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REPRESENTATION OF GENERALIZED CIRCULAR SURFACES AS RIGID BODY MOTIONS

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Abstract. In this paper, we introduce generalized circular surfaces, a generalization of generalized tube surfaces and circular surfaces. Moreover, we define a special dual quaternion by using the moving frame along the spine curve of generalized circular surface. We then show that the screw motion obtained by this dual quaternion can be used to construct generalized circular surfaces. We also prove that these generalized circular surfaces can be expressed by homothetic motions. Finally, we provide some examples of generalized circular surfaces with figures.

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Keywords: tube surfaces, generalized tube surfaces, circular surfaces, dual quaternion surfaces, rigid body motions (screw motions)

1. INTRODUCTION

A tube surface or a pipe surface is formed by the envelope of the spheres whose centers lie on a curve in three-dimensional real vector space $\mathbb{R}^3$ (which is called a spine curve) and whose radius are constant [15]. Generalized tube surfaces [16] and circular surfaces [19] were defined by generalizing tube surfaces according to different variables, then these kind surfaces attracted attention of many researchers. There are many objects in the world take form of these surfaces.

Real quaternions are defined as a four-dimensional number system [17]. This number system has a practical method in the performing of rotation since they rotate a vector around any axis in $\mathbb{R}^3$ [24]. Recently, real quaternionic and matrix representations of surfaces have been extensively studied by many researchers. [2,3,5,6,9,23,26,27]. Similar problem has been also considered in Minkowski and (pseudo-) Galilean spaces by using split quaternions [4,13,21] and (split-) semi quaternions [28].

Dual quaternions are introduced as a number system isomorphic to the tensor product of real quaternions and dual numbers [10]. In the following years, this number system has been used in the rigid body motions (i.e., screw motions) [1,7,14,18, 29]. Dual quaternions have been applied in dynamics, computer graphics, robotics and spacecrafts, etc.

In this paper, we introduce generalized circular surface by generalizing of a generalized tube surface and a circular surface. A generalized circular surface (i.e., a tube surface, a generalized tube surface or a circular surface) is constituted of a spine curve and a rotation part. In [3], the rotation part of a tube surface was generated by a real quaternion. We combine the spine curve and the real quaternion in a dual quaternion. To do this, we define a dual quaternion whose translation part is a spine curve in $\mathbb{R}^3$ and rotation part is a real quaternion. Then, we prove that screw motion obtained by this dual quaternion generates a tube surface, a circular surface, a generalized tube surface or a generalized circular surface in $\mathbb{R}^3$.

2. Preliminaries

In this section, some basic concepts will be given to provide a background to the main results of this paper.

Symbols

a :	Scalar	A :	Dual number
a :	Real vector in $\mathbb{R}^3$	$\hat{A}$:	Dual vector
d :	Line in $\mathbb{R}^3$	q :	Real quaternion
θ :	Real angle	Q :	Dual quaternion

2.1. *Real quaternions*

A real quaternion can be represented as

$$q = a_0 + a_1 i + a_2 j + a_3 k, \tag{2.1}$$

where a_0, a_1, a_2, a_3 are real numbers and i, j, k are mutually perpendicular unit vectors satisfying $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. The set of real quaternions is often represented by H.

A real quaternion q can be also expressed as $q = S(q) + V(q)$, where $S(q) = a_0$ is the scalar part and $V(q) = a_1 i + a_2 j + a_3 k$ is the vector part of q. If $S(q) = 0$, then q is called a pure real quaternion (i.e., a real vector in $\mathbb{R}^3$). Quaternion product of real quaternions $q = S(q) + V(q)$ and $p = S(p) + V(p)$ is

$$q \star p = S(q)S(p) - \langle V(q), V(p)\rangle + S(q)V(p) + S(p)V(q) + V(q) \times V(p), \tag{2.2}$$

where $\langle , \rangle$ and $\times$ denote the usual scalar and vector products in $\mathbb{R}^3$, respectively.

The quaternionic-conjugate of q can be defined as

$$\bar{q} = S(q) - V(q) = a_0 - a_1 i - a_2 j - a_3 k, \tag{2.3}$$

For the real quaternions p and q, the equality

$$\overline{p \star q} = \bar{q} \star \bar{p} \tag{2.4}$$

can be given. The norm of q can be defined as

$$N(q) = q \star \bar{q} = \bar{q} \star q = a_0^2 + a_1^2 + a_2^2 + a_3^2. \tag{2.5}$$

q is called a unit real quaternion if $N(q) = 1$. A unit real quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ for $a_1^2 + a_2^2 + a_3^2 \neq 0$ can be expressed in the form $q = \cos\theta + \sin\theta v$, where $\cos\theta = a_0$, $\sin\theta = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $v = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$.

Let $p = a_0 + a_1 i + a_2 j + a_3 k$ be a unit real quaternion and w be a vector in $\mathbb{R}^3$ (i.e., a pure quaternion). Then,

$$\varphi : \mathbb{R}^3 \to \mathbb{R}^3; \quad \varphi(w) = p \star w \star p^{-1} = p \star w \star \bar{p}. \tag{2.6}$$

is a linear mapping. Matrix representation of this mapping can be given as

$$M = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 + a_2^2 - a_1^2 - a_3^2 & -2a_0a_1 + 2a_2a_3 \\ -2a_0a_2 + 2a_1a_3 & 2a_0a_1 + 2a_2a_3 & a_0^2 + a_3^2 - a_2^2 - a_1^2 \end{bmatrix}, \tag{2.7}$$

which is orthogonal and represents a rotation in $\mathbb{R}^3$. Thus, linear mapping φ can be given as

$$\varphi(w) = p \star w \star \bar{p} = Mw. \tag{2.8}$$

If p is in the form $p = \cos\theta + \sin\theta v$, then v is the rotation axis of the rotation in $\mathbb{R}^3$. Moreover, $\varphi(w) = p \star w \star \bar{p} = Mw$ rotates the vector w in $\mathbb{R}^3$ around the vector v by a real angle 2θ.

For more information about real quaternions, see [7, 17, 24].

A homothetic motion in $\mathbb{R}^3$ can be defined by

$$y(s,t) = h(s,t)M(s,t)x(s,t) + n(s,t), \tag{2.9}$$

where y is the position vector of a point in the fixed space $\mathbb{R}'$ and x is the position vector of the moving space $\mathbb{R}$. M is an orthogonal matrix, n is a translation vector and h is a scalar and s, t are real parameters [8, 20].

2.2. *Dual quaternions*

A dual number $A = a_0 + \varepsilon a_1$ is constituted of two real numbers a_0, a_1 whose real unit is 1 and dual unit is $\varepsilon \neq 0$ satisfying $\varepsilon^2 = 0$. Addition and multiplication rules of $A = a_0 + \varepsilon a_1$ and $B = b_0 + \varepsilon b_1$ are defined by

$$A + B = (a_0 + b_0) + \varepsilon(a_1 + b_1), \tag{2.10}$$

$$AB = a_0b_0 + \varepsilon(a_0b_1 + a_1b_0), \tag{2.11}$$

respectively. The dual conjugate of a dual number $A = a_0 + \varepsilon a_1$ is

$$A^* = a_0 - \varepsilon a_1. \tag{2.12}$$

Module $\mathbb{D}^3$ on dual numbers is given by

$$\mathbb{D}^3 = \left\{ \hat{A} = a_0 + \varepsilon a_1 : a_0, a_1 \in \mathbb{R}^3 \right\}. \tag{2.13}$$

Each element of $\mathbb{D}^3$ is called a dual vector. The scalar and vector products of $\hat{A} = a_0 + \varepsilon a_1$ and $\hat{B} = b_0 + \varepsilon b_1$ are defined respectively by

$$\left\langle \hat{A}, \hat{B} \right\rangle = \langle a_0, b_0 \rangle + \varepsilon \left(\langle a_0, b_1 \rangle + \langle a_1, b_0 \rangle \right), \tag{2.14}$$

$$\hat{A} \times \hat{B} = a_0 \times b_0 + \varepsilon \left(a_0 \times b_1 + a_1 \times b_0 \right). \tag{2.15}$$

For further information about dual numbers, see [10, 22, 25, 29].

A dual quaternion can be represented as

$$Q = A_0 + A_1 i + A_2 j + A_3 k, \tag{2.16}$$

where A_0, A_1, A_2, A_3 are dual numbers and i, j, k are the same mutually perpendicular unit vectors as in real quaternions. The set of dual quaternions is often represented by $\mathbb{H}$.

An alternative representation of a dual quaternion can be given as

$$Q = S(Q) + V(Q), \tag{2.17}$$

where $S(Q) = A_0$ and $V(Q) = A_1 i + A_2 j + A_3 k$ are, respectively, the scalar and the vector parts of Q. If $S(Q) = 0$, then Q is called a pure dual quaternion (i.e., a dual vector in $\mathbb{D}^3$). Quaternion product of any two dual quaternions $Q = S(Q) + V(Q)$ and $P = S(P) + V(P)$ is introduced as

$$\begin{aligned} Q \star P = S(Q)S(P) - \langle V(Q), V(P) \rangle + S(Q)V(P) \\ + S(P)V(Q) + V(Q) \times V(P). \end{aligned} \tag{2.18}$$

The dual conjugate, quaternionic conjugate, quaternionic-dual conjugate and norm of a dual quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k = q_0 + \varepsilon q_1$, where $q_0 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_1 = a_0^* + a_1^* i + a_2^* j + a_3^* k$ for $A_0 = a_0 + \varepsilon a_0^*$, $A_1 = a_1 + \varepsilon a_1^*$, $A_2 = a_2 + \varepsilon a_2^*$ and $A_3 = a_3 + \varepsilon a_3^*$, can be given as

$$Q^* = A_0^* + A_1^* i + A_2^* j + A_3^* k = q_0 - \varepsilon q_1, \tag{2.19}$$

$$\bar{Q} = A_0 - A_1 i - A_2 j - A_3 k = \bar{q}_0 + \varepsilon \bar{q}_1, \tag{2.20}$$

$$\bar{Q}^* = A_0^* - A_1^* i - A_2^* j - A_3^* k = \bar{q}_0 - \varepsilon \bar{q}_1, \tag{2.21}$$

$$N(Q) = Q \star \bar{Q} = \bar{Q} \star Q = A_0^2 + A_1^2 + A_2^2 + A_3^2, \tag{2.22}$$

respectively. If $N(Q) = 1$, then Q is called a unit dual quaternion.

For more information about dual quaternions, see [1, 7, 10, 14, 18, 29].

3. Generalized circular surfaces

In this section, we define generalized circular surfaces as a generalization of generalized tube surfaces and circular surfaces. Then, we show that these surfaces are constituted of a spine curve and a rotation part. Moreover, we show that the rotation part is generated by a real quaternion.

Definition 1. (generalized circular surface) Let $a_1(t)$, $a_2(t)$ and $a_3(t)$ be orthonormal vector fields and $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be an arbitrary frame on a curve $\alpha(t)$ in $\mathbb{R}^3$ such as Frenet frame, Darboux frame, Bishop frame, moving frame, etc. Then a generalized circular surface can be parameterized as

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)(\cos\theta a_1(t) + \sin\theta a_2(t)) \in \mathbb{R}^3, \tag{3.1}$$

where $r(t,\theta) \in \mathbb{R}$ is a real variable determined by the real parameters $t, \theta \in \mathbb{R}$.

A generalized circular surface can be categorized as:

(1) If $r(t,\theta)$ is a real constant $r \in \mathbb{R}$, then

$$\phi(t,\theta) = \alpha(t) + r(\cos\theta a_1(t) + \sin\theta a_2(t)) \tag{3.2}$$

represents a tube surface in $\mathbb{R}^3$ [15].

(2) If $r(t,\theta)$ is a real variable $r(\theta) \in \mathbb{R}$, then

$$\phi(t,\theta) = \alpha(t) + r(\theta)(\cos\theta a_1(t) + \sin\theta a_2(t)) \tag{3.3}$$

represents a generalized tube surface in $\mathbb{R}^3$ [16].

(3) If $r(t,\theta)$ is a real variable $r(t) \in \mathbb{R}$, then

$$\phi(t,\theta) = \alpha(t) + r(t)(\cos\theta a_1(t) + \sin\theta a_2(t)) \tag{3.4}$$

represents a circular surface in $\mathbb{R}^3$ [19].

For further information about tube surfaces, generalized tube surfaces and circular surfaces, see [11, 12, 15, 16, 19].

Remark 1. A generalized circular surface is constituted of a spine curve $\alpha(t)$ and a rotation part $R(t,\theta)$ as

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)R(t,\theta) \in \mathbb{R}^3, \quad r(t,\theta) \in \mathbb{R} \tag{3.5}$$

where $R(t,\theta) = \cos\theta a_1(t) + \sin\theta a_2(t)$. In [3], the rotation part were generated by the unit real quaternion $p(t,\theta) = \cos\theta + \sin\theta a_3(t)$ as

$$R(t,\theta) = p(t,\theta) \star a_1(t), \tag{3.6}$$

where $a_1(t)$ is a pure real quaternion. $R(t,\theta) = p(t,\theta) \star a_1(t)$ represents a rotation performed by the real quaternions. Thus, the generalized circular surface can be expressed as

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)p(t,\theta) \star a_1(t). \tag{3.7}$$

4. Representation of generalized circular surfaces as screw motions

In this section, we define a dual quaternion surface whose rotation part is a unit real quaternion surface and translation part is a spine curve $\alpha(t)$ in $\mathbb{R}^3$. We show that screw motion obtained by this dual quaternion surface constructs a generalized circular surface $\phi(t,\theta)$ in $\mathbb{R}^3$.

Let $Q_w = 1 + \varepsilon w$ be a dual quaternion corresponding to the real vector $w \in \mathbb{R}^3$. A dual quaternion P can be expressed as

$$P = p + \frac{\varepsilon}{2}\alpha \star p, \tag{4.1}$$

where p is a unit real quaternion and α is a real vector (i.e., a pure real quaternion). p and α represent rotation and translation parts of P, respectively. Then the mapping

$$\Psi : \mathbb{H} \to \mathbb{H}, \quad \Psi(Q_w) = P \star Q_w \star \bar{P}^* \tag{4.2}$$

can be given as

$$\begin{aligned} \Psi(Q_w) &= P \star Q_w \star \bar{P}^* \\ &= \left(p + \frac{\varepsilon}{2}\alpha \star p\right) \star (1 + \varepsilon w) \star \left(\bar{p} - \frac{\varepsilon}{2}\bar{p} \star \bar{\alpha}\right) \\ &= 1 + \varepsilon\left(p \star w \star \bar{p} + \alpha\right), \end{aligned} \tag{4.3}$$

where $\Psi(Q_w) = 1 + \varepsilon(p \star w \star \bar{p} + \alpha)$ is a dual quaternion corresponding to the vector $p \star w \star \bar{p} + \alpha$ in $\mathbb{R}^3$. If $p = \cos\theta + \sin\theta v$, then $p \star w \star \bar{p} + \alpha$ means that $p \star w \star \bar{p}$ (which is given by Eqs. (2.6)-(2.8)) rotates the vector w in $\mathbb{R}^3$ around the axis v by an angle 2θ, and afterwards $p \star w \star \bar{p} + \alpha$ translates the vector $p \star w \star \bar{p}$ along the same axis v by translation vector α [1, 7, 14, 18, 29].

Definition 2. Let $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be a moving frame on a curve $\alpha(t)$ in $\mathbb{R}^3$. Then, using unit real quaternion surface $p(t,\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}a_3(t)$, dual quaternion surface can be defined as

$$P(t,\theta) = p(t,\theta) + \frac{\varepsilon}{2}\alpha(t) \star p(t,\theta), \tag{4.4}$$

where $p(t,\theta)$ is the rotation part of $P(t,\theta)$ and $\alpha(t)$ is the translation part of $P(t,\theta)$.

Theorem 1. *Let $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be a moving frame on a curve $\alpha(t)$ in $\mathbb{R}^3$ and $p(t,\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}a_3(t)$ be a unit real quaternion surface. For the dual quaternions $Q_{r(t,\theta)a_1(t)} = 1 + \varepsilon r(t,\theta)a_1(t)$ and $P(t,\theta) = p(t,\theta) + \frac{\varepsilon}{2}\alpha(t) \star p(t,\theta)$, the screw motion*

$$\Psi(Q_{r(t,\theta)a_1(t)}) = P(t,\theta) \star Q_{r(t,\theta)a_1(t)} \star \bar{P}^*(t,\theta) \tag{4.5}$$

generates the generalized circular surface

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)(\cos\theta a_1(t) + \sin\theta a_2(t)) \in \mathbb{R}^3, \;\; r(t,\theta) \in \mathbb{R}. \tag{4.6}$$

Proof. We will not use the parameters t and θ for simplicity.

Using Eq. (4.3), Eq. (4.5) can be expressed as

$$\begin{aligned} \Psi(Q_{ra_1}) &= P \star Q_{ra_1} \star \bar{P}^* \\ &= P \star (1 + \varepsilon r a_1) \star \bar{P}^* \\ &= 1 + \varepsilon\left(r p \star a_1 \star \bar{p} + \alpha\right). \end{aligned} \tag{4.7}$$

Here $\Psi(Q_{ra_1}) = 1+\varepsilon(rp\star a_1\star\bar{p}+\alpha)$ is a dual quaternion corresponding to the following equation

$$rp\star a_1\star\bar{p}+\alpha\in\mathbb{R}^3. \tag{4.8}$$

From Eq. (2.2), we obtain

$$\begin{aligned} p\star a_1\star\bar{p} &= \left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}a_3\right)\star(0+a_1)\star\bar{p} \\ &= \left(-\sin\frac{\theta}{2}\langle a_3,a_1\rangle+\cos\frac{\theta}{2}a_1+\sin\frac{\theta}{2}a_3\times a_1\right)\star\bar{p}. \end{aligned} \tag{4.9}$$

Since $\langle a_3,a_1\rangle=0$ and $a_3\times a_1=a_2$, we get

$$\begin{aligned} p\star a_1\star\bar{p} &= \left(\cos\frac{\theta}{2}a_1+\sin\frac{\theta}{2}a_2\right)\star\bar{p} \\ &= \left(0+\cos\frac{\theta}{2}a_1+\sin\frac{\theta}{2}a_2\right)\star\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}a_3\right) \\ &= -\left\langle\cos\frac{\theta}{2}a_1+\sin\frac{\theta}{2}a_2,-\sin\frac{\theta}{2}a_3\right\rangle \\ &\quad+\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}a_1+\sin\frac{\theta}{2}a_2\right) \\ &\quad-\left(\cos\frac{\theta}{2}a_1+\sin\frac{\theta}{2}a_2\right)\times\sin\frac{\theta}{2}a_3. \end{aligned} \tag{4.10}$$

Since $\langle a_1,a_3\rangle=\langle a_2,a_3\rangle=0$, $a_1\times a_3=-a_2$ and $a_2\times a_3=a_1$, we get

$$\begin{aligned} p\star a_1\star\bar{p} &= \cos^2\frac{\theta}{2}a_1+\cos\frac{\theta}{2}\sin\frac{\theta}{2}a_2 \\ &\quad-\cos\frac{\theta}{2}\sin\frac{\theta}{2}a_1\times a_3-\sin^2\frac{\theta}{2}a_2\times a_3 \\ &= \cos\theta a_1+\sin\theta a_2. \end{aligned} \tag{4.11}$$

Using this equation in Eq. (4.8), we obtain

$$rp\star a_1\star\bar{p}+\alpha=\alpha+r(\cos\theta a_1+\sin\theta a_2). \tag{4.12}$$

It is obvious that this equation represents the generalized circular surface

$$\phi(t,\theta)=\alpha(t)+r(t,\theta)\left(\cos\theta a_1(t)+\sin\theta a_2(t)\right). \tag{4.13}$$

This completes the proof. □

Theorem 2. *Let $\phi(t,\theta)$ be a generalized circular surface corresponding to the screw motion*

$$\Psi(Q_{r(t,\theta)a_1(t)})=P(t,\theta)\star Q_{r(t,\theta)a_1(t)}\star\bar{P}^*(t,\theta), \tag{4.14}$$

where $\{a_1(t),a_2(t),a_3(t)=a_1(t)\times a_2(t)\}$ is any moving frame on $\alpha(t)$, $Q_{r(t,\theta)a_1(t)}=1+\varepsilon r(t,\theta)a_1(t)$, $P(t,\theta)=p(t,\theta)+\frac{\varepsilon}{2}\alpha(t)\star p(t,\theta)$ and $p(t,\theta)=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}a_3(t)$.

Then, the generalized circular surface $\phi(t,\theta)$ can be expressed by the homothetic motion of $a_1(t)$ in $\mathbb{R}^3$ as

$$\phi(t,\theta) = \alpha(t) + h(t,\theta)M(t,\theta)a_1(t), \tag{4.15}$$

where $M(t,\theta)$ is an orthogonal matrix satisfying $p(t,\theta)\star a_1(t)\star \bar{p}(t,\theta) = M(t,\theta)a_1(t)$, $h(t,\theta) = r(t,\theta)$ is a homothetic scalar, $\alpha(t)$ is a translation vector and t, θ are homothetic parameters.

Proof. From Eqs. (4.12) and (4.13), surface generalization $\phi(t,\theta)$ can be expressed as

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)p(t,\theta)\star a_1(t)\star \bar{p}(t,\theta). \tag{4.16}$$

Using Eqs. (2.6)-(2.8), we get

$$p(t,\theta)\star a_1(t)\star \bar{p}(t,\theta) = M(t,\theta)a_1(t). \tag{4.17}$$

Using this equation in Eq. (4.16), we obtain

$$\begin{aligned}\phi(t,\theta) &= \alpha(t) + r(t,\theta)p(t,\theta)\star a_1(t)\star \bar{p}(t,\theta)\\ &= \alpha(t) + h(t,\theta)M(t,\theta)a_1(t).\end{aligned} \tag{4.18}$$

This completes the proof. □

Remark 2. If we take the real variable $r(t,\theta)\in\mathbb{R}$ as the real constant r, as the real variable $r(t)$ related to the real parameter t or as the real variable $r(\theta)$ related to the real parameter θ in Theorem 4.2, then the screw motions

$$\Psi(Q_{ra_1(t)}) = P(t,\theta)\star Q_{ra_1(t)}\star \bar{P}^*(t,\theta), \tag{4.19}$$

$$\Psi(Q_{r(t)a_1(t)}) = P(t,\theta)\star Q_{r(t)a_1(t)}\star \bar{P}^*(t,\theta), \tag{4.20}$$

$$\Psi(Q_{r(\theta)a_1(t)}) = P(t,\theta)\star Q_{r(\theta)a_1(t)}\star \bar{P}^*(t,\theta) \tag{4.21}$$

(4.22)

generates a tube surface, a circular surface and a generalized tube surface, respectively, as

$$\phi(t,\theta) = \alpha(t) + r\left(\cos\theta a_1(t) + \sin\theta a_2(t)\right), \tag{4.23}$$

$$\phi(t,\theta) = \alpha(t) + r(t)(\cos\theta a_1(t) + \sin\theta a_2(t)), \tag{4.24}$$

$$\phi(t,\theta) = \alpha(t) + r(\theta)(\cos\theta a_1(t) + \sin\theta a_2(t)). \tag{4.25}$$

Example 1. Let us take a curve in $\mathbb{R}^3$ as

$$\alpha(t) = \left(\frac{\sqrt{3}t}{2}, \sin\frac{t}{2}, \cos\frac{t}{2}\right) \tag{4.26}$$

and its moving frame vectors as

$$a_1(t) = \left(0, -\sin\frac{t}{2}, -\cos\frac{t}{2}\right), \tag{4.27}$$

$$a_2(t) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\cos\frac{t}{2}, -\frac{\sqrt{3}}{2}\sin\frac{t}{2}\right), \tag{4.28}$$

$$a_3(t) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\cos\frac{t}{2}, -\frac{1}{2}\sin\frac{t}{2}\right). \tag{4.29}$$

Thus, we get the dual quaternion surface

$$\begin{aligned} P(t,\theta) &= p(t,\theta) + \frac{\varepsilon}{2}\alpha(t) \star p(t,\theta) \\ &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}a_3(t)\right) \\ &+ \frac{\varepsilon}{2}\left(0 + \left(\frac{\sqrt{3}t}{2}, \sin\frac{t}{2}, \cos\frac{t}{2}\right)\right) \star \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}a_3(t)\right), \end{aligned} \tag{4.30}$$

where $p(t,\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}a_3(t)$. Then, screw motion $\Psi(Q_{r(t,\theta)a_1(t)})$ can be given as

$$\begin{aligned} \Psi(Q_{r(t,\theta)a_1(t)}) &= P(t,\theta) \star Q_{r(t,\theta)a_1(t)} \star \bar{P}^*(t,\theta) \\ &= 1 + \varepsilon\left(r(t,\theta)p(t,\theta) \star a_1(t) \star \bar{p}(t,\theta) + \alpha(t)\right). \end{aligned} \tag{4.31}$$

Thus, generalized circular surface corresponding to this equation can be obtained as

$$\begin{aligned} \phi(t,\theta) &= \alpha(t) + r(t,\theta)p(t,\theta) \star a_1(t) \star \bar{p}(t,\theta) \\ &= \left(\frac{\sqrt{3}t}{2}, \sin\frac{t}{2}, \cos\frac{t}{2}\right) + r(t,\theta)\left(\cos\theta a_1(t) + \sin\theta a_2(t)\right), \end{aligned} \tag{4.32}$$

where $a_1(t) = \left(0, -\sin\frac{t}{2}, -\cos\frac{t}{2}\right)$ and $a_2(t) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\cos\frac{t}{2}, -\frac{\sqrt{3}}{2}\sin\frac{t}{2}\right)$.

$\phi(t,\theta)$ can be given as tube surface with $r(t,\theta) = r = \frac{1}{3}$ as in Fig. 1a, generalized tube surface with $r(t,\theta) = r(\theta) = \frac{\theta}{3}$ as in Fig. 1b, circular surface with $r(t,\theta) = r(t) = \frac{t}{3}$ as in Fig. 1c and generalized circular surface with $r(t,\theta) = \frac{t\theta}{3}$ as in Fig. 1d for intervals $-10 \leq t \leq 10$ and $-5 \leq \theta \leq 5$.

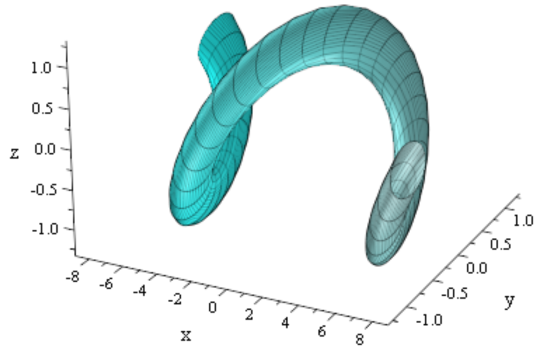

Fig. 1a. Geometric representation of a tube surface

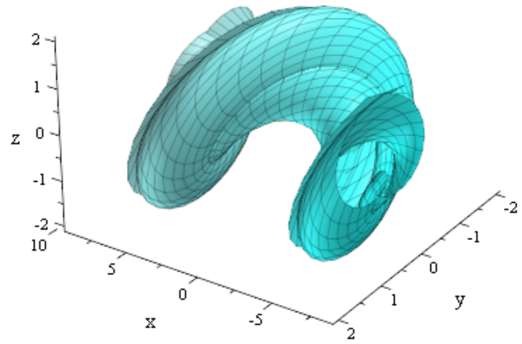

Fig. 1b. Geometric representation of a generalized tube surface

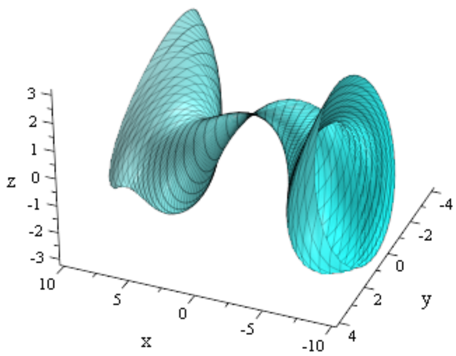

Fig. 1c. Geometric representation of a circular surface

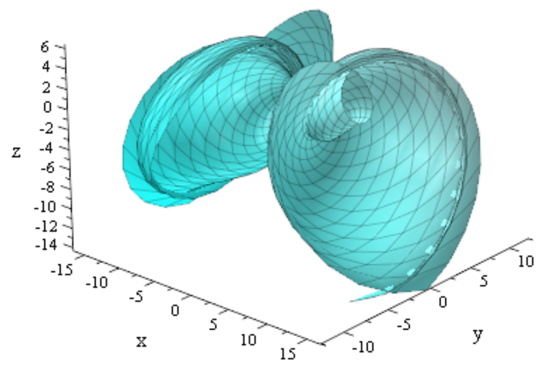

Fig. 1d. Geometric representation of a generalized circular surface

5. Conclusions

We introduce generalized circular surface as

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)(\cos\theta a_1(t) + \sin\theta a_2(t)) \in \mathbb{R}^3, \ \ r(t,\theta) \in \mathbb{R},$$

where $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ is an arbitrary moving frame (i.e., Frenet frame, Darboux frame, Bishop frame, moving frame, etc.) on the spine curve $\alpha(t)$.

We show that the spine curve $\alpha(t)$ is the translation part and $R(t,\theta) = \cos\theta a_1(t) + \sin\theta a_2(t)$ is the rotation part of the generalized circular surface $\phi(t,\theta)$. The rotation part of the generalized circular surface can be generated by a real quaternion as

$$\begin{aligned}\phi(t,\theta) &= \alpha(t) + r(t,\theta)R(t,\theta) \\ &= \alpha(t) + r(t,\theta)p(t,\theta) \star a_1(t),\end{aligned}$$

where $p(t,\theta) = \cos\theta + \sin\theta a_3(t)$ is a real quaternion.

Let $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be a moving frame on the curve $\alpha(t)$ and let $p(t,\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} a_3(t)$ be a unit real quaternion surface. We show that screw motion obtained by using dual quaternions $P(t,\theta) = p(t,\theta) + \frac{\varepsilon}{2}\alpha(t) \star p(t,\theta)$ and $Q_{r(t,\theta)a_1(t)}$ (which is a dual quaternion corresponding to $r(t,\theta)a_1(t) \in \mathbb{R}^3$) constructs a generalized circular surface in $\mathbb{R}^3$ as

$$\begin{aligned}\Psi(Q_{r(t,\theta)a_1(t)}) &= P(t,\theta) \star Q_{r(t,\theta)a_1(t)} \star \bar{P}^*(t,\theta) \\ &= 1 + \varepsilon\left(\alpha(t) + r(t,\theta)\left(\cos\theta a_1(t) + \sin\theta a_2(t)\right)\right).\end{aligned}$$

This equation represents the generalized circular surface

$$\phi(t,\theta) = \alpha(t) + r(t,\theta)\left(\cos\theta a_1(t) + \sin\theta a_2(t)\right).$$

If we take $r(t,\theta)$ as real constant r or as real variables $r(t)$ or $r(\theta)$, then the screw motion Ψ generates a tube surface, a circular surface or a generalized tube surface, respectively, as

$$\phi(t,\theta) = \alpha(t) + r\left(\cos\theta a_1(t) + \sin\theta a_2(t)\right),$$
$$\phi(t,\theta) = \alpha(t) + r(t)(\cos\theta a_1(t) + \sin\theta a_2(t)),$$
$$\phi(t,\theta) = \alpha(t) + r(\theta)(\cos\theta a_1(t) + \sin\theta a_2(t)).$$

Moreover, the generalized circular surface is expressed by homothetic motion as

$$\phi(t,\theta) = \alpha(t) + h(t,\theta)M(t,\theta)a_1(t),$$

where $M(t,\theta)$ is an orthogonal matrix satisfying $p(t,\theta)\star a_1(t)\star \bar{p}(t,\theta) = M(t,\theta)a_1(t)$, $h(t,\theta) = r(t,\theta)$ is a homothetic scalar, $\alpha(t)$ is a translation vector and t, θ are homothetic parameters.

Special dual quaternion and screw motion used in this paper can be studied in the researching of some surfaces in Minkowski, Galilean or pseudo-Galilean spaces.

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