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1D INVERSE AND D1 INVERSE OF SQUARE MATRICES

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Abstract. In this paper we introduce two new classes of inverses for square matrices, which are called 1Drazin (in short, 1D) inverse and Drazin1 (in short, D1) inverse. Next, we investigated the existence and uniqueness of 1D inverse and its dual D1 inverse. Some representation and characterizations of these inverses are derived. In addition to this, we obtain some properties of 1D and D1 inverses through idempotent and binary relations.

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1. INTRODUCTION

The generalized inverse of matrices and its applications was studied by Rao et. al (1971) in [22]. The Drazin inverse in ring theory was established in [9] and the applications of Drazin inverse in solving singular linear control systems were discussed in [3]. The authors of [2, 3] have pointed out the importance of Drazin inverse in matrix theory and computations. Various applications of Drazin inverses can be found in [1]. The Drazin and Moore-Penrose (DMP) inverse was introduced in [13]. An integral representation for the DMP inverse was established in [23]. The representation of DMP inverse for maximal classes of matrices was developed in [7]. In [24], the DMP inverse for an element of a ring with involution was discussed. The DMP inverse for a Hilbert space operator was established as an extension of the corresponding case of DMP on matrices in [19]. Further, the binary relation given by the DMP inverse in relation with some other partial orders are investigated in [5]. Its extension to finite potent endomorphisms on arbitrary vector spaces was studied in [21]. Some extensions of DMP inverse can be found in [19]. Several generalized

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inverses for arbitrary index and square matrix have been discussed in [13–15, 18]. In [17], generalized inverse for matrices of an arbitrary index has been discussed by considering the adjoint matrix. From projectors, 1MP and MP1 generalized inverses along with their induced partial orders were discussed in [10].

Motivated by the work of Hernández et al. [10], in this paper we introduce and study the 1D and D1 inverses. The main contributions of this paper are the following:

- Two new classes of generalized inverses, named 1D inverse and D1 inverse, are introduced. Existence and uniqueness of these inverses of matrices are established.
- We have studied some characterizations and representations of 1D inverse and D1 inverse for square matrices.
- Maximal classes and binary relations based on these inverses are introduced along with some results.

1.1. *Preliminary considerations*

In this section, we present a few notations and definitions, which will be used in the subsequent sections. Throughout this paper, we denote the set of complex matrices of order $m\times n$ by $\mathbb{C}^{m\times n}$, while the range space, null space and the conjugate transpose of a matrix $A\in\mathbb{C}^{n\times n}$ are denoted, respectively, by $R(A)$, $N(A)$, and A^*. The smallest nonnegative integer k that satisfies $rank(A^k)=rank(A^{k+1})$ is called the index of the matrix $A\in\mathbb{C}^{n\times n}$ and denoted $ind(A)$. For a matrix $A\in\mathbb{C}^{m\times n}$, if a matrix X satisfies $AXA=A$ then X is called the generalized or inner inverse of A and it is denoted by A^-. Similarly, if a matrix X satisfies $XAX=X$, then X is called outer inverse of A and is denoted by $A^{(2)}$. If X is an outer inverse of $A\in\mathbb{C}^{m\times n}$ with $R(X)=T$ and $N(X)=S$ (for some T and S subspaces adequately prescribed), then we denote it by $A^{(2)}_{T,S}$. The outer inverses with a prescribed range and null-space are very important in Matrix Theory. It has been extensively used in solving the nonlinear equations [1,20] as well as in statistics [8].

A matrix $X\in\mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of the matrix $A\in\mathbb{C}^{m\times n}$ if it satisfies $AXA=A,\ XAX=X,\ (AX)^*=AX$, and $(XA)^*=XA$. It is denoted by $A^\dagger$. The 1MP inverse [10] of a rectangular matrix $A\in\mathbb{C}^{m\times n}$ is denoted as $A^{-,\dagger}$ and defined by $A^{-,\dagger}:=A^-AA^\dagger$ for a fixed inner inverse A^- of A. Similarly, its dual called MP1 inverse, is defined as $A^{\dagger,-}:=A^\dagger AA^-$ for some fixed inner inverse A^- of A. The Drazin inverse [6] was introduced for an element in associative rings and semigroups. Furthermore, it has been extended from square to rectangular matrices [4]. We now recall the definition of Drazin inverse for square matrices.

Definition 1 ([6]). Let $A\in\mathbb{C}^{n\times n}$ with $ind(A)=k$. A matrix $X\in\mathbb{C}^{n\times n}$ is called the Drazin inverse of A if it satisfies

$$XA^{k+1}=A^k,\ XAX=X,\ AX=XA.$$

The Drazin inverse of A is denoted by A^D. The combination of Drazin (D) inverse and Moore-Penrose (MP) inverse is renamed as DMP inverse, which is recalled below.

Definition 2 ([13])**.** Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. A matrix $X \in \mathbb{C}^{n\times n}$ is called the DMP inverse of A if it satisfies $XAX = X$, $XA = AA^D$ and $A^kX = A^kA^\dagger$. Similarly, Y is called the dual DMP inverse of A if $YAY = Y$, $AY = AA^D$, $YA^k = A^\dagger A^k$.

The following results will be used in what follows.

Lemma 1 ([12], Theorem 2.2)**.** *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then, $(BA)^D = B((AB)^D)^2A$.*

2. 1D INVERSES

In this section, we introduce 1D inverse for square matrices. In addition, we discuss a few characterizations of these inverses along with its interconnection with other generalized inverses.

2.1. *Existence and uniqueness of 1D inverses*

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and let A^- be a fixed inner inverse of A. Then $A^-AA^D \in \mathbb{C}^{n\times n}$ is the unique solution of the following system of matrix equations*

$$XAX = X, \quad XA^k = A^-A^k \text{ and } AX = AA^D. \tag{2.1}$$

Proof. Let $X = A^-AA^D$. Then, we have

$$XAX = A^-AA^DAA^-AA^D = A^-AA^DAA^D = A^-AA^D = X,$$

$$XA^k = A^-AA^DA^k = A^-A^DA^{k+1} = A^-A^k,$$

and $AX = AA^-AA^D = AA^D$. Thus X is a solution of (2.1). Next, we will show the uniqueness of X. Suppose there is another solution, say Y, which satisfies equation (2.1). Now,

$$Y = YAY = YAA^D = YA^k(A^D)^k = A^-A^k(A^D)^k = A^-AA^D = X.$$

□

In view of Proposition 1, we define the following representation of 1D inverse for square matrices.

Definition 3. Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and let A^- be a fixed inner inverse of A. A matrix $X \in \mathbb{C}^{n\times n}$ is called the 1D inverse of A if it satisfies

$$XAX = X, \; XA^k = A^-A^k \text{ and } AX = AA^D.$$

We denote the 1D inverse of A by $A^{-,D}$. Clearly, $A^{-,D} = A^-AA^D$.

Remark 1. It is important to observe that every fixed inner inverse A^- of A may produce a different 1D inverse of A. From now on, when we refer to the 1D inverse of A, we are assuming that an inner A^- of A has been previously fixed and will be not be explicitly indicated.

Example 1. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We can show that $ind(A) = 2$ and $A^D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now, if we fix $A^- = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, we get $A^-AA^D = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

On the other hand one can verify that the Moore-Penrose Drazin (MPD) inverse of A, that is, $A^{\dagger,D} = A^\dagger AA^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is different from the previous one.

If we consider another inner inverse of A, say $A^= = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, then we obtain $A^=AA^D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq A^-AA^D$. Clearly, this example shows that the 1D inverse $A^{-,D} = A^-AA^D$ is different from the MPD inverse $A^{\dagger,D}$, and it depends on the selected A^- for its computation.

The characterization of 1D inverse through outer inverse with a prescribed range and null space is presented below.

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then the following statements are valid.*

(i) *$AA^{-,D}$ is a projector onto $R(A^k)$ along $N(A^k)$.*
(ii) $A^{-,D} = A^{(2)}_{R(A^-A^k),N(A^k)}$.

Proof. (i) From Definition 3, we obtain

$$(AA^{-,D})^2 = A(A^{-,D}AA^{-,D}) = AA^{-,D}$$

and

$$AA^{-,D} = AA^D. \tag{2.2}$$

Using equation (2.2), we get $R(AA^{-,D}) = R(AA^D) = R(A^k)$ and $N(AA^{-,D}) = N(AA^D) = N(A^k)$.

(ii) Clearly $A^{-,D}AA^{-,D} = A^{-,D}$. From the expressions

$$A^{-,D} = A^{-}AA^{D} = A^{-}A^{k}(A^{D})^{k}, \text{ and}$$

$$A^{-}A^{k} = A^{-}A^{k+1}A^{D} = A^{-}A^{k}A^{D}A = A^{-}A^{k}(A^{D})^{k}A^{k} = A^{-}AA^{D}A^{k} = A^{-,D}A^{k},$$

we obtain $R(A^{-,D}) = R(A^{-}A^{k})$. Using the proved in (i), we can verify the null space from the following expression:

$$N(A^{-,D}) = N(AA^{-,D}) = N(AA^{D}) = N(A^{k}).$$

□

2.2. *Characterizations of 1D inverses*

Next, we discuss a few characterizations for a matrix to be a 1D inverse from a geometrical point of view.

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^{-} be a fixed inner inverse of A. Then the following statements are equivalent:*

(i) $X = A^{-,D}$.
(ii) $XA^{k} = A^{-}A^{k}$ *and* $R(X^{*}) = R((A^{k})^{*})$.
(iii) $R(X^{*}) = R((A^{k})^{*})$, $N(X^{*}) = N((A^{-}A^{k})^{*})$ *and* $A^{D}X = (A^{D})^{2}$.
(iv) $A^{k+1}X = A^{k}$ *and* $N(X^{*}) = N((A^{-}A^{k})^{*})$.
(v) $A^{D}AX = A^{D}$, $R(X^{*}) = R((A^{k})^{*})$ *and* $XA^{k} = A^{-}A^{k}$.
(vi) $A^{D}AX = A^{D}$ *and* $N(X^{*}) = N((A^{-}A^{k})^{*})$.
(vii) $XA^{k} = A^{-}A^{k}$ *and* $N(X) = N(A^{k})$.

Proof. (i) ⇒(ii) Let $X = A^{-,D}$. Then $XA^{k} = A^{-}A^{k}$ holds by Definition 3. Moreover,

$$X^{*} = (A^{-}AA^{D})^{*} = (A^{-}(A^{D})^{k}A^{k})^{*} = (A^{k})^{*}((A^{D})^{k})^{*}(A^{-})^{*}, \text{ and}$$

$$(A^{k})^{*} = (A^{k+1}A^{D})^{*} = (A^{k}AA^{D})^{*} = (A^{k}AX)^{*} = X^{*}A^{*}(A^{k})^{*}.$$

Thus, $R(X^{*}) = R((A^{k})^{*})$.
(ii) ⇒ (i) From $R(X^{*}) = R((A^{k})^{*})$, we have $X = ZA^{k}$ for some $Z \in \mathbb{C}^{n\times n}$. Now, using (ii),

$$X = ZA^{k} = ZA^{k+1}A^{D} = (ZA^{k})(AA^{D}) = XA^{k}(A^{D})^{k} = A^{-}A^{k}(A^{D})^{k} = A^{-}AA^{D} = A^{-,D}.$$

(i) ⇒(iii) Let $X = A^{-,D}$. The equality $R(X^{*}) = R((A^{k})^{*})$ can be shown as before. On the other hand, $X^{*} = (A^{-}AA^{D})^{*} = (A^{-}A^{k}(A^{D})^{k})^{*} = ((A^{D})^{k})^{*}(A^{-}A^{k})^{*}$ and $(A^{-}A^{k})^{*} = (A^{-}AA^{D}A^{k})^{*} = (XA^{k})^{*} = (A^{k})^{*}X^{*}$. Thus, $N(X^{*}) = N((A^{-}A^{k})^{*})$. Finally, $A^{D}X = A^{D}A^{-}AA^{D} = A^{D}AA^{D}A^{-}AA^{D} = (A^{D})^{2}AA^{D} = (A^{D})^{2}$.
(iii) ⇒(ii) It is sufficient to show $A^{-}A^{k} = XA^{k}$. From (iii), we assume that $R(X^{*}) = R((A^{k})^{*})$. Then, there exists $Z \in \mathbb{C}^{n\times n}$ such that $X = ZA^{k} = (ZA^{k})(AA^{D}) = XA^{2}(A^{D})^{2} = XA^{2}A^{D}X$. Hence, by (iii), $R(I - XA^{2}A^{D})^{*} \subseteq N(X^{*}) = N((A^{-}A^{k})^{*})$, which implies $(I - XA^{2}A^{D})(A^{-}A^{k}) = O$ and hence $A^{-}A^{k} = XA^{2}A^{D}A^{-}A^{k} = XAA^{D}(AA^{-}A)A^{k-1} = XA^{k}$. Observe that this last equality is valid for any $k \geq 0$.
(iii) ⇒(iv) Pre-multiplying $A^{D}X = (A^{D})^{2}$ by A^{k+2}, it follows that $A^{k+1}X =$

$AA^{k+1}A^DX = A^{k+2}A^DX = A^{k+2}(A^D)^2 = A^k$. The other condition is evident.
(iv) $\Rightarrow$(iii) Let $A^{k+1}X = A^k$. Then $R((A^k)^*) \subseteq R(X^*)$ and

$$(A^D)^2 = (A^D)^3A = (A^D)^{k+2}A^k = (A^D)^{k+2}A^{k+1}X = A^D((A^D)^{k+1}A^{k+1})X = A^DX.$$

From $(A^-A^k)^* = (A^k)^*(A^-)^*$, we have $R((A^-A^k)^*) \subseteq R((A^k)^*) \subseteq R(X^*)$. Using that $N(X^*) = N((A^-A^k)^*)$ and the dimension theorem, we have $dim(R(X^*)) = dim(R((A^-A^k)^*))$. Hence, we arrive at $R((A^k)^*) = R((A^-A^k)^*) = R(X^*)$.
(i) $\Rightarrow$(v) It is clear that $A^DAX = A^D(AA^{-,D}) = A^DAA^D = A^D$.
(v) $\Rightarrow$(ii) It is trivial.
(iv) $\Rightarrow$(vi) Let $A^{k+1}X = A^k$. Then $A^D = (A^D)^2A = (A^D)^{k+1}A^k = (A^D)^{k+1}A^{k+1}X = A^DAX$.
(vi) $\Rightarrow$(iv) Let $A^D = A^DAX$. Then $A^k = A^{k+1}A^D = A^{k+1}A^DAX = A^{k+1}X$.
(ii) $\Leftrightarrow$(vii) It is trivial since $R(X^*) = N(X)^\perp$. Hence, the proof is complete. $\square$

From an algebraic and geometrical point of view we have the following characterizations.

Theorem 3. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then the following statements are equivalent:*

(i) $X = A^{-,D}$.
(ii) $XA = A^-A^DA^2$ *and* $N(A^k) \subseteq N(X)$.
(iii) $AXA = AA^DA$, $N(A^k) \subseteq N(X)$ *and* $R(X) \subseteq R(A^-A^k)$.
(iv) $AX = AA^D$ *and* $R(X) \subseteq R(A^-A^k)$.
(v) $X^2A = X$ *and* $XA^k = A^-A^k$.
(vi) $X^2A = X$ *and* $XA = A^-A^DA^2$.

Proof. (i) $\Rightarrow$(ii) Let $X = A^{-,D}$. Then $XA = A^-AA^DA = A^-A^DA^2$ and the null space inclusion $N(A^k) \subseteq N(X)$ follows from Theorem 2.
(ii) $\Rightarrow$(i) From $N(A^k) \subseteq N(X)$, we have that $X = ZA^k$ for some $Z \in \mathbb{C}^{n\times n}$. Now, from $XA = A^-A^DA^2$ we get

$$X = ZA^k = ZA^{k+1}A^D = (ZA^k)AA^D = XAA^D = A^-A^DA^2A^D = A^-AA^D = A^{-,D}.$$

(ii) $\Rightarrow$(iii) Clearly $AXA = AA^DA$. From the equivalence of (ii) and (i), we have $X = A^-AA^D$. Thus $R(X) \subseteq R(A^-A^k)$ follows from $X = A^-AA^D = A^-A^k(A^D)^k$.
(iii) $\Rightarrow$(ii) For $k \geq 1$, let $R(X) \subseteq R(A^-A^k)$. Then $X = A^-A^kZ$, for some $Z \in \mathbb{C}^{n\times n}$. Now, from $A^-A^k = A^-AA^{k-1} = A^-(AA^-A)A^{k-1} = A^-AA^-A^k$, we get

$$XA = A^-A^kZA = A^-AA^-A^kZA = A^-AXA = A^-AA^DA = A^-A^DA^2. \tag{2.3}$$

If k is either 0, then the result is trivially true.
(iii) $\Rightarrow$(iv) It is trivially true.
(iv) $\Rightarrow$(iii) From $R(X) \subseteq R(A^-A^k)$, we have $X = A^-A^kZ$ for some $Z \in \mathbb{C}^{n\times n}$. For $k \geq 1$, we obtain $X = A^-AA^-A^kZ = A^-AX = A^-AA^D = A^-(A^D)^kA^k$. Thus $N(A^k) \subseteq$

$N(X)$. It can be proved similarly for $k = 0$. The other conditions are trivial.
(i) $\Rightarrow$(v) Let $X = A^{-,D}$. By definition, we obtain $XA^k = A^- A^k$. Then,

$$X^2 A = (A^{-,D})^2 A = A^- AA^D A^- AA^D A = A^- AA^D = X.$$

(v) $\Rightarrow$(i) From $X^* = (X^2 A)^* = (X^3 A^2)^* = \cdots = (X^{k+1} A^k)^* = (A^k)^* (X^{k+1})^*$, we obtain $R(X^*) \subseteq R((A^k)^*)$. By using $R(X^*) \subseteq R((A^k)^*)$ and $XA^k = A^- A^k$ we verify that

$$X = XA^k (A^D)^k = A^- A^k (A^D)^k = A^- AA^D = A^{-,D}.$$

(v) $\Rightarrow$(vi) It follow from the equivalence of (v) $\Leftrightarrow$ (i).
(vi) $\Rightarrow$(v) Let $XA = A^- A^D A^2$. Then $XA^k = A^- A^D A^{k+1} = A^- A^k$.
Now, the implication's chain is closed as (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) and (i) $\Leftrightarrow$(v) $\Leftrightarrow$(vi). $\square$

The 1D inverse can be obtained from the Drazin inverse as given in the next theorem.

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then $A^{-,D} = (A^- A^2)^D$.*

Proof. In the view of the Lemma 1, we have

$$\begin{aligned}(A^- A^2)^D &= (A^- AA)^D = A^- A[(AA^- A)^D]^2 A = A^- A(A^D)^2 A = A^- AA^D AA^D \\ &= A^- AA^D = A^{-,D}.\end{aligned}$$

$\square$

Proposition 2. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then $A^{-,D} = A^D$ if and only if $A^D A^k = A^- A^k$.*

Proof. Only if part follows from $A^D A^k = A^{-,D} A^k = A^- AA^D A^k = A^- A^k$. Conversely, we have

$$A^{-,D} = A^- AA^D = A^- A^k (A^D)^k = A^D A^k (A^D)^k = A^D AA^D = A^D.$$

$\square$

We now discuss the idempotent property for 1D inverses.

Proposition 3. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then*

(i) $(A^{-,D})^2 = A^- A^D$.
(ii) *$A^{-,D}$ is idempotent if and only if $A^{-,D} = A^- A^D$ if and only if $A^{-,D} = A^{-,D} A$.*

Proof. (i)

$$(A^{-,D})^2 = A^{-,D} A^{-,D} = A^- AA^D A^- AA^D = A^- A^D (AA^- A) A^D = A^- (A^D AA^D) = A^- A^D.$$

(ii) The first equivalence is trivial from (i). Let's prove that $A^{-,D}$ is idempotent if and only if $A^{-,D} = A^{-,D} A$. In fact, let $A^{-,D}$ is idempotent. Then, by item (i), we have

$A^{-,D} = A^{-}A^{D}$. Now, $A^{-,D} = A^{-}AA^{D} = A^{-}A^{D}A = A^{-,D}A$. Conversely, let $A^{-,D} = A^{-,D}A$. Then

$$\begin{aligned} A^{-,D} &= A^{-,D}A = A^{-}AA^{D}A = A^{-}AA^{D}AA^{D}A = A^{-}AA^{D}AA^{-}AA^{D}A = A^{-,D}AA^{-,D}A \\ &= A^{-,D}A^{-,D}. \end{aligned}$$

□

Theorem 5. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^{-} be a fixed inner inverse of A. If $A^{-,D}$ is idempotent then:*

(i) $A^{k} = A^{k+1}$. *Consequently,* $A^{D}A^{k} = A^{k}$.
(ii) $(A^{-,D})^{k} = (A^{-,D})^{k}A$. *More generally,* $(A^{-,D})^{m} = (A^{-,D})^{m}A$ *for every* $m \in \mathbb{N}$.
(iii) $A^{-,D} = (A^{-,D})^{m}A^{m}$ *for every* $m \in \mathbb{N}$
(iv) $A^{k}A^{-,D} = A^{k}$.

Proof. (i) Using Proposition 3 (i), we obtain

$$\begin{aligned} A^{k} &= A^{k+1}A^{D} = A^{k}AA^{-}AA^{D} = A^{k}AA^{-}A^{D}A = A^{k}AA^{-,D}A = A^{k}AA^{-}AA^{D}A \\ &= A^{k+1}A^{D}A = A^{k+1}. \end{aligned}$$

(ii) By Proposition 3 (ii), we get $(A^{-,D})^{k} = A^{-,D} = A^{-,D}A = (A^{-,D})^{k}A$.
(iii) Applying Proposition 3 (ii), repetitively, we have

$$\begin{aligned} A^{-,D} &= (A^{-,D})^{m} = (A^{-,D})^{m-1}A^{-,D} = (A^{-,D})^{m-1}A^{-,D}A = (A^{-,D})^{m-1}A^{-,D}A^{2} \\ &= (A^{-,D})^{m-1}A^{-,D}A^{3} = \cdots = (A^{-,D})^{m}A^{m}. \end{aligned}$$

(iv) From part (i), $A^{k}A^{-,D} = A^{k+1}A^{-}AA^{D} = A^{k+1}A^{D} = A^{k}$. □

Next, we characterize 1D inverses by using matrix equalities.

Theorem 6. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^{-} be a fixed inner inverse of A. Then the following statements are equivalent:*

(i) $A^{-,D} = X$.
(ii) $XAX = X$, $XA^{k} = A^{-}A^{k}$, $AXA = AA^{D}A$ *and* $AX = AA^{D}$.
(iii) $A^{-}AX = X$, $XA^{k} = A^{-}A^{k}$ *and* $X = XAA^{D}$.
(iv) $A^{-}AXAA^{D} = X$ *and* $AXA^{k} = A^{k}$.
(v) $A^{-}AA^{D}A = XA$, $A^{k}X = A^{k}A^{D}$ *and* $X = XAA^{D}$.
(vi) $AA^{D}AX = A^{D}A$, $XAA^{D}A = A^{-}AA^{D}A$, $XAA^{D}AX = X$, *and* $AA^{D}AXAA^{D}A = AA^{D}A$.

Proof. (i) ⇒(ii) Let $X = A^{-,D}$. In view of Proposition 1, it is sufficient to show $AXA = AA^{D}A$. Since $AXA = AA^{-,D}A = AA^{-}AA^{D}A = AA^{D}A$, the proof is complete.
(ii) ⇒(iii) It follows from $XAA^{D} = XAX = X$ and

$$A^{-}AX = A^{-}AA^{D} = A^{-}A^{k}(A^{D})^{k} = XA^{k}(A^{D})^{k} = XAA^{D} = X.$$

(iii) $\Rightarrow$(iv) The implication is true since $A^-AX = X$ implies $AXA^k = A^k$ trivially and $A^-AXAA^D = XAA^D = X$.
(iv) $\Rightarrow$(i) Assuming that $A^-AXAA^D = X$ and $AXA^k = A^k$, we get

$$X = A^-AXAA^D = A^-AXA^k(A^D)^k = A^-A^k(A^D)^k = A^-AA^D = A^{-,D}.$$

(i) $\Rightarrow$(v) The first equality is trivial. The following identities hold from

$$XAA^D = A^-AA^DAA^D = A^-AA^D = X$$

and

$$A^kX = A^kA^{-,D} = A^kA^-AA^D = A^kA^D.$$

(v) $\Rightarrow$(i) It follows from $X = XAA^D = A^-AA^DAA^D = A^-AA^D = A^{-,D}$
(i) $\Rightarrow$(vi) We verify that $AA^DAX = AA^DAA^{-,D} = AA^DAA^-AA^D = AA^D$,

$$XAA^DA = A^{-,D}AA^DA = A^-AA^DAA^DA = A^-AA^DA,$$

$$XAA^DAX = A^{-,D}AA^DAA^{-,D} = A^-AA^DAA^DAA^-AA^D = A^-AA^D = A^{-,D} = X,$$

and

$$AA^DAXAA^DA = (AA^DAX)AA^DA = (AA^D)AA^DA = AA^DA.$$

(vi) $\Rightarrow$(i) It is clear from $X = XAA^DAX = A^-AA^DAX = A^-AA^D = A^{-,D}$. □

Next results shows that, among other things, commutative property can not be extended from Drazin inverse to 1D inverses, in general. This is an interesting difference between both classes of matrices. Furthermore, when index of A is 1, the expression $A^{-,D}A$ is the same projector as the one given by the inner inverse: A^-A.

Theorem 7. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then*

(i) $A^{-,D}A = A^-A$ *if and only if* $AA^DA = A$.
(ii) $A^{-,D}A = A^DA$ *if and only if* $A^{-,D} = A^D$.
(iii) $A^kA^-A^k = A^k$ *if and only if* $A^kA^{-,D}A^k = A^k$.
(iv) $A^{-,D} = A^D$ *if and only if* $A^{-,D}A = AA^{-,D}$.

Proof. (i) Let $A^{-,D}A = A^-A$. Then $A = AA^-A = AA^{-,D}A = AA^-AA^DA = AA^DA$. Converse is trivial.
(ii) It follows directly from definition.
(iii) Let $A^kA^-A^k = A^k$. Then

$$A^k = A^kA^-A^k = A^kA^-A^DA^{k+1} = A^kA^-AA^DA^k = A^kA^{-,D}A^k.$$

Converse part follows from $A^kA^-A^k = A^kA^-A^DA^{k+1} = A^kA^-AA^DA^k = A^kA^{-,D}A^k = A^k$.
(iv) Let $A^{-,D} = A^D$. Then $A^{-,D}A = A^DA = AA^D = AA^-AA^D = AA^{-,D}$. Conversely, let $A^{-,D}A = AA^{-,D}$. By definition we have $A^{-,D}AA^{-,D} = A^{-,D}$. On the other hand, $A^{k+1}A^{-,D} = A^kAA^-AA^D = A^{k+1}A^D = A^k$. Hence, by uniqueness of Drazin inverse, $A^D = A^{-,D}$. □

Theorem 5 (ii) can be refined as in the following Theorem.

Theorem 8. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, let A^- be a fixed inner inverse of A and let $m \in \mathbb{N} \setminus \{1\}$. Then*

$$(A^{-,D})^m = \begin{cases} (A^-A^D)^{\frac{m}{2}} & \text{if } m \text{ is even.} \\ A^-(A^D)^{\frac{m+1}{2}} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. If m is even, then $m = 2t$ for some positive integer t. Now, using Proposition 3 (i),

$$(A^{-,D})^m = ((A^{-,D})^2)^t = (A^-A^D)^t = (A^-A^D)^{\frac{m}{2}}.$$

If m is odd, then $m = 2l+1$ for some positive integer l. Again, by Proposition 3 (a),

$$\begin{aligned}(A^{-,D})^m &= ((A^{-,D})^2)^l A^{-,D} = (A^-A^D)^l A^-AA^D = (A^-A^D)^{l-1}A^-A^DA^-AA^D \\ &= (A^-A^D)^{l-1}A^-(A^D)^2AA^-AA^D = (A^-A^D)^{l-1}A^-(A^D)^2 = \cdots = A^-(A^D)^{l+1} \\ &= A^-(A^D)^{\frac{m+1}{2}}.\end{aligned}$$

□

We close this subsection by providing a canonical form for 1D inverses. Moreover, this result allows us to compare 1D inverse to Drazin inverse.

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, let A^- be a fixed inner inverse of A and let $A = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1}$ be a core-nilpotent decomposition of A with $C \in \mathbb{C}^{s\times s}$ nonsingular and $N \in \mathbb{C}^{(n-s)\times(n-s)}$ nilpotent. For an inner inverse of A of the form $A^- = P\begin{bmatrix} C^{-1} & Y \\ Z & T \end{bmatrix}P^{-1}$ (with $YN = 0$, $NZ = 0$ and $NTN = N$), the 1D inverse of A is given by*

$$A^{-,D} = P\begin{bmatrix} C^{-1} & 0 \\ Z & 0 \end{bmatrix}P^{-1} = A^D + P\begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}P^{-1}.$$

Consequently, $A^{-,D} = A^D$ if and only if $Z = 0$.

Proof. It is a straightforward computation by calculating $A^{-,D} = A^-AA^D$. □

2.3. *Maximal classes for 1D inverses*

Maximal classes are an interesting topic in order to determine the whole class of all matrices that can be used instead of A^D in the expression $A^{-,D} = A^-AA^D$. Maximal classes for 1D inverses are investigated in the following result.

Theorem 10. *Let $X \in \mathbb{C}^{n\times n}$ and $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. Then the following statements are equivalent:*

(i) $A^{-,D} = A^-AX$.
(ii) $AA^D = AX$.

(iii) $AA^DA = AXA$ *and* $N(A^k) = N(AX)$.
(iv) $X = A^D + (I - A^-A)Z$ *for any* $Z \in \mathbb{C}^{n\times n}$.

Proof. (i)⇒(ii) Since $A^{-,D} = A^-AX$, we get $AA^D = AA^-AA^D = AA^{-,D} = AA^-AX = AX$.
(ii) ⇒(iii) The null space equality $N(A^k) = N(AX)$ follows from

$$A^k = A^{k+1}A^D = A^kA^DA = A^kAX, \text{ and}$$
$$AX = AA^D = (A^D)^kA^k.$$

(iii)⇒(i) From $N(A^k) = N(AX)$, we obtain $AX = ZA^k$ for some $Z \in \mathbb{C}^{n\times n}$.
Pre-multiplying by A^-, we have

$$A^-AX = A^-ZA^k = A^-ZA^{k+1}A^D = A^-ZA^kAA^D = A^-AXAA^D = A^-AA^DAA^D$$
$$= A^-AA^D = A^{-,D}.$$

(ii) ⇒(iv), The general solution of $AX = 0$ is given by $X = (I - A^-A)Z$ for some $Z \in \mathbb{C}^{n\times n}$. Since A^D is a particular solution of $AX = AA^D$, the general solution of $AX = AA^D$ is given by $A^D + (I - A^-A)Z$, for arbitrary $Z \in \mathbb{C}^{n\times n}$.
(iv) ⇒(i) Pre-multiplying both sides of the equality in (iv) by A^-A, we obtain $A^-AX = A^-AA^D + A^-AZ - A^-AA^-AZ = A^{-,D}$. □

Theorem 11. *Let* $A, X \in \mathbb{C}^{n\times n}$ *with* $ind(A) = k$ *and* A^- *be a fixed inner inverse of* A. *Then*

(i) $A^{-,D} = XAA^D$ *if and only if* $X = A^- + Z(I - AA^D)$ *for some* $Z \in \mathbb{C}^{n\times n}$.
(ii) $A^{-,D} = XAY$ *if and only if* $X = A^- + Z(I - AA^D)$ *and* $Y = A^D + (I - A^-A)W$ *for some* $Y, W \in \mathbb{C}^{n\times n}$.

Proof. (i) If $X = A^- + Z(I - AA^D)$, then $XAA^D = A^{-,D}$. Conversely, let $A^{-,D} = XAA^D$. Clearly A^- is a particular solution of $XAA^D = A^{-,D}$. If Z is any solution of $XAA^D = 0$, then $ZAA^D = O$. Thus we can express Z as $Z = Z - ZAA^D = Z(I - AA^D)$. Hence the general solution of $XAA^D = 0$ is given by $X = Z(I - AA^D)$. Consequently, $X = A^- + Z(I - AA^D)$, $Z \in \mathbb{C}^{n\times n}$, is the general solution of $XAA^D = A^{-,D}$.
(ii) It follows from part (i) and Theorem 10 (iv). □

2.4. *A binary relation based on 1D inverses*

In view of 1MP partial order [10], we introduce the following binary relation for 1D inverse.

Definition 4. Let $A, B \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. We will say that A is below B under the relation $\leq^{-,D}$ if $A^{-,D}A = A^{-,D}B$ and $AA^{-,D} = BA^{-,D}$, and it is denoted by $A \leq^{-,D} B$.

Remark 2. Clearly, the relation $\leq^{-,D}$ is reflexive but is neither symmetric nor anti-symmetric.

The above remark is validated in the following example.

Example 2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and $B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

We can verify that $A^D = 0 = A^{-,D}$ and $B^D = 0 = B^{-,D}$. This leads $A^{-,D}A = A^{-,D}B$ and $AA^{-,D} = BA^{-,D}$. Hence $A \leq^{-,D} B$. Similarly we can verify $B \leq^{-,D} A$ but $A \neq B$.

Next we find $B_1^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and setting $B_1^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$, we get $B_1^{-,D} = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$. Clearly $A \leq^{-,D} B_1$ but $B_1 B_1^{-,D} \neq AB_1^{-,D}$.

Thus, the relation $\leq^{-,D}$ is not symmetric.

In general, the relation $\leq^{-,D}$ is not transitive as shown in the below example.

Example 3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $A^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $B^- = \begin{bmatrix} -0.5 & 1 & -0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$. Then, we evaluate

$$A^D = A^\# = A, \qquad A^{-,D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$B^D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B^{-,D} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we can verify that $A^{-,D}A = A^{-,D} = A^{-,D}B$, $AA^{-,D} = A = BA^{-,D}$, $B^{-,D}B = B^{-,D} = B^{-,D}C$, and $BB^{-,D} = B^D = CB^{-,D}$. Thus $A \leq^{-,D} B$, and $B \leq^{-,D} C$. However, $A \not\leq^{-,D} C$ since

$$AA^{-,D} - CA^{-,D} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next, we investigate conditions under which the binary relation is true. Observe that, by definition, $AA^{-,D} = BA^{-,D}$ if and only if $AA^D = BA^{-,D}$ for any $A, B \in \mathbb{C}^{n\times n}$ with $ind(A) = k$.

Proposition 4. *Let $A,B \in \mathbb{C}^{n\times n}$ with $ind(A)=k$ and A^- be a fixed inner inverse of A. Then*

(i) *$A^{-,D}A = A^{-,D}B$ if and only if $A^DA = A^DB$.*
(ii) *$AA^{-,D} = BA^{-,D}$ if and only if $A^D = BA^-A^D$.*

Proof.
(i) Let $A^{-,D}A = A^{-,D}B$. Then $AA^DA = AA^-AA^DA = AA^{-,D}A = AA^{-,D}B = AA^DB$. Premultiplying by A^D both sides of the last equality we get the result. The converse is trivial.
(ii) By the previous observation, it is sufficient to prove that $AA^D = BA^{-,D}$ if and only if $A^D = BA^-A^D$. Using the assumption $AA^D = BA^{-,D}$, we get

$$A^D = A^DAA^D = AA^DA^D = BA^{-,D}A^D = BA^-A^D.$$

Conversely, $AA^D = BA^-A^DA = BA^-AA^D = BA^{-,D}$. □

Corollary 1. *Let $A,B \in \mathbb{C}^{n\times n}$ with $ind(A)=k$ and A^- be a fixed inner inverse of A. Then the following statements are equivalent:*

(i) *$A \le^{-,D} B$.*
(ii) *$AA^DA = BA^{-,D}A = AA^DB$.*
(iii) *$AA^D = BA^{-,D}$ and $A^DA = A^DB$.*

Proof. (i) ⇒(ii) Let $A \le^{-,D} B$. Then $AA^DA = AA^-AA^DA = AA^{-,D}A = BA^{-,D}A$ and

$$AA^DB = AA^-AA^DB = AA^{-,D}B = AA^{-,D}A = AA^DA.$$

(ii) ⇒(iii) Let $AA^DA = BA^{-,D} = AA^DB$. Then $AA^D = AA^DAA^D = BA^{-,D}AA^D = BA^{-,D}$ and $AA^D = AA^DAA^D = A^DAA^DA = A^DB$.
(iii)⇒(i) It follows from Proposition 4 (i) and $BA^{-,D} = AA^D = AA^-AA^D = AA^{-,D}$. □

Remark 3. On the class of $n\times n$ EP matrices we have that 1D inverses are the same as 1MP inverses (since $A^D = A^\dagger$). Hence, over that class, the 1D relation is a partial order.

Remark 4. Notice that the behaviour of this new binary relation seems to have similarities with the following comparison: if we change from $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$ to $A^DA = A^DB$ and $AA^D = BA^D$, we loss properties and pass from a partial order to a preorder on $\mathbb{C}^{n\times n}$. We will complete this discussion further, and we will see that under certain conditions something similar occurs in this case.

The result given in Theorem 9 allows us to provide all elements B such that $A \le^{-,d} B$ for a given matrix A.

Theorem 12. *Let $A,B \in \mathbb{C}^{n\times n}$ with $ind(A)=k$, let A^- be a fixed inner inverse of A and let $A = P\begin{bmatrix} C & 0 \\ 0 & N\end{bmatrix}P^{-1}$ be a core-nilpotent decomposition of A with*

$C \in \mathbb{C}^{s\times s}$ nonsingular and $N \in \mathbb{C}^{(n-s)\times(n-s)}$ nilpotent. For an inner inverse of A of the form $A^- = P \begin{bmatrix} C^{-1} & Y \\ Z & T \end{bmatrix} P^{-1}$ (with $YN = 0$, $NZ = 0$ and $NTN = N$), the following conditions are equivalent:

(a) $A \leq^{-,D} B$.

(b) $B = P \begin{bmatrix} C & 0 \\ -B_4ZC & B_4 \end{bmatrix} P^{-1}$.

Proof. We consider the partition $B = P \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} P^{-1}$ according to the partition of A. By using the expression of $A^{-,D}$ given in Theorem 9, from $A^{-,D}A = A^{-,D}B$, some easy computations yield $B_1 = C$ and $B_2 = 0$. Similarly, from $AA^{-,D} = BA^{-,D}$ we get $B_3 = -CB_4Z$. Then, $B = P \begin{bmatrix} C & 0 \\ -B_4ZC & B_4 \end{bmatrix} P^{-1}$, with $B_4 \in \mathbb{C}^{(n-s)\times(n-s)}$. The converse is straightforward. $\square$

An interesting result to understand a little bit better this binary relation is to relate it to some known partial order.

Theorem 13. *Let $A, B \in \mathbb{C}^{n\times n}$ be matrices of index at most 1 and let A^- be a fixed inner inverse of A. If $A \leq^{-,D} B$ then $A\# \leq B$, where $\# \leq$ is the left sharp partial order.*

Proof. Suppose that $A \leq^{-,D} B$ for $A, B \in \mathbb{C}^{n\times n}$ being matrices of index at most 1. Then $A^{-,D}A = A^{-,D}B$ and $AA^{-,D} = BA^{-,D}$. Since matrix A has index at most 1, $A^D = A^\#$, where $A^\#$ denotes the group inverse of A (that is, $AA^\#A = A$, $A^\#AA^\#$ and $AA^\# = A^\#A$). Now, from $A^{-,D}A = A^{-,D}B$, we get $A^-AA^\#A = A^-AA^\#B$ and premultiplying it by $A^\#A^3$ we arrive at $A^2 = AB$. On the other hand, similarly from $AA^{-,D} = BA^{-,D}$, we get $A = BA^-A$, and then $R(A) \subseteq R(B)$. By [16, Definition 6.3.1] we have that $A\# \leq B$, that is, A is a predeccessor of B under the left sharp partial order. $\square$

This last result leads us to believe that the binary relation $\leq^{-,D}$ can have a good behaviour for at most index matrices. However, the binary relation $\leq^{-,D}$ is not a partial order even on the set of matrices of index at most 1 as shown in the next example.

Example 4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.
Then, $A^- = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$, $A^D = A$, $A^{-,D} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ $B^- = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$,

$B^D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & -3 & -2 \end{bmatrix}$, $B^{-,D} = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 3 & 2 \end{bmatrix}$. Further, we calculate

$$A^{-,D}A = A^{-,D} = A^{-,D}B,\ AA^{-,D} = A = BA^{-,D},\ B^{-,D}B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} = B^{-,D}C$$

and $BB^{-,D} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} = CB^{-,D}$.

However, $AA^{-,D} = A \neq \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ -1/3 & -1/3 & -1/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} = CA^{-,D}$.

Hence, we conclude that $A \leq^{-,D} B, B \leq^{-,D} C$ but $A \nleq^{-,D} C$.

After all these considerations, finally, we are able to present an interesting class of matrices over which transitivity for the relation $\leq^{-,D}$ hold. Consider

$$\mathcal{PO}_k^\ell = \left\{ A \in \mathbb{C}^{n\times n} : ind\ (A) = k, \|A^D\| \leq 1, \lim_{m\to\infty} \|A^m A^D - A^- A^m\| = 0 \right\}.$$

Theorem 14. *Let $A, B \in \mathcal{PO}_k^\ell$ and let A^- be a fixed inner inverse of A. Then $\leq^{-,D}$ is a pre-order on $\mathcal{PO}_k^\ell$.*

Proof. We are going to prove that $A \leq^{-,D} B$ if and only if $A \leq^D B$ for matrices in $\mathcal{PO}_k^\ell$.

Let $A \leq^{-,D} B$ and $m \geq k$. Then $AA^D = BA^-AA^D$ and $A^-AA^DA = A^-AA^DB$. Now,

$$A^DA = A^DAA^DA = A^DAA^-AA^DA = A^DAA^-AA^DB = A^DAA^DB = A^DB.$$

On the other hand,

$$\begin{aligned} AA^D - BA^D &= BA^-AA^D - BA^D = BA^-A^{m+1}(A^D)^{m+1} - BA^m(A^D)^{m+1} \\ &= B(A^-A^{m+1}A^D - A^mA^D)(A^D)^m = B(A^-A^m - A^mA^D)(A^D)^m \\ &= B(A^-A^m - A^DA^m)(A^D)^m. \end{aligned}$$

Therefore, $\|AA^D - BA^D\| \leq \|B\|\|A^-A^m - A^DA^m\|\|(A^D)^m\| \to 0$ as $m \to \infty$, and hence $A \leq^D B$.

Conversely, let $A \leq^D B$ and $m \geq k$. Then $A^DB = A^DA = AA^D = BA^D$. Further, $A^{-,D}A = A^-AA^DA = A^-AA^DB = A^{-,D}B$ and

$$\begin{aligned} AA^{-,D} - BA^{-,D} &= AA^-AA^D - BA^-AA^D = AA^D - BA^-AA^D \\ &= BA^D - BA^-AA^D = B(A^D)^{m+1}A^m - BA^-A^m(A^D)^m \\ &= B(A^mA^D - A^-A^m)(A^D)^m. \end{aligned}$$

Therefore, $\|AA^{-,D} - BA^{-,D}\| \leq \|B\| \|A^m A^D - A^- A^m\| \|(A^D)^m\| \to 0$ as $m \to \infty$. Hence, $A \leq^{-,D} B$. The fact that $\leq^D$ is a pre-order completes the proof. □

3. D1 INVERSES

In this section, we will introduce the dual inverses (called D1 inverses) of 1D inverses of square matrices. The detailed results will be not included but only we shall present its definition and some comments since all of them can be directly obtained similarly to the previous ones.

Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and let A^- be a fixed inner inverse of A. Then $A^D AA^- \in \mathbb{C}^{n\times n}$ is the unique solution of the system of matrix equations $XAX = X$, $A^k X = A^k A^-$, and $XA = A^D A$. Such a matrix will be called the D1 inverse of A and denoted by $A^{D,-} = A^D AA^-$.

As well as for 1D inverses, it is easy to verify that the DMP inverse, GDMP inverse [11], and D1 inverse all are different in general. We recall that a matrix $X \in \mathbb{C}^{n\times n}$ is called a G-Drazin of $A \in \mathbb{C}^{n\times n}$ if $AXA = A$ and $A^k X = XA^k$. As before, it can be found examples where $A^{D,-} \neq A^{D,\dagger}$ and $A^{GD\dagger} = A^{GD}AA^\dagger \neq A^{D,-}$.

A dual result similar to Theorem 1, related to outer inverses with prescribed range and null space, can be stated. Moreover, Theorems 2, 3, and 4, corresponding to characterizations and representations, can be deduced for D1 inverses. Results related to the equality with the Drazin inverses and to the idempotent property for D1 inverses can be similarly obtained as in Propositions 2 and 3 and Theorem 5, respectively. New characterizations and new properties can be obtained similarly to those obtained in Theorems 6, 7, and 8 can be obtained also in this case. A canonical form for D1 inverses can be obtained as in Theorem 9. From those results we conclude: $A^{-,D} = A^D = A^{D,-} \Leftrightarrow AA^{-,D} = A^{-,D}A$ and $AA^{D,-} = A^{D,-}A$.

Now, a binary relation $A \leq^{D,-} B$ can be defined as $AA^{D,-} = BA^{D,-}$ and $A^{D,-}A = A^{D,-}B$.

A similar study may be done for this binary relation. For instance, if $A, B \in \mathbb{C}^{n\times n}$ are matrices of index at most 1 such that $A \leq^{D,-} B$ then $A \leq \#B$, where $\leq \#$ is the right sharp partial order.

Consider the matrix class

$$\mathcal{PO}_k^r = \left\{ A \in \mathbb{C}^{n\times n} : ind\ (A) = k, \|A^D\| \leq 1, \lim_{m\to\infty} \|A^m A^- - A^m A^D\| = 0 \right\}.$$

If $A, B \in \mathcal{PO}_k^r$ then the relation $\leq^{D,-}$ is a pre-order on $\mathcal{PO}_k^r$. Now, we combine both 1D and D1 inverses to relate them to the common inner inverse used in their definitions.

Theorem 15. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A, both expressed as in (9). The following statements are valid.*

(a) *If $A^-A = AA^-$ then $k \leq 1$ and $A^{-,D} = A^{D,-} = A^\#$.*
(b) *If $A^{-,D} = A^{D,-}$ and $NT = TN$ (in decomposition given in (9)) then $A^-A = AA^-$ (and $k \leq 1$).*

3.1. *An application: Solving linear systems of equations*

This last subsection is about linear systems whose solution is given by the D1 inverse introduced before.

Theorem 16. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$ and A^- be a fixed inner inverse of A. For any $\ell \geq k$, the following equation $A^\ell x = A^\ell A^- b, \ \ b \in \mathbb{C}^n$, is consistent and its general solution is given by $x = A^{D,-}b + (I - A^D A)y, \quad$ for arbitrary $y \in \mathbb{C}^n$.*

Proof. Let $x = A^{D,-}b + (I - A^D A)y$, with $y \in \mathbb{C}^n$. Then,

$$A^\ell x = A^\ell (A^{D,-}b + (I - A^D A)y) = A^\ell (A^D A A^- b + (I - A^D A)y) = A^\ell (A^D)^\ell A^\ell A^- b$$
$$= A^\ell A^- b.$$

Therefore, x is a solution of $A^\ell x = A^\ell A^- b$.

Let z be any arbitrary solution of $A^\ell x = A^\ell A^- b$. Next, we will claim that $z = A^{D,-}b + (I - A^D A)y$ for some $y \in \mathbb{C}^n$. In fact, from $A^D A z = (A^D)^\ell A^\ell z = (A^D)^\ell A^\ell A^- b$, we obtain $z = A^{D,-}b + z - A^{D,-}b = A^{D,-}b + z - A^D A z = A^{D,-}b + (I - A^D A)z$. This completes the proof. □

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