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CERTAIN INVESTIGATIONS OF PSEUDO Z-SYMMETRIC SPACETIMES

SEZGIN ALTAY DEMIRBAG, UDAY CHAND DE, AND FUSUN OZEN ZENGIN

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Abstract. The section 1 of this paper deals with the definition for a pseudo symmetric spacetime. In section 2, it is proved that a $(PZS)_4$ spacetime satisfying Codazzi type of Z-tensor does not exist. In which condition a $(PZS)_4$ spacetime can be a perfect fluid has been found as a necessary and sufficient condition. After that, special properties are obtained if the $(PZS)_4$ spacetime has harmonic conformal curvature tensor.

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1. INTRODUCTION

Symmetric spaces have been studied because of their importance in differential geometry. They are used in various brances of mathematics such as compact Lie groups, Grassmannian and bounded symmetric fields. Every symmetric space has its own special geometry such as Euclidean, elliptical and hyperbolic geometry, etc. The initial study on symmetric spaces was made by E. Cartan, [11].

On the other hand, these fields have a lot in common properties, and they have a beautiful theory. Symmetric spaces can be considered from many perspectives. They can be viewed as Riemannian manifolds with point reflection or with parallel curvature tensor or as a homogeneous space with a special isotropy or with special holonomy or special Killing vector fields or a particular Lie group involution.

Let us assume that (M, g, ∇) denotes an n-dimensional Riemannian manifold admitting the Levi-Civita connection ∇. If the curvature tensor R of this manifold satisfies the condition $\nabla R = 0$, then this manifold is named locally symmetric. This condition is equivalent form to the fact that every point in $P \in M$ is the isometry of the local symmetry $F(P)$, [21]. Because of the importance of the symmetric manifolds in differential geometry, many authors have studied some properties of them, [12, 22, 27], etc. A pseudo symmetric spacetime is a four-dimensional time-oriented

Lorentzian manifold if the curvature tensor satisfies the condition, [26]

$$R_{hijk,l} = 2A_l R_{hijk} + A_h R_{lijk} + A_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl} \tag{1.1}$$

such that A_l is a non-zero 1-form, R_{hijk} is the curvature tensor obeying the property $R_{lijk} = g_{hl} R^h_{ijk}$, and the notation "," is the covariant derivative with respect to the metric tensor g. These manifolds are indicated by $(PS)_n$. In (1.1), for $A_l = 0$, a pseudo symmetric manifold is reduced to a symmetric manifold in the sense of Cartan.

(M,g) is called pseudo-Ricci symmetric $(PRS)_n$ manifold if its non-vanishing Ricci tensor R_{ij} obeys the condition

$$R_{ij,l} = 2A_l R_{ij} + A_i R_{lj} + A_j R_{il} \tag{1.2}$$

here A_l is a non-vanishing 1-form. As a particular case, if we consider the condition $R_{ij,k} = 0$ in (1.2) then this manifold is said to be Ricci symmetric.

In a Riemannian or a semi-Riemannian manifold of dimension n $(n > 3)$, the conformal curvature tensor C^h_{ijk} of type $(1,3)$ is given by the following form

$$\begin{aligned} C^h_{ijk} = R^h_{ijk} &- \frac{1}{n-2}\left[R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j\right] \\ &+ \frac{R}{(n-1)(n-2)}\left[g_{ij}\delta^h_k - g_{ik}\delta^h_j\right] \end{aligned} \tag{1.3}$$

where R^h_{ijk} is the curvature tensor, R_{ij} is the Ricci tensor and R is the scalar curvature of this manifold, [28].

Assume that M is a semi-Riemannian manifold of dimension n and the metric g having the signature (p,q) such that $p+q = n$. If we choose $(p,q) = (1,n-1)$ or $(n-1,1)$ then M is called a Lorentzian manifold, [21].

A generalized Robertson-Walker spacetime (GRW) is an n-dimensional spacetime $(n \geq 3)$ written as a warped product $-I \times \psi^2 M^*$. Here, it is assumed that the warping function or scaling factor $(\psi > 0)$ is a smooth function, M^* is considered as a Riemannian manifold with $(n-1)$ dimensional and an open interval of the real line is denoted by I, [1,2]. If we consider M^* as a three-dimensional Riemannian manifold with constant curvature, then the the GRW spacetime reduces to a RW spacetime. Examples of special spaces contained in these GRW spacetimes are de Sitter space-time, Einstein-de Sitter space-time, static Einstein space-time and Friedmann cosmological models. These spaces have been studied in many papers. Some of them are [4, 25] etc.

A perfect fluid is a four-dimensional spacetime whose non-vanishing Ricci tensor R_{ij} is of the form

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j \tag{1.4}$$

assuming that α and β are smooth functions, g is the Lorentzian metric and A_i is the velocity vector satisfying the conditions $A_i A^i = -1$ and $A^i = g^{ij} A_j$ [12, 22, 27].

We know that a *RW* spacetime is a perfect fluid spacetime [21] and also every four-dimensional *GRW* spacetime is a perfect fluid spacetime if and only if it is a *RW* spacetime. By using these results, some remarkable theorems are proved in [2–6, 8–17]. The energy-momentum tensor (briefly EMT) in general relativity theory defines the matter content. This content is considered a content that has dynamic and kinematic quantities such as density, pressure and velocity, acceleration, vortex, shear and expansion. It is known that a perfect fluid has no heat conduction terms and it has no EMT related to viscosity.

In a perfect fluid spacetime, EMT is given as

$$T_{ij} = (\sigma + p)A_i A_j + p g_{ij} \tag{1.5}$$

where σ is the energy density and p is the isotropic pressure of this spacetime, [21].

We know that the Einstein's field equations with the cosmological constant Λ are in the following form

$$R_{ij} - \frac{R}{2} + \Lambda g_{ij} = kT_{ij} \tag{1.6}$$

where k denotes the gravitational constant and T_{ij} is the energy momentum tensor satisfying the relation (1.5). And the Einstein's field equations (in short, EFE) without cosmological constant are in the following form

$$R_{ij} - \frac{R}{2} = kT_{ij}.$$

The dark energy in modern cosmology is considered as a candidate to accelerate the expansion of the universe, and the σ and p as scalar functions are hypothetically constrained by an equation of state that regulates the quality of the ideal fluid.

This is actually an equation of the form $p = p(\sigma, T_0)$, where T_0 represents the absolute temperature. To reduce the equation of state to $p = p(\sigma)$, the condition that T_0 must be constant. The perfect fluid with this property is called as isentropic [5]. If we choose as $\sigma = p$, then the perfect fluid is called as stiff matter, [7],[8], [9], [10], [13], [15], [16],[17], [18], [19], [20], [21].

If the energy density σ and the isotropic pressure p of the fluid are related by $p = w\sigma$, then w is said to be (EoS) parameter of the equation of state. (EoS) parameter plays an important role in examining the energy density and investigating the expansion of the universe. Using this parameter, different phases of the universe can be observed. The accelerating phase of the universe discussed in recent studies is portrayed when we consider the condition $w < -\frac{1}{3}$. If the conditions $-1 < w < 0$ and $w < -1$ hold, then it includes the quintes sense phase and phantom regime, respectively.

The vector U is called as K-compatible vector field (where K can be the Riemann, the Weyl or a generalized tensor)

$$\left(U_i K^m_{jkl} + U_j K^m_{kil} + U_k K^m_{ijl}\right) U_m = 0. \tag{1.7}$$

And, U is a vector field which is called Riemann or Weyl permutable if the following relations hold

$$R^m_{kl[i}U_{j]}U_m = 0 \quad \text{or} \quad C^m_{kl[i}U_{j]}U_m = 0.$$

For a Riemannian manifold or a semi-Riemannian manifold, a vector field U is said to be a torse-forming if it satisfies

$$\nabla_X U = \rho X + \mu(X)U,$$

when $X \in TM$, $\mu(X)$ and ρ denote a linear form and a scalar function, respectively. If we consider the local transcription, the above relation is written as

$$U_{i,j} = \rho g_{ij} + \mu_j U_i \tag{1.8}$$

here U_i and μ_i denote the covariant components of U and μ. Considering the equation (1.8), if the vector field U_i satisfies the condition

$$U_{i,j} = \rho g_{ij} \tag{1.9}$$

then U_i is said to be a concircular vector field, [8, 15].

A generalized symmetric tensor of type $(0,2)$ called Z-tensor was defined by Mantica and Molinari [16] in the form

$$Z_{ij} = R_{ij} + \varphi g_{ij}, \tag{1.10}$$

φ being a scalar function. The trace of this tensor shown by Z is a scalar function and we have from (1.10)

$$Z = g^{ij}Z_{ij} = R + n\varphi. \tag{1.11}$$

Taking $\varphi = -\frac{R}{n}$, we obtain the classical Z-tensor. The generalized Z-tensor is called as the Z-tensor, shortly. When special cases of the Z-tensor are chosen, some well-known structures of Riemannian manifolds are found. For example,

(i) if $Z_{ij} = 0$ (i.e, Z-flat), then this manifold reduces to an Einstein manifold [10],
(ii) if $Z_{ij} = \lambda_l Z_{ij}$ (Z-recurrent), then this manifold reduces to a generalized Ricci recurrent manifold, where $Z_{ij} \neq 0$,
(iii) if $Z_{ij,l} = Z_{il,j}$ (Codazzi tensor), then we find,

$$R_{ij,l} - R_{il,j} = \frac{1}{2(n-1)}(g_{ij}R_l - g_{il}R_j).$$

This result defines the nearly conformal symmetric manifold $(NCS)_n$, [23].
(iv) By using (1.6) and (1.10), the Z-tensor is related by EMT of EFE with the cosmological constant Λ is in the form, [13]

$$Z_{ij} = kT_{ij} \tag{1.12}$$

where

$$\varphi = -\frac{1}{2}R + \Lambda \tag{1.13}$$

and k is the gravitional constant and φ denotes an arbitrary scalar function. So, the Z-tensor may be considered as a generalized Einstein gravitational tensor. If we take $Z = 0$ which means the vacuum solution then we get an Einstein space with the condition $\Lambda = (\frac{n-2}{2n})R$ and the conservation of the total EMT ($T^l_{i,l} = 0$) gives $Z_{ij,l} = 0$. Thus, this spacetime has the conserved energy-momentum density.

These manifolds are studied in considerable details by many authors because of their importance [19, 20], etc. In section 2, we will discuss the properties of pseudo Z-symmetric spacetimes shown by $(PZS)_4$.

2. Pseudo Z-symmetric spacetimes

In this section, using (1.2), we consider four-dimensional pseudo Z symmetric spacetime $(PZS)_4$ satisfying the following condition

$$Z_{ij,l} = 2A_l Z_{ij} + A_i Z_{jl} + A_j Z_{il} \tag{2.1}$$

where Z_{ij} is a symmetric tensor defined by (1.10), A_i is a non-null covector and it is called the associated 1-form of this spacetime.

If the Z-tensor satisfies EFE without cosmological constant, from (1.13), we have

$$\varphi = -\frac{R}{2}. \tag{2.2}$$

Theorem 1. *Let us consider a $(PZS)_4$ spacetime. At a point P of this spacetime, if the scalar function Z is non-zero, then the Ricci tensor is of eigenvector A_k with the eigenvalue $\frac{5}{6}R$.*

Proof. Assume that M be a $(PZS)_4$ spacetime. If we multiply (2.1) by g^{il}, then we get

$$Z^l_{j,l} = 3A^m Z_{mj} + A_j Z. \tag{2.3}$$

On the other hand, differentiating the equation (1.10), it can be seen that

$$Z_{ij,l} = R_{ij,l} + \varphi_l g_{ij}. \tag{2.4}$$

Now, multiplying (2.4) by g^{il}, one can obtain that

$$Z^l_{j,l} = R^l_{j,l} + \varphi_j. \tag{2.5}$$

If we put the Ricci identity $R^l_{j,l} = \frac{1}{2}R_j$ in (2.5), then we find

$$Z^l_{j,l} = \frac{1}{2}R_j + \varphi_j. \tag{2.6}$$

With the help of the equation (2.2), (2.6) can be reduced to

$$Z^{l}_{j,l} = 0. \tag{2.7}$$

Hence, if we compare the equations (2.3) and (2.7), then we can see that

$$A^m Z_{mj} = -\frac{Z}{3} A_j. \tag{2.8}$$

From (1.11) and (2.2), the equation (2.8) reduces to

$$A^m Z_{mj} = -\frac{2}{3} \varphi A_j. \tag{2.9}$$

By the aid of (1.10) and (2.9), we obtain

$$A^m R_{mj} = -\frac{5}{3} \varphi A_j. \tag{2.10}$$

Finally, from (2.2) and (2.10), we find

$$A^m R_{mj} = \frac{5}{6} R A_j.$$

Thus, we complete the proof. □

Theorem 2. *A $(PZS)_4$ spacetime with Codazzi type Z-tensor does not exist.*

Proof. If we assume that $(PZS)_4$ spacetime is of Codazzi type, then we get from (2.1)

$$A_k Z_{ij} - A_j Z_{ik} = 0. \tag{2.11}$$

Multiplying (2.11) by A^k and assuming that $A^k A_k = \|A\|^2$,

$$Z_{ij} = \frac{1}{\|A\|^2} Z_{ik} A^k A_j. \tag{2.12}$$

Considering (2.9) in (2.12), we find

$$Z_{ij} = -\frac{2}{3\|A\|^2} \varphi A_i A_j. \tag{2.13}$$

If we use the equation (1.10), (2.13) can be written as

$$R_{ij} = -\frac{2}{3\|A\|^2} \varphi A_i A_j - \varphi g_{ij}. \tag{2.14}$$

Multiplying (2.14) by g^{ij} and using the equation (2.2), one can easily seen that

$$\varphi = 0.$$

From this result and the equation (2.14), we can say that $R_{ij} = 0$. Finally, we get from (1.10), $Z_{ij} = 0$. Hence, the proof is completed. □

For a $(PZS)_4$ spacetime, considering the equations (1.3) and (2.1), the conformal curvature tensor is obtained as the following relation, [20]

$$C^m_{ijl,m} = \frac{1}{2}\left[A_j Z_{il} - \frac{4Z}{9} g_{il} A_j - A_i Z_{jl} + \frac{4Z}{9} g_{jl} A_i\right]. \tag{2.15}$$

In this case, assuming that $C^m_{jkl,m} = 0$ in (2.15), then we also have the following equation, [20]

$$Z_{ij} = \frac{4Z}{9} g_{ij} + \frac{7Z}{9} A_i A_j. \tag{2.16}$$

Now, considering the equations (1.10), (1.11), (2.2) and (2.16), we can write the Ricci tensor, [20]

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j \tag{2.17}$$

taking $\alpha = \frac{R}{18}$ and $\beta = -\frac{7R}{9}$ in (1.4).

Theorem 3. *A necessary and sufficient condition for a $(PZS)_4$ spacetime to be a perfect fluid is that the conformal curvature tensor is harmonic.*

Proof. Assume that our spacetime is a $(PZS)_4$ spacetime satisfying the property $C^m_{jkl,m} = 0$. Then, this spacetime is quasi-Einstein, [18] and it satisfies the condition (2.17). By using (2.2) and (2.17) in (1.10), we get

$$Z_{ij} = -\frac{4}{9} R g_{ij} - \frac{7}{9} R A_i A_j. \tag{2.18}$$

Also, we have from (1.11) and (2.2)

$$Z = -R. \tag{2.19}$$

By putting (2.18) and (2.19) in (2.15), we get $C^m_{ijl,m} = 0$. Hence, the proof is completed. □

Theorem 4 ([20, Theorem 3.3])**.** *Let M be a $(PZS)_4$ spacetime with the property $C^m_{ijl,m} = 0$. If the condition $A_k A^k < 0$ holds, then EMT is of the perfect fluid form with the properties*

$$\sigma = \frac{T}{3}, \quad p = \frac{4T}{9}. \tag{2.20}$$

The condition $R_{ij} A^i A^j > 0$ is said to be timelike convergence condition for every timelike vector field A_i and the Ricci tensor R_{ij} of type $(0,2)$ of a spacetime, [24].

Theorem 5. *If a $(PZS)_4$ spacetime with harmonic conformal curvature tensor admits the timelike vector field with the convergence condition, then this spacetime obeys cosmic strong energy condition.*

Proof. Assume that $(PZS)_4$ spacetime with harmonic conformal curvature tensor admits the timelike convergence condition. Multiplying (2.17) by A^iA^j and using $A_iA^i = -1$, we find

$$R_{ij}A^iA^j = -\frac{5}{6}R. \tag{2.21}$$

Since A_i is a timelike vector field and convergence condition $R_{ij}A^iA^j > 0$ holds in (2.21) then we have $R < 0$. Now, if we multiply (1.12) by g^{ij}, then we obtain

$$Z = kT. \tag{2.22}$$

Hence, using (2.19) and (2.22), one can easily seen that

$$T = -\frac{R}{k}. \tag{2.23}$$

Remembering that the sign of R is negative and using (2.23), we have $T > 0$. In this case, from (2.20)

$$\sigma > 0, \quad p > 0. \tag{2.24}$$

Finally, we get that $\sigma + 3p > 0$. The inequality $\sigma + 3p > 0$ shows that this spacetime obeys cosmic strong energy condition, [5]. This result completes the proof. □

Theorem 6. *A $(PZS)_4$ spacetime with harmonic conformal curvature tensor contains pure matter if this spacetime admits the timelike convergence condition.*

Proof. From the timelike convergence condition, we say that a $(PZS)_4$ spacetime has the negative scalar curvature. Also, we know from (2.24), $\sigma > 0$. Then, we can say that our spacetime under this consideration is of pure matter. □

Proposition 1 ([20, Propositon 5.10]). *Let M be a $(PZS)_4$ spacetime with the property $C^m_{ijl,m} = 0$. Then the spacetime is a GRW-spacetime and we have*

$$U_{l,k} = f(g_{kl} + U_kU_l) \tag{2.25}$$

assuming that f is a suitable scalar function and U_k is a concircular vector field.

Theorem 7. *In a $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor, the electric part of the conformal curvature tensor tensor is vanishing.*

Proof. In a spacetime, the Weyl tensor is described by the symmetric and traceless tensors E_{ij} and H_{ij} and they have 10 independent components. If we consider the vector field A_i with the condition $A_iA^i = -1$, the electric and magnetic parts for the Weyl tensors are:

$$E_{ij} = A^kA^lC_{kijl}, \quad H_{ij} = A^kA^l\tilde{C}_{kijl} \tag{2.26}$$

where $\tilde{C}_{kijl} = \frac{1}{2}\varepsilon_{kimn}C^{mn}_{jl}$ is the dual, [9]. E_{ij} and H_{ij} satisfy the conditions $E_{ij}A^i = 0$ and $H_{ij}A^i = 0$. Hence, each of them has 5 independent components and they completely describe the Weyl tensor.

If a $(PZS)_4$ spacetime obeys the property $C^m_{ijk,m} = 0$ then, from Proposition 1, [20], this spacetime is a GRW-spacetime. Hence, we have from (2.25)

$$A_{i,j} = f(g_{ij} + A_i A_j). \tag{2.27}$$

This equation means that A_k is in the form (1.8). Taking the covariant derivative of (2.27) and using the equation (2.27) again, we get

$$A_{i,jk} = f_k(g_{ij} + A_i A_j) + f^2\left[(g_{ik} + A_i A_k)A_j + (g_{jk} + A_j A_k)A_i\right] \tag{2.28}$$

where $f_k = f_{,k}$. From [29], we have $f_k = \mu A_k$.

Interchanging the indices in (2.28), subtracting these two equations and assuming the following equation

$$w_k = f_k - f^2 A_k \tag{2.29}$$

where μ is a scalar function, then we obtain from (2.28) and (2.29)

$$A_{i,jk} - A_{i,kj} = w_k g_{ij} - w_j g_{ik}. \tag{2.30}$$

The equation (2.30) reduces to

$$R^h_{ijk} A_h = (\mu - f^2)[g_{ij} A_k - g_{ik} A_j]. \tag{2.31}$$

With the help of (2.26), we can write

$$\begin{aligned} E_{ij} = R_{kijl} A^k A^l - \frac{1}{2}\left(g_{ij} R_{kl} - g_{il} R_{kj} + g_{kl} R_{ij} - g_{kj} R_{il}\right) A^k A^l \\ + \frac{R}{6}\left(g_{ij} g_{kl} - g_{il} g_{jk}\right) A^k A^l. \end{aligned} \tag{2.32}$$

From (2.17), we have

$$R_{ij} = \frac{R}{18} g_{ij} - \frac{7R}{9} A_i A_j. \tag{2.33}$$

By using (2.32), we get

$$E_{ij} = R_{kijl} A^k A^l + \frac{5R}{18}(g_{ij} + A_i A_j). \tag{2.34}$$

Considering the equations (2.31) and (2.34), it can be found that

$$E_{ij} = \left[-(\mu - f^2) + \frac{5R}{18}\right](g_{ij} + A_i A_j). \tag{2.35}$$

On the other hand, from (2.31), we have

$$R_{hk} A^h = 3(\mu - f^2) A_k. \tag{2.36}$$

Finally, by using (2.33), (2.36) reduces to

$$\mu - f^2 = \frac{5R}{18}. \tag{2.37}$$

In this case, with the help of (2.35) and (2.37), it can be found that the electric part of the Weyl tensor is vanishing. Then, we complete the proof. □

Theorem 8. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor has a Riemannian compatible vector field.*

Proof. Considering a $(PZS)_4$ obeying harmonic conformal curvature tensor, using the equation (2.31) and the properties of the curvature tensor known as $R^h_{ijk} = g^{hm}R_{mijk}$, we find

$$R_{mijk}A^m = (\mu - f^2)\left[g_{ij}A_k - g_{ik}A_j\right]. \tag{2.38}$$

In this case, considering the equation (2.38) and $R_{mijk} = R_{kjim}$, we can write

$$(R_{jklm}A^m)A_i + (R_{kilm}A^m)A_j + (R_{ijlm}A^m)A_k = 0. \tag{2.39}$$

Finally, from (2.39), we can write

$$(R_{jklm}A_i + R_{kilm}A_j + R_{ijlm}A_k)A^m = 0. \tag{2.40}$$

Comparing the equations (1.7) and (2.40) gives that this spacetime has a Riemann compatible vector field A_k. □

Theorem 9. *In a $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor, the magnetic part for the Weyl tensor is vanishing.*

Proof. For a $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor, by the aid of the equations (1.12) and (2.18), we have EMT in the form

$$T_{ij} = \gamma g_{ij} + \eta A_i A_j$$

where $\gamma = -\frac{4R}{9k}$ and $\eta = -\frac{7R}{9k}$.

We know that for a $(PZS)_4$ spacetime, if the trace of Z-tensor is non-zero , then this spacetime is Weyl compatible and it is of Weyl compatible EMT, [19]. In addition, on a four-dimensional spacetime admitting a Weyl compatible EMT of the form $T_{jl} = \alpha g_{jl} + \beta A_j A_l$, the magnetic part for the Weyl tensor is vanishing [19].

From the above results, we complete the proof. □

Theorem 10. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor is conformally flat.*

Proof. Theorem 7 and Theorem 9 accomplish the proof. □

Theorem 11. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor is of Petrov type O.*

Proof. The proof follows from the Theorem 10. □

Theorem 12. *A $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor is a warped product written as -$I \times_{q^2} M^*$ such that $M*$ is an 3-dimensional Einstein manifold.*

Proof. Let us consider a $(PZS)_4$ with the property $C^m_{ijl,m} = 0$. Then, this spacetime is a GRW-spacetime, [20].

Since the GRW-spacetime has the proper concircular vector field, then it is the necessary and sufficient condition that a coordinate system exists with respect to which the fundamental quadratic differential form may be given in the following

$$ds^2 = -(dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta \tag{2.41}$$

where $(\alpha, \beta = 2, 3, \ldots, n)$ and q is a scalar function of x^1 only, i.e.,
$q = q(x^1) \neq const$. The first fundamental form for a GRW-spacetime is the form as in (2.41).

Since e^q is a positive function for all q, from (2.41), this spacetime is a warped product space written by

$$g = -dt^2 \oplus f^2(t) g^*$$

here we assume that g^* is the metric tensor of an 3-dimensional Riemannian manifold M^* and f is a smooth function defined by $f : I \to (0, \infty)$ on M^*. The warped product is conformally conservative if and only if M^* is an Einstein manifold, [14]. Hence, the proof is completed. □

Theorem 13. *In a $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor, the metric of this spacetime is conformal to a RW-spacetime.*

Proof. From Theorem 12, a $(PZS)_4$ spacetime admitting harmonic curvature tensor is a warped product spacetime. The warped product is conformally conservative if and only if M^* is an Einstein manifold, [14].

For a warped product spacetime if g^* is a 3-dimensional Riemannian manifold admitting constant curvature, then we get the manifold (M^4, g) and this manifold is called a RW-spacetime. Thus, this result completes the proof. □

Theorem 14. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor is a subprojective space.*

Proof. Assume that $(PZS)_4$ obeys harmonic curvature tensor. Thus, from Theorem 10, our spacetime is conformally flat. Since the associated covector A_k is a timelike vector field, then this space is subprojective, [20]. From this result, we complete the proof. □

Theorem 15. *In a $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor, the flow is geodesic, irrotational and has no shear.*

Proof. From [6], it is known that a four-dimensional perfect fluid spacetime admitting an equation of state given by $p = p(\sigma)$ and divergence-free Weyl conformal curvature tensor is conformally flat, and this spacetime is endowed with the RW metric. Also, the flow is irrotational, geodesic and it has no shear, [6]. From Theorem 4 and the equation (1.5), we complete the proof. □

Theorem 16. *For a perfect fluid $(PZS)_4$ spacetime, if the conditions $\sigma = \frac{T}{3}$ and $p = \frac{4T}{9}$ are satisfied, then the following facts hold for this spacetime:*
It is

(1) *conformally flat and also Petrov type O.*
(2) *a GRW-spacetime and also a RW-spacetime.*
(3) *a subprojective spacetime.*

Proof. We know from [20, Propopositon 3.6], that in a perfect fluid $(PZS)_4$ spacetime, if the conditions $\sigma = \frac{T}{3}$ and $p = \frac{4T}{9}$ are satisfied, then $C^m_{ijl,m} = 0$ on any coordinate domain of $(PZS)_4$. Also, we know from [20], [Prop.5.10] that if an n-dimensional $(n \geq 4)$ spacetime satisfies the property $C^m_{ijl,m} = 0$, then this spacetime is a GRW-spacetime.

From these results and by using Theorem 10, Theorem 11, Theorem 13 and Theorem 14, the proof is clear. □

REFERENCES

[1] L. J. Alias, A. Romero , and M. Sánchez, *Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes* . VII River Edge NJ,USA: World Sci.Publ., 1995. doi: 10.1007/BF02105675.

[2] L. J. Alias, A. Romero, and M. Sanchez, "Uniqueness of complete spacelike hypersurfaces of mean curvature in generalized Robertson-Walker spacetimes ." *Gen.Relativ.Gravit*, vol. 27, no. 1, pp. 71–84, 1995, doi: 10.1007/BF02105675.

[3] A. M. Blaga, "Solitons and geometrical structures in a perfect fluid spacetime," *Rocky Mountain J.Math.*, vol. 50, no. 1, pp. 41–53, 2020, doi: 10.1216/rmj.2020.50.41.

[4] B. Y. Chen, "A simple characterization of generalized Robertson-Walker spacetimes," *Gen.Relativ.Gravit.*, vol. 46, p. 1833(5 pages), 2014, doi: 10.1007/s10714-014-1833-9.

[5] S. W. Hawking and G. F. R. Ellis, *The large-scale structures of spacetimes*. Cambridge: Cambridge Univ.Press, 1973. doi: 10.1017/CB09780511524646.

[6] L. C. Shepley and A. H. Taub, "Spacetimes containing perfect fluids and having a vanishing conformal divergence," *Commun.Math.Phys.*, vol. 5, pp. 237–256, 1967, doi: 10.1007/BF01646477.

[7] H. Stephani, M. Maccallum, C. Hoenselaers, and E. Herit, *Exact solutions of Einstein's field equations.* Cambridge Monographs on Mathematical Phys.: Cambridge Univ.Press, 2003. doi: 10.1017/CB09780511535185.

[8] A. V. Aminova, "Groups of transformations of Riemannian manifolds," *J.Sov.Math.*, vol. 55, no. 5, pp. 1996–2041, 1991, doi: 10.1007/BF01095673.

[9] E. Bertschinger and A. J. Hamilton, "Lagrangian evolution of the Weyl tensor," *Astroph.J.*, vol. 435, pp. 1–7, 1994.

[10] A. L. Besse, *Einstein manifolds*. Berlin: Springer Verlag, 1987. doi: 10.1007/978-3-540-74311-8.

[11] E. Cartan, "Sur une classe remaquable d'espaces de Riemann," *Bull.Soc.Math.France*, vol. 54, pp. 214–264, 1926, doi: 10.24033/bsmf.1105.

[12] M. C. Chaki, "On pseudo symmetric manifolds ." *An.Stiint.Univ.Al.I.Cuza Iaşi.Sct.I. a Mat*, vol. 33, no. 1, pp. 53–58, 1987.

[13] F. de Felice and C. J. S. Clarke, *Relativity on curved manifolds*. Cambridge: Cambridge University Press, 1990.

[14] A. Gebarowski, “Nearly conformally symmetric warped product manifolds,” *Bull.Inst.Math.Acad.Sinica*, vol. 20, pp. 359–371, 1992, doi: 1573668924633338240.
[15] J. Kowolik, “On some Riemannian manifolds admitting torse-forming vector fields,” *Dem.Math.*, vol. 18, no. 3, pp. 885–891, 1985, doi: 10.1515/dema-1985-0319.
[16] C. A. Mantica and L. G. Molinari, “Weakly Z-symmetric manifold,” *Acta Math.Hungar.*, vol. 135, pp. 80–96, 2012, doi: 10.1007/s10474-011-0166-3.
[17] C. A. Mantica, L. G. Molinari, Y. J. Suh, and S. Shenawy, “Perfect fluid generalized Robertson-Walker space-times and Gray’s decomposition,” *J.Math.Phys.*, vol. 60, p. 052506, 2019, doi: 10.1063/1.5089040.
[18] C. A. Mantica and Y. J. Suh, “Pseudo Z-symmetric riemannian manifolds with harmonic curvature tensors,” *Int.J.Geom.Meth.Mod.Phys.*, vol. 9, no. 1, pp. 278–286, 2012, doi: 10.1142/S0219887812500041.
[19] C. A. Mantica and Y. J. Suh, “Pseudo Z-symmetric spacetimes,” *J.Math.Phys.*, vol. 55, no. 4, p. Article ID.042502, 2014, doi: 10.1063/1.4871442.
[20] C. A. Mantica and Y. J. Suh, “Pseudo Z-symmetric space-times with divergence-free Weyl tensor and pp-waves,” *Int.J.Geom.Meth.Mod.Phys.*, vol. 13, no. 2, p. Article ID.1650015, 2016, doi: 10.1142/S0219887816500158.
[21] O’Neil.
[22] F. Ozen Zengin, “On riemannian manifolds admitting w_2 curvature tensor,” *Miskolc Math.Notes*, vol. 12, no. 2, pp. 289–296, 2011, doi: 10.18514/MMN.2011.332.
[23] W. Roter, “On a generalization of conformally symmetric metrics,” *Tensor(N.S)*, vol. 46, pp. 278–286, 1987.
[24] R. K. Sachs and W. Hung-Hsi, *General relativity for a mathematician*. New-York: Springer-Verlag, 1977. doi: 10.1007/978-1-4612-9903-5.
[25] M. Sanchez, “On the geometry of generalized Robertson-Walker spacetimes: geodesics,” *Gen.Relativ.Gravit.*, vol. 30, pp. 915–932, 1998, doi: 10.1023/A:1026664209847.
[26] R. N. Sen and M. C. Chaki, “On curvature restrictions of a certain kind of conformally-flat Riemannian space of class one.” *Proc.Nat.Ins.Sci.,India Part A*, vol. 33, pp. 100–102, 1967.
[27] G. Soos, “Über die geodátischen abbildungen von Riemanschen ráumen and projectiv-symmetrische Riemannsche ráume.” *Acta Math.Acad.Sci.Hungar*, vol. 9, pp. 359–361, 1958, doi: 10.1007/BF02020266.
[28] K. Yano and T. Adati, “On certain spaces admitting concircular transformation.” *Proc.Japan Acad.*, vol. 25, pp. 188–195, 1949, doi: 10.3792/pja/1195571941.
[29] P. Zhao, U. C. De, B. Ünal, and K. De, “Sufficient conditions for a pseudo symmetric spacetime to be a perfect fluid spacetime,” *Int.J.Geom.Meth.Mod.Phys.*, vol. 18, no. 13, p. 2150217, 2021, doi: 10.1142/S0219887821502170.

Authors’ addresses

Sezgin Altay Demirbag
Istanbul Technical University, Department of Mathematics, 34349 Istanbul, Turkey
E-mail address: saltay@email.com

Uday Chand De
University of Calcutta, Department of Pure Mathematics, 700019 West Bengal, India
E-mail address: uc_de@yahoo.com

Fusun Ozen Zengin
(**Corresponding author**) Istanbul Technical University, Department of Mathematics 34349, Istanbul Turkey
E-mail address: fozen@itu.edu.tr