

TRIMMED LEAST SQUARE ESTIMATORS FOR STABLE AR(1) PROCESSES

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ABSTRACT

We prove the weak consistency of the trimmed least square estimator of the covariance parameter of an AR(1) process with stable errors.

KEYWORDS

autoregressive processes, least square estimator, consistency, trimming, stable errors

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 62F12; Secondary 62G30

1. INTRODUCTION

Trimming is a standard method to decrease the effect of large sample elements in statistical procedures. For example, by a result of Csörgő, Horváth and Mason [7], partial sums of moderately trimmed i.i.d. random variables in the domain of attraction of a stable law are asymptotically normal. This fact that was used in Berkes, Horváth and Schauer [3] to construct CUSUM tests for the change of location parameters of i.i.d. stable sequences. For an extension for AR(1) processes, see Bazarova, Berkes and Horváth [1]. The purpose of the present paper is to investigate the asymptotic behavior of the trimmed least square estimator of AR(1) processes.

Let $X_i = \rho X_{i-1} + \varepsilon_i$, $-\infty < i < \infty$ be an AR(1) process, where ρ is the autoregressive parameter of the process and the innovations ε_i are i.i.d. random variables. Assume that

$$\lim_{t \rightarrow \infty} \frac{P\{\varepsilon_0 > t\}}{L(t)t^{-\alpha}} = u, \quad \lim_{t \rightarrow \infty} \frac{P\{\varepsilon_0 < -t\}}{L(t)t^{-\alpha}} = v, \quad (1.1)$$

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for some numbers $u \geq 0$, $v \geq 0$, $u + v = 1$, where L is slowly varying at infinity. Assume also that ε_0 has a density h satisfying

$$\int_{-\infty}^{\infty} |h(t+s) - h(t)| dt \leq C|s| \quad (1.2)$$

with some $C > 0$. Let $d = d(n) \geq n^\gamma$ for some $\gamma > 0$, $d(n)/n \rightarrow 0$ and let $\eta_{d,n}$ denote the d^{th} largest among $|X_1|, |X_2|, \dots, |X_n|$. We will prove

THEOREM. We have, as $n \rightarrow \infty$

$$\frac{\sum_{k=1}^n X_k I\{|X_k| \leq \eta_{d,n}\} X_{k+1} I\{|X_{k+1}| \leq \eta_{d,n}\}}{\sum_{k=1}^n X_k^2 I\{|X_k| \leq \eta_{d,n}\}} \rightarrow \rho \quad (1.3)$$

in probability.

In other words, the least square estimator (1.3) of the covariance parameter of the process $\{X_n, n \in \mathbb{Z}\}$ is weakly consistent. In analogy with the results of [1] it is natural to expect that the LS estimator is strongly consistent and satisfies the central limit theorem, but this is a considerably harder problem we will investigate elsewhere.

2. PROOF OF THE THEOREM

We will prove the theorem in two steps. First we show that the analogue of (1.3) holds if we replace the random truncation level $\eta_{d,n}$ in (1.3) by the nonrandom level $tH^{-1}(d/n)$, where $H(x) = P\{|X_0| > x\}$ denotes the survival function of X_0 . Then we return to the original truncation level $\eta_{d,n}$ (i.e. trimming) by a tightness argument.

Nonrandom truncation

According to Cline [6], we have

$$H(x) = x^{-\alpha} L_*(x), \quad (2.1)$$

where L_* is slowly varying at ∞ and

$$\lim_{x \rightarrow \infty} \frac{H(x)}{P\{|\varepsilon_0| > x\}} = \lim_{x \rightarrow \infty} \frac{L_*(x)}{L(x)} = \frac{1}{1 - |\rho|^\alpha}. \quad (2.2)$$

LEMMA 1. For any $t > 0$ we have

$$\frac{\sum_{k=1}^n X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\}}{nEX_0^2 I\{|X_0| \leq tH^{-1}(d/n)\}} \rightarrow 1 \quad \text{in probability.}$$

Proof. Let

$$T_k = X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\}, \quad 1 \leq k \leq n.$$

Using (1.2) and Withers [9] we get that X_k and thus also T_k is a strongly mixing stationary sequence with mixing rate $\alpha(k) \leq C_1 \exp(-\lambda k)$ for some $C_1 > 0$ and $\lambda > 0$. Integration by parts (similarly as in (2.4) below) shows that

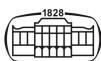
$$E|T_k|^p \sim C(d/n)H^{-1}(d/n)^{2p} \quad p \geq 2$$

for some constant C that depends also on t . (Since t is a constant in the whole proof, dependence on t presents no difficulty.) By a correlation inequality in Davydov [8] we get for any $p > 2$ that

$$\begin{aligned} |ET_0 T_k - ET_0 ET_k| &\leq (\alpha(k))^{(p-1)/p} (ET_0^p)^{1/p} (ET_k^p)^{1/p} \\ &\leq C_1 \exp(-\lambda k(p-1)/p) (ET_0^p)^{2/p}. \end{aligned}$$

Hence setting $\tilde{T}_k = T_k - ET_k$ we conclude that

$$\begin{aligned} E \left(\sum_{k=1}^n \tilde{T}_k \right)^2 &= nE\tilde{T}_0^2 + 2 \sum_{k=1}^{n-1} (n-k) E\tilde{T}_0 \tilde{T}_k \\ &\leq n \left(E\tilde{T}_0^2 + 2 \sum_{k=1}^{n-1} |E\tilde{T}_0 \tilde{T}_k| \right) \end{aligned}$$



$$\begin{aligned} &\leq n \left(ET_0^2 + C_3 \sum_{k=1}^{n-1} \exp(-\lambda k(p-1)/p) (ET_0^p)^{2/p} \right) \\ &\leq n (ET_0^2 + C_5 (ET_0^p)^{2/p}). \end{aligned}$$

Thus Markov's inequality and simple calculations yield

$$P \left\{ \sum_{k=1}^n \tilde{T}_k \geq nE|T_0| \right\} \rightarrow 0,$$

using $d \geq n^\gamma$ and choosing p near 2. This proves Lemma 1. \square

LEMMA 2. We have, as $n \rightarrow \infty$

$$\frac{\sum_{k=1}^n X_k I\{|X_k| \leq tH^{-1}(d/n)\} X_{k+1} I\{|X_{k+1}| \leq tH^{-1}(d/n)\}}{\sum_{k=1}^n X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\}} \rightarrow \rho \quad (2.3)$$

in probability.

Proof. Let

$$\begin{aligned} A_n(t) &:= \sum_{k=1}^n X_k I\{|X_k| \leq tH^{-1}(d/n)\} X_{k+1} I\{|X_{k+1}| \leq tH^{-1}(d/n)\} \\ B_n(t) &:= \sum_{k=1}^n X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} \end{aligned}$$

We have

$$\begin{aligned} &|A_n(t) - \rho B_n(t)| \\ &= \left| \sum_{k=1}^n X_k (\rho X_k + \varepsilon_{k+1}) I\{|X_k| \leq tH^{-1}(d/n), |\rho X_k + \varepsilon_{k+1}| \leq tH^{-1}(d/n)\} \right. \\ &\quad \left. - \rho \sum_{k=1}^n X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} \right| \\ &= \left| - \sum_{k=1}^n \rho X_k^2 I\{|X_k| \leq tH^{-1}(d/n), |\rho X_k + \varepsilon_{k+1}| > tH^{-1}(d/n)\} \right. \\ &\quad \left. + \sum_{k=1}^n X_k \varepsilon_{k+1} I\{|X_k| \leq tH^{-1}(d/n), |\rho X_k + \varepsilon_{k+1}| \leq tH^{-1}(d/n)\} \right| \\ &\leq \sum_{k=1}^n \rho X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} (I\{|\rho X_k| > \rho tH^{-1}(d/n)\} + I\{|\varepsilon_{k+1}| > (1-\rho)tH^{-1}(d/n)\}) \\ &\quad + \sum_{k=1}^n |X_k| |\varepsilon_{k+1}| I\{|X_k| \leq tH^{-1}(d/n)\} I\{|\varepsilon_{k+1}| \leq (\rho+1)tH^{-1}(d/n)\} \\ &=: Z_1 + Z_2. \end{aligned}$$

Recall that H is regularly varying with exponent α . Thus using integration by parts and well known properties of regularly varying functions (see e.g. [5]) we get

$$\begin{aligned} E|X_0| I\{|X_0| \leq tH^{-1}(d/n)\} &= \\ &= xP(|X_0| > x) \Big|_0^{tH^{-1}(d/n)} + \int_0^{tH^{-1}(d/n)} P(|X_0| > x) dx \sim C_0(d/n)H^{-1}(d/n), \end{aligned} \quad (2.4)$$

where C_0 is some constant. Similarly,

$$E|\varepsilon_0| I\{|\varepsilon_0| \leq tH^{-1}(d/n)\} \sim C_1(d/n)H_*^{-1}(d/n), \quad (2.5)$$



where H_*^{-1} is the generalized inverse of the distribution function of ε_0 . Another integration by parts yields

$$EX_0^2 I\{|X_0| \leq tH^{-1}(d/n)\} \sim C'_0(d/n)(H^{-1}(d/n))^2. \quad (2.6)$$

Thus using the independence of X_k and ε_{k+1} we get

$$\begin{aligned} EZ_1 &= \rho \sum_{k=1}^n EX_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} I\{|X_k| > tH^{-1}(d/n)\} + I\{|\varepsilon_{k+1}| > (1-\rho)tH^{-1}(d/n)\} \\ &= \rho \sum_{k=1}^n EX_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} I\{|\varepsilon_{k+1}| > (1-\rho)tH^{-1}(d/n)\} \\ &= \rho \sum_{k=1}^n EX_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} P\{|\varepsilon_{k+1}| > (1-\rho)tH^{-1}(d/n)\} \\ &= \rho n EX_1^2 I\{|X_1| \leq tH^{-1}(d/n)\} \cdot P\{|\varepsilon_1| > (1-\rho)tH^{-1}(d/n)\} \\ &= n EX_1^2 I\{|X_1| \leq tH^{-1}(d/n)\} o(1). \end{aligned}$$

On the other hand, using (2.5), (2.6) and again the independence of X_k and ε_{k+1} we get

$$\begin{aligned} EZ_2 &= \sum_{k=1}^n E|X_k| |\varepsilon_{k+1}| I\{|X_k| \leq tH^{-1}(d/n)\} I\{|\varepsilon_{k+1}| \leq (\rho+1)tH^{-1}(d/n)\} \\ &= \sum_{k=1}^n E|X_0| I\{|X_0| \leq tH^{-1}(d/n)\} E|\varepsilon_0| I\{|\varepsilon_0| \leq (\rho+1)tH^{-1}(d/n)\} \\ &= n E|X_0| I\{|X_0| \leq tH^{-1}(d/n)\} E|\varepsilon_0| I\{|\varepsilon_0| \leq (\rho+1)tH^{-1}(d/n)\} \\ &\leq Cn(d/n)^2 H^{-1}(d/n) H_*^{-1}(d/n). \end{aligned}$$

Since $L(x)/L_*(x)$ tends to a constant as $x \rightarrow \infty$, it is easy to show that $H_*^{-1}(d/n)/H^{-1}(d/n)$ also converges to a constant as $n \rightarrow \infty$. Using these relations and the Markov inequality we get

$$\frac{A_n(t) - \rho B_n(t)}{n EX_0^2 I\{|X_0| \leq tH^{-1}(d/n)\}} \xrightarrow{P} 0. \quad (2.7)$$

By Lemma 1 we have

$$\frac{B_n(t)}{n EX_0^2 I\{|X_0| \leq tH^{-1}(d/n)\}} \xrightarrow{P} 1 \quad (2.8)$$

and Lemma 2 follows. \square

Tightness

The following lemma is a special case of a tightness condition of Davydov [8].

LEMMA 3. Let $\{\zeta_i(s), 0 \leq s \leq 1, i \geq 1\}$ be non-decreasing processes in $D[0, 1]$, let $\zeta(s), 0 \leq s \leq 1$ be a non-decreasing function and define

$$K_n(s) = \frac{1}{B_n} \sum_{1 \leq i \leq n} (\zeta_i(s) - \zeta(s)),$$

where $B_n \rightarrow \infty$. If the finite dimensional distributions of $K_n(s)$ converge, as $n \rightarrow \infty$, to those of a Gaussian process and there exist $\tau > \mu > 1$ and a sequence (a_n) such that

$$E|K_n(s_2) - K_n(s_1)|^\tau \leq C_6 |s_2 - s_1|^\mu \quad \text{if } |s_2 - s_1| \geq a_n$$

and

$$\frac{n}{B_n} \sup_{|s_2 - s_1| \leq a_n} |\zeta(s_2) - \zeta(s_1)| \rightarrow 0$$

then the sequence $\{K_n(s), 0 \leq s \leq 1, n \geq 1\}$ is tight.



Let now

$$m_n(t) = EX_0 X_1 I\{|X_0| \leq tH^{-1}(d/n)\} I\{|X_1| \leq tH^{-1}(d/n)\}$$

and

$$b_n = EX_0^2 I\{|X_0| \leq tH^{-1}(d/n)\}. \quad (2.9)$$

We prove the tightness of the sequence

$$L_n(t) = \frac{1}{nb_n} \sum_{i=1}^n (X_i X_{i+1} I\{|X_i| \leq tH^{-1}(d/n)\} I\{|X_{i+1}| \leq tH^{-1}(d/n)\} - m_n(t)) \quad (n \geq 1)$$

of processes in $D[0, 1]$. As a first step, we replace $X_i = \sum_{k=0}^{\infty} \rho^k \varepsilon_{i-k}$ by

$$Y_i = \sum_{k=0}^{\lfloor K \log n \rfloor} \rho^k \varepsilon_{i-k},$$

where K is a large constant. Using the exponential decay of the coefficients in the series for X_i , simple calculations show that it suffices to prove the tightness of

$$\hat{L}_n(t) = \frac{1}{nb_n} \sum_{i=1}^n (Y_i Y_{i+1} I\{|Y_i| \leq tH^{-1}(d/n)\} I\{|Y_{i+1}| \leq tH^{-1}(d/n)\} - m_n(t)).$$

To prove the latter, we use Lemma 3 above and to guarantee the monotonicity assumptions there, we split the variables $Y_i Y_{i+1}$ as

$$Y_i Y_{i+1} = Z_i + Z_i^* \quad \text{where } Z_i = \max(Y_i Y_{i+1}, 0), \quad Z_i^* = \min(Y_i Y_{i+1}, 0).$$

Put

$$\begin{aligned} m_{n,1}(t) &= EZ_0 I\{|Y_0| \leq tH^{-1}(d/n)\} I\{|Y_1| \leq tH^{-1}(d/n)\}, \\ m_{n,2}(t) &= EZ_0^* I\{|Y_0| \leq tH^{-1}(d/n)\} I\{|Y_1| \leq tH^{-1}(d/n)\}, \end{aligned}$$

and

$$\hat{L}_{n,1}(t) = \frac{1}{nb_n} \sum_{i=1}^n (Z_i I\{|Y_i| \leq tH^{-1}(d/n)\} I\{|Y_{i+1}| \leq tH^{-1}(d/n)\} - m_{n,1}(t)),$$

and define $\hat{L}_{n,2}$ analogously; clearly $\hat{L}_n(t) = \hat{L}_{n,1}(t) + \hat{L}_{n,2}(t)$. Let further

$$g_n = \frac{1}{\log \log n}.$$

The following lemma establishes the properties needed for the application of Lemma 3.

LEMMA 4. $\hat{L}_{n,1}(t)$ and $m_{n,1}(t)$ satisfy

$$m_{n,1}(t) \text{ is nondecreasing on } [1/2, 3/2], \quad (2.10)$$

$$\frac{1}{b_n} \sup_{|t_2 - t_1| \leq g_n} |m_{n,1}(t_2) - m_{n,1}(t_1)| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.11)$$

$$E|\hat{L}_{n,1}(t_2) - \hat{L}_{n,1}(t_1)|^6 \leq C_1 |t_2 - t_1|^\tau, \text{ if } |t_2 - t_1| \geq g_n, \quad (2.12)$$

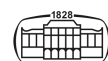
where $\tau > 2$ and C_1 is a constant.

Proof. The definition of $m_{n,1}(t)$ and $Z_0 \geq 0$ imply immediately (2.10). By the definition of $m_{n,1}(t)$ we get for all $1/2 \leq t_1 < t_2 \leq 3/2$, setting $v_1 = t_1 H^{-1}(d/n)$, $v_2 = t_2 H^{-1}(d/n)$,

$$\begin{aligned} 0 &\leq m_{n,1}(t_2) - m_{n,1}(t_1) \\ &= EZ_0 I\{|Y_0| \leq v_2\} I\{|Y_1| \leq v_2\} - EZ_0 I\{|Y_0| \leq v_1\} I\{|Y_1| \leq v_1\} \\ &= EZ_0 I\{v_1 < |Y_0| \leq v_2\} I\{|Y_1| \leq v_2\} + EZ_0 I\{|Y_0| \leq v_1\} I\{v_1 < |Y_1| \leq v_2\} \\ &=: W_1 + W_2. \end{aligned} \quad (2.13)$$

Clearly $|Z_0| \leq |Y_0||Y_1|$ and thus by the Cauchy-Schwarz inequality we get

$$|W_1| \leq E|Y_0||Y_1| I\{v_1 < |Y_0| \leq v_2\} I\{|Y_1| \leq v_2\}$$



$$\begin{aligned} &\leq (EY_0^2 I\{v_1 < |Y_0| \leq v_2\})^{1/2} (EY_1^2 I\{|Y_1| \leq v_2\})^{1/2} \\ &\leq C'(d/n)H^{-1}(d/n)^2(t_2 - t_1)^{1/2} \end{aligned}$$

where in the last step we used integration by parts similar to (2.4). A similar estimate holds for W_2 , establishing (2.11) in view of (2.6), (2.9). To verify (2.12), we note that

$$\hat{L}_{n,1}(t_2) - \hat{L}_{n,1}(t_1) = \frac{1}{nb_n} \sum_{i=1}^n \xi_i,$$

where $\xi_i = \eta_i - E\eta_i$ with

$$\begin{aligned} \eta_i &= \eta_i(t_1, t_2) \\ &= Z_i(I\{|Y_i| \leq v_2\}I\{v_1 < |Y_{i+1}| \leq v_2\} + I\{|Y_{i+1}| \leq v_2\}I\{v_1 < |Y_i| \leq v_2\}) \end{aligned} \quad (2.14)$$

by using a decomposition as in (2.13). Thus

$$E(\hat{L}_{n,1}(t_2) - \hat{L}_{n,1}(t_1))^6 = \frac{1}{n^6 b_n^6} \sum_{1 \leq i_1, \dots, i_6 \leq n} E\xi_{i_1} \dots \xi_{i_6}.$$

We note that $\{\xi_i\}$ is a stationary, $[K \log n]$ -dependent sequence with zero mean. Let us divide the indices i_1, \dots, i_6 into groups so that the difference between the indices within a group are less than $[K \log n]$ and between groups is larger than $[K \log n]$. Clearly $E\xi_{i_1} \dots \xi_{i_6} = 0$, if there is at least one group containing a single element. So it suffices to consider the cases when all groups contain at least two elements. This allows the cases of one single group with 6 elements (D_1), two groups with 3+3 (D_2) or 4+2 (D_3) elements and finally 3 groups with 2 elements in each (D_4). If there is only one group, then via Hölder's inequality we have

$$|E\xi_{i_1} \xi_{i_2} \dots \xi_{i_6}| \leq E|\xi_0|^6 \leq 2^6 (E|\eta|^6 + (E|\eta|)^6).$$

Since the cardinality of D_1 is bounded by constant times $n(\log n)^5$ we conclude, by the decomposition in (2.14),

$$\begin{aligned} &\left| \frac{1}{n^6 b_n^6} \sum_{D_1} E\xi_{i_1} \xi_{i_2} \dots \xi_{i_6} \right| \\ &\leq C_6 \frac{n(\log n)^5}{n^6 b_n^6} \times \left(EY_0^6 Y_1^6 I\{|Y_0| \leq v_1\} I\{v_1 < |Y_1| \leq v_2\} \right. \\ &\quad + EY_0^6 Y_1^6 I\{|Y_1| \leq v_2\} I\{v_1 < |Y_0| \leq v_2\} \\ &\quad + (E|Y_0||Y_1|I\{|Y_0| \leq v_1\} I\{v_1 < |Y_1| \leq v_2\})^6 \\ &\quad \left. + (E|Y_0||Y_1|I\{|Y_1| \leq v_2\} I\{v_1 < |Y_0| \leq v_2\})^6 \right). \end{aligned}$$

The Cauchy–Schwarz inequality and integration by parts yield

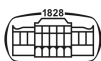
$$EY_0^6 Y_1^6 I\{|Y_0| \leq v_2\} I\{v_1 < |X_1| \leq v_2\} \leq C_7 |t_2 - t_1| \frac{d}{n} (H^{-1}(d/n))^{12},$$

resulting in

$$\left| \frac{1}{n^6 b_n^6} \sum_{D_1} E\xi_{i_1} \xi_{i_2} \dots \xi_{i_6} \right| \leq C_8 \frac{(\log n)^5}{d^5} (t_2 - t_1).$$

Using again the $[K \log n]$ dependence of ξ_i and the fact that the cardinality of D_2 is constant times $n^2(\log n)^4$ we conclude via Hölder's and Cauchy–Schwarz inequalities

$$\begin{aligned} &\left| \frac{1}{n^6 b_n^6} \sum_{D_2} E\xi_{i_1} \xi_{i_2} \dots \xi_{i_6} \right| = \left| \frac{1}{n^6 b_n^6} \sum_{D_2} E\xi_{i_1} \xi_{i_2} \xi_{i_3} E\xi_{i_4} \xi_{i_5} \xi_{i_6} \right| \\ &\leq C_8 \frac{n^2(\log n)^4}{n^6 b_n^6} \left((EY_0^3 Y_1^3 I\{|Y_0| \leq v_2\} I\{v_1 < |Y_1| \leq v_2\} \right. \\ &\quad \left. + E|Y_0|^3 |Y_1|^3 I\{|Y_1| \leq v_2\} I\{v_1 < |Y_0| \leq v_2\} \right) \end{aligned}$$



$$\begin{aligned}
& + (E|Y_0||Y_1|I\{|Y_0| \leq v_2\}I\{v_1 < |Y_1| \leq v_2\})^3 \\
& + (EY_0 Y_1 I\{|Y_1| \leq v_2\}I\{v_1 < |Y_0| \leq v_2\})^3)^2 \\
& \leq C_9 \frac{(\log n)^4}{d^4} (t_2 - t_1)^2.
\end{aligned}$$

Similar arguments give

$$\left| \frac{1}{n^6 b_n^6} \sum_{D_3} E \xi_{i_1} \xi_{i_2} \dots \xi_{i_6} \right| \leq C_{10} \frac{(\log n)^4}{d^4} (t_2 - t_1)^2.$$

For D_4 we have

$$\begin{aligned}
& \left| \frac{1}{n^6 b_n^6} \sum_{D_3} E \xi_{i_1} \xi_{i_2} \dots \xi_{i_6} \right| \\
& \leq C_{11} \frac{1}{n^6 b_n^6} \left(K \log n \sum_{i=0}^{\lfloor K \log n \rfloor} \xi_0 \xi_i \right)^3 \\
& \leq C_{12} \frac{(\log n)^6}{(nd)^3} (t_2 - t_1)^3.
\end{aligned}$$

Putting together our estimates and using the choice of g_n we conclude for all $|t_2 - t_1| \geq g_n$

$$\begin{aligned}
E|\hat{L}_{n,1}(t_2) - \hat{L}_{n,1}(t_1)|^6 & \leq C_{13} \left(\frac{(\log n)^5}{d^5} (t_2 - t_1) + \frac{(\log n)^4}{d^4} (t_2 - t_1)^2 + \frac{(\log n)^6}{(nd)^3} (t_2 - t_1)^3 \right) \\
& \leq C_{14} |t_2 - t_1|^\tau
\end{aligned}$$

with any $2 < \tau \leq 3$. Hence Lemma 4 and with that the tightness of the sequence

$$L_n(t) = \frac{1}{nb_n} \sum_{i=1}^n (X_i X_{i+1} I\{|X_i| \leq tH^{-1}(d/n)\} I\{|X_{i+1}| \leq tH^{-1}(d/n)\} - m_n(t)) \quad (2.15)$$

in $D[0, 1]$ has been proved.

We can now easily complete the proof of the Theorem. Lemma 2 shows that $m_n(t)/b_n \rightarrow \rho$ which, together with the tightness of $L_n(t)$ in (2.15), implies the tightness of

$$\frac{1}{nb_n} \sum_{i=1}^n X_i X_{i+1} I\{|X_i| \leq tH^{-1}(d/n)\} I\{|X_{i+1}| \leq tH^{-1}(d/n)\}.$$

Together with the statement

$$\frac{1}{nb_n} \sum_{k=1}^n X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\} \xrightarrow{P} 1 \quad (2.16)$$

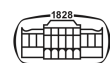
of Lemma 1 this implies that the sequence

$$L_n^*(t) = \frac{\sum_{i=1}^n X_i X_{i+1} I\{|X_i| \leq tH^{-1}(d/n)\} I\{|X_{i+1}| \leq tH^{-1}(d/n)\}}{\sum_{k=1}^n X_k^2 I\{|X_k| \leq tH^{-1}(d/n)\}}, \quad (n \geq 1) \quad (2.17)$$

is also tight. The proof of Lemma 2 shows that the finite dimensional distributions of L_n^* converge to the finite dimensional distributions of the process $L_\infty \equiv \rho$ and thus $L_n^* \rightarrow \rho$ weakly in $D[0, 1]$. On the other hand, in [1] it is proved that

$$\frac{\eta_{d,n}}{H^{-1}(d/n)} \xrightarrow{P} 1.$$

Thus by Billingsley [4], p. 144–145, the $D[0, 1]$ convergence of the processes L_n^* in (2.17) to ρ remains valid for $t = \eta_{d,n}/H^{-1}(d/n)$, which implies the statement of the Theorem. \square



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