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EXISTENCE AND UNIQUENESS RESULTS FOR A COUPLED SYSTEM OF NONLINEAR Ψ-FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we establish some new existence results for certain coupled systems of nonlinear Ψ-fractional differential equations with four-point fractional integral boundary conditions. The proofs are derived from Banach fixed point theorem. As an application, a nontrivial example is presented to illustrate our theoretical results.

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1. INTRODUCTION

In this article, we are concerned with the study of the following $\Psi-$Caputo fractional coupled system

$$\begin{cases} {}^C D_{0^+}^{q_1,\Psi} x(t) = f_1(t,x(t),y(t)) \\ \\ {}^C D_{0^+}^{q_2,\Psi} y(t) = f_2(t,x(t),y(t)), \end{cases} \tag{1.1}$$

with conditions

$$\begin{cases} x(0) = \varphi_1(x), \\ \\ y(0) = \varphi_2(y), \\ \\ x(1) = I_{0^+}^{\alpha,\Psi} x(\theta);\ \theta \in]0,1[;\ \alpha > 0, \\ \\ y(1) = I_{0^+}^{\beta,\Psi} y(\delta);\ \delta \in]0,1[;\ \beta > 0, \end{cases} \tag{1.2}$$

where, ${}^{C}D_{0^+}^{q_1,\Psi}$ and ${}^{C}D_{0^+}^{q_1,\Psi}$ are Ψ-Caputo derivatives at order $q_1 \in (0,1)$ and $q_2 \in (0,1)$ of x and y respectively, $I_{0^+}^{\alpha,\Psi}$ and $I_{0^+}^{\beta,\Psi}$ are Ψ-fractional integrals and φ_1 and φ_2 are two continuous given functions. This type of derivative was introduced by R. Almeida in [2]. In our survey, to prove the existence of result, we exploit the topological method, which have proven to be useful in the investigation of a variety of nonlinear analysis problems. The a priori estimate method has been used to verify the existence of solutions for some nonlinear differential equations or nonlinear partial differential equations boundary value problems. Due to Isaia [9], we employ the fixed point theorem, which was determined using coincidence degree theory for condensing maps. In [9] the author investigate the problem $x(t) = \varphi(t,x(t)) + \int_a^b \Psi(t,s,x(s))\,ds$ by using the "a priori estimate method" which derived from the invariance under homotopy of the degree defined for $\alpha-$ condensing perturbations of the identity. The authors of [5] analyze a nonlinear fractional q-difference subject to nonlinear more generic four-point boundary conditions using two approaches: a prior estimate method for condensing maps and the Banach contraction principle fixed point theorem. The fractional Cauchy problem results for non-nonlinear Ψ-Caputo fractional differential equations with non-local conditions are investigated by El Mfadel et al. [7]. They used fixed point theorems, topological degree theory for condensing maps, and fractional analysis techniques to develop some novel existence theorems. The significance of fractional calculus can be seen in the modulation of a variety of phenomena in electromagnetic, electrochemistry and viscoelasticity. The choice of a Ψ function under certain conditions is not arbitrary. However, it is a solution for unifying the kernels of several forms of fractional operators, namely, Riemann-Liouville, Hadamard and Erdelyi-Kober. The purpose of this research is to investigate a coupled Ψ-fraction system with boundary conditions, which generalizes the Dirichlet problem, for two mixed situations.

Our paper is organized as follows. In Section 2, we give preliminaries concerning $\Psi-$fractional integral and $\Psi-$Caputo fractional derivative which are necessary in this study. In Section 3, we will study the existence of solutions for (1.1) by using the concept Lipschitz and Carathéodory conditions. As application, we will apply the theoretical results on an example in Section 4 followed by a conclusion in Section 5.

2. Preliminaries

We assume the following notations throughout the rest of our paper.

Let $\Delta = [0,1]$ be an interval of $\mathbb{R}$. We denote the set of continuous functions from Δ into $\mathbb{R}$ by $\mathcal{C} = \mathcal{C}(\Delta,\mathbb{R})$ endowed the the supremum norm $\|x\| = \sup\limits_{t\in\Delta} |x(t)|$.

In this section, we give some definitions and properties of Ψ-fractional derivatives and Ψ-fractional integrals, (for more details see [1, 2, 4, 6, 8] and references therein).

Definition 1 ([3]). Let $q > 0$, $g \in L^1([\Delta,\mathbb{R})$ and $\Psi \in C^n(\Delta,\mathbb{R})$ such that $\Psi'(t) > 0$ for all $t \in \Delta$. The Ψ-Riemann-Liouville fractional integral at order q of the function

g is given by

$$I_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)}\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{q-1}g(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function defined by $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}dt, \quad \lambda > 0.$

Definition 2 ([3])**.** Let $\Psi \in C^n(\Delta,\mathbb{R})$ such that $\Psi'(t) > 0$ for all $t \in \Delta$. The $\Psi-$Caputo fractional derivative at order $q > 0$ of the function $g \in C^{n-1}(\Delta,\mathbb{R})$ is given by

$${}^{C}D_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(n-q)}\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{n-q-1}g_\Psi^{[n]}(s)ds,$$

where

$$g_\Psi^{[n]}(s) = \left(\frac{1}{\Psi'(s)}\frac{d}{ds}\right)^n g(s) \text{ and } n = [q]+1,$$

with $[q]$ is the integer part of the real number q.

Remark 1. In particular, if $q \in]0,1[$, then we have

$${}^{C}D_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)}\int_0^t (\Psi(t)-\Psi(s))^{q-1}g'(s)ds$$

and

$${}^{C}D_{0^+}^{q,\Psi}g(t) = I_{0^+}^{1-q,\Psi}\left(\frac{g'(t)}{\Psi'(t)}\right).$$

Proposition 1 ([3])**.** *Let $q > 0$, if $g \in C^{n-1}(\Delta,\mathbb{R})$, then we have*

1) ${}^{C}D_{0^+}^{q,\Psi}I_{0^+}^{q,\Psi}g(t) = g(t)$.
2) $I_{0^+}^{q,\Psi}\,{}^{C}D_{0^+}^{q,\Psi}g(t) = g(t) - \sum_{k=0}^{n-1}\frac{g_\Psi^{[k]}(0)}{k!}(\Psi(t)-\Psi(0))^k$.
3) $I_{a^+}^{q,\Psi}$ *is linear and bounded from $C(\Delta,\mathbb{R})$ to $C(\Delta,\mathbb{R})$.*

Lemma 1. *The Ψ-Riemann-Liouville fractional integral at order q of the constant function* 1 *is given by:*

$$I_{0^+}^{q,\Psi}(1) = \frac{1}{\Gamma(q+1)}(\Psi(t)-\Psi(0))^q.$$

Proof. We have:

$$I_{0^+}^{q,\Psi}g(t) = \frac{1}{\Gamma(q)}\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{q-1}g(s)ds.$$

Then,

$$\begin{aligned} I_{0^+}^{q,\Psi}(1) &= \frac{1}{\Gamma(q)}\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{q-1}ds = \frac{1}{\Gamma(q+1)}\int_0^t ((\Psi(t)-\Psi(s))^q)'ds \\ &= \frac{1}{\Gamma(q+1)}(\Psi(t)-\Psi(0))^q. \end{aligned}$$

□

3. Main results

For the existence of solution for the problem (1.1)-(1.2), we need the following auxiliary lemmas.

Lemma 2. *For each $i = 1,2$, let f_i be a continuous function from $[0,1] \times \mathbb{R}^2$ into $\mathbb{R}$. So the problem (1.1)-(1.2) is equivalent to the general solution:*

$$\begin{cases} x(t) = I_{0^+}^{q_1,\Psi} f_1(t,x(t),y(t)) + \left(1 - \dfrac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right)(\varphi_1(x) - f_1(0,\varphi_1(x),\varphi_2(y))) \\ \qquad + \dfrac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\left(I_{0^+}^{\alpha,\Psi} x(\theta) - f_1(1,x(1),y(1))\right), \\ \\ y(t) = I_{0^+}^{q_2,\Psi} f_2(t,x(t),y(t)) + \left(1 - \dfrac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right)(\varphi_2(y) - f_2(0,\varphi_1(y),\varphi_2(y))) \\ \qquad + \dfrac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\left(I_{0^+}^{\beta,\Psi} y(\delta) - f_2(1,x(1),y(1))\right). \end{cases} \tag{3.1}$$

Proof. For some constants $c_1, c_2, m_1, m_2 \in \mathbb{R}$ and for $i = 1,2,\ q_i > 0$, the general solution of

$$\begin{cases} {}^C D_{0^+}^{q_1,\Psi} x(t) = f_1(t,x(t),y(t)) \\ \\ {}^C D_{0^+}^{q_2,\Psi} y(t) = f_2(t,x(t),y(t)) \end{cases} \tag{3.2}$$

is given by

$$\begin{aligned} x(t) &= I_{0^+}^{q_1,\Psi} f_1(t,x(t),y(t)) + c_1 + (\Psi(t)-\Psi(0))\ c_2, \\ y(t) &= I_{0^+}^{q_2,\Psi} f_2(t,x(t),y(t)) + m_1 + (\Psi(t)-\Psi(0))\ m_2. \end{aligned}$$

Using conditions (1.2) in this form we obtain

$$x(0) = \varphi_1(x) = I_{0^+}^{q_1,\Psi} f_1(0,x(0),y(0)) + c_1,$$

thus

$$\begin{aligned} c_1 &= \varphi_1(x) - I_{0^+}^{q_1,\Psi} f_1(0,x(0),y(0)) = \varphi_1(x) - I_{0^+}^{q_1,\Psi} f_1(0,\varphi_1(x),\varphi_2(y)), \\ x(1) &= I_{0^+}^{\alpha,\Psi} x(\theta) = I_{0^+}^{q_1,\Psi} f_1(1,x(1),y(1)) + c_1 + (\Psi(1)-\Psi(0))\ c_2. \end{aligned}$$

We obtain also $c_2 = \dfrac{1}{\Psi(1)-\Psi(0)}\left(I_{0^+}^{\alpha,\Psi} x(\theta) - I_{0^+}^{q_1,\Psi} f_1(1,x(1),y(1)) - c_1\right)$.

Hence

$$x(t) = I_{0^+}^{q_1,\Psi} f_1(t,x(t),y(t)) + \varphi_1(x) - I_{0^+}^{q_1,\Psi} f_1(0,\varphi_1(x),\varphi_2(y))$$

$$
\begin{aligned}
&+\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)} I_{0^+}^{\alpha,\Psi} x(\theta)-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)} I_{0^+}^{q_1,\Psi} f_1(1,x(1),y(1)) \\
&-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)} \varphi_1(x)+\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)} I_{0^+}^{q_1,\Psi} f_1(0,\varphi_1(x),\varphi_2(y)),
\end{aligned}
$$

therefore

$$
\begin{aligned}
x(t)= & I_{0^+}^{q_1,\Psi} f_1(t,x(t),y(t))+\left(1-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right)\left(\varphi_1(x)-f_1(0,\varphi_1(x),\varphi_2(y))\right) \\
& +\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\left(I_{0^+}^{\alpha,\Psi} x(\theta)-f_1(1,x(1),y(1))\right).
\end{aligned}
$$

In the same way, we check the formula of $y(t)$ as follows

$$
\begin{aligned}
y(t)= & I_{0^+}^{q_2,\Psi} f_2(t,x(t),y(t))+\left(1-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right)\left(\varphi_2(y)-f_2(0,\varphi_1(x),\varphi_2(y))\right) \\
& +\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\left(I_{0^+}^{\beta,\Psi} y(\delta)-f_2(1,x(1),y(1))\right).
\end{aligned}
$$

□

We need the following hypothesis in the rest of our paper.

(H_1) For $t \in [0,1]$ and $x_i, y_i \in \mathcal{C}$ for $i=1,2$, there exist two strictly positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that:

$$
\begin{cases}
\|f_1(t,x_1(t),y_1(t))-f_1(t,x_2(t),y_2(t))\| \le \mathcal{L}_1\left(\| x_1-x_2 \| + \| y_1-y_2 \|\right), \\
\|f_2(t,x_1(t),y_1(t))-f_2(t,x_2(t),y_2(t))\| \le \mathcal{L}_2\left(\| x_1-x_2 \| + \| y_1-y_2 \|\right).
\end{cases} \tag{3.3}
$$

(H_2) There exist three constants $K_i, M_i, N_i > 0$, $i=1,2$ and $n \in [0,1]$ such that, for $t \in [0,1]$ and $\forall x, y \in \mathcal{C}$ we have

$$
\begin{aligned}
&\| f_1(t,x(t),y(t)) \| \le K_1 \| x \|^n + M_1 \| y \|^n + N_1 \\
&\| f_2(t,x(t),y(t)) \| \le K_2 \| x \|^n + M_2 \| y \|^n + N_2.
\end{aligned}
$$

Moreover, we put

$$
\varepsilon_i = 1+\left(\frac{(\Psi(1)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)^{(2-i)}+\left(\frac{(\Psi(1)-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)^{(i-1)}+2\,\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)}.
$$

Now, we consider two operators $\mathcal{P}, \mathcal{R}\colon \mathcal{C} \to \mathcal{C}$ as follows:

$$
\begin{aligned}
\mathcal{P}(x(t)) = & I_{0^+}^{q_1,\Psi} f_1(t,x(t),y(t)),\ t \in [0,1]. \\
\mathcal{R}(x(t)) = & \left(1-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right)\left(\varphi_1(x)-I_{0^+}^{q_1,\Psi} f_1(0,\varphi_1(x),\varphi_2(y))\right) \\
& +\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\left(I_{0^+}^{\alpha,\Psi} x(\theta)-I_{0^+}^{q_1,\Psi} f_1(1,x(1),y(1))\right),\ t \in [0,1].
\end{aligned}
$$

Then, the general solution in Lemma 2 can be written using these operators as

$$\mathcal{H}(x(t)) = \mathcal{P}(x(t)) + \mathcal{R}(x(t)),$$
$$\mathcal{H}(y(t)) = \mathcal{P}(y(t)) + \mathcal{R}(y(t)).$$

For continuity of f_i for $i = 1,2$, let us show that the operator $\mathcal{P}$ is well defined and admits a fixed point, to deduce the existence of the solution which is the fixed point.

Lemma 3. *The operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ is Lipschitz with the constant*

$$\sum_{i=1}^{2} k_i = \sum_{i=1}^{2} \frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} \mathcal{L}_i.$$

Moreover, $\mathcal{P}$ satisfies the following inequality:

$$\| \mathcal{P}(x,y) \| \leq \sum_{i=1}^{2} \frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} \mathcal{L}_i \left(K_i \| x \|^n + M_i \| y \|^n + N_i \right),$$

for every $(x,y) \in \mathcal{C}$.

Proof. Let $x_i, y_i \in \mathcal{C}$, for $i = 1,2$, we have:

$$\begin{aligned} \mid \mathcal{P}(x_1(t)) - \mathcal{P}(x_2(t)) \mid &= \mid I_{0^+}^{q_1,\Psi} f_1(t,x_1(t),y(t)) - I_{0^+}^{q_1,\Psi} f_1(t,x_2(t),y(t)) \mid \\ &\leq I_{0^+}^{q_1,\Psi} \mid f_1(t,x_1(t),y(t)) - f_1(t,x_2(t),y(t)) \mid \\ &\leq I_{0^+}^{q_1,\Psi}(1) \mathcal{L}_1 \left(\| x_1 - x_2 \| + \| y - y \| \right) \\ &\leq \frac{(\Psi(t)-\Psi(0))^{q_1}}{\Gamma(q_1+1)} \mathcal{L}_1 \| x_1 - x_2 \| \\ &\leq \frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)} \mathcal{L}_1 \| x_1 - x_2 \| . \end{aligned}$$

In a similar way,

$$\mid \mathcal{P}(y_1(t)) - \mathcal{P}(y_2(t)) \mid \leq \frac{(\Psi(1)-\Psi(0))^{q_2}}{\Gamma(q_2+1)} \mathcal{L}_2 \| y_1 - y_2 \| .$$

Hence, $\mathcal{P}$ is a Lipschitz with the constant $k_i = \dfrac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} \mathcal{L}_i$.

Moreover, for the condition, we have

$$\begin{aligned} \mid \mathcal{P}(x(t)) \mid &\leq I_{0^+}^{q_1,\Psi} \mid f_1(t,x(t),y(t)) \mid \\ &\leq I_{0^+}^{q_1,\Psi}(1) \left(K_1 \| x \|^n + M_1 \| y \|^n + N_1 \right) \\ &\leq \frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)} \left(K_1 \| x \|^n + M_1 \| y \|^n + N_1 \right) \end{aligned}$$

and

$$\mid \mathcal{P}(y(t)) \mid \leq \frac{(\Psi(1)-\Psi(0))^{q_2}}{\Gamma(q_2+1)} \left(K_2 \| x \|^n + M_2 \| y \|^n + N_2 \right),$$

which gives

$$\| \mathcal{P}(x(t),y(t)) \| \leq \sum_{i=1}^{i=2} \frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} \left(K_i \| x \|^n + M_i \| y \|^n + N_i\right).$$

□

Lemma 4. *$\mathcal{R}$ is a continuous operator and satisfies the growth condition defined by*

$$\| \mathcal{R}(x(t)) \| \leq \left(1+\frac{(\Psi(1)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}+2\,\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}\right)\left(K_1 \| x \|^n + M_1 \| y \|^n + N_1\right),$$

and

$$\| \mathcal{R}(y(t)) \| \leq \left(1+\frac{(\Psi(1)-\Psi(0))^{\beta}}{\Gamma(\beta+1)}+2\,\frac{(\Psi(1)-\Psi(0))^{q_2}}{\Gamma(q_2+1)}\right)\left(K_2 \| x \|^n + M_2 \| y \|^n + N_2\right),$$

for every $x, y \in \mathcal{C}$.

Proof. Consider a subset $\mathcal{D}_p = \{(x,y) \in \mathcal{C}, \| (x,y) \| \leq p\} \subset \mathcal{C}$, and consider a sequence $\{z_n = (x_n, y_n)\} \in \mathcal{D}_p$ such that $z_n \to z = (x,y)$.

Now to show that $\mathcal{R}$ is continuous just show that $\| \mathcal{R}(z_n) - \mathcal{R}(z) \| \to 0$. From the continuity of f_i and φ_i, we have $f_{i,n} \to f_i$, $\varphi_1(x_n) \to \varphi_1(x)$ and $\varphi_2(y_n) \to \varphi_2(y)$ when $n \to \infty$.

According to the hypothesis (H_1), we obtain

$$\begin{aligned}\Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{q_1-1} &\| f_1(t,x_n,y_n) - f_1(t,x,y) \| \\ &\leq \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{q_1-1} \mathcal{L}_1\left(\| x_n - x \| + \| y_n - y \|\right),\end{aligned}$$

and

$$\begin{aligned}\Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{q_2-1} &\| f_2(t,x_n,y_n) - f_2(t,x,y) \| \\ &\leq \Psi'(s)\left(\Psi(t)-\Psi(s)\right)^{q_2-1} \mathcal{L}_2\left(\| x_n - x \| + \| y_n - y \|\right).\end{aligned}$$

By the Lebesgue dominated convergent theorem, we obtain

$$\begin{aligned} I_{0^+}^{q_1,\Psi} &\mid f_1(t,x_n,y_n) - f_1(t,x,y) \mid \to 0,\\ I_{0^+}^{q_2,\Psi} &\mid f_2(t,x_n,y_n) - f_2(t,x,y) \mid \to 0,\\ I_{0^+}^{\alpha,\Psi} &\mid x_n - x \mid \to 0,\\ I_{0^+}^{\beta,\Psi} &\mid y_n - y \mid \to 0,\end{aligned}$$

as $n \to \infty$.

It follows that $\| \mathcal{R}(z_n) - \mathcal{R}(z) \| \to 0$. Hence, the operator $\mathcal{R}$ is continuous. Using (H2) to show the condition, we have

$$\mid \mathcal{R}(x(t)) \mid \leq \mid \left(1-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right)\varphi_1(x) \mid$$

$$
\begin{aligned}
&+ \mid \left(1-\frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)}\right) I_{0^+}^{q_1,\Psi} f_1(0,\varphi_1(x),\varphi_2(y)) \mid \\
&+ \mid \frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)} I_{0^+}^{\alpha,\Psi} x(\theta) \mid + \mid \frac{\Psi(t)-\Psi(0)}{\Psi(1)-\Psi(0)} I_{0^+}^{q_1,\Psi} f_1(1,x(1),y(1)) \mid \\
\leq{}& \mid \varphi_1(x) \mid + I_{0^+}^{q_1,\Psi}(1) \mid f_1(0,\varphi_1(x),\varphi_2(y)) \mid + I_{0^+}^{\alpha,\Psi}(1) \mid x(\theta) \mid \\
&+ I_{0^+}^{q_1,\Psi}(1) \mid f_1(1,x(1),y(1)) \mid \\
\leq{}& \left(1+I_{0^+}^{\alpha,\Psi}(1)\right) \parallel x \parallel + 2\, I_{0^+}^{q_1,\Psi}(1)\,(K_1 \parallel x \parallel^n + M_1 \parallel y \parallel^n + N_1) \\
\leq{}& \left(1+\frac{(\Psi(1)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)(K_1 \parallel x \parallel^n + M_1 \parallel y \parallel^n + N_1) \\
&+ 2\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}(K_1 \parallel x \parallel^n + M_1 \parallel y \parallel^n + N_1) \\
\leq{}& \left(1+\frac{(\Psi(1)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)} + 2\,\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}\right) \\
&\cdot (K_1 \parallel x \parallel^n + M_1 \parallel y \parallel^n + N_1).
\end{aligned}
$$

The same for,

$$
\parallel \mathcal{R}(y(t)) \parallel \leq \left(1+\frac{(\Psi(1)-\Psi(1))^{\beta}}{\Gamma(\beta+1)} + 2\,\frac{(\Psi(1)-\Psi(1))^{q_2}}{\Gamma(q_2+1)}\right)(K_2 \parallel x \parallel^n + M_2 \parallel y \parallel^n + N_2).
$$

Which implies that

$$
\parallel \mathcal{R}(x(t),y(t)) \parallel \leq \sum_{i=1}^{i=2} (K_i \parallel x \parallel^n + M_i \parallel y \parallel^n + N_i)\,\varepsilon_i.
$$

□

Lemma 5. *If $\mathcal{R}$ is a compact operator, then $\mathcal{R}$ is Lipschitz with zero constant.*

Proof. Let $\Omega \subset \mathcal{B}_r$ a bounded set. Show that $\mathcal{R}(\Omega)$ is relatively compact in $\mathcal{C}$, which implies that $\mathcal{R}$ is compact.
For $x_i \in \Omega$ we have

$$
\parallel \mathcal{R}(x_i) \parallel \leq (K_i \parallel x \parallel^n + M_i \parallel y \parallel^n + N_i)\,\varepsilon_i.
$$

According to this inequality, the set $\mathcal{R}(\Omega)$ is uniformly bounded. For equi-continuity of $\mathcal{R}$, take $s_1, s_2 \in [0,1]$ with $s_1 < s_2$ and let $x \in \Omega$, then we have

$$
\begin{aligned}
&\mid \mathcal{R}(x(s_1)) - \mathcal{R}(x(s_2)) \mid \\
&\leq (\Psi(s_2)-\Psi(s_1)) \\
&\quad \cdot \left(\mid \varphi_1(x) \mid + \mid I_{0^+}^{q_1,\Psi} f_1(0,\varphi_1(x),\varphi_2(y)) \mid + \mid I_{0^+}^{\alpha,\Psi} x(\theta) \mid + I_{0^+}^{q_1,\Psi} f_1(1,x(1),y(1)) \mid\right) \\
&\leq (\Psi(s_2)-\Psi(s_1))
\end{aligned}
$$

$$\cdot\left(1+\frac{(\Psi(1)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}+2\,\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}\right)(K_1 \parallel x \parallel^n +M_1 \parallel y \parallel^n +N_1)\,.$$

Which implies that

$$\mid \mathcal{R}((x,y)(s_1))-\mathcal{R}((x,y)(s_2)) \mid\leq \sum_{i=1}^{i=2}\varepsilon_i\,(K_i \parallel x \parallel^n +M_i \parallel y \parallel^n +N_i)\,(s_2-s_1).$$

We deduce that $\parallel \mathcal{R}((x,y)(s_1))-\mathcal{R}((x,y)(s_2)) \parallel\to 0$ when $s_2\to s_1$.
Therefore, $\mathcal{R}$ is equi-continuous. Thus, by Ascoli-Arzela theorem, $\mathcal{R}$ is a compact operator, then $\mathcal{R}$ is Lipschitz with zero constant. □

Theorem 1. *The problem* $(1.1)-(1.2)$ *has at least one solution* $(x,y)\in\mathcal{C}$, *if* $(H_1)-(H_2)$ *are satisfied and* $\sum_{i=1}^{i=2}k_i<1$. *And the set of solutions is bounded in* $\mathcal{C}$.

Proof. Let the operators $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{H}$ defined previously are continuous and bounded, and we have that $\mathcal{P}$ and $\mathcal{R}$ are Lipschitz with zero constant.

Hence, $\mathcal{H}$ is Lipschitz with constant k_i, then $\mathcal{H}$ is strict $\kappa-$contraction with constant k_i. Since, $\sum_{i=1}^{i=2}k_i<1$, so $\mathcal{H}$ is $\kappa-$condensing. Define the set

$$\pi=\left\{(x,y)\in\mathcal{C},\ \text{there exists } \lambda\in[0,1] \text{ such that } x=\lambda\mathcal{H}(x) \text{ and } y=\lambda\mathcal{H}(y)\right\},$$

and show that π is bounded.

For $x\in\pi$, we have $x=\lambda\mathcal{H}(x)=\lambda(\mathcal{P}(x)+\mathcal{R}(x))$, which implies that

$$\begin{aligned}\parallel x\parallel &\leq \lambda(\parallel \mathcal{P}(x)\parallel+\parallel \mathcal{R}(x)\parallel)\\ &\leq\left(\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}+\varepsilon_1\right)(K_1 \parallel x \parallel^n +M_1 \parallel y \parallel^n +N_1)\,,\end{aligned}$$

hence we get:

$$\begin{aligned}\parallel (x,y)\parallel &\leq \lambda(\parallel \mathcal{P}(x,y)\parallel+\parallel \mathcal{R}(x,y)\parallel)\\ &\leq\sum_{i=1}^{i=2}\left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)}+\varepsilon_i\right)(K_i \parallel x \parallel^n +M_i \parallel y \parallel^n +N_i)\,.\end{aligned}$$

Then, π is bounded in $\mathcal{C}$.

On the other hand, suppose that π is not bounded, then we divide the inequality by $b=\parallel x\parallel$ and letting $b\to\infty$, arriving at

$$1\leq\sum_{i=1}^{i=2}\left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)}+\varepsilon_i\right)\lim_{b\to\infty}\frac{K_ib^n+M_ib^n+N_i}{b}=0,$$

which is a contradiction.

Then, π is a bounded set in $\mathcal{C}$ and the operator $\mathcal{H}$ admits at least one fixed point and it is the solution of problem $(1.1)-(1.2)$. □

Now show the existence and uniqueness of the solution by the following theorem.

Theorem 2. *The problem* $(1.1)-(1.2)$ *has a unique solution if* (H_1) *is satisfied and*

$$\sum_{i=1}^{i=2}\left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)}+\varepsilon_i\right)\mathcal{L}_i<1.$$

Proof. Let $x_1,x_2\in\mathcal{C}$ and $t\in[0,1]$, we have:

$$\begin{aligned}
\mid\mathcal{H}(x_1(t))-\mathcal{H}(x_2(t))\mid &\leq\mid\mathcal{P}(x_1(t))-\mathcal{P}(x_2(t))\mid+\mid\mathcal{R}(x_1(t))-\mathcal{R}(x_2(t))\mid\\
&\leq\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}\mathcal{L}_1\parallel x_1-x_2\parallel+\varepsilon_1\mathcal{L}_1\parallel x_1-x_2\parallel\\
&\leq\left(\frac{(\Psi(1)-\Psi(0))^{q_1}}{\Gamma(q_1+1)}+\varepsilon_1\right)\mathcal{L}_1\parallel x_1-x_2\parallel.
\end{aligned}$$

If $\sum_{i=1}^{i=2}\left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)}+\varepsilon_i\right)\mathcal{L}_i<1$, hence $\mathcal{H}$ is contraction. Then, by the Banach contraction principle, the problem $(1.1)-(1.2)$ admits a unique solution, it is the only fixed point of the operator $\mathcal{H}$. □

4. ILLUSTRATIVE EXAMPLE

In this section, we will apply the previous results to the following example. Consider the following equation:

$$\begin{cases}
{}^{C}D_{0^+}^{\frac{4}{3},t^2}x(t)=\dfrac{e^{-\pi t}}{10}+\dfrac{sin\mid x(t)\mid+sin\mid y(t)\mid}{9+t}, & t\in I=[0,1]\\[2ex]
{}^{C}D_{0^+}^{\frac{7}{5},t^2}y(t)=\dfrac{e^{-15t}}{13}+\dfrac{sin\mid x(t)\mid+sin\mid y(t)\mid}{16+(t+1)^2}.
\end{cases}\tag{4.1}$$

with the conditions:

$$\begin{cases}
x(0)=\frac{2}{5}sin(x)+\frac{1}{2},\\[1ex]
y(0)=\frac{3}{5}cos(y)+\frac{1}{3},\\[1ex]
x(1)=I_{0^+}^{\frac{1}{4},t^2}x(\frac{1}{5}),\\[1ex]
y(1)=I_{0^+}^{\frac{4}{5},t^2}y(\frac{2}{5}),
\end{cases}\tag{4.2}$$

with the constant in problem $(1.1)-(1.2)$:

$$q_1=\frac{4}{3},\quad q_2=\frac{7}{5},\quad \alpha=\frac{1}{4},\quad \beta=\frac{4}{5},\quad \theta=\frac{1}{5},\quad \delta=\frac{2}{5},$$

and with functions

$$f_1(t,x(t),y(t))=\frac{e^{-\pi t}}{10}+\frac{sin\mid x(t)\mid+sin\mid y(t)\mid}{9+t},$$

$$f_2(t,x(t),y(t)) = \frac{e^{-15t}}{13} + \frac{sin \mid x(t) \mid + sin \mid y(t) \mid}{16+(t+1)^2},$$

$$\Psi(t) = t^2 \ , \ \varphi_1(x) = \frac{2}{5} sin(x) + \frac{1}{2} \ , \ \varphi_2(y) = \frac{3}{5} cos(y) + \frac{1}{3}.$$

Using the parameter values provided, we get

$$\sum_{i=1}^{i=2} \left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} + \varepsilon_i \right) = 9,1114.$$

We can easily show that

$$\|f_1(t,x_1(t),y_1(t)) - f_1(t,x_2(t),y_2(t))\| \leq \frac{1}{9} (\| x_1 - x_2 \| + \| y_1 - y_2 \|),$$

$$\|f_2(t,x_1(t),y_1(t)) - f_2(t,x_2(t),y_2(t))\| \leq \frac{1}{16} (\| x_1 - x_2 \| + \| y_1 - y_2 \|).$$

Hence the condition (H_1) holds with $\mathcal{L}_1 = \frac{1}{9}$, $\mathcal{L}_2 = \frac{1}{16}$.

In addition we have

$$\sum_{i=1}^{i=2} k_i = \sum_{i=1}^{i=2} \frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} \mathcal{L}_i = 0,1436.$$

For any $t \in [0,1]$ and $x,y \in \mathbb{R}$, we have

$$\mid f_1(t,x(t),y(t)) \mid \leq \frac{1}{9} \mid x \mid + \frac{1}{9} \mid y \mid + 1,$$

$$\mid f_2(t,x(t),y(t)) \mid \leq \frac{1}{16} \mid x \mid + \frac{1}{16} \mid y \mid + 2.$$

Condition (H_2) holds with $M_1 = K_1 = \frac{1}{9}$, $M_2 = K_2 = \frac{1}{16}$, $n = N_1 = 1$ and $N_2 = 2$. In view of Theorem 1, the solution set:

$$\pi = \left\{ (x,y) \in \mathcal{C}, \text{ there exists } \lambda \in [0,1] \text{ such that } x = \lambda \mathcal{H}(x) \text{ and } y = \lambda \mathcal{H}(y) \right\}.$$

Then

$$\begin{aligned} \| (x,y) \| &\leq \lambda (\| \mathcal{P}(x,y) \| + \| \mathcal{R}(x,y) \|) \\ &\leq \sum_{i=1}^{i=2} \left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} + \varepsilon_i \right) (K_i \| x \| + M_i \| y \| + N_i). \end{aligned}$$

By using Theorem 1, the problem $(4.1)-(4.2)$ has at least a solution $(x,y) \in \mathcal{C}$.

Furthermore, we have:

$$\sum_{i=1}^{i=2} \left(\frac{(\Psi(1)-\Psi(0))^{q_i}}{\Gamma(q_i+1)} + \varepsilon_i \right) \mathcal{L}_i = 0,77 < 1.$$

Hence from Theorem 2, it follows that $(4.1)-(4.2)$ has a unique solution.

5. CONCLUSION

In this paper, we studied the existence results for a coupled system of nonlinear fractional differential equations with four-point fractional integral boundary conditions involving $\Psi-$Caputo fractional derivatives. The existence theorems are proved by using some fixed point theorems. As an application, an example is presented to illustrate the applicability of our main result.

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