

A universal right tail upper bound for supercritical Galton-Watson processes with bounded offspring

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Abstract

We consider a supercritical Galton-Watson process Z_n whose offspring distribution has mean $m > 1$ and is bounded by some $d \in \{2, 3, \dots\}$. As is well-known, the associated martingale $W_n = Z_n/m^n$ converges a.s. to some nonnegative random variable W_∞ . We provide a universal upper bound for the right tail of W_∞ and W_n , which is uniform in n and in all offspring distributions with given m and d , namely:

$$\mathbb{P}(W_n \geq x) \leq c_1 \exp \left\{ -c_2 \frac{m-1}{m} \frac{x}{d} \right\}, \quad \forall n \in \mathbb{N} \cup \{+\infty\}, \forall x \geq 0,$$

for some explicit constants $c_1, c_2 > 0$. For a given offspring distribution, our upper bound decays exponentially as $x \rightarrow \infty$, which is actually suboptimal, but our bound is *universal*: it provides a single *effective* expression – which does not require large x – and is valid simultaneously for all supercritical bounded offspring distributions.

1 Introduction

Consider a Galton-Watson process $(Z_n)_{n \geq 0}$ with $Z_0 = 1$ where the offspring distribution has expectation $m = \mathbb{E}[Z_1] > 1$ and $d = \sup\{k \in \mathbb{N}, \mathbb{P}(Z_1 = k) > 0\} < +\infty$. In other words, the GW process is supercritical with bounded offspring distribution. By definition,

$$Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n,i},$$

where $(\xi_{n,i})$ is a collection of independent identically distributed random variables with the same distribution as Z_1 . The martingale $W_n = Z_n/m^n$ converges almost surely and in L^2 to some random variable $W_\infty \geq 0$, see for example [AN72]. The seminal work of Harris [Har48]

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shows that W_∞ has finite exponential moments and provides the following estimates for the generating function:

$$\varphi(x) := \log \mathbb{E} e^{xW_\infty} = x^\gamma C(x) + O(1) \quad \text{as } x \rightarrow \infty,$$

where C is a positive multiplicatively periodic function (in particular it is bounded above and below in $(0, +\infty)$), γ is defined by $m^\gamma = d$ or $\gamma = \ln d / \ln m$, and $O(1)$ is the standard Landau notation. Harris' work is based on the Poincaré functional equation satisfied by φ , namely $\varphi(mx) = f(\varphi(x))$ where f is the generating function of Z_1 . Biggins and Bingham [BB93] could use this result to provide asymptotic estimates for the tail of W_n for both cases $n < +\infty$ or $n = +\infty$. This was later refined by Fleischmann and Wachtel in [FW09], who were mainly studying the left tail of W_∞ , but also expressed for the right tail the following asymptotics (see Remark 3 in their paper):

$$\mathbb{P}(W_\infty > x) = (1 + o(1)) x^{\frac{-\gamma}{2(\gamma-1)}} C_1(x) e^{-C_2(x) x^{\frac{\gamma}{\gamma-1}}}, \quad (1)$$

where C_1 and C_2 are again positive multiplicatively periodic functions. In 2013, Denisov, Korshunov, and Wachtel [WDK13] provide similar results when the offspring distribution is no more bounded but on the contrary heavy tailed, relying on probabilistic techniques for sums of i.i.d. random variables rather than on the Poincaré functional equation. Their article is also a good reference for an overview on the topic.

In this note, we seek for a more effective upper bound for the tail of W_n . We seek for a bound that would be *nonasymptotic*, available for both n finite or infinite, and above all *uniform* over the possible choices of the offspring distribution, or at least only depending on m and d . To motivate this, observe that the optimal strategy leading to large values of W_∞ often stems in just asking the first generations to have maximal degree (see [FW09]). For example, if you assume $\eta = \mathbb{P}(Z_1 = d) > 0$, you have $W_k = x = d^k/m^k$ with probability

$$\eta^{1+d+\dots+d^{k-1}} = \eta^{\Theta(d^{k-1})} = \eta^{\Theta(x^{\gamma/(\gamma-1)} d^{-1})},$$

which already somehow resembles the asymptotics given in (1). The maximal value of η being of course m/d , we obtain a uniform upper bound for the event that W_k equals its maximal possible value d^k/m^k . It is natural to ask whether we can obtain a similar uniform upper bound for the event $\{W_\infty > x\}$, possibly an upper bound that would feature the term $\exp(-cx^{\gamma/(\gamma-1)} d^{-1})$, or be as close as possible to (1).

We didn't find any such result in the literature. In this note, we thus provide, up to our knowledge, a first result in that direction.

Theorem 1.1. *There are universal constants $c_1, c_2 > 0$ such that for any $m > 1$, any $d \in \{2, 3, \dots\}$ and any offspring distribution on $\{0, \dots, d\}$ with mean m , for all $n \in \mathbb{N} \cup \{+\infty\}$ and $x \geq 0$, we have*

$$\mathbb{P}(W_n \geq x) \leq c_1 \exp \left\{ -c_2 \frac{m-1}{m} \frac{x}{d} \right\}. \quad (2)$$

Remark 1.2. *Our original motivation for proving Theorem 1.1 lies in the study of the contact process on power-law trees in [FJ23]. In that work the other existing upper bounds were not really useful, and it required a more effective upper bound like (2).*

Remark 1.3. *We didn't seek optimal constants, but in our proof we can take $c_2 = (\log 3 - 1)/40$. Less importantly, we can take $c_1 = 2$, or $c_1 = 1$ if we discard small values of x .*

2 Proof of Theorem 1.1

Heuristically, we obtain the upper bound by arguing that in order to have $\{W_n > xd\}$, we need to have simultaneous occurrence of $\Theta(x)$ unlikely events, corresponding to the existence of unusually fecund individuals. For each individual x of the branching process, we define $|x|$ its generation, so the ancestor has generation 0, its direct children generation 1, and so on. For $n \geq |x|$ we define Z_n^x the number of descendants of x of generation n , so $Z_{|x|}^x = 1$ and for each individual x , we have that $(Z_{n+|x|}^x)_{n \geq 0}$ has the same law as $(Z_n)_{n \geq 0}$. We further define

$$n_x := \inf_n \{n > |x|, Z_n^x > adm^n\},$$

with a some parameter that we will fix later on, and we say x is *fecund* if n_x is finite, or in other words if the descendants of x alone can give a contribution to W_n (for some $n > |x|$) larger than ad .

In order to determine that an individual x is fecund, there is no need to reveal the whole progeny of x , or event the whole progeny up to generation n_x . It suffices to reveal its progeny up to generation $n_x - 1$ (which automatically satisfies $Z_{n_x-1}^x \leq adm^{n_x-1}$ by the definition of n_x), and then reveal one by one the offspring of these individuals of generation $n_x - 1$, up to a point where we have revealed the presence of at least adm^{n_x} individuals of generation n_x . Note that at this point:

- There is a number A_x of descendants of x of generation $n_x - 1$ with still unrevealed offspring, for some A_x satisfying $0 \leq A_x < Z_{n_x-1}^x \leq adm^{n_x-1}$.
- We have already revealed the presence of a number B_x of descendants of x of generation n_x , with

$$adm^{n_x} \leq B_x < adm^{n_x} + d.$$

The upper bound on B_x here is a simple consequence of the fact that the offspring of each individual is bounded by d . It does not prevent that we might still have $Z_{n_x}^x \geq adm^{n_x} + d$.

We now construct recursively a multitype branching tree T as follows¹:

¹In order to improve readability of this note, we try and reserve the terminology of individuals/population and so on to the original branching process, while we reserve the graph/tree terminology to this second multitype branching process.

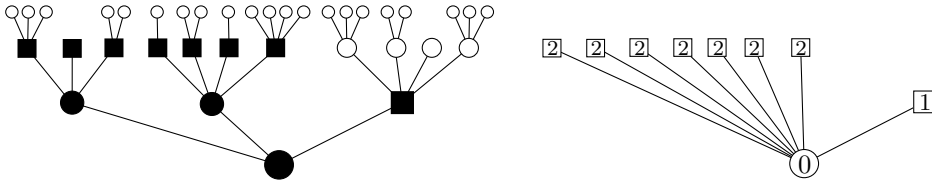


Figure 1: On the left we represented the first three generations of an infinite GW process which has $d = 4$ and $m = 1.75$. We also choose $a = 0.5$, so an individual is fecund if it has at least $\lceil adm \rceil = 4$ descendants in the first generation, or $\lceil adm^2 \rceil = 7$ descendants in the second generation, or $\lceil adm^3 \rceil = 11$ in the third, etc. Here only the ancestor \circ is fecund, with $n_\circ = 2$. This can be observed by revealing only the individuals drawn in black, with in particular $B_\circ = 7$ individuals of generation 2 and $A_\circ = 1$ individual of generation 1 with unrevealed offspring, all represented by squares. On the right, the corresponding multitype tree T , with \circ as single internal vertex, as the other vertices with type 1 and 2 correspond to non-fecund individuals of the GW process.

- The root of T is the ancestor of the branching process, and has type 0. More generally, each vertex of the tree corresponds to or is actually identified with some individual x of the branching process, and has type $|x|$.
- If a vertex of T corresponds to a fecund individual x of the branching process, then in T it has A_x children of type $n_x - 1$ and B_x children of type n_x , which correspond of course to the previously described descendants of x in the branching process of generation $n_x - 1$ and n_x .
- Otherwise, the vertex is a leaf of T .

Note that the internal nodes of T are necessarily fecund individuals of the branching process, and in particular T contains only one vertex if the ancestor is not fecund. The next two lemmas allow to bound the number of internal nodes of T , and then the size of Z_n .

Lemma 2.1. *Set $a = \frac{8m}{m-1}$. Then, for any positive integer r , the probability that T contains at least r internal nodes is at most $e^{-(\log 3 - 1)r}$.*

Lemma 2.2. *With $a = \frac{8m}{m-1}$, if you condition on T containing less than r internal nodes, then the probability of $\{W_n > 5adr\}$ is bounded by $e^{-\frac{2}{3}r}$.*

With these two lemmas at hand, we obtain for any positive integer r the inequality

$$\mathbb{P}(W_n > 5adr) \leq e^{-(\log 3 - 1)r} + e^{-\frac{2}{3}r},$$

which translates into Theorem 1.1 when taking x nonnegative real number.

Proof of Lemma 2.1. Recall that the martingale $W_n = Z_n/m^n$ converges almost surely and in L^2 to W_∞ with $\mathbb{E}[W_\infty] = 1$ and $\text{Var } W_\infty = \frac{\text{Var } Z_1}{m(m-1)}$. Stopping the submartingale W_n^2 at the first time it hits $[x, +\infty)$, we then obtain a bound for the tail distribution of $\sup_l W_l$ as

$$\begin{aligned} \mathbb{P}(\sup_l W_l \geq x) &\leq x^{-2} \mathbb{E}[W_\infty^2 \mathbb{1}_{\{\sup_l W_l \geq x\}}] \leq \left(1 + \frac{\text{Var } Z_1}{m(m-1)}\right) x^{-2} \\ &\leq \left(1 + \frac{d}{(m-1)}\right) x^{-2}, \end{aligned}$$

where in the last inequality we use $\text{Var } Z_1 \leq \mathbb{E}[Z_1]d = md$.

Thus, a vertex of T of type n is an internal node with probability

$$p_n := \mathbb{P}\left(\sup_l W_l \geq adm^n\right) \leq \frac{1}{a^2 m^{2n} d^{2n}} + \frac{1}{a^2 m^{2n} d(m-1)}.$$

For any n , we have

$$adm^n p_n \leq \frac{1}{a} \left(\frac{1}{d} + \frac{1}{m-1}\right) \leq \frac{m}{a(m-1)} \leq \frac{1}{8},$$

as well as $p_n \leq p_0 \leq 1/16$. We now look at the subtree of internal nodes of T . Note that if an internal node has in T (at most) $adm^{n+1} + d$ children of type $n+1$ and adm^n of type n , amongst these children, the number of internal nodes is bounded by

$$\begin{aligned} &\text{Bin}(adm^{n+1} + d, p_{n+1}) \oplus \text{Bin}(adm^n, p_n) \\ &\preceq \text{Pois}\left(2adm^{n+1} \frac{p_{n+1}}{1-p_{n+1}}\right) \oplus \text{Pois}\left(adm^n \frac{p_n}{1-p_n}\right) \\ &\preceq \text{Pois}\left(\frac{3}{8} \cdot \frac{16}{15}\right) = \text{Pois}\left(\frac{1}{3}\right), \end{aligned}$$

where $\text{Bin}(n, p)$ and $\text{Pois}(\lambda)$ stand here for random variables with binomial (respectively Poisson) distribution, \oplus means we sum independent random variables, and \preceq stands for stochastic domination. The probability that the root is an internal node is also bounded by the probability that a $\text{Pois}(1/3)$ is positive. Thus in order to have at least r internal nodes, you need to have a $\text{Pois}(mr)$ larger than r where $m = 1/3$, which we bound by the usual Chernoff bound

$$\mathbb{P}(\text{Pois}(mr) \geq r) \leq e^{-mr}(em)^r \leq (em)^r = e^{-(\log 3 - 1)r}.$$

□

Proof of Lemma 2.2. Suppose the tree T is finite, with $|i_T|$ internal nodes and a set of leaves L_T . For given $n \geq 0$, we bound the size of Z_n by splitting the individuals with respect to their most recent ancestor belonging to T (this most recent ancestor is always well-defined, as the ancestor of the branching process is in T). Note that for any vertex x of T , at most adm^n individuals of generation n can share x as their

most recent ancestor in T . We thus obtain the following upper bound for Z_n :

$$Z_n \leq |i_T|adm^n + \sum_{\ell \in L_T, |\ell| \leq n} Z_n^\ell, \quad (3)$$

where we recall the notation Z_n^ℓ for the number of descendants of the individual ℓ at generation n . If we condition on T , the processes Z^ℓ , for $\ell \in L_T$, are independent, and $(Z_k^\ell)_{k \geq |\ell|}$ is of course the branching process, initiated at time $k = |\ell|$ with $Z_{|\ell|}^\ell = 1$, and conditioned² on $\{\forall k \geq |\ell|, Z_k^\ell \leq adm^k\}$. Its value at time n is stochastically dominated by $Z_{n-|\ell|} \wedge adm^n$. Dividing by m^n , we get

$$W_n \leq |i_T|ad + \sum_{\ell \in L_T, |\ell| \leq n} W_n^\ell, \quad (4)$$

where the random variables $W_n^\ell := m^{-n}Z_n^\ell$ are independent and stochastically dominated by $m^{-|\ell|}W_{n-|\ell|} \wedge ad$. Hence,

$$\begin{aligned} \mathbb{E}[W_n^\ell] &\leq m^{-|\ell|}, \\ \text{Var } W_n^\ell &\leq m^{-2|\ell|} \mathbb{E}[W_\infty^2] \leq m^{-2|\ell|} \left(1 + \frac{d}{m-1}\right) \\ &\leq m^{-|\ell|} \left(1 + \frac{d}{m-1}\right). \end{aligned}$$

Summing over the leaves of T , we obtain

$$\sum_{\ell \in L_T} \mathbb{E}[W_n^\ell] \leq \sum_{\ell \in L_T} m^{-|\ell|} \leq 3ad|i_T|,$$

using that the sum of the $m^{-|\ell|}$ over the leaves which are the children of a given internal node, is necessarily bounded by $3ad$. Similarly,

$$\sum_{\ell \in L_T} \text{Var } W_n^\ell \leq \left(1 + \frac{d}{m-1}\right) 3ad|i_T| \leq \frac{m}{m-1} 3ad^2|i_T| = \frac{3}{8}a^2d^2|i_T|.$$

Now we have $\mathbb{E}[W_n \mid T] \leq 4ad|i_T|$, and Bernstein's inequality states

$$\begin{aligned} \mathbb{P}\left(W_n \geq 4ad|i_T| + adx \mid T\right) &\leq \exp\left(-\frac{\frac{1}{2}(adx)^2}{\sum \text{Var } W_n^\ell + \frac{1}{3}(ad)(adx)}\right) \\ &\leq \exp\left(-\frac{x}{\frac{3}{4}\frac{|i_T|}{x} + \frac{2}{3}}\right), \end{aligned}$$

and hence

$$\mathbb{P}\left(W_n \geq 5adx \mid |i_T| \leq x\right) \leq e^{-\frac{12}{17}x} \leq e^{-\frac{2}{3}x}.$$

□

²Indeed, the conditioning on T and on ℓ being a leaf of T , translates into the conditioning on the individual ℓ not being fecund.

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