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ON A SEQUENCE OF RATIONAL NUMBERS WITH UNUSUAL DIVISIBILITY BY A POWER OF 2

ARTŪRAS DUBICKAS

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Abstract. In this note we consider the sequence of rational numbers $b_n = \sum_{k=1}^{n} 2^k/k$. We show that the power of 2 in the expansion of b_n is unusually large, at least $n+1-\log_2(n+1)$, and that this bound is best possible. The sequence b_n, $n = 1,2,3,\ldots$, is related to the sequence A0031449 in the On-Line Encyclopedia of Integer Sequences.

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1. Introduction

In [6] Farhi considered the following sequence $a_1 = 1$,

$$a_n = \frac{na_{n-1}}{2} + (n-1)! \tag{1.1}$$

for $n = 2,3,\ldots$. Set also

$$b_n = \sum_{k=1}^{n} \frac{2^k}{k} \tag{1.2}$$

for $n \in \mathbb{N}$. Then, by $b_n - b_{n-1} = 2^n/n$ and (1.1), we obtain

$$a_n = \frac{n!}{2^n} b_n \tag{1.3}$$

for each $n \in \mathbb{N}$.

In [6], expressing a_n in terms of Genocchi numbers and Stirling numbers of the first kind, Farhi showed that

$$a_n \in \mathbb{N} \tag{1.4}$$

for each $n \in \mathbb{N}$. This, according to the definition of a_n in (1.1), is nontrivial and in some sense reminds the surprising integrality conditions of so-called Somos sequences (see [14] and also some subsequent work in [8, 11, 17, 18]). The fractional parts of the sequence $\frac{2^n}{n}$, $n = 1,2,3,\ldots$, were considered in [4, 5].

Of course, there are several alternative ways to prove (1.4) which are simpler than that in [6]. This was observed by Farhi in a subsequent paper [7]. For example, by the identity

$$\sum_{k=0}^{n} \frac{k!(n-k)!}{n!} = \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1} \tag{1.5}$$

(see [12, 15]), using (1.2) and (1.3) we find that

$$\sum_{k=0}^{n} k!(n-k)! = \frac{(n+1)!}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1} = \frac{(n+1)!}{2^{n+1}} b_{n+1} = a_{n+1},$$

which implies (1.4). In fact, $\sum_{k=0}^{n} k!(n-k)!$, $n = 1, 2, 3, \ldots$, is the sequence A0031449 in [13].

For a prime number p and a positive integer u by $\nu_p(u)$ we denote the largest nonnegative integer k for which p^k divides u. Likewise, for a rational $r = u/v$, where $u, v \in \mathbb{N}$ are relatively prime integers, we set $\nu_p(r) = \nu_p(u) - \nu_p(v)$.

With this notation in [6, Corollary 2.5] it was also shown that

$$\nu_2\Big(\sum_{k=1}^{n} \frac{2^k}{k}\Big) \geq s_2(n), \tag{1.6}$$

where $s_2(n)$ is the sum of digits of n in base 2. This was improved in [7, Theorem 2.5], where it was shown that

$$\nu_2\Big(\sum_{k=1}^{n} \frac{2^k}{k}\Big) \geq n - \lfloor \log_2 n \rfloor \tag{1.7}$$

for each $n \in \mathbb{N}$.

Now, we will refine the estimates (1.6), (1.7) and obtain a sharp bound.

Theorem 1. *For each $n \in \mathbb{N}$ we have*

$$\nu_2\Big(\sum_{k=1}^{n} \frac{2^k}{k}\Big) \geq n + 1 - \log_2(n+1), \tag{1.8}$$

with equality if and only if $n = 2^k - 1$ for $k \in \mathbb{N}$.

Note that

$$n + 1 - \log_2(n+1) \geq n - \lfloor \log_2 n \rfloor. \tag{1.9}$$

Indeed, choose a unique integer $k \geq 0$ satisfying $2^k \leq n < 2^{k+1}$. Then, $1 + \lfloor \log_2 n \rfloor = k + 1$ and $\log_2(n+1) \leq \log_2(2^{k+1}) = k + 1$, which proves (1.9).

From the above proof of (1.9) we see that the right hand sides of (1.7) and (1.8) are equal only if n has the form $n = 2^k - 1$. We will derive (1.8) from the inequality (2.3) below, which is stronger than (1.8) for many $n \in \mathbb{N}$ that are not of the form $2^k - 1$. The nontrivial part is to show that for $n = 2^k - 1$ one has equality in (1.8). (The proof of (1.7) in [7] is entirely different: it uses (1.5) and some other identity.) The proof of Theorem 1 is self-contained except that we need any version of the fact that the sequence $\nu_2(b_n)$, $n = 1,2,3,\dots$, is unbounded as $n \to \infty$.

Let D_n be the least common multiple of $1,2,3,\dots,n$. By (1.2), it is clear that $D_n b_n \in \mathbb{N}$ for each $n \in \mathbb{N}$. From Theorem 1 we will derive the following corollary which strengthens (1.4) (since (1.4) holds even if the factor $n!$ in (1.3) is replaced by D_n).

Corollary 1. *For each $n \in \mathbb{N}$ we have*

$$\frac{D_n b_n}{2^n} = D_n \sum_{k=1}^{n} \frac{2^{k-n}}{k} \in \mathbb{N}. \tag{1.10}$$

Finally, we remark that in [15] Sury proved not only (1.5) but also the identity

$$\sum_{k=1}^{n} \frac{2^k}{k} = 2\left(\binom{n}{1} + \frac{1}{3}\binom{n}{3} + \frac{1}{5}\binom{n}{5} + \frac{1}{7}\binom{n}{7} + \dots\right). \tag{1.11}$$

See also [1–3, 9, 10, 16] for some related identities. For example, in [10] Meštrović showed that

$$\sum_{k=1}^{n} \frac{2^k - 1}{k} = \sum_{k=1}^{n} \frac{1}{k}\binom{n}{k}. \tag{1.12}$$

We conclude with a simple generalization of (1.11) and (1.12).

Theorem 2. *For each $n \in \mathbb{N}$ and each $z \in \mathbb{C}$ we have*

$$\sum_{k=1}^{n} \frac{(1+z)^k - (1-z)^k}{k} = 2\left(z\binom{n}{1} + \frac{z^3}{3}\binom{n}{3} + \frac{z^5}{5}\binom{n}{5} + \frac{z^7}{7}\binom{n}{7} + \dots\right)$$

and

$$\sum_{k=1}^{n} \frac{(1+z)^k - 1}{k} = \sum_{k=1}^{n} \frac{z^k}{k}\binom{n}{k}. \tag{1.13}$$

In particular, we give a very simple proof of (1.13), which shows that the identities (1.11), (1.12) and [10, Corollary 1.2] are not so mysterious as it may appear from their proofs in [10]. They come by inserting $z = 1$ into the natural identities of Theorem 2.

2. Proofs

Proof of Theorem 1. For $n \in \mathbb{N}$ put

$$d_n = \nu_2(b_n),$$

where b_n is defined in (1.2). Then, $b_n = 2^{d_n} u_n / v_n$, where u_n and v_n are odd coprime positive integers. Note that $d_1 = 1$, $d_2 = 2$, $\dots$, etc. We have

$$b_{n+1} - b_n = \frac{2^{n+1}}{n+1} = \frac{2^{n+1-\nu_2(n+1)}}{(n+1)2^{-\nu_2(n+1)}},$$

where $n+1-\nu_2(n+1)>0$. From

$$b_{n+1}=\frac{2^{d_n}u_n}{v_n}+\frac{2^{n+1-\nu_2(n+1)}}{(n+1)2^{-\nu_2(n+1)}}$$

and the fact that the integer $(n+1)2^{-\nu_2(n+1)}$ is odd, we find that

$$d_{n+1}=\min(d_n,n+1-\nu_2(n+1)) \tag{2.1}$$

if

$$d_n\neq n+1-\nu_2(n+1), \tag{2.2}$$

and

$$d_{n+1}>d_n$$

if

$$d_n=n+1-\nu_2(n+1).$$

We claim that for each $n\in\mathbb{N}$

$$d_n\geq\min_{k\geq n}(k+1-\nu_2(k+1)). \tag{2.3}$$

Indeed, if for some $n\in\mathbb{N}$ the inequality opposite to (2.3) holds then d_n is less than $k+1-\nu_2(k+1)$ for $k=n,n+1,n+2,\dots$. Then, by (2.1) and (2.2), we should have $d_n=d_{n+1}=d_{n+2}=\dots$. Thus, the sequence d_k, $k=1,2,3,\dots$, is bounded, which is impossible by (1.6) or (1.7). This proves (2.3).

Next, since $u\geq 2^{\nu_2(u)}$, for any $u\in\mathbb{N}$ we have $\nu_2(u)\leq\log_2(u)$. Hence, from (2.3) we get $d_n\geq\min_{k\geq n}(k+1-\log_2(k+1))$. The function $f(x)=x+1-\log_2(x+1)$ is increasing for $x\geq 1$. So, for any $n\in\mathbb{N}$, the smallest value of the function $f(x)$ in the set $x\in\{n,n+1,n+2,\dots\}$ is attained at $x=n$. Thus, $d_n\geq n+1-\log_2(n+1)$, which is (1.8).

Further, equality in (1.8) can only hold if $\log_2(n+1)$ is an integer. For $n\in\mathbb{N}$ this happens for $n=2^k-1$, where $k\in\mathbb{N}$, only. So the values $n=2^k-1$, $k=1,2,3,\dots$, are the only values for which equality in (1.8) can possibly be attained. We will show that it is always attained, namely,

$$d_{2^k-1}=\nu_2(b_{2^k-1})=2^k-k \tag{2.4}$$

for every $k\in\mathbb{N}$.

Fix any $k\in\mathbb{N}$. For a contradiction assume that $d_{2^k-1}\neq 2^k-k$, so that (2.2) is true for $n=2^k-1$. Then, by (2.1), we must have

$$d_{2^k}=\min(d_{2^k-1},2^k-k)\leq 2^k-k.$$

However, by (1.8) and $2^k+1<2^{k+1}$, it follows that

$$d_{2^k}\geq 2^k+1-\log_2(2^k+1)>2^k+1-(k+1)=2^k-k,$$

which contradicts to the previous inequality. This rules out the possibility $d_{2^k-1}\neq 2^k-k$ and so proves (2.4). □

Proof of Corollary 1. Let L_n be the least common multiple of all odd integers between 1 and n. By (1.2), it is clear that $L_n b_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. Furthermore, since L_n is an odd integer, this implies $L_n b_n 2^{-\nu_2(b_n)} \in \mathbb{N}$. Next, in view of $D_n = L_n 2^{\lfloor \log_2 n \rfloor}$ we obtain

$$D_n b_n 2^{-\nu_2(b_n) - \lfloor \log_2 n \rfloor} \in \mathbb{N},$$

which implies (1.10) provided that $\nu_2(b_n) + \lfloor \log_2 n \rfloor \geq n$. However, by Theorem 1, we already know that $\nu_2(b_n) \geq n + 1 - \log_2(n+1)$. This completes the proof of the corollary by (1.9). (Of course, as observed by the referee, $\nu_2(b_n) + \lfloor \log_2 n \rfloor \geq n$ already holds by Farhi's inequality (1.7).) □

Proof of Theorem 2. In order to prove (1.13) we fix $n \in \mathbb{N}$ and set

$$f(z) = \sum_{k=1}^{n} \frac{(1+z)^k - 1}{k} - \sum_{k=1}^{n} \frac{z^k}{k} \binom{n}{k}.$$

Then,

$$f'(z) = \sum_{k=1}^{n} (1+z)^{k-1} - \sum_{k=1}^{n} z^{k-1} \binom{n}{k} = \frac{(1+z)^n - 1}{(1+z) - 1} - \frac{(1+z)^n - 1}{z} = 0$$

for $z \neq 0$. Inserting $z = 0$ we obtain $f'(0) = n - \binom{n}{1} = 0$ as well. Hence, $f'(z) = 0$ for each $z \in \mathbb{C}$, which implies that $f(z)$ is a constant. From $f(0) = 0$ we conclude that $f(z) = 0$ for each $z \in \mathbb{C}$. This proves (1.13).

Clearly, (1.13) also implies the first identity of this theorem by subtracting (1.13) with z replaced by $-z$ from (1.13) with z itself. □

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Author's address

Artūras Dubickas

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania

E-mail address: arturas.dubickas@mif.vu.lt