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Tolerances as images of congruences in varieties defined by linear identities

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ABSTRACT. An identity $s = t$ is *linear* if each variable occurs at most once in each of the terms s and t . Let T be a tolerance relation of an algebra \mathcal{A} in a variety defined by a set of linear identities. We prove that there exist an algebra \mathcal{B} in the same variety and a congruence θ of \mathcal{B} such that a homomorphism from \mathcal{B} onto \mathcal{A} maps θ onto T .

An identity $s = t$ is *linear* if each variable occurs at most once in each of the terms s and t , see, for example, M. N. Bleicher, H. Schneider and R. L. Wilson [1, Theorem 4.19], W. Taylor [8], I. Bošnjak and R. Madarász [2], A. Pilitowska [7], and their references. In the particular case where every variable occurs exactly twice, once in s and once in t , we speak of a *balanced linear* identity, see M. V. Lawson [6]. For example, the variety of semigroups and that of commutative semigroups are defined by balanced linear identities. Binary reflexive, symmetric, and compatible relations are called *tolerances*; see I. Chajda [3]. If $\varphi: \mathcal{B} \rightarrow \mathcal{A}$ is a surjective homomorphism and θ is a congruence of the algebra \mathcal{B} , then $\varphi(\theta) = \{(\varphi(x), \varphi(y)) : (x, y) \in \theta\}$ is a tolerance of \mathcal{A} . Each tolerance of \mathcal{A} is obtained this way; this follows from our result below (applied for the variety defined by the empty set of linear identities). Sometimes, like in I. Chajda, G. Czédli, and R. Halaš [4] or G. Czédli and G. Grätzer [5], we can choose an appropriate \mathcal{B} from a given variety. We have the following additional result of this kind.

Theorem. *Assume that \mathcal{V} is a variety defined by a set of linear identities, that $\mathcal{A} = (A, F) \in \mathcal{V}$, and that T is a tolerance of \mathcal{A} . Then there exist an algebra $\mathcal{B} \in \mathcal{V}$, a congruence θ of \mathcal{B} , and a surjective homomorphism $\varphi: \mathcal{B} \rightarrow \mathcal{A}$ such that $T = \varphi(\theta)$.*

Proof. We generalize the idea of G. Czédli and G. Grätzer [5].

If \mathcal{D} is an arbitrary algebra (not necessarily in \mathcal{V}), then the *complex algebra* \mathcal{C} of \mathcal{D} , in other words the *algebra of complexes* of \mathcal{D} , has the underlying set $\{X \subseteq D : X \neq \emptyset\}$, and for each basic operation f of \mathcal{D} , the corresponding operation of \mathcal{C} is defined by

$$f(X_1, \dots, X_n) = \{f(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}.$$

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If s is a linear term, which means that each variable occurs in s at most once, then it can be shown that

$$s(X_1, \dots, X_n) = \{s(x_1, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}$$

holds for arbitrary $X_i \in \mathcal{C}$ (but this does not hold for arbitrary terms in general). This implies, as proved in [1] and [8], that if a variety is defined by linear identities, then it contains the complex algebra of each of its members.

Next, let E denote the set $\{X \subseteq A : X^2 \subseteq T \text{ and } X \neq \emptyset\}$. Since it is clearly a subalgebra of the complex algebra of \mathcal{A} , the paragraph above implies that $\mathcal{E} = (E, F)$ belongs to \mathcal{V} . Let $B = \{(x, Y) \in A \times E : x \in Y\}$. Then $\mathcal{B} = (B, F)$ also belongs to \mathcal{V} since it is a subalgebra of $\mathcal{A} \times \mathcal{E}$. Define $\theta = \{((x_1, Y_1), (x_2, Y_2)) \in B^2 : Y_1 = Y_2\}$. As the kernel of the second projection from \mathcal{B} to \mathcal{E} , it is a congruence of \mathcal{B} . The first projection $\varphi: \mathcal{B} \rightarrow \mathcal{A}$, $(x, Y) \mapsto x$, is a surjective homomorphism since, for every $x \in A$, $x = \varphi(x, \{x\})$.

Clearly, if $((x_1, Y_1), (x_2, Y_2)) \in \theta$, then $\{x_1, x_2\} \subseteq Y_1 = Y_2 \in E$ implies that $(\varphi(x_1, Y_1), \varphi(x_2, Y_2)) = (x_1, x_2) \in T$. Hence $\varphi(\theta) \subseteq T$. Conversely, let $(x_1, x_2) \in T$. Then, with $Y = \{x_1, x_2\}$, we have that $(x_1, Y), (x_2, Y) \in B$, $((x_1, Y), (x_2, Y)) \in \theta$, and $x_i = \varphi(x_i, Y)$. This implies that $(x_1, x_2) \in \varphi(\theta)$, and we conclude that $T \subseteq \varphi(\theta)$. \square

REFERENCES

- [1] Bleicher, M. N., Schneider, H., Wilson, R. L.: Permanence of identities on algebras. *Algebra Universalis* **3**, 72–93 (1973)
- [2] Bošnjak, I., Madarász R.: On power structures. *Algebra and Discr. Math.* **2**, 14–35 (2003)
- [3] Chajda, I.: Algebraic Theory of Tolerance Relations. Palacký University Olomouc, Olomouc (1991)
- [4] Chajda, I., Czédli, G., Halaš, R.: Independent joins of tolerance factorable varieties. *Algebra Universalis* (submitted)
- [5] Czédli, G., Grätzer, G.: Lattice tolerances and congruences. *Algebra Universalis* **66**, 5–6 (2011)
- [6] Lawson, M. V.: A correspondence between balanced varieties and inverse monoids. *Int. J. Algebra Comput.* **16**, 887–924 (2006)
- [7] Pilitowska, A.: Linear identities in graph algebras. *Comment. Math. Univ. Carol.* **50**, 11–24 (2009)
- [8] Taylor, W.: Equational logic. *Houston J. Math.*, Survey 1979. Abridged version in: G. Grätzer, *Universal Algebra*, 2nd ed., Appendix 4, Springer-Verlag (1979)

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