2 to the case when $\phi_0(0,0) = (0,0)$. Similarly, by composing ϕ on both sides with appropriate translations one can reduce Problem 3.3 to the case when $(0,0) \in U$ and $\phi(0,0) = (0,0)$.

Let $T \cong \mathbb{R}^2$ denote the group of translations of $\mathbb{R}^2$. If we conjugate T with any of these ϕ_0 then by Fact 2.3 we arrive at a subgroup of $\mathrm{Aff}(2,\mathbb{R})$. Since T is connected, these subgroups are uniquely determined by their Lie algebras, which are 2-dimensional commutative subalgebras of $\mathfrak{aff}(2,\mathbb{R})$.

In the more general setup, if we conjugate a small neighbourhood of the identity in T with any of these ϕ then by Lemma 2.4 we arrive at a small neighbourhood of the identity in a connected subgroup of $\mathrm{PGL}(2,\mathbb{R})$. Again, these subgroups are uniquely determined by their Lie algebras, which are, in this case, 2-dimensional commutative subalgebras of $\mathfrak{pgl}(2,\mathbb{R})$. By Lemma 2.5 these subalgebras, after a base change, become subalgebras of $\mathfrak{aff}(2,\mathbb{R})$.

So in both problems, we need to classify 2-dimensional commutative subalgebras of $\mathfrak{aff}(2,\mathbb{R})$, and analyze whether the corresponding connected subgroups are isomorphic to T or not.

The embedding of a Lie algebra $\mathfrak{g}$ into $\mathfrak{aff}(2,\mathbb{R})$ is also called an *affine structure on* $\mathfrak{g}$. Such an embedding gives rise to an affine action on $\mathbb{R}^2$ of the simply connected Lie group with Lie algebra $\mathfrak{g}$, which we call the *corresponding action*. In particular, a 2-dimensional abelian subalgebra of $\mathfrak{aff}(2,\mathbb{R})$ is called an affine structure on the 2-dimensional abelian Lie algebra, and the corresponding action is an affine action of T on $\mathbb{R}^2$.

Affine structures on the 2-dimensional abelian Lie algebra were analyzed in the work of Rem and Goze [10] where they proved that there are six affinely non-equivalent affine structures. They listed the affine structures on the 2-dimensional Lie algebra and the corresponding action. Based on their list, there are six actions we have to check for a potential ϕ or ϕ_0.

3.1. The six affine actions

In what follows, we are checking the affine actions listed in [10] whether they give rise to a solution of Problem 3.2 or Problem 3.3. By definition, these affine actions are homomorphisms from the group of translations $T \cong \mathbb{R}^2$ into $\mathrm{Aff}(2, \mathbb{R})$. Let $A(s,t) \in \mathrm{Aff}(2, \mathbb{R})$ denote the homomorphic image of the element $(s,t) \in T$. Recall from Notation 2.2 that $A(s,t)$ is a 3×3 matrix of certain special form.

For every affine action, we are looking for a diffeomorphism $\phi : U \to V$ between connected open neighbourhoods U, V of the origin in $\mathbb{R}^2$ such that $\phi(0,0) = (0,0)$ and

$$\phi^{-1}(\phi(x,y) + (s,t)) = A^*(s,t) \cdot [x,y,1]^T, \tag{3}$$

where $A^*(s,t)$ is the 2×3 submatrix of $A(s,t)$ containing the first two rows. We will not specify U and V, only their existence is important to us. However, by studying the formulas we have for ϕ the reader can easily find appropriate U and V.

Let us denote the translation by (s,t) as $T(s,t) : \mathbb{R}^2 \to \mathbb{R}^2$ and let $a(s,t) : \mathbb{R}^2 \to \mathbb{R}^2$ denote the affine transformation as defined above with the $A(s,t)$ matrix. With this notation equation (3) becomes

$$\phi^{-1} \circ T(s,t) \circ \phi = a(s,t)$$

where $\circ$ denotes the function composition. After rearranging it, we have

$$\phi^{-1} \circ T(s,t) = a(s,t) \circ \phi^{-1}.$$

Applying both sides to $(x,y) = (0,0)$ we obtain

$$\phi^{-1}(s,t) = a(s,t)(0,0) = A^*(s,t) \cdot [0,0,1]^T,$$

so we can recover ϕ from the first two entries in the last column of $A(s,t)$.

In the first two cases our ϕ is actually an $\mathbb{R}^2 \to \mathbb{R}^2$ diffeomorphism, in the last four cases, we get local maps giving solutions in the selected range.

1. Identity (case A_5 in [10])

$$A(s,t) = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Here ϕ is the identity map, it won't define the translates of a single curve.

2. Parabola (case A_4 in [10])

$$A(s,t) = \begin{pmatrix} 1 & 0 & s \\ s & 1 & t + \frac{s^2}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\phi : (x, y) \mapsto \left(x, y + \frac{x^2}{2}\right)$. Every line $(x, ax+b)$ maps to a translate of a parabola $y = \frac{x^2}{2}$. Its image is $\left(x, \frac{(x+a)^2}{2} + \frac{a^2}{2} + b\right)$. This is Pudlák's map in [9] we mentioned in the introduction.

3. Log curve (case A_6 in [10])

$$A(s,t) = \begin{pmatrix} e^s & 0 & e^s - 1 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\phi : (x, y) \mapsto (\ln(x + 1), y)$. It is more convenient to work with the map ϕ' : $(x, y) \mapsto (x, \ln(y))$, giving the same family of curves. For every line $y = ax + b$ with $a > 0$, the part $x > \frac{b}{a}$ maps to a translate of a log curve, $y = \ln(x)$. Its image is $\left(x, \ln\left(x + \frac{b}{a}\right) + \ln(a)\right)$.

4. (case A_1 in [10])

$$A(s,t) = \begin{pmatrix} e^s & 0 & e^s - 1 \\ e^s(e^t - 1) & e^s e^t & e^s(e^t - 1) \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\phi : (x, y) \mapsto \left(\ln(x + 1), \ln\left(1 + \frac{y}{x+1}\right)\right)$.

The image of the part $x > \max\left(-\frac{b+1}{a+1}, -1\right)$ of the $(x, ax+b)$ line is a curve. However, different lines map into different types of curves. Lines of the form $y = c(x+1)$ map to a horizontal line $y = \ln(1 + c)$. This is still an interesting map. Let us consider a line $y = ax + b$ where $a \neq -1$ and $\frac{b-a}{1+a} > 0$. If we set $\ln(x + 1) = X$, then the image curve is

$$\left(X, \ln\left(1 + a + (b - a)e^{-X}\right)\right) = \left(X, \ln\left(1 + e^{-\left(X - \ln\left(\frac{b-a}{1+a}\right)\right)}\right) + \ln(1 + a)\right),$$

which is a translate of the curve $y = \ln(1 + e^{-x})$.

5. (case A_2 in [10])

$$A(s,t) = \begin{pmatrix} e^s & 0 & e^s - 1 \\ e^s t & e^s & e^s t \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\phi : (x, y) \mapsto \left(\ln(x + 1), \frac{y}{x+1}\right)$.

As in the previous case, lines of the form $y = c(x+1)$ map to a horizontal line. But if we consider lines $ax + b$, where $b > a \neq 0$, and set $\ln(x+1) = X$, then the image curve is

$$\left(X, a + \frac{b-a}{e^X}\right) = \left(X, e^{-(X-\ln(b-a))} + a\right),$$

which is a translate of the curve $y = e^{-x}$.

6. Rotations (case A_3 in [10])

$$A(s,t) = \begin{pmatrix} e^s \cos t & -e^s \sin t & 1 - e^s \cos t \\ e^s \sin t & e^s \cos t & e^s \sin t \\ 0 & 0 & 1 \end{pmatrix}$$

Images of translations with $(s, t + 2k\pi)$ would give the same line for any integer k. Also, lines of the form $y = c(x+1)$ map to a horizontal line.
In this last case, the map is given by

$$\phi : (x,y) \mapsto \left(\frac{\ln((1-x)^2 + y^2)}{2}, \arctan\left(\frac{y}{1-x}\right)\right),$$

or, equivalently by

$$\phi : (x,y) \mapsto (\Re(\ln(1 - x + iy)), \Im(\ln(1 - x + iy))).$$

If a line is given by the $x = t, y = at + b$ equations then its image after the map is

$$x(t) = \Re\left(\ln\left(t - \frac{1+ib}{1-ia}\right) + \ln(ia - 1)\right),$$
$$y(t) = \Im\left(\ln\left(t - \frac{1+ib}{1-ia}\right) + \ln(ia - 1)\right),$$

which is the real and the imaginary part of a translate of the complex logarithm function (Fig. 1).
From the six affine actions, we found four different curves such that arrangements of their translates are (locally) combinatorically equivalent to arrangements of lines.

4. Concluding remarks and open problems

- One can ask the same problem in higher dimensions. What are the surfaces such that any finite arrangement of hyperplanes is combinatorically equivalent to translates of the surface? Remm and Goze classified the three-dimensional commutative, associative real algebras in [10], so based on their work, one can characterize such surfaces in three-space. One of these (an extension of Pudlák's map [9]) was used by Zahl in

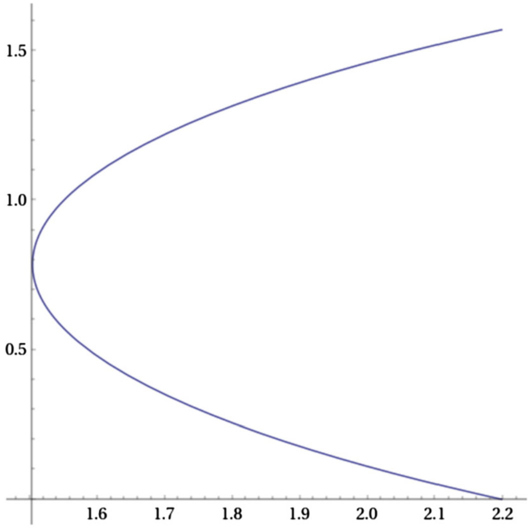

Fig. 1. Part of the curve from Case 6.

[17] to construct a norm determining $\Omega(n^{3/2})$ unit distances among n points. In the same paper, Zahl was able to break the $n^{3/2}$ barrier, showing that for the Euclidean norm, the number of unit distances determined by n points is $O(n^{3/2-c})$ for some $c > 0$. In dimension four and higher there are n-element pointsets with $\Omega(n^2)$ unit distances (see more about such constructions in [7]).

- It follows from Székely's proof of the Szemerédi-Trotter theorem in [15] that m translates of a convex curve and n points determine $O(n^{2/3}m^{2/3}+n+m)$ incidences (see also in [3]). In this paper, we listed four curves where the above incidence bound is sharp, i.e. for each curve, there are arrangements of m translates of the curve and n points with $\Omega(n^{2/3}m^{2/3}+n+m)$ incidences for arbitrarily large n and m. Are there such planar curves significantly different from the ones listed above?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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