

# Ranking by pairwise comparisons for Swiss-system tournaments

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## Abstract

Pairwise comparison matrices are widely used in Multicriteria Decision Making. This article applies incomplete pairwise comparison matrices in the area of sport tournaments, namely proposing alternative rankings for the 2010 Chess Olympiad Open tournament. It is shown that results are robust regarding scaling technique. In order to compare different rankings, a distance function is introduced with the aim of taking into account the subjective nature of human perception. Analysis of the weight vectors implies that methods based on pairwise comparisons have common roots. Visualization of the results is provided by Multidimensional Scaling on the basis of the defined distance. The proposed rankings give in some cases intuitively better outcome than currently used lexicographical orders.

*Keywords:* Multicriteria decision making, Incomplete pairwise comparison matrix, Ranking for Swiss-system tournaments, Multidimensional Scaling

## 1 Introduction

There are some sport tournaments where the ranking of the competitors is based on the results of games played against each other. In the world of sport there is no consensus in using a particular ranking method. Various evaluation methods have been applied to different events taking into account the specifics of the particular sport activity. Further complication emerges when participants are teams, and the final ranking should also reflect individual results. One characteristic example for that situation is a chess team championship.

In this article it is assumed that the final ranking of participants of a sport tournament is based on the outcome of the games that have been played against each other. The result of a game is given according to the rules of the particular sport, however, reasonable data transformation is also allowed. Ranking of the participants will be made by applying the pairwise comparison models of Multicriteria Decision Making methodology. It has two main advantages: the obvious interpretation of the games played against each other and the ability to address the problem of intransitivity (cyclical preferences regarding three alternatives: A is better than B, B is better than C, but C is better than A), a common feature of subjective evaluations by individuals, but a well-known phenomenon in objective sport results, too.

The 'alternatives' to be compared are the participants of the tournament, their game results will be incorporated into a pairwise comparison matrix, and the final result will follow the

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ranking of the priority vector derived from the matrix with an estimation method. It is worth mentioning that this approach can be applied not only for some real competitions (national soccer, basketball, hockey, etc. championships, chess tournaments), but for the ranking of teams or individuals having historic data of their rivalry (tennis or chess players, athletes).

Although the application of complete pairwise comparison matrices (i.e. everybody met all the other competitors at least once) can raise interesting questions to be analyzed (comparison of the 'official' rankings and those obtained by using the pairwise comparison method, for instance), the focus of this article is to use incomplete pairwise comparison matrices for the final ranking. This is the reason why the analyzed case is a Chess Olympiad: it is an ideal example of the potential application of incomplete pairwise comparison matrices.

Chess competitions are often organized as a Swiss-system tournament. All participants face each other for a determined number of games (rounds are often organized at the same time), while there is no knockout phase. It means that a loss in the first rounds does not make impossible for the contestant to win the championship, as well as all participants play the same number of matches. The final rank of the players (teams) is determined mainly by the application of different lexicographical orders, but it lacks a well established solution for determining the final ranking – taking into account the performance of opponents of a team is a central issue as not all possible matchings materialized, which poses a serious challenge. It will be shown that an alternative way based on the results against each other could give a ranking which is in some sense intuitively better than the currently used methods. The more detailed analysis will take the results of the 39th Chess Olympiad Open tournament as a basis. The competition took place from September 20th to October 3rd, 2010 in Khanty-Mansiysk, Russia.

This choice was supported by the following arguments:

- It was an important sport event recently;
- Results were easy to collect and to adapt to the pairwise comparison method;
- Not only the winner, but the position of other participants were of interest;
- The reciprocity of the pairwise comparison matrix was a reasonable assumption;
- Participants were interested in the size of their win or lose;
- The official ranking method is debated.

All of these issues will be thoroughly discussed.

The article is organized as follows. In Section 2 the applied theory and some methods for the consistent approximation of incomplete pairwise comparison matrices will be described. Section 3 deals with the representation of the tournament results in a pairwise comparison format. Special features of the chess olympiad which make the necessary assumptions reasonable will also be presented here. Section 4 analyzes different rankings with a focus on comparing the official result with those of calculated from pairwise comparisons. For this purpose a distance measure will be introduced. Section 5 highlights some interesting properties of the example. Section 6 uses Multidimensional Scaling (MDS) to draw the rankings on a two-dimensional map based on the distance defined in Section 4. It reveals that the different rankings obtained from pairwise comparisons are close to each other. In Section 7 the summary of the results and the outline of further research are given.

## 2 Incomplete pairwise comparison matrices

Pairwise comparisons have been widely used in decision making since Saaty published the AHP method [15].

In a Multicriteria Decision Making problem the  $n \times n$  real matrix  $\mathbf{A} = (a_{ij})$  is a pairwise comparison matrix if it is positive and reciprocal that is  $a_{ij} > 0$  and  $a_{ji} = 1/a_{ij}$  for all  $i, j = 1, 2, \dots, n$ . The reciprocity condition also means that  $a_{ii} = 1$  for all  $i = 1, 2, \dots, n$ .

Matrix element  $a_{ij}$  is the numerical answer to the question 'How many times is the  $i$ th alternative more important/better/favourable than the  $j$ th?' The final aim of the use of pairwise comparisons is to determine a weight vector  $\mathbf{w} = (w_i)$  for the alternatives, where  $w_i/w_j$  somehow approximates the pairwise comparison  $a_{ij}$ . The solution is obvious if matrix  $\mathbf{A}$  is consistent, namely  $a_{ik} = a_{ij}a_{jk}$  for all  $i, j, k = 1, 2, \dots, n$ , because there exists a positive  $n$ -dimensional vector  $\mathbf{w}$  such that  $a_{ij} = w_i/w_j$  for all  $i, j = 1, 2, \dots, n$ .

In the inconsistent case the real values of the decision maker can only be estimated. Saaty proposed the Eigenvector Method (EM) for this purpose, which is based on the Perron theorem [13], as a positive matrix has a dominant eigenvalue with multiplicity one and an associated strictly positive eigenvector.

Distance-minimization techniques, like Logarithmic Least Squares Method (LLSM) [3, 4, 5] minimize the function  $\sum_i \sum_j d(a_{ij}, w_i/w_j)$  where  $d(a_{ij}, w_i/w_j)$  is the proper difference of  $a_{ij}$  from its approximation  $w_i/w_j$  (in case of LLSM  $d(a_{ij}, w_i/w_j) = (\log a_{ij} - \log(w_i/w_j))^2$ ).

It may happen that a subset of pairwise comparisons are unknown due to the lack of available data, uncertain evaluations, or other problems. Incomplete pairwise comparison matrices were introduced in [9], for example (missing elements are denoted by \*):

$$\mathbf{A} = \begin{pmatrix} 1 & * & a_{13} & a_{14} \\ * & 1 & a_{23} & * \\ 1/a_{13} & 1/a_{23} & 1 & a_{34} \\ 1/a_{14} & * & 1/a_{34} & 1 \end{pmatrix}.$$

In order to handle incomplete pairwise comparison matrices, introduce the variables  $x_1, x_2, \dots, x_d \in \mathbb{R}_+$  for the  $d$  missing elements in the upper triangle of  $\mathbf{A}$  supposing reciprocity (in all there are  $2d$  unknown entries in the matrix). The new matrix is denoted by  $\mathbf{A}(\mathbf{x})$  as its elements are the functions of the variables:

$$\mathbf{A}(\mathbf{x}_1, \mathbf{x}_2) = \begin{pmatrix} 1 & x_1 & a_{13} & a_{14} \\ 1/x_1 & 1 & a_{23} & x_2 \\ 1/a_{13} & 1/a_{23} & 1 & a_{34} \\ 1/a_{14} & 1/x_2 & 1/a_{34} & 1 \end{pmatrix}.$$

(In)complete pairwise comparison matrices can be represented by graphs [7, 10]. Let  $\mathbf{A}$  be a pairwise comparison matrix of size  $n \times n$ . Then  $G := (V, E)$ , where  $V = \{1, 2, \dots, n\}$ , vertices correspond to the alternatives, and  $E = \{e(i, j) : a_{ij} \text{ is known and } i \neq j\}$ , thus  $E$  represents the structure of known elements.

A recent result in this field is the extension of EM and LLSM to the incomplete case [2]. The EM solution is arising from the idea that inconsistency could be measured by the maximal eigenvalue, so it is a natural approach to choose the unknown elements for minimizing the dominant eigenvalue. Therefore the task is:  $\lambda_{max}(\mathbf{A}(\mathbf{x})) \rightarrow \min$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_d)^T \in \mathbb{R}_+^d$ . As proved in [2], the solution is unique if and only if graph  $G$  is connected, thus two alternatives could be compared via directly or indirectly through other alternatives. An algorithm is also proposed for the  $\lambda_{max}$ -optimal completion based on cyclic coordinates.

Another opportunity is the LLSM method. In this case only the known elements of  $\mathbf{A}$  are examined:  $\{\sum_{e(i,j) \in E} [\log a_{ij} - \log(w_i/w_j)]^2\} \rightarrow \min$ . The solution is unique again if and only if graph  $G$  is connected, thus it depends strictly on the position of known elements, not on their exact values [2]. Calculation of the optimal weights requires only the inverse of the upper-left  $(n-1) \times (n-1)$  submatrix in the  $\mathbf{L}$  Laplace-matrix of graph  $G$  and some additional matrix multiplication. The problem can be solved quickly by any commonly used office softwares, like MS Excel.

### 3 Application to the Chess Olympiad

An example for an incomplete pairwise comparison matrix is the result matrix of a chess olympiad described in the introduction. In the 39th Chess Olympiad 2010 Open tournament officially 149 teams participated. All of them played 11 matches, except for some teams due to their late arrivals.<sup>1</sup>

The matrix consists of 810 results from played matches, the ratio of known elements is small (7.3%) as 11,026 elements is placed in the upper triangle of a  $149 \times 149$  matrix. The diagonal elements of the matrix contains unities. There was no significant tendency in number of draws during the 11 rounds and it is the least frequent result. It implies that weaker teams played no 'simple' draws in the final rounds, when they still had no chances to reach a good position. This is important in chess, as draws could have arranged by the mutual agreement of players. However, that possibility is not totally excluded. The distribution of results is presented in Figure 1.

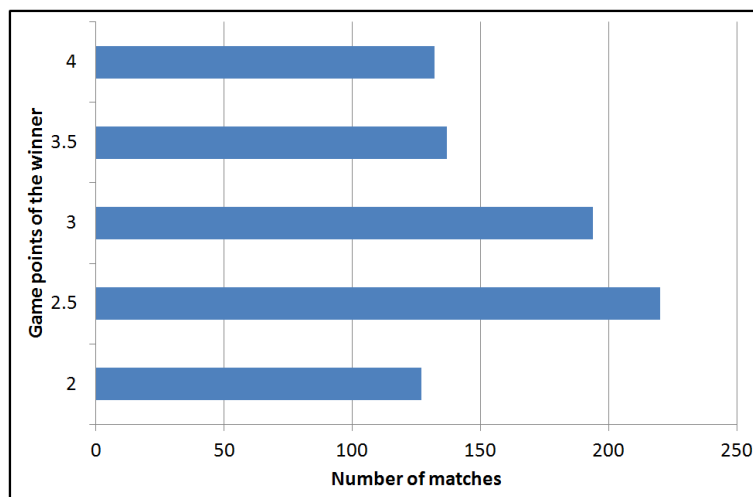


Figure 1: Distribution of game points of the winner (draws appear as 2)

In the tournament, all teams constituted by 5 players (4 and 1 reserve), a match between two of them includes 4 games with 3 different results (white win or loss, draw). In both teams 2 players have white, implying symmetricity as the chances of the match are a priori equal.<sup>2</sup> The

<sup>1</sup>Results are available in [14] in MS Excel or pdf-files. The data were organized into pairwise comparison matrices by the author; these are available on request.

<sup>2</sup>This is not true in individual chess tournaments, which is an important argument for examining the olympiad and similar team championships. Maybe the 5 Russian teams enjoyed some advantages due to the location, but it is not significant. In some sports it is false to suppose the reciprocity of the pairwise comparison matrix, like in

winner of the game gets 1 game point, the loser 0, while draw means 0.5 game points for each player. Therefore the final result of the match for one team, the sum of players' game points, ranges from 0 to 4, by 0.5. If a team achieves in the match minimum 2.5 game points, it gets 2 match points. If it scores 2 game points (thus its opponent has likewise 2 game points), it gets 1 match point. If it secures 1.5 or fewer game points, it counts as 0 match points for this match. The sum of allocated match points is always 2.

The official ranking method is a lexicographical order determined by the application of 4 tie-breaking procedures in sequence, proceeding from TB1 to TB4 to the extent required.<sup>3</sup> The primary criteria of the lexicographical order is TB1, the number of match points. However, as at most 22 match points could be scored in 11 matches, TB2 rules certainly should be applied, which means that teams are strongly interested in increasing their game points, players cannot be satisfied with a simple 2.5:1.5 win. Consequently, the size of wins reliably reflects the difference of teams' performance and it is justified to give higher weights to bigger wins. This is not the case in a lot of sport events.<sup>4</sup>

It was presented that match results correspond to the main conception of ratios used in pairwise comparison matrices, and they could be transformed into values (ratios). Draws (2:2) are obviously converted to 1, but in the other 4 sort of results 4 different rules were applied (reciprocity ensures that it is enough to see the results from the viewpoint of the winner). The variants are presented in Table 1.

Table 1: Transformation of match results into pairwise comparison values (ratios)

Game points	Number of matches	A variant	B variant	C variant	D variant
0	132	1/5	1/8	1/3	1/5
0.5	137	1/4	1/6	2/5	1/4
1	194	1/3	1/4	1/2	2/7
1.5	220	1/2	1/2	2/3	1/3
2	127	1	1	1	1
2.5	220	2	2	1.5	3
3	194	3	4	2	3.5
3.5	137	4	6	2.5	4
4	132	5	8	3	5

The size of wins weighs most in B variant, while in D ratios are not heavily effected by the number of game points. C variant contracts the scale, while A variant means the baseline scenario.

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soccer (home or away) or tennis (hard, carpet, clay or grass). The examination of pairwise comparison matrices without the reciprocity condition is out of the scope of the paper.

<sup>3</sup>The official ranking method is available in [8]. The position of teams that finish with the same number of match points shall be determined by application of the following tie-breaking procedures in sequence, proceeding from TB1 to TB4 to the extent required:

- TB1: Number of match points.
- TB2: Olympiad Sonneborn-Berger points. The number of each opponent's match points, excluding the opponent who scored the lowest number of matchpoints, multiplied by the number of game points achieved against this opponent.
- TB3: Number of game points. Their sum is always 4 in one match.
- TB4: Buchholz points. The sum of the match points of all the teams opponents, excluding the lowest one.

<sup>4</sup>It is true for tennis where players want to end the match as soon as possible, but not in football where goal difference usually does not count so much.

Now a method should be chosen to solve the problem, namely, to find a weight vector which approximates the incomplete pairwise comparison matrix relatively well. As it was mentioned in the previous section, EM and LLSM will be used. From a computational point of view, LLSM seems to be more favourable: the algorithm proposed by [2] for the calculation of the right eigenvector is an iteration process in which a lot of univariable minimization problems should be solved. Therefore, in Section 4 the EM ranking will be discussed only for C variant. It was checked that the EM for other variants gives similar results to the LLSM, their difference is negligible compared to the official final ranking.

The appropriate result has another criteria, that graph  $G$  corresponding to the pairwise comparison matrix should be connected. It depends only on the position of known elements, which is the same for the 4 variants. It cannot be decided in advance, but the solution of [2] operates only for connected graphs. The pairing algorithm described in [8] suggests it will be connected and it was proved by the implementation of LLSM.

## 4 Comparison of different rankings

The approximation of the incomplete pairwise comparison matrix is a 149-dimensional positive vector with normalized weights:  $\mathbf{w} = (w_1, w_2, \dots, w_{149})^T \in \mathbb{R}_+^{149}$  where  $\sum_{i=1}^{149} w_i = 1$ . Elements are difficult to evaluate because of the large size of vectors, however, some features of the optimal weights will be discussed in Section 5. This section focuses on the relation of different rankings, gained from ordering of the elements of the priority vectors (full lists of countries sorted by positions according to the final ranking are presented in Table 5). In the following, A-LLSM/B-LLSM/C-LLSM/D-LLSM corresponds to the ranking derived from the LLSM method for A/B/C/D variants. C-EM is the ranking obtained by the right eigenvector (EM) for C variant. Final ranking is the official final result of the Chess Olympiad. Start serves as a reference: it is the ranking of teams before the first round of the tournament based on the average FIDE (Fédération Internationale des Échecs or World Chess Federation) rating of a team's players, which reflects their former performance.<sup>5</sup>

One of the most known index for comparing rankings is Spearman's rank correlation coefficient [16]. Let  $X_i$  denote the rank of alternative  $i$  according to ranking  $X$  and  $Y_i$  denote the rank of alternative  $i$  according to ranking  $Y$ , then Spearman's rank correlation coefficient is:

$$\rho = 1 - \frac{6 \sum_{i=1}^n (X_i - Y_i)^2}{n(n^2 - 1)}$$

where  $n$  is the maximal rank number.  $\rho$  is the element of  $[-1, 1]$ . These limits are reached when the two rankings are the same (+1) or entirely opposite (-1).  $\rho = 0$  signs that there is no relation between the two rankings.  $\rho^2$  can be interpreted as the fraction of variance shared between the two rankings. In this case there are no ties.<sup>6</sup>

Rank correlations are collected into a symmetric matrix, where the element of the  $i$ th row and the  $j$ th column is the Spearman's rank correlation coefficient between the corresponding rankings (see Table 2).

The minimal  $\rho$  among the 5 proposed rankings based on pairwise comparisons (in the  $5 \times 5$  submatrix of Table 2) is much bigger than 0.99: one of them explains the others at least in 99.5% according to the interpretation of  $\rho^2$ . It seems to be a high value, however, there are some

<sup>5</sup>See details in [8].

<sup>6</sup>The official Start and Final rankings exclude ties by definition. Vectors approximating pairwise comparison matrices have no equal coordinates.

Table 2: Spearman’s rank correlation coefficients between rankings

	Start	Final	A-LLSM	B-LLSM	C-LLSM	D-LLSM	C-EM
Start	1	0.9353	0.9683	0.9684	0.9686	0.9654	0.9680
Final	0.9353	1	0.9688	0.9686	0.9689	0.9699	0.9690
A-LLSM	0.9683	0.9688	1	0.9997	0.9998	0.9987	0.9998
B-LLSM	0.9684	0.9686	0.9997	1	0.9998	0.9978	0.9998
C-LLSM	0.9686	0.9689	0.9998	0.9998	1	0.9983	0.9999
D-LLSM	0.9654	0.9699	0.9987	0.9978	0.9983	1	0.9983
C-EM	0.9680	0.9690	0.9998	0.9998	0.9999	0.9983	1

differences between them. For example, France is 6th in D-LLSM, while 10th in B-LLSM, or Jordan is 67th in D-LLSM, but 73rd in B-LLSM.

Rank correlations among the Final and the proposed rankings are above 0.96, the knowledge of the official result decrease the uncertainty regarding one of the latters at least by 93% suggested by  $\rho^2$ . Nevertheless, great part of this is a simple fiction: an expert totally uninformed about this championship could still estimate the final result with more than 86% certainty. It seems that Start is nearer to LLSM rankings than to Final. It is not positive as the tournament will lose its curiosity if there are no surprises. On the other hand, the official method distorts in the direction of weaker teams confirmed by subsequent observations.

Spearman’s rank correlation coefficient was used because it is a commonly accepted measure. In this case it has some disadvantages:  $\rho$  is based on the square of rank differences, which means its value is determined mainly by the positions of weaker teams: the sum of rank differences between Start and Final rankings is 35,864, from which Pakistan (occupying the 123rd and 62nd positions, respectively) accounts for 3,721, more than 10%. However, for the public the first 20 teams are more important than teams from the 80th to 99th positions.

The contextual factors of the situation should also be considered. Now it is more satisfactory to place the best team in the first position than to place the worst contestant last, known as ceiling effect [17]. There are a lot of measures of rank correlation which takes into account similar nonlinear effects. It would be a logical solution to use weights, but it still enlarges the differences in ranks due to squares and the chosen measure is intended to be used for another purpose, to visualize the rankings, which implies a rank correlation index is not appropriate.

In order to compare the above defined rankings, these problems should be addressed with a method satisfying the following soft properties:

- It regards the allocation of first places more important than the positions of weaker teams;
- It is symmetric and strictly monotonically increasing at every point, it increases if teams are positioned far from each other regardless to their exact positions;
- It possibly avoids the enlarging effect of square differences.

Only rankings without ties will be discussed, i.e. supposing that all ties were broken by an arbitrary method like alphabetical order. It is a reasonable assumption in case of sport tournaments.

**Definition 1** *The  $\tau$  measure between rankings  $X$  and  $Y$  is*

$$\tau(X, Y) = \sqrt{\sum_{i=1}^n \left( \log \frac{X_i}{Y_i} \right)^2}.$$

*Due to the properties of logarithm, it is the log-Euclidean metrics of the two rankings [1].*

**Proposition 1**  $\tau$  measure is a distance.

PROOF:

Denote  $\epsilon(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (x_i - y_i)^2$  the standard Euclidean distance of vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Now  $\tau(X, Y) = \epsilon(\log(X), \log(Y))$  where  $\log(X)$  is a vector in  $\mathbb{R}^n$  and  $(\log(X))_i = \log X_i$ . It means  $\tau$  is a distance, too.  $\square$

It is not a new result, for a more general discussion see [12]. In the following, it will be referred to as  $\tau$  distance.

**Proposition 2**  $\tau$  has a maximum depending on the length of the ranking.  $\tau$  has a unique maximum.

PROOF:

Take two rankings  $X$  and  $Y$ . Suppose the  $i$ th alternative is better than the  $j$ th in both rankings:  $X_i < X_j$  and  $Y_i < Y_j$ . It will be shown that the value of  $\tau$  can be higher. In this analysis the final square root in the formula is irrelevant due to its monotonically increasing property.

Change the positions of  $i$  and  $j$  in  $Y$ , denote the new ranking by  $Z$ . Examine  $\tau(X, Y)$  and  $\tau(X, Z)$ . The positions of all  $k \neq i, j$  alternatives is the same in  $Y$  and  $Z$ , the sum of squares for all  $k \neq i, j$  is the same. Denote  $\log X_i$  by  $a_i$ ,  $\log X_j$  by  $a_j$ ,  $\log Y_i$  by  $b_i$  and  $\log Y_j$  by  $b_j$ , then some calculation shows ( $a_i < a_j$  and  $b_i < b_j$  from the assumption:

$$\begin{aligned} \tau(X, Z) - \tau(X, Y) &= (a_i - \log Z_i)^2 + (a_j - \log Z_j)^2 - (a_i - b_i)^2 - (a_j - b_j)^2 \\ &= (a_i - b_j)^2 + (a_j - b_i)^2 - (a_i - b_i)^2 - (a_j - b_j)^2 \\ &= -2a_i b_j - 2a_j b_i + 2a_i b_i + 2a_j b_j \\ &= 2b_j(a_j - a_i) + 2b_i(a_i - a_j) \\ &= 2(b_j - b_i)(a_j - a_i) \geq 0. \end{aligned}$$

It implies that  $\tau(X, Y)$  can be higher if there exists two alternatives  $i$  and  $j$  that  $X_i < X_j$  and  $Y_i < Y_j$ . It is true for every  $Y$  except the opposite ranking  $X^{-1}$ . Suppose that  $X_i = i$  for all  $i$  as indexing of the alternatives is arbitrary. If there exists no  $i$  and  $j$  that  $X_i < X_j$  and  $Y_i < Y_j$ , then  $X_1 = 1 \Rightarrow Y_1 = n$  because of  $i = 1$  and  $j = 2, 3, \dots, n$ . Similarly,  $X_2 = 2 \Rightarrow Y_2 = n - 1$  because of  $i = 1$  and  $j = 3, 4, \dots, n$ . It leads to the final conclusion that  $Y = X^{-1}$ . As the number of position changes is limited by  $\binom{n}{2}$ , the iteration ends, and argument of the maximum is the two opposite rankings.  $\square$

**Remark 1** The value of  $\tau$  depends on the base of logarithm, which corresponds to a multiplying factor. In the following, the natural logarithm will be used.

**Remark 2** The  $\tau$  distance satisfies the required conditions:

- It differentiates stronger in first places: if the first and second teams change their positions,  $\tau^2$  evaluates it by  $2 \log(2) \approx 1.3863$ , if the 80th and 90th teams switch places,  $\tau^2$  increases only by  $2 \log(\frac{9}{8}) \approx 0.2357$ . It is quite significant difference, but not meaningless – people tend to record the best teams, while they do not bother about teams with average performance.
- It is symmetric and increases if teams positioned far from each other regardless to their exact positions as it is a distance.



- The logarithmic transformation contracts the scale rather than enlarging it. After that, Euclidean distance enlarges somewhat the differences but the concavity of logarithm is dominant for large differences. Among Start, Final and the other 5 rankings most  $\max\{X_i/Y_i; Y_i/X_i\}$  ratios are near to 1, where the logarithm can be approximated with the identity function.<sup>7</sup> It means that the enlarging effect of squares remains high. For example, nearly 26-26% of the total  $\tau^2$  between A-LLSM and C-LLSM is due to Armenia and China occupying the 5th and 6th positions in both rankings.

$\tau$  distances between rankings are recorded in Table 3. The proposed rankings are more or less at the same distance from Start and Final. Start and Final rankings are also the farthest as Spearman's rank correlation coefficient show. The difference between A-LLSM, B-LLSM and C-LLSM is negligible, while D-LLSM is somewhat far (it rewards mainly the victories, not their size). The EM method for C variant is nearly the same as LLSM rankings. Some countries have again great influence on the numbers: the  $\tau^2$  between B-LLSM and D-LLSM is 1.013, from which 0.2609 derives from France (10th and 6th, respectively). Notably, some countries performed much better or worse compared to the knowledge of their players reflected by Start.

Table 3:  $\tau$  distances between rankings

	Start	Final	A-LLSM	B-LLSM	C-LLSM	D-LLSM	C-EM
Start	0	4.008	2.928	2.926	2.909	3.150	2.923
Final	4.008	0	2.806	2.817	2.795	2.788	2.806
A-LLSM	2.928	2.806	0	0.489	0.359	0.672	0.383
B-LLSM	2.926	2.817	0.489	0	0.262	1.007	0.259
C-LLSM	2.909	2.795	0.359	0.262	0	0.896	0.142
D-LLSM	3.150	2.788	0.672	1.007	0.896	0	0.913
C-EM	2.923	2.806	0.383	0.259	0.142	0.913	0

**Remark 3**  $\tau$  has a maximal value of  $\sqrt{\sum_{i=1}^n [\log(i) - \log(n+1-i)]^2}$ , here  $\tau_{\max}^{149} \approx 21.12$ . It makes possible to normalize it, however, we have rankings with the same length, so it seems to be unnecessary. The theoretical maximum reflects the proximity of different rankings with respect to the contracting effect of logarithm dominant for high  $\max\{X_i/Y_i; Y_i/X_i\}$  ratios.

The relation of different rankings could be examined by other statistical tools, as well. The position of teams in Final and A-LLSM rankings are drawn on Figure 2 with linear regression analysis (the coefficient of explanatory variable  $x$  is equal to Spearman's correlation coefficient between Final and A-LLSM rankings). Due to the similarity of the proposed rankings, the substitution of A-LLSM with another rankings calculated from the incomplete pairwise comparison matrix results in a similar chart. There is a remarkable tendency: teams with lower TB1 (match points), but higher TB2 (Sonneborn-Berger points) tend to achieve better positions than teams with the opponent performance benefit from the official lexicographical order.<sup>8</sup> It derives from lack of continuity of the lexicographical order.<sup>9</sup> For example, Zambia is officially the 47th, but

<sup>7</sup>The maximum of these ratios is 3.875 for Bulgaria, which is 8th in Start, but 31th in Final and D-LLSM. Start and Final are the farthest rankings, but in this relation only 7 are above 2 (among them the one for Bulgaria), while other 5 equals to 2. In the  $[1, 2]$  interval the linear approximation of the logarithmic function is acceptable.

<sup>8</sup>See footnote 3.

<sup>9</sup>It could be proved by simple intuition: let  $a$  an alternative with values  $(a_1, a_2, a_3, \dots, a_n)$  according to the  $n$  criteria and choose a sequence of alternatives, where  $b_k$  has ratings  $(a_1 - 1/k, a_2 + 1, a_3 + 1, \dots, a_n + 1)$ . Let the limit of the sequence  $b$ , that is an alternative with values  $(a_1, a_2 + 1, a_3 + 1, \dots, a_n + 1)$ . Then  $a$  is better than  $b_k$  for all  $k$  according to the lexicographical order (if it prefers higher values), but  $b$  is better than  $a$ . It contradicts to continuity, as the set of alternatives worse than  $a$  is not closed.

at most 89th in LLSM and EM rankings, while Georgia stands at the 30th position in Final and is at least the 16th according to the proposed solutions. Other examples are given in Table 5.

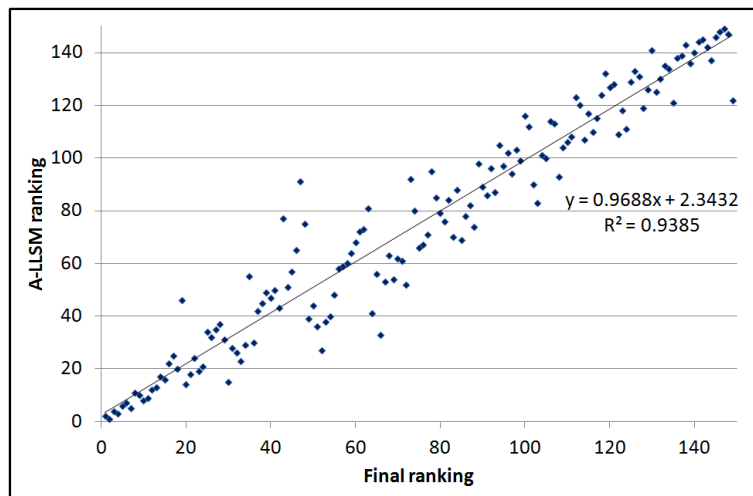


Figure 2: Relation of Final and A-LLSM rankings

Another important issue is the allocation of first places. In this respect, the results are robust as the first four positions are occupied by Ukraine, Russia 1, Hungary and Israel in Final and all of the proposed rankings.

## 5 Some characteristics of the example

Despite the similar rankings derived from LLSM vectors, the priority vectors are different for the 4 variants; Table 4 shows some details. The means of optimal weights are equal due to the normalization, while standard deviations confirm intuition: it is the largest in B variant which uses the widest scale and the smallest in 'narrow' C variant. For B and D variants the weight of the winner is similar, but its superiority (Max / Min ratio) to the last team is more than double for B. However, a factor of 250 or 500 refers to significant differences among teams.

Table 4: Features of LLSM vectors

	A-LLSM	B-LLSM	C-LLSM	D-LLSM
Maximum	0.0336	0.0422	0.0210	0.0417
Minimum	0.0002	0.0001	0.0007	0.0002
Max / Min	158.62	576.08	28.02	249.83
Mean	0.0067	0.0067	0.0067	0.0067
Standard deviation	0.0068	0.0082	0.0045	0.0078
Average win's ratio	3.2650	4.5300	2.1325	3.7291
Power	4.2818	4.2074	4.4012	4.1946

The ratio between the maximal and minimal weights has remarkable implications highlighted

in the last 2 rows of Table 4. Average win's ratio corresponds to the mean of wins in each coding:

$$\sum_{i=2,5,3,3,5,4} \frac{\text{Number of matches where winner's game points is } i * \text{Ratio corresponding to result } i}{\text{Number of matches without draws}}$$

For example, it is  $(220 * 2 + 194 * 4 + 137 * 6 + 132 * 8) / 683$  for B variant. Finally, power equals to  $\log(\text{Max} / \text{Min}) / \log(\text{Average win's ratio})$ , which reflects a kind of 'order of magnitude' in the tournament. For example, in A variant the factor between the first and the last team is 150-fold, while a standard victory in a match corresponds to a pairwise comparison ratio of 3.2. As  $3.2^4 \approx 150$ , a 'virtual' chain of 5 teams should exist, where each team defeated the following with this average difference. However, if one of the first and last teams played against each other, the ratio as a known element in the incomplete pairwise comparison matrix is at most 5 according to a 4:0 win in A variant. It implies participants should be put into 4-5 groups. Figure 3 presents the weights derived from LLSM vectors.

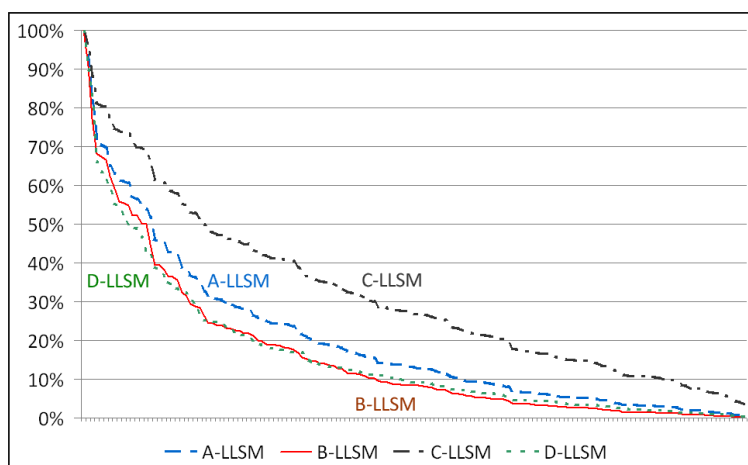


Figure 3: Weights of LLSM vectors relative to the first

Due to the pairing method, the classification of teams partly happens. The structure of known elements in the pairwise comparison matrix shows an impressive picture; Figure 4 indicates them with a filled box covering approximately 7% of the whole area, which is the ratio of known elements relative to all possible matchings. On the two charts matrix rows (teams) are ordered according to Final (a, left) and A-LLSM (b, right) rankings, respectively. Known elements representing matches knit around the diagonal, this effect is more stronger in the second case. It is affirmed by some calculations: the average difference between the rank of two teams played against each other is 28.70 in Final and 25.32 in A-LLSM, while the median is 22.5 and 19. Thus matches were taken place between contestants with a similar performance, the matching algorithm operates not randomly. Despite the fact that the pairing method was not examined, the above numbers suggest that the proposed pairing based on pairwise comparisons is better regarding the classification of teams – if the number of matches is limited, they should be played between participants whose internal ranking is difficult to decide.

Amid these circumstances, it is favourable that the number of rounds in the tournament remains small compared to the number of participants, as it makes possible the appearance of significant ratios not permitted by the transformation of match results into pairwise comparison ratios. It raises the question whether all countries should participate in the same tournament which is not the case in almost all team sports like soccer or ice hockey.

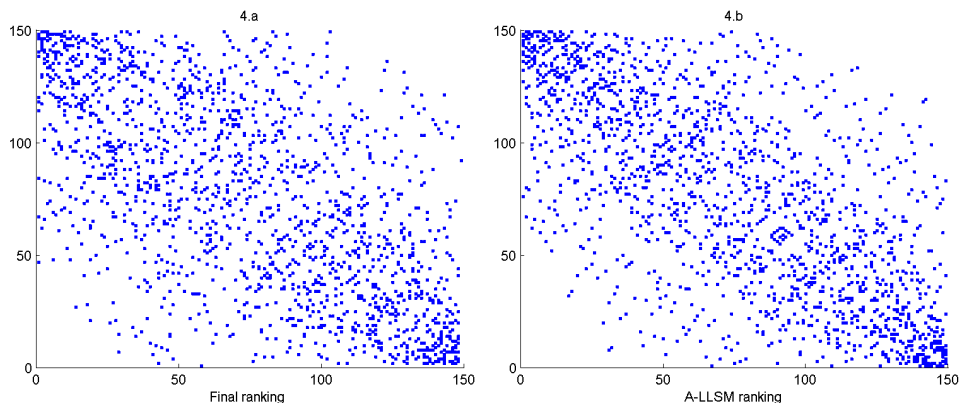


Figure 4: Map of known elements in the pairwise comparison matrix, teams sorted by positions according to Final (a, left) and A-LLSM (b, right) rankings

## 6 Visualization

Rankings based on pairwise comparison matrices have revealed some interesting facts about the performance of certain teams, nonetheless, the long lists do not give an impressive picture about their relation. The  $\tau$  distance seems to be a good starting point, as it makes possible to plot  $n$  rankings in a  $n - 1$  dimensional space. In addition, it is possible to decrease number of dimensions in order to give a better overview if it goes without significant loss of information. Multidimensional Scaling (MDS) is a statistical method in information visualization for exploring similarities or dissimilarities in data; a basic book in this field is [11]. A classical application of MDS is to draw cities on a map from the matrix consisting of their air distance.

For sufficiently small  $n$ , it requires only 1 or 2 dimensions to visualize all points representing different rankings. The distance matrix of the 7 rankings is still calculated in Section 4, but the applied software (SPSS v18.0) requires at least 10 cases to be evaluated. Thus, regarding former observations, 3 further rankings (Sonneborn-Berger, Buchholz, Mix) were defined on the basis of

different components of the original lexicographical order.<sup>10</sup>

MDS maps the original distances in interval, ratio or ordinal scales; the most general interval scale were applied. Here  $\delta$  discrepancies on the reduced-dimension map are related to the original  $d$  distances by a linear function:  $\delta = a + bd$ , which means it is indifferent to multiplying the distances by a constant factor caused by choosing a different base of logarithm for  $\tau$ . The goodness of mapping (the information loss derives from dimension reduction) is measured by Kruskal's Stress and RSQ.

The pairwise  $\tau$  distances 10 rankings could be mapped appropriately in two dimensions, but one dimension is not enough. The coordinates on the MDS map has no further meaning, distances on the map reflects the original distances as well as possible, while the mean of  $x$  and  $y$  coordinates is 0. Kruskal's Stress is 0.1431, a middle-strength relation. RSQ reaches 0.9468, approximately 94.5% of variance of the scaled data can be accounted for by the MDS procedure. The reduced map is plotted in Figure 5.

Start and Final rankings are the farthest from each other, the 5 rankings derived from the incomplete pairwise comparison matrices are in a small cluster approximately at the same distance from Start and Final. The proposed rankings are almost the same, only D-LLSM differs from the others marginally (it gives a great weight to all wins, but hardly rewards its size). The 3 newly defined rankings (Sonneborn-Berger, Buchholz and Mix) are nearer to the Final than to the Start which is reassuring, while Buchholz and Mix are still nearer to the proposed rankings. It backs the former observation that methods based on pairwise comparison matrices overweighs TB4 (Buchholz points) to TB2 (Sonneborn-Berger points), while the official lexicographical order prefers teams with higher TB2 (Sonneborn-Berger points).<sup>11</sup> It is not by chance, as TB4 (Buchholz points) rather reflects the performance of opponents, the key idea beyond pairwise comparison matrices.

## 7 Conclusion

This paper presents an alternative method to determine the final ranking of teams participated in the 39th Chess Olympiad 2010 Open tournament based on incomplete pairwise comparison matrices. In lack of former experience, 4 arbitrary scales were used to transform the match

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<sup>10</sup>See footnote 3.

- Sonneborn-Berger: a lexicographical order based on the Sonneborn-Berger points (TB2) of teams, then match points (TB1) and game points (TB3). It still gives a complete order.
- Buchholz: another lexicographical order firstly based on the product of Buchholz points (TB4) and match points (TB1) divided by the number of matches. The idea is that the Buchholz points reflect only the force of opponents, so it should be modified. It is pointless to multiply it with game points (TB3) as it gives almost the Sonneborn-Berger points (TB2) and teams are interested in the increase of their match points (TB1). The subsequent tie-breaking rule is number of match points (TB1), then Sonneborn-Berger points (TB2).
- Mix: a composite index based on Sonneborn-Berger (TB2) and Buchholz points (TB4). In order to measure them in a similar scale, Buchholz points are multiplied by a correcting factor:

$$F = \frac{3 * \text{Number of wins} + 2 * \text{Number of draws} + 1 * \text{Number of losses}}{\text{Number of matches}}$$

Then Sonneborn-Berger (with an average of 203) and modified Buchholz points (average 232) are added. The unique tie-break between Scotland and Faroe Islands is decided for the latter because of higher Sonneborn-Berger points.

The 3 rankings partly moderate the oddities of the final ranking. For example, the officially 47th Zambia is 72nd, 75th and 79th in the Sonneborn-Berger, Buchholz and Mix rankings, while the 64th Germany is 37th, 60th and 45th, respectively.

<sup>11</sup>See footnote 3.

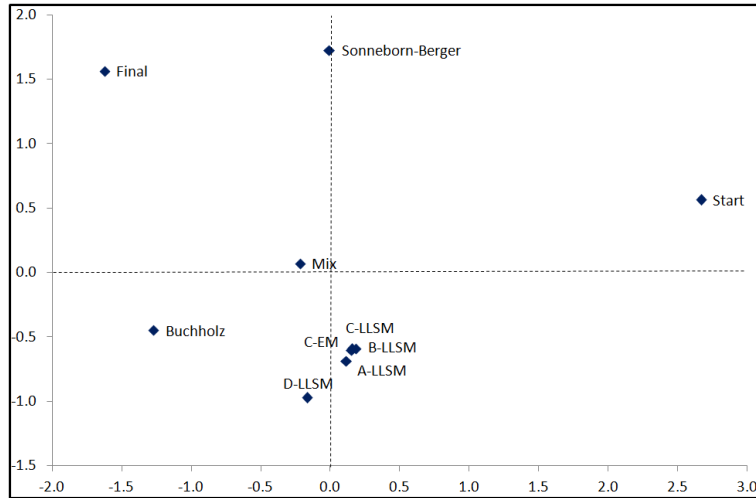


Figure 5: Rankings on a two-dimensional map by  $\tau$  distances

results into ratios. Two well-known techniques are used, the Eigenvector Method (EM) and the Logarithmic Least Squares Method (LLSM). The decision between them is far from trivial, while other methods deserve testing, like the one given by Fedrizzi and Giove in [6].

Results show that in some cases the incompleteness of the pairwise comparison matrix has favourable effects if it has a special structure, namely, if ordering on the priority vector derived from the matrix, known elements are located near the diagonal with high probability: some ratios given by the optimal completion can significantly exceed the original scale.

The chosen variant for transforming match results into ratios affects the absolute priorities significantly, but final rankings served by weights are relatively robust. It was confirmed by the application of a distance based on the asymmetry of committed mistakes in rankings, however, it was strongly influenced by some particular teams. The construction of better indices is a topic of future research.

Matches with the same result were represented by the same ratios. It means some simplification as a 4:0 win in the final round against a strong opponent seems to be more valuable than a similar result in the first rounds against underdogs. However, the official ranking does not take into account the opponent's strength in awarding victories, and a priori it is not reasonable to transform these outcomes differently. Nevertheless, further investigations are needed how this aspect could be incorporated into the pairwise comparison method.

Alternative proposals revealed some oddities of the current FIDE olympiad tie-breaking rules. As there is no commonly accepted ranking method in chess, other techniques are worth examining. Visualization implies that certain lexicographical orders, despite the lack of continuity, approximates the results of the proposed method well. While it is really impossible to find a perfect final ranking for a similarly complex Swiss-system tournament, it was justified that in some cases the use of pairwise comparison matrices give robust and intuitively better results than currently used ranking techniques. This approach can be extended to other sport championships.

## Acknowledgements

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## Appendix

Table 5: Positions of teams according to different rankings

<b>Team</b>	<b>Start</b>	<b>Final</b>	<b>A-LLSM</b>	<b>B-LLSM</b>	<b>C-LLSM</b>	<b>D-LLSM</b>
Ukraine	2	1	2	2	2	2
Russia 1	1	2	1	1	1	1
Israel	11	3	4	4	4	4
Hungary	5	4	3	3	3	3
China	3	5	6	5	5	7
Russia 2	4	6	7	7	7	9
Armenia	6	7	5	6	6	5
Spain	16	8	11	9	10	10
United States of America	9	9	10	11	11	8
France	10	10	8	10	9	6
Poland	15	11	9	8	8	11
Azerbaijan	7	12	12	12	12	13
Russia 3	14	13	13	14	13	12
Belarus	35	14	17	17	17	17
Netherlands	13	15	16	16	16	14
Slovakia	22	16	22	22	22	22
Brazil	24	17	25	26	25	24
India	19	18	20	21	20	20
Denmark	44	19	46	47	47	42
Czech Republic	17	20	14	13	14	15
Italy	30	21	18	19	18	18
Greece	25	22	24	23	24	25
Cuba	18	23	19	18	19	19
England	12	24	21	20	21	21
Argentina	26	25	34	34	34	33
Estonia	48	26	32	31	32	32
Kazakhstan	41	27	35	36	36	35
Moldova	31	28	37	38	37	36
Iran	38	29	31	32	31	30
Georgia	20	30	15	15	15	16
Bulgaria	8	31	28	28	28	31
Croatia	28	32	26	25	26	26
Serbia	21	33	23	24	23	23
Sweden	34	34	29	30	30	28
Lithuania	39	35	55	55	55	58
Slovenia	29	36	30	29	29	29
Canada	53	37	42	42	42	41
Austria	45	38	45	43	46	48
Russia 4	52	39	49	49	49	51
Iceland	54	40	47	46	45	44
Egypt	40	41	50	50	50	50
Montenegro	56	42	43	44	43	43
Qatar	55	43	77	77	76	79
Peru	46	44	51	52	51	49
Turkey	50	45	57	58	58	54
Uruguay	74	46	65	66	65	65
Zambia	121	47	91	89	91	92
ICSC	72	48	75	76	75	73
Uzbekistan	33	49	39	39	39	40
Philippines	37	50	44	45	44	46
Norway	23	51	36	35	35	37
Vietnam	27	52	27	27	27	27

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Table 5 – continued from previous page

<b>Team</b>	<b>Start</b>	<b>Final</b>	<b>A-LLSM</b>	<b>B-LLSM</b>	<b>C-LLSM</b>	<b>D-LLSM</b>
Chile	51	53	38	37	38	38
Colombia	57	54	40	40	40	39
Australia	49	55	48	48	48	47
Former YUG Rep of Macedonia	43	56	58	57	57	61
Albania	68	57	59	59	59	57
Singapore	73	58	60	60	60	55
Finland	60	59	64	64	64	63
Belgium	71	60	68	69	69	70
United Arab Emirates	88	61	72	70	71	72
Pakistan	123	62	73	74	74	71
IPCA	70	63	81	81	81	80
Germany	42	64	41	41	41	45
Switzerland	47	65	56	54	54	62
Bosnia & Herzegovina	32	66	33	33	33	34
Indonesia	67	67	53	53	53	56
Kyrgyzstan	77	68	63	61	63	64
Latvia	58	69	54	56	56	53
Russia 5	61	70	62	63	62	60
Mongolia	66	71	61	62	61	59
Mexico	36	72	52	51	52	52
Bangladesh	82	73	92	92	92	89
South Africa	81	74	80	80	80	78
Portugal	59	75	66	65	66	66
Turkmenistan	69	76	67	68	68	69
Jordan	79	77	71	73	72	67
Libya	105	78	95	95	94	95
Paraguay	84	79	85	85	85	84
Faroe Islands	83	80	79	79	79	75
Venezuela	64	81	76	78	78	74
Costa Rica	80	82	84	84	84	83
Scotland	63	83	70	72	70	68
Yemen	85	84	88	90	89	85
Ecuador	65	85	69	67	67	76
Tajikistan	62	86	78	75	77	81
Andorra	89	87	82	82	82	82
Ireland	75	88	74	71	73	77
Algeria	91	89	98	97	98	99
Dominican Republic	87	90	89	87	87	93
New Zealand	92	91	86	86	86	87
Malaysia	86	92	96	96	96	96
Thailand	94	93	87	88	88	91
Panama	107	94	105	105	104	101
Barbados	96	95	97	98	97	97
Japan	101	96	102	102	102	103
Luxembourg	90	97	94	94	95	94
Cyprus	109	98	103	103	103	100
Guatemala	103	99	99	100	99	98
Malta	106	100	116	116	116	114
Nigeria	139	101	112	114	113	109
IBCA	78	102	90	91	90	88
Iraq	76	103	83	83	83	86
Sri Lanka	115	104	101	101	101	104
Jamaica	97	105	100	99	100	107
Uganda	127	106	114	115	114	113
Nepal	114	107	113	110	112	116
Puerto Rico	100	108	93	93	93	90
Lebanon	99	109	104	104	105	102
Monaco	95	110	106	106	106	106
Honduras	125	111	108	108	108	112

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Table 5 – continued from previous page

<b>Team</b>	<b>Start</b>	<b>Final</b>	<b>A-LLSM</b>	<b>B-LLSM</b>	<b>C-LLSM</b>	<b>D-LLSM</b>
Palestine	134	112	123	123	122	124
Korea	116	113	120	120	120	120
Bolivia	104	114	107	107	107	105
Trinidad & Tobago	108	115	117	117	117	115
Botswana	102	116	110	112	110	110
Bahrain	112	117	115	113	115	117
Mauritius	110	118	124	122	123	126
Chinese Taipei	131	119	132	132	132	132
Kenya	133	120	127	126	126	127
Aruba	120	121	128	128	128	130
Wales	93	122	109	111	111	108
Jersey	113	123	118	118	118	111
Angola	98	124	111	109	109	118
Mali	145	125	129	129	129	128
Namibia	129	126	133	133	133	134
Malawi	141	127	131	131	131	131
Ethiopia	132	128	119	119	119	119
Hongkong	124	129	126	127	127	123
Guernsey	128	130	141	142	142	141
Mauritania	146	131	125	124	124	125
Surinam	111	132	130	130	130	129
Macau	122	133	135	135	135	136
Mozambique	130	134	134	134	134	133
Madagascar	140	135	121	121	121	122
Netherlands Antilles	118	136	138	139	139	138
Cameroon	144	137	139	137	137	139
Sao Tome and Principe	138	138	143	143	143	142
Haiti	135	139	136	136	136	137
Ghana	137	140	140	140	140	140
Bermuda	126	141	144	144	144	144
Sierra Leone	148	142	145	145	145	145
Papua New Guinea	117	143	142	141	141	143
San Marino	119	144	137	138	138	135
Burundi	143	145	146	146	146	146
Rwanda	142	146	148	148	148	148
U.S. Virgin Islands	149	147	149	149	149	149
Seychelles	136	148	147	147	147	147
Senegal	147	149	122	125	125	121