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NEWTON-TYPE INEQUALITIES ASSOCIATED WITH CONVEX FUNCTIONS VIA QUANTUM CALCULUS

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Abstract. In this paper, we firstly establish an identity by using the notions of quantum derivatives and integrals. Using this quantum identity, quantum Newton-type inequalities associated with convex functions are proved. We also show that the newly established inequalities can be recaptured into some existing inequalities by taking $q \to 1^-$. Finally, we give mathematical examples of convex functions to verify the newly established inequalities.

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1. INTRODUCTION

A function $f\colon [a,b] \to \mathbb{R}$ is convex if it satisfies an inequality:

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),$$

where $x,y \in [a,b]$ and $t \in [0,1]$.

The most famous inequalities related to the integral inequalities for convex functions are Simpson- and Newton-type inequalities. Simpson's rules, famous techniques for numerical integration and approximations of definite integrals, were discovered by Thomas Simpson (1710-1761). These techniques are also known as Kepler's rule because Johannes Kepler used a similar estimation about 100 years ago. Simpson's rule consists of three-point Newton-Cotes quadrature rule, so estimations based on three steps quadratic kernel are sometimes called Newton-type inequalities.

(1) Simpson's quadrature formula (Simpson's $1/3$ rule) is as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right],$$

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see [6] for more details.

(2) Newton-Cotes quadrature formula or Simpson's second formula (Simpson's $3/8$ rule) is as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right],$$

see [15] for more details.

The estimations of Simpson- and Newton-type inequalities are as follows:

Theorem 1 ([6]). *Suppose that $f\colon [a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Theorem 2 ([15]). *Suppose that $f\colon [a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$, then*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{6480}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Currently, many researchers have focused on the Newton-type inequalities, see [4, 9–12, 16–18] and the references cited therein. Particularly, some researchers have studied on the Newton-type inequalities by using quantum calculus.

Quantum calculus, also known as q-calculus, gains q-analoques of mathematical objects which can be recaptured by letting $q\to 1^-$. The q-calculus has wide applications in various fields of physics and mathematics such as relativity theory, mechanics, quantum theory, orthogonal polynomials, number theory, and hypergeometric functions [8, 14]. In the beginning study of the q-calculus, the concept was revealed by renowned mathematician Euler (1707-1783), who introduced the q-parameter in Newton's infinite series. In 1910, Jackson [13] studied the concept of Euler to define q-integral and q-derivative of continuous functions over the interval $(0,\infty)$, also known as calculus without limits. In 1966, Al-Salam [1] studied the concepts of q-fractional integral inequalities and q-Riemann-Liouville fractional integral inequalities. In particular, in 2013, Tariboon and Ntouyas [20] presented the q-integral and the q-derivative of continuous functions over finite intervals. Some new results of q-calculus in Newton-type inequalities can be found in [2, 3, 5, 7, 19, 21, 22] and the references cited therein.

Inspired by the ongoing studies, we propose to prove new versions of quantum Newton-type inequalities associated with convex functions. We also prove that the newly established inequalities are the generalization of the existing Newton-type inequalities.

2. Preliminaries

The definitions and fundamental concepts of q-calculus are presented in this section. Throughout this paper, let q be a constant with $0 < q < 1$ and $[a,b] \subseteq \mathbb{R}$ be an interval with $a < b$. The q-number of n is given by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

Definition 1 ([20])**.** For a continuous function $f\colon [a,b] \to \mathbb{R}$, the q-derivative on $[a,b]$ is defined as:

$$_aD_q f(x) = \begin{cases} \dfrac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, & \text{if } x \neq a; \\ \lim\limits_{x \to a} {_aD_q f(x)}, & \text{if } x = a. \end{cases} \tag{2.1}$$

The function f is called a q_a-differentiable function if $_aD_q f(x)$ exists.

In Definition 1, if $a = 0$, then (2.1) is recaptured as follows:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)(x)},$$

which is the q-Jackson derivative, see [13] for more details.

Definition 2 ([20])**.** For a continuous function $f\colon [a,b] \to \mathbb{R}$, the q_a-integral on $[a,b]$ is defined as:

$$\int_a^x f(t)\ {_ad_q t} = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n) a) \tag{2.2}$$

for $x \in [a,b]$. The function f is called a q_a-integrable function if $\int_a^x f(t)\ d_q t$ for all $x \in [a,b]$ exists.

In Definition 2, if $a = 0$, then (2.2) is recaptured as follows:

$$\int_0^x f(t)\ d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x), \tag{2.3}$$

which is the q-Jackson integral, see [13] for more details. Moreover, Jackson [13] gave the q-Jackson integral on the interval $[a,b]$ as follows:

$$\int_a^b f(t)\ d_q t = \int_0^b f(t)\ d_q t - \int_0^a f(t)\ d_q t.$$

Lemma 1 ([19])**.** *For continuous functions $f,\ g \to \mathbb{R}$, the following expression holds:*

$$\int_0^c g(t)\ {_aD_q f(tb + (1-t)a)}\ d_q t = \frac{g(t) f(tb + (1-t)a)}{b-a}\bigg|_0^c - \frac{1}{b-a} \int_0^c D_q g(t) f(qtb + (1-qt)a)\ d_q t. \tag{2.4}$$

Lemma 2 ([21]). *The following expression holds:*

$$\int_a^b (x-a)^{\alpha} \ {}_a d_q x = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q},$$

where $\alpha \in \mathbb{R} - \{-1\}$.

3. Main Results

In this section, we will derive Newton-type inequalities for convex functions by using the q-derivative and q-integral. The following lemma is required to obtain the main results.

Lemma 3. Suppose that $f\colon [a,b] \to \mathbb{R}$ is a q_a-differentiable function on (a,b) such that ${}_aD_q f$ is continuous and integrable on $[a,b]$. Then, we have the following identity:

$$\begin{aligned}
&\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] \\
&\quad -\frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}} f(x) \ {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) \ {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^{b} f(x) \ {}_{\frac{a+2b}{3}} d_q x\right] \\
&= \frac{b-a}{9}\left[\int_0^1 \left(qt-\frac{3}{8}\right) {}_aD_q f\left(t\frac{2a+b}{3}+(1-t)a\right) d_q t\right. \\
&\quad + \int_0^1 \left(qt-\frac{1}{2}\right) {}_aD_q f\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right) d_q t \\
&\quad \left. + \int_0^1 \left(qt-\frac{5}{8}\right) {}_aD_q f\left(tb+(1-t)\frac{a+2b}{3}\right) d_q t\right].
\end{aligned} \tag{3.1}$$

Proof. Let

$$\begin{aligned}
&\frac{b-a}{9}\left[\int_0^1 \left(qt-\frac{3}{8}\right) {}_aD_q f\left(t\frac{2a+b}{3}+(1-t)a\right) d_q t\right. \\
&\quad + \int_0^1 \left(qt-\frac{1}{2}\right) {}_aD_q f\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right) d_q t \\
&\quad \left. + \int_0^1 \left(qt-\frac{5}{8}\right) {}_aD_q f\left(tb+(1-t)\frac{a+2b}{3}\right) d_q t\right] \\
&= \frac{(b-a)}{9}\left[I_1+I_2+I_3\right].
\end{aligned} \tag{3.2}$$

Using Lemma 1, we have

$$\int_0^1 \left(qt-\frac{3}{8}\right) {}_aD_q f\left(t\frac{2a+b}{3}+(1-t)a\right) d_q t$$

$$
\begin{aligned}
&= \frac{3}{b-a}\left(qt-\frac{3}{8}\right) f\left(t\frac{2a+b}{3}+(1-t)a\right)\Big|_0^1 \\
&\quad - \frac{3}{b-a}\int_0^1 qf\left(qt\frac{2a+b}{3}+(1-qt)a\right)\, d_qt \\
&= \frac{3}{b-a}\left(q-\frac{3}{8}\right) f\left(\frac{2a+b}{3}\right)+\frac{9f(a)}{8(b-a)} \\
&\quad - \frac{3}{b-a}(1-q)\sum_{n=0}^{\infty} q^{n+1} f\left(q^{n+1}\frac{2a+b}{3}+(1-q^{n+1})a\right) \\
&= \frac{3}{b-a}\left(q-\frac{3}{8}\right) f\left(\frac{2a+b}{3}\right)+\frac{9f(a)}{8(b-a)} \\
&\quad - \frac{3}{b-a}(1-q)\sum_{n=1}^{\infty} q^{n} f\left(q^{n}\frac{2a+b}{3}+(1-q^{n})a\right) \\
&= \frac{3}{b-a}\left(q-\frac{3}{8}\right) f\left(\frac{2a+b}{3}\right)+\frac{9f(a)}{8(b-a)} \\
&\quad - \frac{3}{b-a}(1-q)\left[\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\frac{2a+b}{3}+(1-q^{n})a\right)-f\left(\frac{2a+b}{3}\right)\right] \\
&= \frac{9f(a)}{8(b-a)}+\frac{15}{8(b-a)}f\left(\frac{2a+b}{3}\right)-\frac{9}{(b-a)^2}\int_a^{\frac{2a+b}{3}} f(x)\ {}_ad_qx. \qquad (3.3)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_0^1 \left(qt-\frac{1}{2}\right)\ {}_aD_qf\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right)\, d_qt \\
&= \frac{3}{2(b-a)}f\left(\frac{2a+b}{3}\right)+\frac{3}{2(b-a)}f\left(\frac{a+2b}{3}\right)-\frac{9}{(b-a)^2}\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x)\ {}_{\frac{2a+b}{3}}d_qx, \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \left(qt-\frac{5}{8}\right)\ {}_aD_qf\left(tb+(1-t)\frac{a+2b}{3}\right)\, d_qt \\
&= \frac{15}{8(b-a)}f\left(\frac{a+2b}{3}\right)+\frac{9}{8(b-a)}f(b)-\frac{9}{(b-a)^2}\int_{\frac{a+2b}{3}}^{b} f(x)\ {}_{\frac{a+2b}{3}}d_qx. \qquad (3.5)
\end{aligned}
$$

Substituting the inequalities (3.3) - (3.5) in inequality (3.2), we get the required inequality (3.1). Hence, the proof is accomplished. □

Theorem 3. *Under the conditions of Lemma 3, if $|{}_aD_qf|$ is a convex function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right. \\
&\quad \left. -\frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}} f(x)\ {}_ad_qx+\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x)\ {}_{\frac{2a+b}{3}}d_qx+\int_{\frac{a+2b}{3}}^{b} f(x)\ {}_{\frac{a+2b}{3}}d_qx\right]\right| \\
&\le \frac{b-a}{9}\left[(\Lambda_1(q)+\Lambda_3(q)+\Lambda_5(q))\left|\ {}_aD_qf(a)\right|\right. \\
&\quad \left.+(\Lambda_2(q)+\Lambda_4(q)+\Lambda_6(q))\left|\ {}_aD_qf(b)\right|\right], \qquad (3.6)
\end{aligned}$$

where $\Lambda_i(q), i=1,2,...,6$ are defined by

$$\Lambda_1(q)=\begin{cases}\frac{6-q-q^2-15q^3}{24[2]_q[3]_q}, & 0<q<\frac{3}{8};\\ \frac{480q^3+248q^2+248q-3}{768[2]_q[3]_q}, & \frac{3}{8}\le q<1,\end{cases}$$

$$\Lambda_2(q)=\begin{cases}\frac{3-5q-5q^2}{24[2]_q[3]_q}, & 0<q<\frac{3}{8};\\ \frac{160q^2+160q-69}{768[2]_q[3]_q}, & \frac{3}{8}\le q<1,\end{cases}$$

$$\Lambda_3(q)=\begin{cases}\frac{1+q+q^2-2q^3}{6[2]_q[3]_q}, & 0<q<\frac{1}{2};\\ \frac{4q^3+2q^2+2q+1}{12[2]_q[3]_q}, & \frac{1}{2}\le q<1,\end{cases}$$

$$\Lambda_4(q)=\begin{cases}\frac{2-q-q^2-q^3}{6[2]_q[3]_q}, & 0<q<\frac{1}{2};\\ \frac{2q^3+4q^2+4q-1}{12[2]_q[3]_q}, & \frac{1}{2}\le q<1,\end{cases}$$

$$\Lambda_5(q)=\begin{cases}\frac{5q+5q^2-3q^3}{24[2]_q[3]_q}, & 0<q<\frac{5}{8};\\ \frac{96q^3+40q^2-160q+275}{768[2]_q[3]_q}, & \frac{5}{8}\le q<1,\end{cases}$$

$$\Lambda_6(q)=\begin{cases}\frac{15+q+q^2-6q^3}{24[2]_q[3]_q}, & 0<q<\frac{5}{8};\\ \frac{192q^3+368q^2+368q+45}{768[2]_q[3]_q}, & \frac{5}{8}\le q<1.\end{cases}$$

Proof. By taking modulus in Lemma 3, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x){}_ad_qx\right| \\
&\le \frac{b-a}{9}\left[\int_0^1\left|qt-\frac{3}{8}\right|\left|\ {}_aD_qf\left(t\frac{2a+b}{3}+(1-t)a\right)\right|\ d_qt\right. \\
&\quad +\int_0^1\left|qt-\frac{1}{2}\right|\left|\ {}_aD_qf\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right)\right|\ d_qt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \left|qt-\frac{5}{8}\right| \left| {}_aD_q f\left(tb+(1-t)\frac{a+2b}{3}\right)\right| d_qt\Bigg] \\
&\le \frac{b-a}{9}\Bigg[\int_0^1 \left|qt-\frac{3}{8}\right| \left(\frac{3-t}{3}\left|{}_aD_qf(a)\right|+\frac{t}{3}\left|{}_aD_qf(b)\right|\right) d_qt \\
&+\int_0^1 \left|qt-\frac{1}{2}\right| \left(\frac{2-t}{3}\left|{}_aD_qf(a)\right|+\frac{1+t}{3}\left|{}_aD_qf(b)\right|\right) d_qt \\
&+\int_0^1 \left|qt-\frac{5}{8}\right| \left(\frac{1-t}{3}\left|{}_aD_qf(a)\right|+\frac{2+t}{3}\left|{}_aD_qf(b)\right|\right) d_qt\Bigg].
\end{aligned}
$$

Using Lemma 2, it can easily compute the integrals as follows:

$$
\Lambda_1(q)=\int_0^1 \left|qt-\frac{3}{8}\right|\frac{3-t}{3}\,d_qt=\begin{cases}\frac{6-q-q^2-15q^3}{24[2]_q[3]_q}, & 0<q<\frac{3}{8};\\ \frac{480q^3+248q^2+248q-3}{768[2]_q[3]_q}, & \frac{3}{8}\le q<1,\end{cases}
$$

$$
\Lambda_2(q)=\int_0^1 \left|qt-\frac{3}{8}\right|\frac{t}{3}\,d_qt=\begin{cases}\frac{3-5q-5q^2}{24[2]_q[3]_q}, & 0<q<\frac{3}{8};\\ \frac{160q^2+160q-69}{768[2]_q[3]_q}, & \frac{3}{8}\le q<1,\end{cases}
$$

$$
\Lambda_3(q)=\int_0^1 \left|qt-\frac{1}{2}\right|\frac{2-t}{3}\,d_qt=\begin{cases}\frac{1+q+q^2-2q^3}{6[2]_q[3]_q}, & 0<q<\frac{1}{2};\\ \frac{4q^3+2q^2+2q+1}{12[2]_q[3]_q}, & \frac{1}{2}\le q<1,\end{cases}
$$

$$
\Lambda_4(q)=\int_0^1 \left|qt-\frac{1}{2}\right|\frac{1+t}{3}\,d_qt=\begin{cases}\frac{2-q-q^2-q^3}{6[2]_q[3]_q}, & 0<q<\frac{1}{2};\\ \frac{2q^3+4q^2+4q-1}{12[2]_q[3]_q}, & \frac{1}{2}\le q<1,\end{cases}
$$

$$
\Lambda_5(q)=\int_0^1 \left|qt-\frac{5}{8}\right|\frac{1-t}{3}\,d_qt=\begin{cases}\frac{5q+5q^2-3q^3}{24[2]_q[3]_q}, & 0<q<\frac{5}{8};\\ \frac{96q^3+40q^2-160q+275}{768[2]_q[3]_q}, & \frac{5}{8}\le q<1,\end{cases}
$$

$$
\Lambda_6(q)=\int_0^1 \left|qt-\frac{5}{8}\right|\frac{2+t}{3}\,d_qt=\begin{cases}\frac{15+q+q^2-6q^3}{24[2]_q[3]_q}, & 0<q<\frac{5}{8};\\ \frac{192q^3+368q^2+368q+45}{768[2]_q[3]_q}, & \frac{5}{8}\le q<1.\end{cases}
$$

Hence, the proof is accomplished. □

Remark 1. If we take the limit $q\to 1^-$ in Theorem 3, then inequality (3.6) becomes

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \\
&\le \frac{25(b-a)}{576}\left[|f'(a)|+|f'(b)|\right],
\end{aligned}
$$

which is proven in [12].

Theorem 4. *Under the conditions of Lemma 3 and $r>1$, if $|{}_aD_qf|^r$ is a convex function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}&\Bigg|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\\&\quad-\frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}}f(x)\;{}_ad_qx+\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}f(x)\;{}_{\frac{2a+b}{3}}d_qx+\int_{\frac{a+2b}{3}}^{b}f(x)\;{}_{\frac{a+2b}{3}}d_qx\right]\Bigg|\\&\leq\frac{b-a}{9}\Big[(\Lambda_7(q))^{1-\frac{1}{r}}\left(\Lambda_1(q)|{}_aD_qf(a)|^r+\Lambda_2(q)|{}_aD_qf(b)|^r\right)^{\frac{1}{r}}\\&\quad+(\Lambda_8(q))^{1-\frac{1}{r}}\left(\Lambda_3(q)|{}_aD_qf(a)|^r+\Lambda_4(q)|{}_aD_qf(b)|^r\right)^{\frac{1}{r}}\\&\quad+(\Lambda_9(q))^{1-\frac{1}{r}}\left(\Lambda_5(q)|{}_aD_qf(a)|^r+\Lambda_6(q)|{}_aD_qf(b)|^r\right)^{\frac{1}{r}}\Big],\end{aligned}\tag{3.7}$$

where $\Lambda_i(q), i=1,2,\dots,6$ are given in Theorem 3, and $\Lambda_i(q), i=7,8,9$ are defined by

$$\Lambda_7(q)=\int_0^1\left|qt-\frac{3}{8}\right|\,d_qt=\begin{cases}\frac{3-5q}{8[2]_q}, & 0<q<\frac{3}{8};\\ \frac{20q-3}{32[2]_q}, & \frac{3}{8}\leq q<1,\end{cases}$$

$$\Lambda_8(q)=\int_0^1\left|qt-\frac{1}{2}\right|\,d_qt=\begin{cases}\frac{1-q}{2[2]_q}, & 0<q<\frac{1}{2};\\ \frac{q}{2[2]_q}, & \frac{1}{2}\leq q<1,\end{cases}$$

$$\Lambda_9(q)=\int_0^1\left|qt-\frac{5}{8}\right|\,d_qt=\begin{cases}\frac{5-3q}{8[2]_q}, & 0<q<\frac{5}{8};\\ \frac{12q+5}{32[2]_q}, & \frac{5}{8}\leq q<1.\end{cases}$$

Proof. By taking modulus in Lemma 3, applying the power mean inequality, and using the convexity of $|{}_aD_qf|^r$, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x){}_ad_qx\right|\\&\leq\frac{b-a}{9}\Bigg[\int_0^1\left|qt-\frac{3}{8}\right|\left|{}_aD_qf\left(t\frac{2a+b}{3}+(1-t)a\right)\right|\,d_qt\\&\quad+\int_0^1\left|qt-\frac{1}{2}\right|\left|{}_aD_qf\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right)\right|\,d_qt\\&\quad+\int_0^1\left|qt-\frac{5}{8}\right|\left|{}_aD_qf\left(tb+(1-t)\frac{a+2b}{3}\right)\right|\,d_qt\Bigg]\\&\leq\frac{b-a}{9}\Bigg[\left(\int_0^1\left|qt-\frac{3}{8}\right|\,d_qt\right)^{1-\frac{1}{r}}\\&\quad\times\left(\int_0^1\left|qt-\frac{3}{8}\right|\left|{}_aD_qf\left(t\frac{2a+b}{3}+(1-t)a\right)\right|^r\,d_qt\right)^{\frac{1}{r}}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{1}{2}\right|\,d_qt\right)^{1-\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\left|qt-\frac{1}{2}\right|\left|\,{}_aD_qf\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right)\right|^r\,d_qt\right)^{\frac{1}{r}}\right]\\
&+\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{5}{8}\right|\,d_qt\right)^{1-\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\left|qt-\frac{5}{8}\right|\left|\,{}_aD_qf\left(tb+(1-t)\frac{a+2b}{3}\right)\right|^r\,d_qt\right)^{\frac{1}{r}}\right]\\
\le\,&\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{3}{8}\right|\,d_qt\right)^{1-\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\left|qt-\frac{3}{8}\right|\left(\frac{3-t}{3}\left|\,{}_aD_qf(a)\right|^r+\frac{t}{3}\left|\,{}_aD_qf(b)\right|^r\right)\,d_qt\right)^{\frac{1}{r}}\right]\\
&+\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{1}{2}\right|\,d_qt\right)^{1-\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\left|qt-\frac{1}{2}\right|\left(\frac{2-t}{3}\left|\,{}_aD_qf(a)\right|^r+\frac{1+t}{3}\left|\,{}_aD_qf(b)\right|^r\right)\,d_qt\right)^{\frac{1}{r}}\right]\\
&+\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{5}{8}\right|\,d_qt\right)^{1-\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\left|qt-\frac{5}{8}\right|\left(\frac{1-t}{3}\left|\,{}_aD_qf(a)\right|^r+\frac{2+t}{3}\left|\,{}_aD_qf(b)\right|^r\right)\,d_qt\right)^{\frac{1}{r}}\right].
\end{aligned}
$$

Using Lemma 2, the integrals can be easily computed as follows:

$$
\Lambda_7(q)=\int_0^1\left|qt-\frac{3}{8}\right|\,d_qt=\begin{cases}\frac{3-5q}{8[2]_q}, & 0<q<\frac{3}{8};\\ \frac{20q-3}{32[2]_q}, & \frac{3}{8}\le q<1,\end{cases}
$$

$$
\Lambda_8(q)=\int_0^1\left|qt-\frac{1}{2}\right|\,d_qt=\begin{cases}\frac{1-q}{2[2]_q}, & 0<q<\frac{1}{2};\\ \frac{q}{2[2]_q}, & \frac{1}{2}\le q<1,\end{cases}
$$

$$
\Lambda_9(q)=\int_0^1\left|qt-\frac{5}{8}\right|\,d_qt=\begin{cases}\frac{5-3q}{8[2]_q}, & 0<q<\frac{5}{8};\\ \frac{12q+5}{32[2]_q}, & \frac{5}{8}\le q<1.\end{cases}
$$

Thus, the proof is accomplished. □

Remark 2. If we take the limit $q \to 1^-$ in Theorem 4, then inequality (3.7) becomes

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{36}\left\{\left(\frac{17}{16}\right)^{1-1/r}\left(\frac{251\left|f'(a)\right|^r+937\left|f'(b)\right|^r}{1152}\right)^{1/r}+\left(\frac{\left|f'(a)\right|^r+\left|f'(b)\right|^r}{2}\right)^{1/r}\right.$$
$$\left.+\left(\frac{17}{16}\right)^{1-1/r}\left(\frac{937\left|f'(a)\right|^r+251\left|f'(b)\right|^r}{1152}\right)^{1/r}\right\},$$

which is proven in [18].

Theorem 5. *Under the conditions of Lemma 3 and $r>1$ with $s^{-1}+r^{-1}=1$, if $|{}_aD_qf|^r$ is a convex function on $[a,b]$, then we have the following inequality:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right.$$
$$\left.-\frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}} f(x)\ {}_ad_qx+\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x)\ {}_{\frac{2a+b}{3}}d_qx+\int_{\frac{a+2b}{3}}^{b} f(x)\ {}_{\frac{a+2b}{3}}d_qx\right]\right|$$
$$\leq \frac{b-a}{9}\left[\frac{5}{8}\left(\frac{(3q+2)\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right.$$
$$+\frac{1}{2}\left(\frac{(2q+1)\left|{}_aD_qf(a)\right|^r+(q+2)\left|{}_aD_qf(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}$$
$$\left.+\frac{3}{8}\left(\frac{q\left|{}_aD_qf(a)\right|^r+(2q+3)\left|{}_aD_qf(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right]. \tag{3.8}$$

Proof. By taking modulus in Lemma 3, applying the Hölder's inequality, and using the convexity of $|{}_aD_qf|^r$, we have

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x){}_ad_qx\right|$$
$$\leq \frac{b-a}{9}\left[\int_0^1\left|qt-\frac{3}{8}\right|\left|{}_aD_qf\left(t\frac{2a+b}{3}+(1-t)a\right)\right|\,d_qt\right.$$
$$+\int_0^1\left|qt-\frac{1}{2}\right|\left|{}_aD_qf\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right)\right|\,d_qt$$
$$\left.+\int_0^1\left|qt-\frac{5}{8}\right|\left|{}_aD_qf\left(tb+(1-t)\frac{a+2b}{3}\right)\right|\,d_qt\right]$$

$$\leq \frac{b-a}{9}\left[\left[\left(\int_0^1\left|qt-\frac{3}{8}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_aD_qf\left(t\frac{2a+b}{3}+(1-t)a\right)\right|^r d_qt\right)^{\frac{1}{r}}\right]\right.$$
$$+\left[\left(\int_0^1\left|qt-\frac{1}{2}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_aD_qf\left(t\frac{a+2b}{3}+(1-t)\frac{2a+b}{3}\right)\right|^r d_qt\right)^{\frac{1}{r}}\right]$$
$$\left.+\left[\left(\int_0^1\left|qt-\frac{5}{8}\right|^s\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_aD_qf\left(tb+(1-t)\frac{a+2b}{3}\right)\right|^r d_qt\right)^{\frac{1}{r}}\right]\right]$$
$$\leq \frac{b-a}{9}\left[\left[\left(\int_0^1\left|qt-\frac{3}{8}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{3-t}{3}\left|{}_aD_qf(a)\right|^r+\frac{t}{3}\left|{}_aD_qf(b)\right|^r\right)d_qt\right)^{\frac{1}{r}}\right]\right.$$
$$+\left[\left(\int_0^1\left|qt-\frac{1}{2}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{2-t}{3}\left|{}_aD_qf(a)\right|^r+\frac{1+t}{3}\left|{}_aD_qf(b)\right|^r\right)d_qt\right)^{\frac{1}{r}}\right]$$
$$\left.+\left[\left(\int_0^1\left|qt-\frac{5}{8}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{1-t}{3}\left|{}_aD_qf(a)\right|^r+\frac{2+t}{3}\left|{}_aD_qf(b)\right|^r\right)d_qt\right)^{\frac{1}{r}}\right]\right].$$

By using inequality (2.3), we have

$$\int_0^1\left|qt-\frac{3}{8}\right|^s d_qt=(1-q)\sum_{n=0}^{\infty}q^n\left|q^{n+1}-\frac{3}{8}\right|^s\leq(1-q)\sum_{n=0}^{\infty}q^n\left|1-\frac{3}{8}\right|^s$$
$$=(1-q)\frac{5^s}{8^s}\frac{1}{(1-q)}=\frac{5^s}{8^s}.$$

So, we find that

$$\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{3}{8}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{3-t}{3}\left|{}_aD_qf(a)\right|^r+\frac{t}{3}\left|{}_aD_qf(b)\right|^r\right)d_qt\right)^{\frac{1}{r}}\right]$$
$$=\frac{b-a}{9}\left[\left(\frac{5^s}{8^s}\right)^{\frac{1}{s}}\left(\frac{(3q+2)\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right].$$

Similarly, we obtain

$$\frac{b-a}{9}\left[\left(\int_0^1\left|qt-\frac{1}{2}\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{2-t}{3}\left|{}_aD_qf(a)\right|^r+\frac{1+t}{3}\left|{}_aD_qf(b)\right|^r\right)d_qt\right)^{\frac{1}{r}}\right]$$
$$=\frac{b-a}{9}\left[\left(\frac{1}{2^s}\right)^{\frac{1}{s}}\left(\frac{(2q+1)\left|{}_aD_qf(a)\right|^r+(q+2)\left|{}_aD_qf(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right],$$

and

$$\frac{b-a}{9}\left[\left(\int_0^1 \left|qt-\frac{5}{8}\right|^s d_qt\right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{1-t}{3}\left|{}_aD_qf(a)\right|^r+\frac{2+t}{3}\left|{}_aD_qf(b)\right|^r\right) d_qt\right)^{\frac{1}{r}}\right]$$
$$=\frac{b-a}{9}\left[\left(\frac{3^s}{8^s}\right)^{\frac{1}{s}}\left(\frac{q\left|{}_aD_qf(a)\right|^r+(2q+3)\left|{}_aD_qf(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right].$$

Thus, the proof is accomplished. □

4. Examples

In this section, we give examples to support the main results.

Example 1. Let $f\colon [1,5]\to\mathbb{R}$ be defined by $f(x)=x^2$. From Theorem 3 with $q=\frac{3}{4}$, the left-hand side of inequality (3.6) becomes

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right.$$
$$\left.-\frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}} f(x)\ {}_ad_qx+\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x)\ {}_{\frac{2a+b}{3}}d_qx+\int_{\frac{a+2b}{3}}^{b} f(x)\ {}_{\frac{a+2b}{3}}d_qx\right]\right|$$
$$=\left|\frac{1}{8}\left[f(1)+3f\left(\frac{7}{3}\right)+3f\left(\frac{11}{3}\right)+f(5)\right]\right.$$
$$\left.-\frac{1}{4}\left[\int_1^{\frac{7}{3}} x^2\ {}_1d_{\frac{3}{4}}x+\int_{\frac{7}{3}}^{\frac{11}{3}} x^2\ {}_{\frac{7}{3}}d_{\frac{3}{4}}x+\int_{\frac{11}{3}}^{5} x^2\ {}_{\frac{11}{3}}d_{\frac{3}{4}}x\right]\right|$$
$$\approx 0.6206,$$

and the right-hand side of inequality (3.6) becomes

$$\frac{b-a}{9}\left[(\Lambda_1(q)+\Lambda_3(q)+\Lambda_5(q))\left|{}_aD_qf(a)\right|+(\Lambda_2(q)+\Lambda_4(q)+\Lambda_6(q))\left|{}_aD_qf(b)\right|\right]$$
$$=\frac{4}{9}\left[\left(\Lambda_1\left(\frac{3}{4}\right)+\Lambda_3\left(\frac{3}{4}\right)+\Lambda_5\left(\frac{3}{4}\right)\right)\left|{}_1D_{\frac{3}{4}}f(1)\right|\right.$$
$$\left.+\left(\Lambda_2\left(\frac{3}{4}\right)+\Lambda_4\left(\frac{3}{4}\right)+\Lambda_6\left(\frac{3}{4}\right)\right)\left|{}_1D_{\frac{3}{4}}f(5)\right|\right]$$
$$\approx 1.6945.$$

It is clear that

$$0.6206 \le 1.6945,$$

which shows that inequality (3.6) is valid.

Example 2. Let $f\colon [1,5] \to \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 4 with $q = \frac{3}{4}$, the left-hand side of inequality (3.7) becomes

$$\begin{aligned}
&\left| \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right. \\
&\quad \left. - \frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}} f(x)\ {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x)\ {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^{b} f(x)\ {}_{\frac{a+2b}{3}} d_q x \right] \right| \\
&= \left| \frac{1}{8}\left[f(1) + 3f\left(\frac{7}{3}\right) + 3f\left(\frac{11}{3}\right) + f(5) \right] \right. \\
&\quad \left. - \frac{1}{4}\left[\int_1^{\frac{7}{3}} x^2\ {}_1 d_{\frac{3}{4}} x + \int_{\frac{7}{3}}^{\frac{11}{3}} x^2\ {}_{\frac{7}{3}} d_{\frac{3}{4}} x + \int_{\frac{11}{3}}^{5} x^2\ {}_{\frac{11}{3}} d_{\frac{3}{4}} x \right] \right| \\
&\approx 0.6206,
\end{aligned}$$

and the right-hand side of inequality (3.7) becomes

$$\begin{aligned}
&\frac{b-a}{9}\left[(\Lambda_7(q))^{1-\frac{1}{r}} \left(\Lambda_1(q) |{}_a D_q f(a)|^r + \Lambda_2(q) |{}_a D_q f(b)|^r \right)^{\frac{1}{r}} \right. \\
&\quad + (\Lambda_8(q))^{1-\frac{1}{r}} \left(\Lambda_3(q) |{}_a D_q f(a)|^r + \Lambda_4(q) |{}_a D_q f(b)|^r \right)^{\frac{1}{r}} \\
&\quad \left. + (\Lambda_9(q))^{1-\frac{1}{r}} \left(\Lambda_5(q) |{}_a D_q f(a)|^r + \Lambda_6(q) |{}_a D_q f(b)|^r \right)^{\frac{1}{r}} \right] \\
&= \frac{4}{9}\left[(\Lambda_7\left(\frac{3}{4}\right))^{1-\frac{1}{2}} \left(\Lambda_1\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \Lambda_2\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(5)|^2 \right)^{\frac{1}{2}} \right. \\
&\quad + (\Lambda_8\left(\frac{3}{4}\right))^{1-\frac{1}{2}} \left(\Lambda_3\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \Lambda_4\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(5)|^2 \right)^{\frac{1}{2}} \\
&\quad \left. + (\Lambda_9\left(\frac{3}{4}\right))^{1-\frac{1}{2}} \left(\Lambda_5\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \Lambda_6\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(5)|^2 \right)^{\frac{1}{2}} \right] \\
&\approx 3.3900.
\end{aligned}$$

It is clear that

$$0.6206 \leq 3.3900,$$

which shows that inequality (3.7) is valid.

Example 3. Let $f\colon [1,5] \to \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 5 with $q = \frac{3}{4}$, the left-hand side of inequality (3.8) becomes

$$\left| \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right.$$

$$
\begin{aligned}
&-\frac{1}{(b-a)}\left[\int_a^{\frac{2a+b}{3}} f(x)\ {}_a d_q x+\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x)\ {}_{\frac{2a+b}{3}} d_q x+\int_{\frac{a+2b}{3}}^{b} f(x)\ {}_{\frac{a+2b}{3}} d_q x\right]\Bigg|\\
&=\left|\frac{1}{8}\left[f(1)+3f\left(\frac{7}{3}\right)+3f\left(\frac{11}{3}\right)+f(5)\right]\right.\\
&\quad\left.-\frac{1}{4}\left[\int_1^{\frac{7}{3}} x^2\ {}_1 d_{\frac{3}{4}} x+\int_{\frac{7}{3}}^{\frac{11}{3}} x^2\ {}_{\frac{7}{3}} d_{\frac{3}{4}} x+\int_{\frac{11}{3}}^{5} x^2\ {}_{\frac{11}{3}} d_{\frac{3}{4}} x\right]\right|\\
&\approx 0.6206,
\end{aligned}
$$

and the right-hand side of inequality (3.8) becomes

$$
\begin{aligned}
&\frac{b-a}{9}\left[\frac{5}{8}\left(\frac{(3q+2)\left|{}_aD_q f(a)\right|^r+\left|{}_aD_q f(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right.\\
&\quad+\frac{1}{2}\left(\frac{(2q+1)\left|{}_aD_q f(a)\right|^r+(q+2)\left|{}_aD_q f(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\\
&\quad\left.+\frac{3}{8}\left(\frac{q\left|{}_aD_q f(a)\right|^r+(2q+3)\left|{}_aD_q f(b)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right]\\
&=\frac{4}{9}\left[\frac{5}{8}\left(\frac{\left(\frac{9}{4}+2\right)\left|{}_1D_{\frac{3}{4}} f(1)\right|^2+\left|{}_1D_{\frac{3}{4}} f(5)\right|^2}{3[2]_{\frac{3}{4}}}\right)^{\frac{1}{2}}\right.\\
&\quad+\frac{1}{2}\left(\frac{\left(\frac{3}{2}+2\right)\left|{}_1D_{\frac{3}{4}} f(1)\right|^2+\left(\frac{3}{4}+2\right)\left|{}_1D_{\frac{3}{4}} f(5)\right|^2}{3[2]_{\frac{3}{4}}}\right)^{\frac{1}{2}}\\
&\quad\left.+\frac{3}{8}\left(\frac{\frac{3}{4}\left|{}_1D_{\frac{3}{4}} f(1)\right|^2+\left(\frac{3}{2}+3\right)\left|{}_1D_{\frac{3}{4}} f(5)\right|^2}{3[2]_{\frac{3}{4}}}\right)^{\frac{1}{2}}\right]\\
&\approx 4.0741.
\end{aligned}
$$

It is clear that

$$
0.6206\leq 4.0741,
$$

which shows that inequality (3.8) is valid.

5. Conclusions

In this work, we proved new versions of quantum Newton-type inequalities associated with convex functions. We also demonstrated that the newly established inequalities can be recaptured into classical Newton-type inequalities by taking the limit $q \to 1^{-}$. Mathematical examples were given to verify the newly established inequalities. In future works, researchers can obtain similar inequalities of Newton-type inequalities associated with convex functions by using post quantum calculus.

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