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SOLVABILITY OF A THIRD-ORDER SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS VIA A GENERALIZED FIBONACCI SEQUENCE

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Abstract. In this paper, we solve in closed-form the following third-order system of nonlinear difference equations

$$x_{n+1}=\frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q+b_n y_n y_{n-1})},\quad y_{n+1}=\frac{x_n x_{n-1} y_{n-1}^q}{y_n(c_n x_{n-2}^p+d_n x_n x_{n-1})},\quad p,q\in\mathbb{N}, n\in\mathbb{N}_0$$

where the initial values $x_{-i}, y_{-i}, i=0,1,2$ and the parameters $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers. The form of the solutions of the one dimensional case of our system and a more general system defined by one to one functions are also presented.

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1. INTRODUCTION

Difference equations represent an important topic from discrete mathematics nowadays, which generally describes the discrete quantities in phenomena in different disciplines, such as biology and computer sciences. Recently, there has been a noticeable development in the study of nonlinear difference equations and their systems, and within this area, an essential part ought to be reserved for the investigations of the closed-form formulas of solutions, although it is often difficult to achieve. We refer the interested reader to [1–7, 9–16, 18, 19]. In this context, as a generalization of the system

$$x_{n+1}=\frac{x_{n-1}y_n}{y_n\pm y_{n-2}},\quad y_{n+1}=\frac{y_{n-1}x_n}{x_n\pm x_{n-2}},\quad n\in\mathbb{N}_0. \tag{1.1}$$

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studied in [17], the authors of [8] considered the following system of difference equations

$$x_{n+1}=\frac{x^{p}_{n-k+1}y_n}{ay^{p}_{n-k}+by_n},\quad y_{n+1}=\frac{y^{p}_{n-k+1}x_n}{\alpha x_{n-k}+\beta x_n},\quad n,p\in\mathbb{N}_0,k\in\mathbb{N},\tag{1.2}$$

where the coefficients a,b,α,β and the initial values $x_{-i},y_{-i},i\in 0,1,...,k$ are real numbers.

Our goal in this paper is to find the solution form of the following system of difference equations

$$x_{n+1}=\frac{y_ny_{n-1}x^{p}_{n-1}}{x_n(a_ny^{q}_{n-2}+b_ny_ny_{n-1})},\quad y_{n+1}=\frac{x_nx_{n-1}y^{q}_{n-1}}{y_n(c_nx^{p}_{n-2}+d_nx_nx_{n-1})},\quad n\in\mathbb{N}_0,\ p,\ q\in\mathbb{N}\tag{1.3}$$

where the parameters $(a_n)_{n\in\mathbb{N}_0},(b_n)_{n\in\mathbb{N}_0},(c_n)_{n\in\mathbb{N}_0},(d_n)_{n\in\mathbb{N}_0}$ and the initial values $x_{-i},y_{-i},i=0,1,2,$ are non-zero real numbers. We will also determine the form of the solution of the following system which is a generalization of System (1.3).

$$\begin{aligned}x_{n+1}&=f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q+b_ng(y_n)g(y_{n-1})\right]}\right),\\ y_{n+1}&=g^{-1}\left(\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p+d_nf(x_n)f(x_{n-1})\right]}\right),\quad n\in\mathbb{N}_0,p,q\in\mathbb{N},\end{aligned}\tag{1.4}$$

where $f,g:D\to\mathbb{R}$ are one to one continuous functions on $D\subseteq R$, the initial values $x_{-i},y_{-i},i=0,1,2,$ are real numbers in D and the parameters $(a_n)_{n\in\mathbb{N}_0},(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0},(d_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers.

Let us recall the following basic and well known lemma.

Lemma 1. *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers. Consider the linear difference equation*

$$y_{n+k}=a_ny_n+b_n,\quad k=2,3,\quad n\in\mathbb{N}_0.$$

Then,

$$y_{kn+i}=\left[\prod_{j=0}^{n-1}a_{kj+i}\right]y_i+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{kj+i}\right]b_{kr+i},\quad \text{for } i=0,1,\cdots,k-1.$$

Moreover, if $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constants (i.e. $a_n=a,b_n=b$), then

$$y_{kn+i}=\begin{cases}y_i+bn, & a=1,\\ a^ny_i+\left(\dfrac{a^n-1}{a-1}\right)b, & otherwise,\end{cases}\quad \text{for } i=0,1,\cdots,k-1,\quad n\in\mathbb{N}_0.$$

where, as usual, $\prod_{j=i}^{k}A_j=1$ and $\sum_{j=i}^{k}A_j=0$, for all $k<i$.

We will write the formulas of the solutions of our System (1.3) using the terms of the sequence $\{F_n\}_{n=0}^{\infty}$ defined by the second-order linear difference equation

$$F_{n+2} = F_{n+1} + rF_n, \, r = p, q, \, F_0 = F_1 = 1, \quad n \in \mathbb{N}_0. \tag{1.5}$$

This is one of the generalizations of the Fibonacci sequence and it is called the r-Fibonacci sequence. In the following, we give the first thirteen terms of it.

$$F_0 = 1,$$
$$F_1 = 1,$$
$$F_2 = 1 + r,$$
$$F_3 = 1 + 2r,$$
$$F_4 = 1 + 3r + r^2,$$
$$F_5 = 1 + 4r + 3r^2,$$
$$F_6 = 1 + 5r + 6r^2 + r^3,$$
$$F_7 = 1 + 6r + 10r^2 + 4r^3,$$
$$F_8 = 1 + 7r + 15r^2 + 10r^3 + r^4,$$
$$F_9 = 1 + 8r + 21r^2 + 20r^3 + 5r^4,$$
$$F_{10} = 1 + 9r + 28r^2 + 35r^3 + 15r^4 + r^5,$$
$$F_{11} = 1 + 10r + 36r^2 + 56r^3 + 35r^4 + 6r^5,$$
$$F_{12} = 1 + 11r + 45r^2 + 84r^3 + 70r^4 + 21r^5 + r^6.$$

2. The solutions of system (1.3)

In this part, we show the solvability of our System (1.3). In fact we will give the closed form of the well-defined solutions of our system.

Definition 1. A solution $\{x_n, y_n\}_{n \geq -2}$ of System (1.3) is said to be well-defined if

$$x_n \left(a_n y_{n-2}^q + b_n y_n y_{n-1}\right) y_n \left(c_n x_{n-2}^p + d_n x_n x_{n-1}\right) \neq 0, \, n \in \mathbb{N}_0.$$

Remark 1. We want to note that the sequence $\{F_n\}_{n=0}^{\infty}$ will be defined by

$$F_{n+2} = F_{n+1} + pF_n, \, F_0 = F_1 = 1, n \in \mathbb{N}_0$$

for the formulas of the x_n-component of the solutions and

$$F_{n+2} = F_{n+1} + qF_n, \, F_0 = F_1 = 1, n \in \mathbb{N}_0$$

for the formulas of the y_n-component of the solutions.

Now, we will start the resolution of our system. Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution of System (1.3). We have

$$\begin{aligned} x_{n+1} &= \frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q + b_n y_n y_{n-1})}, & y_{n+1} &= \frac{x_n x_{n-1} y_{n-1}^q}{y_n(c_n x_{n-2}^p + d_n x_n x_{n-1})}, \\ \frac{x_{n+1}x_n}{x_{n-1}^p} &= \frac{y_n y_{n-1}}{a_n y_{n-2}^q + b_n y_n y_{n-1}}, & \frac{y_{n+1}y_n}{y_{n-1}^q} &= \frac{x_n x_{n-1}}{c_n x_{n-2}^p + d_n x_n x_{n-1}}, \\ \frac{x_{n-1}^p}{x_{n+1}x_n} &= \frac{a_n y_{n-2}^q + b_n y_n y_{n-1}}{y_n y_{n-1}}, & \frac{y_{n-1}^q}{y_{n+1}y_n} &= \frac{c_n x_{n-2}^p + d_n x_n x_{n-1}}{x_n x_{n-1}}, \\ \frac{x_{n-1}^p}{x_{n+1}x_n} &= a_n \frac{y_{n-2}^q}{y_n y_{n-1}} + b_n, & \frac{y_{n-1}^q}{y_{n+1}y_n} &= c_n \frac{x_{n-2}^p}{x_n x_{n-1}} + d_n. \end{aligned}$$

Taking the change of variables

$$u_n = \frac{x_{n-2}^p}{x_n x_{n-1}}, \quad v_n = \frac{y_{n-2}^q}{y_n y_{n-1}}, \tag{2.1}$$

System (1.3) can be written as

$$u_{n+1} = a_n v_n + b_n, \quad v_{n+1} = c_n u_n + d_n, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Hence, we have

$$\begin{aligned} u_{n+2} &= a_{n+1} v_{n+1} + b_{n+1} = a_{n+1}[c_n u_n + d_n] + b_{n+1} = a_{n+1} c_n u_n + (a_{n+1} d_n + b_{n+1}), \\ v_{n+2} &= c_{n+1} u_{n+1} + d_{n+1} = c_{n+1}[a_n v_n + b_n] + d_{n+1} = c_{n+1} a_n v_n + (c_{n+1} b_n + d_{n+1}). \end{aligned}$$

From this, we get, for all $n \in \mathbb{N}_0$, the following linear second order nonhomogeneous difference equations,

$$\begin{cases} u_{n+2} = a_{n+1} c_n u_n + (a_{n+1} d_n + b_{n+1}), \\ v_{n+2} = c_{n+1} a_n v_n + (c_{n+1} b_n + d_{n+1}). \end{cases} \tag{2.3}$$

From Lemma 1, we have for all $n \in \mathbb{N}_0$ and for $i = 0, 1$, the solutions of equations in (2.3) are

$$u_{2n+i} = \left[\prod_{j=0}^{n-1} a_{2j+i+1} c_{2j+i}\right] u_i + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+i+1} c_{2j+i}\right] (a_{2r+i+1} d_{2r+i} + b_{2r+i+1}), \tag{2.4}$$

$$v_{2n+i} = \left[\prod_{j=0}^{n-1} c_{2j+i+1} a_{2j+i}\right] v_i + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} c_{2j+i+1} a_{2j+i}\right] (c_{2r+i+1} b_{2r+i} + d_{2r+i+1}). \tag{2.5}$$

From (2.1) and Equations (2.4) and (2.5), it follows that for all $n \in \mathbb{N}_0$

$$u_{2n} = \left[\prod_{j=0}^{n-1} a_{2j+1} c_{2j}\right] \frac{x_{-2}^p}{x_0 x_{-1}} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+1} c_{2j}\right] (a_{2r+1} d_{2r} + b_{2r+1}), \tag{2.6}$$

$$u_{2n+1}=\frac{\left[\prod_{j=0}^{n-1}a_{2j+2}c_{2j+1}\right][a_0y_{-2}^{q}+b_0y_0y_{-1}]}{y_0y_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{2j+2}c_{2j+1}\right](a_{2r+2}d_{2r+1}+b_{2r+2}) \tag{2.7}$$

$$v_{2n}=\frac{\left[\prod_{j=0}^{n-1}c_{2j+1}a_{2j}\right]y_{-2}^{q}}{y_0y_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{2j+1}a_{2j}\right](c_{2r+1}b_{2r}+d_{2r+1}), \tag{2.8}$$

and

$$v_{2n+1}=\frac{\left[\prod_{j=0}^{n-1}c_{2j+2}a_{2j+1}\right][c_0x_{-2}^{p}+d_0x_0x_{-1}]}{x_0x_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{2j+2}a_{2j+1}\right](c_{2r+2}b_{2r+1}+d_{2r+2}). \tag{2.9}$$

Now we give the solution form of Equations (2.3) when all the coefficients in System (1.3) are constant. To do this, we suppose that $a_n=a, b_n=b, c_n=c$ and $d_n=d$, for every $n\in\mathbb{N}_0$. Then Equations (2.3) becomes

$$\begin{cases} u_{n+2}=acu_n+ad+b,\\ v_{n+2}=cav_n+cb+d. \end{cases} \tag{2.10}$$

From Lemma 1, we have for all $n\in\mathbb{N}_0$ and for $i=0,1$, the solutions of equations in (2.10) are

$$u_{2n+i}=\begin{cases} u_i+(ad+b)n, & ac=1,\\ (ac)^n u_i+\left(\dfrac{(ac)^n-1}{ac-1}\right)(ad+b), & otherwise, \end{cases} \tag{2.11}$$

and

$$v_{2n+i}=\begin{cases} v_i+(csb+ct+d)n, & ac=1,\\ (ca)^n v_i+\left(\dfrac{(ca)^n-1}{ca-1}\right)(cb+d), & otherwise. \end{cases} \tag{2.12}$$

From (2.1) and Equations (2.11) and (2.12), it follows that for all $n\in\mathbb{N}_0$

$$u_{2n}=\begin{cases} \dfrac{x_{-2}^{p}}{x_0x_{-1}}+(ad+b)n, & ac=1,\\[2ex] \dfrac{(ac)^n x_{-2}^{p}}{x_0x_{-1}}+\left(\dfrac{(ac)^n-1}{ac-1}\right)(ad+b), & otherwise, \end{cases} \tag{2.13}$$

$$u_{2n+1}=\begin{cases}\dfrac{ay_{-2}^{q}+by_0y_{-1}}{y_0y_{-1}}+(ad+b)n, & ac=1,\\[2ex] \dfrac{(ac)^n(ay_{-2}^{q}+by_0y_{-1})}{y_0y_{-1}}+\left(\dfrac{(ac)^n-1}{ac-1}\right)(ad+b), & otherwise,\end{cases} \tag{2.14}$$

$$v_{2n}=\begin{cases}\dfrac{y_{-2}^{q}}{y_0y_{-1}}+(cb+d)n, & ac=1,\\[2ex] \dfrac{(ca)^ny_{-2}^{q}}{y_0y_{-1}}+\left(\dfrac{(ca)^n-1}{ca-1}\right)(cb+d), & otherwise,\end{cases} \tag{2.15}$$

$$v_{2n+1}=\begin{cases}\dfrac{cx_{-2}^{p}+dx_0x_{-1}}{x_0x_{-1}}+(cb+d)n, & ac=1,\\ (ca)^n\dfrac{cx_{-2}^{p}+dx_0x_{-1}}{x_0x_{-1}}+\left(\dfrac{(ca)^n-1}{ca-1}\right)(cb+d), & otherwise.\end{cases} \tag{2.16}$$

Now, from (2.1) it follows that

$$x_n=\frac{x_{n-2}^{p}}{u_nx_{n-1}},\quad y_n=\frac{y_{n-2}^{q}}{v_ny_{n-1}}, \tag{2.17}$$

So we have

$$x_0=\frac{x_{-2}^{p}}{u_0x_{-1}},$$

hence

$$x_0=\frac{x_{-2}^{pF_0}}{u_0^{F_0}x_{-1}^{F_1}}.$$

Moreover,

$$x_1=\frac{x_{-1}^{p}}{u_1x_0}=\frac{u_0x_{-1}x_{-1}^{p}}{u_1x_{-2}^{p}}=\frac{u_0x_{-1}^{p+1}}{u_1x_{-2}^{p}},$$

so

$$x_1=\frac{u_0^{F_1}x_{-1}^{F_2}}{u_1^{F_0}x_{-2}^{pF_1}}$$

and

$$x_2=\frac{x_0^{p}}{u_2x_1}=\frac{x_{-2}^{p^2}u_1x_{-2}^{p}}{u_0^{p}x_{-1}^{p}u_2u_0x_{-1}^{p+1}}=\frac{u_1x_{-2}^{p^2+p}}{u_2u_0^{p+1}x_{-1}^{2p+1}},$$

thus

$$x_2=\frac{u_1^{F_1}x_{-2}^{pF_2}}{u_2^{F_0}u_0^{F_2}x_{-1}^{F_3}}.$$

Similarly, we obtain

$$x_3 = \frac{u_2 u_0^{2p+1} x_{-1}^{p^2+3p+1}}{u_3 u_1^{p+1} x_{-2}^{2p^2+p}} = \frac{u_2^{F_1} u_0^{F_3} x_{-1}^{F_4}}{u_3^{F_0} u_1^{F_2} x_{-2}^{pF_3}},$$

$$x_4 = \frac{u_3 u_1^{2p+1} x_{-2}^{p^3+3p^2+p}}{u_4 u_2^{p+1} u_0^{p^2+3p+1} x_{-1}^{3p^2+4p+1}} = \frac{u_3^{F_1} u_1^{F_3} x_{-2}^{pF_4}}{u_4^{F_0} u_2^{F_2} u_0^{F_4} x_{-1}^{F_5}},$$

$$x_5 = \frac{u_4 u_2^{2p+1} u_0^{3p^2+4p+1} x_{-1}^{p^3+6p^2+5p+1}}{u_5 u_3^{p+1} u_1^{p^2+3p+1} x_{-2}^{3p^3+4p^2+p}} = \frac{u_4^{F_1} u_2^{F_3} u_0^{F_5} x_{-1}^{F_6}}{u_5^{F_0} u_3^{F_2} u_1^{F_4} x_{-2}^{pF_5}},$$

$$x_6 = \frac{u_5 u_3^{2p+1} u_1^{3p^2+4p+1} x_{-2}^{p^4+6p^3+5p^2+p}}{u_6 u_4^{p+1} u_2^{p^2+3p+1} u_0^{p^3+6p^2+5p+1} x_{-1}^{4p^3+10p^2+6p+1}} = \frac{u_5^{F_1} u_3^{F_3} u_1^{F_5} x_{-2}^{pF_6}}{u_6^{F_0} u_4^{F_2} u_2^{F_4} u_0^{F_6} x_{-1}^{F_7}},$$

$$x_7 = \frac{u_6^{F_1} u_4^{F_3} u_2^{F_5} u_0^{F_7} x_{-1}^{F_8}}{u_7^{F_0} u_5^{F_2} u_3^{F_4} u_1^{F_6} x_{-2}^{pF_7}} = \frac{\prod_{i=0}^{3} u_{2i}^{F_{2(3-i)+1}} x_{-1}^{F_8}}{\prod_{i=0}^{3} u_{2i+1}^{F_{2(3-i)}} x_{-2}^{pF_7}},$$

$$x_8 = \frac{u_7^{F_1} u_5^{F_3} u_3^{F_5} u_1^{F_7} x_{-2}^{pF_8}}{u_8^{F_0} u_6^{F_2} u_4^{F_4} u_2^{F_6} u_0^{F_8} x_{-1}^{F_9}} = \frac{\prod_{i=0}^{3} u_{2i+1}^{F_{2(4-i)-1}} x_{-2}^{pF_8}}{\prod_{i=0}^{4} u_{2i}^{F_{2(4-i)}} x_{-1}^{F_9}},$$

$$x_9 = \frac{u_8^{F_1} u_6^{F_3} u_4^{F_5} u_2^{F_7} u_0^{F_9} x_{-1}^{F_{10}}}{u_9^{F_0} u_7^{F_2} u_5^{F_4} u_3^{F_6} u_1^{F_8} x_{-2}^{pF_9}} = \frac{\prod_{i=0}^{4} u_{2i}^{F_{2(4-i)+1}} x_{-1}^{F_{10}}}{\prod_{i=0}^{4} u_{2i+1}^{F_{2(4-i)}} x_{-2}^{pF_9}},$$

$$x_{10} = \frac{u_9^{F_1} u_7^{F_3} u_5^{F_5} u_3^{F_7} u_1^{F_9} x_{-2}^{pF_{10}}}{u_{10}^{F_0} u_8^{F_2} u_6^{F_4} u_4^{F_6} u_2^{F_8} u_0^{F_{10}} x_{-1}^{F_{11}}} = \frac{\prod_{i=0}^{4} u_{2i+1}^{F_{2(5-i)-1}} x_{-2}^{pF_{10}}}{\prod_{i=0}^{5} u_{2i}^{F_{2(5-i)}} x_{-1}^{F_{11}}}.$$

By induction, it follows that

$$x_{2n} = \frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}} x_{-2}^{pF_{2n}}}{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)}} x_{-1}^{F_{2n+1}}}, \qquad x_{2n+1} = \frac{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)+1}} x_{-1}^{F_{2(n+1)}}}{\prod_{i=0}^{n} u_{2i+1}^{F_{2(n-i)}} x_{-2}^{pF_{2n+1}}}.$$

Similarly we have

$$y_0 = \frac{y_{-2}^{q}}{v_0 y_{-1}},$$

hence

$$y_0 = \frac{y_{-2}^{qF_0}}{v_0^{F_0} y_{-1}^{F_1}}.$$

Moreover,

$$y_1 = \frac{y_{-1}^{q}}{v_1 y_0} = \frac{v_0 y_{-1} y_{-1}^{q}}{v_1 y_{-2}^{q}} = \frac{v_0 y_{-1}^{q+1}}{v_1 y_{-2}^{q}},$$

so

$$y_1 = \frac{v_0^{F_1} y_{-1}^{F_2}}{v_1^{F_0} y_{-2}^{qF_1}}$$

and

$$y_2 = \frac{y_0^{q}}{v_2 y_1} = \frac{y_{-2}^{q^2} v_1 y_{-2}^{q}}{v_0^{q} y_{-1}^{q} v_2 v_0 y_{-1}^{q+1}} = \frac{v_1 y_{-2}^{q^2+q}}{v_2 v_0^{q+1} y_{-1}^{2q+1}},$$

thus

$$y_2 = \frac{v_1^{F_1} y_{-2}^{qF_2}}{v_2^{F_0} v_0^{F_2} y_{-1}^{F_3}}.$$

Similarly, we obtain

$$y_3 = \frac{v_2 v_0^{2q+1} y_{-1}^{q^2+3q+1}}{v_3 v_1^{q+1} y_{-2}^{2q^2+q}} = \frac{v_2^{F_1} v_0^{F_3} y_{-1}^{F_4}}{v_3^{F_0} v_1^{F_2} y_{-2}^{qF_3}},$$

$$y_4 = \frac{v_3 v_1^{2q+1} y_{-2}^{q^3+3q^2+q}}{v_4 v_2^{q+1} v_0^{q^2+3q+1} y_{-1}^{3q^2+4q+1}} = \frac{v_3^{F_1} v_1^{F_3} y_{-2}^{qF_4}}{v_4^{F_0} v_2^{F_2} v_0^{F_4} y_{-1}^{F_5}},$$

$$y_5 = \frac{v_4 v_2^{2q+1} v_0^{3q^2+4q+1} y_{-1}^{q^3+6q^2+5q+1}}{v_5 v_3^{q+1} v_1^{q^2+3q+1} y_{-2}^{3q^3+4q^2+q}} = \frac{v_4^{F_1} v_2^{F_3} v_0^{F_5} y_{-1}^{F_6}}{v_5^{F_0} v_3^{F_2} v_1^{F_4} y_{-2}^{qF_5}},$$

$$y_6 = \frac{v_5 v_3^{2q+1} v_1^{3q^2+4q+1} y_{-2}^{q^4+6q^3+5q^2+q}}{v_6 v_4^{q+1} u_2^{q^2+3q+1} v_0^{q^3+6q^2+5q+1} y_{-1}^{4q^3+10q^2+6q+1}} = \frac{v_5^{F_1} v_3^{F_3} v_1^{F_5} y_{-2}^{qF_6}}{v_6^{F_0} v_4^{F_2} v_2^{F_4} v_0^{F_6} y_{-1}^{F_7}},$$

$$y_7 = \frac{v_6^{F_1} v_4^{F_3} v_2^{F_5} v_0^{F_7} y_{-1}^{F_8}}{v_7^{F_0} v_5^{F_2} v_3^{F_4} v_1^{F_6} y_{-2}^{pF_7}} = \frac{\prod_{i=0}^{3} v_{2i}^{F_{2(3-i)+1}} y_{-1}^{F_8}}{\prod_{i=0}^{3} v_{2i+1}^{F_{2(3-i)}} y_{-2}^{qF_7}},$$

$$y_8 = \frac{v_7^{F_1} v_5^{F_3} v_3^{F_5} v_1^{F_7} y_{-2}^{pF_8}}{v_8^{F_0} v_6^{F_2} v_4^{F_4} v_2^{F_6} v_0^{F_8} y_{-1}^{F_9}} = \frac{\prod_{i=0}^{3} v_{2i+1}^{F_{2(4-i)-1}} y_{-2}^{qF_8}}{\prod_{i=0}^{4} v_{2i}^{F_{2(4-i)}} y_{-1}^{F_9}},$$

$$y_9 = \frac{v_8^{F_1} v_6^{F_3} v_4^{F_5} v_2^{F_7} v_0^{F_9} y_{-1}^{F_{10}}}{v_9^{F_0} v_7^{F_2} v_5^{F_4} v_3^{F_6} v_1^{F_8} y_{-2}^{qF_9}} = \frac{\prod_{i=0}^{4} v_{2i}^{F_{2(4-i)+1}} y_{-1}^{F_{10}}}{\prod_{i=0}^{4} v_{2i+1}^{F_{2(4-i)}} y_{-2}^{qF_9}},$$

$$y_{10} = \frac{v_9^{F_1} v_7^{F_3} v_5^{F_5} v_3^{F_7} v_1^{F_9} y_{-2}^{qF_{10}}}{v_{10}^{F_0} v_8^{F_2} v_6^{F_4} v_4^{F_6} v_2^{F_8} v_0^{F_{10}} y_{-1}^{F_{11}}} = \frac{\prod_{i=0}^{4} v_{2i+1}^{F_{2(5-i)-1}} y_{-2}^{qF_{10}}}{\prod_{i=0}^{5} v_{2i}^{F_{2(5-i)}} y_{-1}^{F_{11}}}.$$

By induction, it follows that

$$y_{2n} = \frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{2(n-i)-1}} y_{-2}^{qF_{2n}}}{\prod_{i=0}^{n} v_{2i}^{F_{2(n-i)}} y_{-1}^{F_{2n+1}}}, \qquad y_{2n+1} = \frac{\prod_{i=0}^{n} v_{2i}^{F_{2(n-i)+1}} y_{-1}^{F_{2(n+1)}}}{\prod_{i=0}^{n} v_{2i+1}^{F_{2(n-i)}} y_{-2}^{qF_{2n+1}}}.$$

From the above calculations, we summarize in the following theorem the form of the solutions of System (1.3).

Theorem 1. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution of System (1.3). Then, for all $n \in \mathbb{N}_0$, we have*

$$x_{2n} = \left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)}}} \right) \frac{x_{-2}^{pF_{2n}}}{x_{-1}^{F_{2n+1}}}, \qquad x_{2n+1} = \left(\frac{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^{n} u_{2i+1}^{F_{2(n-i)}}} \right) \frac{x_{-1}^{F_{2(n+1)}}}{x_{-2}^{pF_{2n+1}}},$$

$$y_{2n} = \left(\frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^{n} v_{2i}^{F_{2(n-i)}}} \right) \frac{y_{-2}^{qF_{2n}}}{y_{-1}^{F_{2n+1}}} \quad \text{and} \quad y_{2n+1} = \left(\frac{\prod_{i=0}^{n} v_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^{n} v_{2i+1}^{F_{2(n-i)}}} \right) \frac{y_{-1}^{F_{2(n+1)}}}{y_{-2}^{qF_{2n+1}}},$$

where the terms of the sequences $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ are given by formulas (2.6) - (2.9) *in the case of variables coefficients and* (2.13) - (2.16) *in the case of constant coefficients.*

Remark 2. If we take $c_n = a_n$, $d_n = b_n$, $n \in \mathbb{N}_0$ $q = p$ and $y_{-i} = x_{-i}$, $i = 0,1,2$, then, we obtain the one dimensional version of System (1.3), that is the equation

$$x_{n+1} = \frac{x_{n-1}^{p+1}}{a_n x_{n-2}^p + b_n x_n x_{n-1}}, \quad p \in \mathbb{N}, n \in \mathbb{N}_0. \tag{2.18}$$

As a consequence the solutions of Equations (2.18) can be obtained from Theorem 1, and their formulas are given in the the following result.

Corollary 1. *Let* $\{x_n\}_{n\geq -2}$ *be a well-defined solution of Equation* (2.18)*, then for* $n \in \mathbb{N}_0$ *we have*

$$x_{2n} = \left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)}}} \right) \frac{x_{-2}^{pF_{2n}}}{x_{-1}^{F_{2n+1}}}, \qquad x_{2n+1} = \left(\frac{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^{n} u_{2i+1}^{F_{2(n-i)}}} \right) \frac{x_{-1}^{F_{2(n+1)}}}{x_{-2}^{pF_{2n+1}}},$$

the terms of the sequence $(u_n)_{n\in\mathbb{N}_0}$ *are given by formulas* (2.6), (2.7) *in the case of variables coefficients and* (2.13), (2.14) *in the case of constant coefficients.*

3. The form of the solutions of a more general system defined by one to one functions

In this part, we will show the solvability of the following system

$$\begin{aligned} x_{n+1} &= f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right]} \right), \\ y_{n+1} &= g^{-1}\left(\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})\right]} \right), \quad n \in \mathbb{N}_0, p, q \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where $f, g : D \to \mathbb{R}$ are one to one continuous functions on $D \subseteq R$, the initial values $x_{-i}, y_{-i}, i = 0,1,2$, are real numbers in D and the parameters $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers.

Definition 2. A solution $\{x_n, y_n\}_{n\geq -2}$ of System (3.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right] \neq 0,$$

$$g(y_n)\left[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})\right] \neq 0,$$

$$\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right]} \in D_{f^{-1}}$$

and

$$\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})\right]} \in D_{g^{-1}}.$$

Since f and g are one to one continuous functions, then we get

$$f(x_{n+1}) = \frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)(a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1}))},$$

$$g(y_{n+1}) = \frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)(c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1}))}.$$

Taking the change of variables

$$X_n = f(x_n), \quad Y_n = g(y_n), \quad n \in \mathbb{N}_0 \tag{3.2}$$

it follows that System (3.1) can be transformed to the following system

$$X_{n+1} = \frac{Y_n Y_{n-1} X_{n-1}^p}{X_n(a_n Y_{n-2}^q + b_n Y_n Y_{n-1})}, \quad Y_{n+1} = \frac{X_n X_{n-1} Y_{n-1}^q}{Y_n(c_n X_{n-2}^p + d_n X_n X_{n-1})}, \quad n \in \mathbb{N}_0,$$

which is in the form (1.3). So, using (3.2), the fact that

$$x_n = f^{-1}(X_n),\, y_n = g^{-1}(Y_n),\, n \in \mathbb{N}_0$$

and Theorem (1), we get the following result which describes the form of the solutions of System (3.1).

Theorem 2. *Let $\{x_n, y_n\}_{n \geq -2}$ be a well-defined solution of System (3.1). Then, for all $n \in \mathbb{N}_0$, we have*

$$x_{2n} = f^{-1}\left(\left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)}}}\right)\frac{(f(x_{-2}))^{pF_{2n}}}{(f(x_{-1}))^{F_{2n+1}}}\right),$$

$$x_{2n+1} = f^{-1}\left(\left(\frac{\prod_{i=0}^{n} u_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^{n} u_{2i+1}^{F_{2(n-i)}}}\right)\frac{(f(x_{-1}))^{F_{2(n+1)}}}{(f(x_{-2}))^{pF_{2n+1}}}\right),$$

$$y_{2n} = g^{-1}\left(\left(\frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^{n} v_{2i}^{F_{2(n-i)}}}\right)\frac{(g(y_{-2}))^{qF_{2n}}}{(g(y_{-1}))^{F_{2n+1}}}\right)$$

and

$$y_{2n+1} = g^{-1}\left(\left(\frac{\prod_{i=0}^{n} v_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^{n} v_{2i+1}^{F_{2(n-i)}}}\right)\frac{(g(y_{-1}))^{F_{2(n+1)}}}{(g(y_{-2}))^{qF_{2n+1}}}\right),$$

where the terms of the sequences $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ are given by the following formulas

$$u_{2n} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+1}c_{2j}\right](f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+1}c_{2j}\right](a_{2r+1}d_{2r}+b_{2r+1}), \quad (3.3)$$

$$u_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+2}c_{2j+1}\right][a_0(g(y_{-2}))^q + b_0 g(y_0)g(y_{-1})]}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+2}c_{2j+1}\right](a_{2r+2}d_{2r+1}+b_{2r+2}), \quad (3.4)$$

$$v_{2n} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+1}a_{2j}\right](g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+1}a_{2j}\right](c_{2r+1}b_{2r}+d_{2r+1}), \quad (3.5)$$

$$v_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+2}a_{2j+1}\right][c_0(f(x_{-2}))^p + d_0 f(x_0)f(x_{-1})]}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+2}a_{2j+1}\right](c_{2r+2}b_{2r+1}+d_{2r+2}), \quad (3.6)$$

if the coefficients are variables and by the following formulas

$$u_{2n} = \begin{cases} \frac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})} + (ad+b)n, & ac = 1, \\ \frac{(ac)^n(f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \left(\frac{(ac)^n - 1}{ac-1}\right)(ad+b), & \text{otherwise}, \end{cases} \quad (3.7)$$

$$u_{2n+1} = \begin{cases} \frac{a(g(y_{-2}))^q + bg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})} + (ad+b)n, & ac = 1, \\ \frac{(ac)^n(a(g(y_{-2}))^q + bg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})} + \left(\frac{(ac)^n - 1}{ac-1}\right)(ad+b), & \text{otherwise}, \end{cases} \quad (3.8)$$

$$v_{2n} = \begin{cases} \dfrac{(g(y_{-2}))^q}{g(y_0)g(y_{-1})} + (cb+d)n, & ac = 1, \\ \dfrac{(ca)^n(g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \left(\dfrac{(ca)^n - 1}{ca-1}\right)(cb+d), & otherwise, \end{cases} \tag{3.9}$$

$$v_{2n+1} = \begin{cases} \dfrac{c(f(x_{-2}))^p + df(x_0)f(x_{-1})}{f(x_0)f(x_{-1})} + (cb+d)n, & ac = 1, \\ \dfrac{(ca)^n(c(f(x_{-2}))^p + df(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})} + \left(\dfrac{(ca)^n - 1}{ca-1}\right)(cb+d), & otherwise, \end{cases} \tag{3.10}$$

if the coefficients are constants, again the sequence $\{F_n\}_{n=0}^{\infty}$ *is defined by*

$$F_{n+2} = F_{n+1} + pF_n,\, F_0 = F_1 = 1, n \in \mathbb{N}_0$$

for the formulas of the x_n*-component of the solutions and*

$$F_{n+2} = F_{n+1} + qF_n,\, F_0 = F_1 = 1, n \in \mathbb{N}_0$$

for the formulas of the y_n*-component of the solutions.*

Remark 3. Clearly if we take the functions f and g such that $f(x) = x$ and $g(x) = x$, then System (3.1) will be nothing other than System (1.3).

4. CONCLUSION

In this paper, a third-order system of nonlinear difference equations was proposed. Using some changes of variables and p-Fibonacci numbers, the solutions of this system were written in closed-form. It is showed also that a more general system defined by one to one functions is also solvable in closed form.

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