

# Stable solution of the Logarithmic Minkowski problem in the case of hyperplane symmetries

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## Abstract

In the case of symmetries with respect to a Coxeter group  $G \subset O(n)$  acting without non-zero fixed points on  $\mathbb{R}^n$ , the stability of the solution of the Logarithmic Minkowski problem on  $S^{n-1}$  is established.

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## 1. Introduction

The so called Minkowski problem form the core of various areas in fully nonlinear partial differential equations and convex geometry (see Trudinger, Wang [64] and Schneider [61]), which was extended to the  $L_p$ -Minkowski theory by Lutwak [47–49]. The classical Minkowski's existence theorem due to Minkowski and Aleksandrov describes the so called surface area measure  $S_K$  of a convex body  $K$  (the case  $p = 1$ ) where the regularity of the solution is well investigated by Lewy [46], Nirenberg [55], Cheng and Yau [24], Pogorelov [57] and Caffarelli [20].

First major results about the  $L_p$ -Minkowski problem for  $p \neq 1$  have been obtained by Chou, Wang [25] and Hug, Lutwak, Yang, Zhang [36], and more recently the papers Boröczky, Lutwak, Yang, Zhang [18], Andrews, Guan, Ni [2], Guan, Ni [29], Kolesnikov, Milman [42], Bianchi,

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Boroczky, Colesanti, Yang [11], Chen, Li, Zhu [22], Chen, Huang, Li [21], Bryan, Ivaki, Scheuer [19], Li, Sheng, Wang [45], Lutwak, Yang, Zhang [51] and Xi, Leng [65] present new developments and approaches. Cases of multiple solutions are discussed in [18], [22], He, Li, Wang [31], Li [43], Huang, Liu, Xu [35] and Stancu [63].

For a compact convex set  $K$  in  $\mathbb{R}^n$ , we write  $V(K)$  to denote to  $n$ -dimensional Lebesgue measure. We say that a compact convex set  $K$  in  $\mathbb{R}^n$  is a convex body if  $V(K) > 0$ ; or equivalently, the interior of  $K$  is non-empty. The cone volume measure or  $L_0$ -surface area measure  $V_K$  on  $S^{n-1}$ , whose study was initiated independently by Firey [27] and Gromov and Milman [28], has become an indispensable tool in the last decades (see say Barthe, Guédon, Mendelson, Naor [8], Naor [53], Paouris, Werner [56], Boroczky, Henk [14]). If a convex body  $K$  contains the origin, then its cone volume measure is  $dV_K = \frac{1}{n} h_K dS_K$  where  $h_K$  is the support function of  $K$  and the total measure is the volume of  $K$ . In particular, the Monge-Ampère equation on the sphere  $S^{n-1}$  corresponding to the logarithmic (or  $L_0$ -) Minkowski problem is

$$h \det(\nabla^2 h + h \operatorname{Id}) = n f \quad (1)$$

where  $\nabla h$  and  $\nabla^2 h$  are the gradient and the Hessian of  $h$  with respect to a moving orthonormal frame. We recall that for a given finite Borel measure  $\mu$  on  $S^{n-1}$ , a positive  $h$  on  $S^{n-1}$  that is the restriction of a convex homogeneous function on  $\mathbb{R}^n$  is the solution of (1) in the Alexandrov sense if the corresponding Monge-Ampère measure satisfies

$$\det(\nabla^2 h + h \operatorname{Id}) d\sigma = \frac{n}{h} \cdot \mu \quad (2)$$

where  $\sigma$  is the Lebesgue measure on  $S^{n-1}$ . In particular, for any Alexandrov solution  $h$  of (1) (or equivalently of (2)), there exists a unique convex body  $K$  with  $o \in \operatorname{int} K$  such that  $h = h_K$  where  $h_K(u) = \max_{x \in K} \langle u, x \rangle$  is the support function of  $K$  for  $u \in \mathbb{R}^n$ ,  $\mu = V_K$  is the cone-volume measure of  $K$  and the corresponding Monge-Ampère measure is  $S_K$ .

We observe that the Monge-Ampère equation (1) is homogeneous in the sense that replacing  $f$  by  $\lambda f$  for  $\lambda > 0$  is equivalent replacing  $h$  by  $\lambda^{1/n} h$ . Therefore, we may assume that  $V(K) = 1$ ; or in other words, the  $f$  in (1) is a probability density, or the  $\mu$  in (2) is a probability measure.

Following partial and related results by Andrews [1], Chou, Wang [25], He, Leng, Li [30], Henk, Schürman, Wills [33], Stancu [62], Xiong [66] the paper Boroczky, Lutwak, Yang, Zhang [18] characterized even cone volume measures by the so called subspace concentration condition. Recently, breakthrough results have been obtained by Chen, Li, Zhu [23], Chen, Huang, Li [21], Kolesnikov [40], Nayar, Tkocz [54], Kolesnikov, Milman [42], Putterman [58] about the uniqueness of the solution, which is intimately related to the conjectured log-Minkowski inequality Conjecture 3.1. As it turns out, subspace concentration condition also holds for the cone-volume measure  $V_K$  if the centroid of a general convex body  $K$  is the origin (see Henk, Linke [32] and Böröczky, Henk [14,15]).

We note that the conjectured uniqueness of the solution of the Logarithmic, or  $L_0$ -Minkowski problem (1) for even positive  $C^\infty$   $f$  has a special role within the  $L_p$ -Minkowski Problems as if  $p < 0$ , then it is known that the solution may not be unique (see Jian, Lu, Wang [39], Li, Liu, Lu [44], Milman [52]). On the positive side, extending the work of Kolesnikov, Milman [42], Chen, Huang, Li, Liu [21] verified the uniqueness of the solution if  $1 - \frac{c}{n^{3/2}} < p < 1$  for some absolute constant  $c \in (0, 1)$  (see also Putterman [58]).

Concerning possibly non-even measures, the logarithmic Minkowski problem (2) is wide open, as the best sufficient condition for a measure being a cone-volume measure is provided

by Chen, Li, Zhu [23] (solving for example the case of absolutely continuous measures), and some obstruction (necessary condition) is provided by Boroczky, Hegedus [13].

Boroczky, Kalantzopoulos [16] proved the following characterization of cone-volume measures under hyperplane symmetry assumption. We note that for any group  $G \subset O(n)$  acting on  $\mathbb{R}^n$  without non-zero fixed points, there exist only finitely many  $G$  invariant linear subspaces of  $\mathbb{R}^n$  where  $G$  is a Coxeter group if it is generated by reflections through  $n$  independent hyperplanes.

**Theorem 1.1** (Boroczky, Kalantzopoulos). *Let  $G \subset O(n)$  be a Coxeter group acting on  $\mathbb{R}^n$  without non-zero fixed points. For a finite non-trivial Borel measure  $\mu$  on  $S^{n-1}$  invariant under  $G$ , there exists a  $G$  invariant Alexandrov solution of the logarithmic Minkowski equation (2) if and only if*

- (i)  $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot \mu(S^{n-1})$  for any proper linear subspace  $L$  invariant under  $G$ ;
- (ii)  $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$  in (i) for a proper invariant linear subspace  $L$  is equivalent with  $\text{supp } \mu \subset L \cup L^\perp$ .

*In addition, if strict inequality holds in (i) for each proper linear subspace  $L$  invariant under  $G$ , then the  $G$  invariant solution is unique.*

We note that the measure in Theorem 1.1 may not be even; for example, possibly  $\mu = V_K$  for a regular simplex  $K$  whose centroid is the origin.

For compact convex sets  $M$  and  $N$ , we write  $M \oplus N$  to denote  $M + N$  if  $\langle x, y \rangle = 0$  holds for  $x \in M$  and  $y \in N$ . In addition, we say that a linear subspace  $L$  of  $\mathbb{R}^n$  is proper if  $1 \leq \dim L \leq n - 1$ . We note that [16] proved that  $V_K(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot V(K)$  holds in Theorem 1.1 (i) for a proper invariant subspace  $L$  if and only if  $K = (K \cap L) \oplus (K \cap L^\perp)$ .

According to [16],  $V_K = V_C$  holds for convex bodies  $K$  and  $C$  in  $\mathbb{R}^n$  invariant under a Coxeter group  $G \subset O(n)$  acting on  $\mathbb{R}^n$  without non-zero fixed points if and only if  $V(K) = V(C)$ , and  $K = K_1 \oplus \dots \oplus K_m$  and  $C = C_1 \oplus \dots \oplus C_m$  for compact convex sets  $K_1, \dots, K_m, C_1, \dots, C_m$  of dimension at least one and invariant under  $G$  where  $K_i$  and  $C_i$  are dilates for  $i = 1, \dots, m$ . Naturally, if  $m = 1$ , then  $K = C$ .

In order to prepare for the stability version Theorem 1.2 of Theorem 1.1, for any compact  $X \subset S^{n-1}$  and  $\varrho \in [0, 2]$ , we consider the tube

$$\Psi(X, \varrho) = \{u \in S^{n-1} : \exists x \in X, \|x - u\| \leq \varrho\}.$$

The cone volume measure  $V_K$  of a convex body  $K$  readily satisfies  $dV_{tK} = t^n dV_K$  for  $t > 0$ . Therefore, when comparing the cone volume measures of convex bodies  $K$  and  $C$ , we may assume that  $V(K) = V(C) = 1$ , and hence  $V_K$  and  $V_C$  are probability measures on  $S^{n-1}$ . In turn, one natural distance between two probability measures  $\mu$  and  $\nu$  on  $S^{n-1}$  is the  $I_1$  Wasserstein distance. First, we consider the family of Lipschitz functions on  $S^{n-1}$ ; namely, for  $\theta > 0$ , let

$$\text{Lip}_\theta = \{f : S^{n-1} \rightarrow \mathbb{R} : \forall a, b \in S^{n-1}, |f(a) - f(b)| \leq \theta \|a - b\|\}. \quad (3)$$

Now the Wasserstein distance of the Borel probability measures  $\mu$  and  $\nu$  on  $S^{n-1}$  is

$$d_W(\mu, \nu) = \sup \left\{ \int_{S^{n-1}} f d\mu - \int_{S^{n-1}} f d\nu : f \in \text{Lip}_1 \right\}.$$

It is known that convergence of a sequence of probability measures with respect to the Wasserstein distance is equivalent with weak convergence.

We note that as  $\mu(S^{n-1}) = \nu(S^{n-1})$  in the definition of  $d_W(\mu, \nu)$ , we may assume that  $\min f = -1$ ; therefore,  $f \in \text{Lip}_1$  implies that

$$\|f\|_\infty = \max_{u \in S^{n-1}} |f(u)| \leq 1. \quad (4)$$

In turn, we observe that if  $d\mu(u) = \varphi(u) du$  and  $d\nu(u) = \psi(u) du$ , then

$$d_W(\mu, \nu) \leq \int_{S^{n-1}} |\varphi(u) - \psi(u)| du. \quad (5)$$

**Theorem 1.2.** *Let  $G \subset O(n)$  be a Coxeter group acting on  $\mathbb{R}^n$  without non-zero fixed points. If  $\mu_1$  and  $\mu_2$  are Borel probability measures on  $S^{n-1}$  invariant under  $G$ , and*

$$\begin{aligned} \mu_1(\Psi(L \cap S^{n-1}, \delta)) &\leq (1 - \tau) \cdot \frac{\dim L}{n}, \\ \mu_2(\Psi(L \cap S^{n-1}, \delta)) &\leq (1 - \tau) \cdot \frac{\dim L}{n} \end{aligned} \quad (6)$$

for  $\delta, \tau \in (0, \frac{1}{2})$  and for any proper subspace  $L$  invariant under  $G$ , then the unique  $G$  invariant Alexandrov solution  $h_i$  of the logarithmic Minkowski problem (2) for  $\mu = \mu_i$ ,  $i = 1, 2$ , satisfies

$$\|h_1 - h_2\|_\infty \leq \gamma_0 \cdot d_W(\mu_1, \mu_2)^{\frac{1}{95n}} \quad (7)$$

$$r_0 \leq h_1, h_2 \leq R_0 \quad (8)$$

where for some absolute constant  $c > 1$ , we have

- $R_0 = n$ ,  $r_0 = \frac{1}{e}$ ,  $\gamma_0 = c^n$  and the condition (6) is irrelevant provided the action of  $G$  is irreducible;
- $R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$ ,  $r_0 = \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$  and  $\gamma_0 = \frac{c^n}{\tau} \cdot \delta^{-\frac{3n}{\tau}} n^{\frac{12n}{\tau}}$  provided the action of  $G$  is reducible.

Actually, Theorem 1.2 can be extended to the case when  $\mu_1(S^{n-1}) \neq \mu_2(S^{n-1})$  (see Corollary 1.3). In this case, we need the bounded Lipschitz distance  $d_{\text{bL}}(\mu, \nu)$  of two Borel measures  $\mu$  and  $\nu$  on  $S^{n-1}$  (see Dudley [26]); namely,

$$d_{\text{bL}}(\mu, \nu) = \sup \left\{ \int_{S^{n-1}} f d\mu - \int_{S^{n-1}} f d\nu : f \in \text{Lip}_1 \text{ and } \|f\|_\infty \leq 1 \right\}.$$

Using the test function constant 1 shows that

$$|\mu(S^{n-1}) - \nu(S^{n-1})| \leq d_{\text{bL}}(\mu, \nu). \quad (9)$$

We observe that if  $\mu(S^{n-1}) = \nu(S^{n-1}) = 1$ , then  $d_{\text{bL}}(\mu, \nu) = d_W(\mu, \nu)$ . On the other hand, if  $\lambda > 0$  and  $\mu$  is any finite non-trivial Borel measure on  $S^{n-1}$ , then

$$d_{\text{bL}}(\mu, \lambda\mu) \leq |\lambda - 1| \cdot \mu(S^{n-1}). \quad (10)$$

**Corollary 1.3.** *Let  $G \subset O(n)$  be a Coxeter group acting on  $\mathbb{R}^n$  without non-zero fixed points. If  $\mu_1$  and  $\mu_2$  are finite Borel measures on  $S^{n-1}$  invariant under  $G$  satisfying  $d_{\text{bL}}(\mu_1, \mu_2) \leq M = \min\{\mu_1(S^{n-1}), \mu_2(S^{n-1})\} > 0$  and*

$$\begin{aligned} \mu_1(\Psi(L \cap S^{n-1}, \delta)) &\leq (1 - \tau) \cdot \frac{\dim L}{n}, \\ \mu_2(\Psi(L \cap S^{n-1}, \delta)) &\leq (1 - \tau) \cdot \frac{\dim L}{n} \end{aligned} \quad (11)$$

for  $\delta, \tau \in (0, \frac{1}{2})$  and for any proper subspace  $L$  invariant under  $G$ , then the unique  $G$  invariant Alexandrov solution  $h_i$  of the logarithmic Minkowski problem (2) for  $\mu = \mu_i$ ,  $i = 1, 2$ , satisfies

$$\|h_1 - h_2\|_\infty \leq \gamma_0 M^{\frac{1}{n}} \cdot d_{\text{bL}}(\mu_1, \mu_2)^{\frac{1}{95n}} \quad (12)$$

$$r_0 M^{\frac{1}{n}} \leq h_1, h_2 \leq R_0 M^{\frac{1}{n}} \quad (13)$$

where for some absolute constant  $c > 1$ , we have

- $R_0 = 2n$ ,  $r_0 = \frac{1}{e}$ ,  $\gamma_0 = c^n$  and the condition (11) is irrelevant provided the action of  $G$  is irreducible;
- $R_0 = 2 \left( \frac{n^6}{\delta} \right)^{\frac{1}{\tau}}$ ,  $r_0 = \frac{n^2}{5^n} \left( \frac{\delta}{n^6} \right)^{\frac{n-1}{\tau}}$  and  $\gamma_0 = \frac{c^n}{\tau} \cdot \delta^{-\frac{3n}{\tau}} n^{\frac{12n}{\tau}}$  provided the action of  $G$  is reducible.

Geometric inequalities under  $n$  independent hyperplane symmetries were first considered by Barthe, Fradelizi [7] and Barthe, Cordero-Erausquin [6]. These papers verified the classical Mahler conjecture and Slicing conjecture, respectively, for these type of bodies.

We observe that the error term in Theorem 1.2 in terms of  $\varepsilon$  is not far from being optimal. We provide an unconditional example; namely, when  $G$  is generated by the reflections through the coordinate hyperplanes. Let  $K$  be the unit cube  $K = [-\frac{1}{2}, \frac{1}{2}]^n$ , and the unconditional  $C$  be obtained from  $K$  by chopping off vertices of  $K$  using simplices of volume  $\varepsilon$  and rescaling (to ensure  $V(C) = 1$ ). Then  $d_W(V_K, V_C) < \gamma_1 \cdot \varepsilon$ , while  $(1 - \gamma_2 \varepsilon^{\frac{1}{n}})K \not\subset C$  for suitable  $\gamma_1, \gamma_2 > 0$  depending on  $n$ .

The stable solution Theorem 1.2 of the logarithmic Minkowski problem under hyperplane symmetry does use the metric structure on  $S^{n-1}$ . The next example shows that we can't expect an "affine invariant" stability version of Theorem 1.2 even if the cone volume measure is affine invariant in certain sense.

**Example 1.4.** If  $e \in S^{n-1}$ , and  $K$  and  $C$  are any convex bodies in  $\mathbb{R}^n$  containing the origin in their interiors with  $V(K) = V(C) = 1$  and  $V_K(e^\perp \cap S^{n-1}) = V_C(e^\perp \cap S^{n-1}) = 0$ , and  $\Phi_s$  is the diagonal transformation with  $\Phi_s(e) = s^{-(n-1)}e$  and  $\Phi_s(x) = sx$  for  $x \in e^\perp$ , then both  $V_{\Phi_s K}$  and

$V_{\Phi_s C}$  tend weakly to  $\mu_0$  as  $s$  tends to infinity where  $\mu_0$  denotes the probability measure on  $S^{n-1}$  with  $\mu_0(\{\pm e\}) = \frac{1}{2}$ . In particular,  $V_{\Phi_s K}$  and  $V_{\Phi_s C}$  are arbitrarily close if  $s$  is large.

Next, we consider two partial converses of Theorem 1.2 to show that concerning Theorem 1.2, both the conditions involved and the conclusion are of the right kind. The first result does not require any symmetry assumption.

**Theorem 1.5.** *Let  $\mu_1$  and  $\mu_2$  be finite Borel measures on  $S^{n-1}$  such that there exists Alexandrov solution  $h_i$  of the logarithmic Minkowski problem (2) for  $\mu = \mu_i$  and  $i = 1, 2$ . If  $h_1, h_2 < R$  for  $R > 0$ , then*

$$d_{\text{bL}}(\mu_1, \mu_2) \leq \gamma(R, n) \cdot \sqrt{\|h_1 - h_2\|_\infty}$$

where  $\gamma(R, n) > 0$  depends on  $R$  and  $n$ .

Secondly, we show that if we have almost equality in Theorem 1.1 (ii) for measures  $\mu_1$  and  $\mu_2$  and a proper linear subspace  $L$  invariant under reflections through independent hyperplanes  $H_1, \dots, H_n$ , then even if  $\mu_1$  and  $\mu_2$  are close, it is possible that the solutions  $h_1$  and  $h_2$  of (2) are arbitrarily far away.

**Theorem 1.6.** *Let  $G \subset O(n)$  be a group acting without non-zero fixed points on  $\mathbb{R}^n$ , let  $R > \sqrt{n}$ , and let  $h$  be a positive  $G$  invariant Alexandrov solution of (2) for a probability measure  $\mu$  on  $S^{n-1}$  with  $h < R$  such that*

$$\mu(\Psi(L \cap S^{n-1}, \delta)) \geq (1 - \varepsilon) \cdot \frac{\dim L}{n}$$

for  $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$ ,  $\delta \in (0, \varepsilon]$  and a proper subspace  $L$  invariant under  $G$  where  $\varepsilon_0 > 0$  depends on  $n$ . Then for any  $t > 1$ , there exists a positive  $G$  invariant Alexandrov solution  $h_t$  of (2) for a probability measure  $\mu_t$  on  $S^{n-1}$  such that

$$\begin{aligned} \|h - h_t\|_\infty &\geq t \\ d_W(\mu, \mu_t) &\leq \gamma(R, n) \varepsilon^{\frac{1}{10n}} \end{aligned}$$

where  $\gamma(R, n) > 0$  depends on  $R$  and  $n$ .

Concerning the set-up of the paper, Section 2 proves the lower and upper bounds (8) on  $h_i$  in Theorem 1.2. Next Section 3 reviews the logarithmic Minkowski conjecture whose stability version Theorem 3.3 in the case of convex bodies with many hyperplane symmetries is essential in proving Theorem 1.2 in Section 4, leading also to Corollary 1.3. Finally, the two partial converses Theorem 1.5 and Theorem 1.6 of Theorem 1.2 are proved in Section 5.

## 2. Bounding the diameter of $K$ in terms of $V_K$

First we point out a simple relation for balls contained in and containing a convex body. As  $\kappa_n = V(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ , we frequently need lower and upper estimates for the  $\Gamma$  function on

positive reals. We use the following version of Stirling's formula due to Artin [3] (3.9). For any  $x \geq 1$ , there exists  $\theta \in (0, 1)$  such that

$$\Gamma(x+1) = \left(\frac{x+1}{e}\right)^x \sqrt{2\pi(x+1)} \cdot e^{-1+\frac{\theta}{12(x+1)}}.$$

If  $x \geq 1$ , then  $(\frac{x+1}{x})^x e^{\frac{1}{12x}} < e$  follows from  $\log(1+t) < t - \frac{t^2}{2} + \frac{t^3}{3}$  for  $t \in (0, 1]$ , and  $(\frac{x+1}{x})^{x+\frac{1}{2}} > e$  follows from the fact that  $(\frac{x+1}{x})^{x+\frac{1}{2}}$  is monotone decreasing; therefore, we have

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} < \Gamma(x+1) < \left(\frac{x}{e}\right)^x \sqrt{2\pi(x+1)}. \quad (14)$$

Much more precise lower and upper bounds are proved by Batir [9], Theorem 1.6 and [10], Theorem 2.2.

**Lemma 2.1.** *If  $K$  is a convex body in  $\mathbb{R}^n$  whose centroid is the origin, and  $K \subset RB^n$  for  $R > 0$ , then  $rB^n \subset K$  for some*

$$r \geq \frac{n^{\frac{n}{2}}}{6^n} \cdot \frac{V(K)}{R^{n-1}}.$$

**Proof.** We set  $r > 0$  be maximal with the property  $rB^n \subset K$ . Since the origin is the centroid of  $K$ , we have  $-K \subset nK$ , and hence  $K$  is contained in a cylinder whose height is  $(n+1)r \leq 2nr$  and base is an  $(n-1)$ -ball of radius  $R$ . Therefore,

$$V(K) \leq 2n\kappa_{n-1}R^{n-1}r.$$

As  $\Gamma(t+1) > (\frac{t}{e})^t \sqrt{2\pi t}$  for  $t \geq 1$  (see (14)) and  $\kappa_{n-1} < \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \kappa_n$ , we have

$$\kappa_{n-1} < \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \kappa_n = \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} < \frac{\sqrt{n+1}}{\sqrt{2\pi}} \cdot \frac{(2e\pi)^{\frac{n}{2}}}{n^{\frac{n}{2}}\sqrt{\pi n}} < \frac{(2e\pi)^{\frac{n}{2}}}{n^{\frac{n}{2}}\pi}.$$

In turn, we deduce

$$r \geq \frac{V(K)}{2n\kappa_{n-1}R^{n-1}} > \frac{n^{\frac{n}{2}}}{n \cdot (2e\pi)^{\frac{n}{2}}} \cdot \frac{V(K)}{R^{n-1}},$$

completing the proof of Lemma 2.1 as  $n \cdot (2e\pi)^{\frac{n}{2}} < 6^n$ .  $\square$

For a convex body  $K$  in  $\mathbb{R}^n$ , we write  $R(K)$  to denote the minimal radius of a Euclidean ball containing  $K$ , and  $r(K)$  to denote the radius of largest ball contained in  $K$ . We observe that if the convex body  $K$  is invariant under the reflections through the hyperplanes  $H_1, \dots, H_n$  with  $H_1 \cap \dots \cap H_n = \{o\}$ , then its centroid is the origin, and

$$r(K)B^n \subset K \subset R(K)B^n.$$

For Proposition 2.2 and Lemma 2.3, let  $\tilde{B}$  denote the Euclidean ball centered at the origin with  $V(\tilde{B}) = 1$ .

**Proposition 2.2.** *Let  $n \geq 2$  and  $\delta, \tau \in (0, \frac{1}{2})$ , let  $G \subset O(n)$  be a Coxeter group acting reducibly and without non-zero fixed points on  $\mathbb{R}^n$ , and let the Borel probability measure  $\mu$  on  $S^{n-1}$  be invariant under  $G$  and satisfy*

$$\mu(\Psi(L \cap S^{n-1}, \delta)) < (1 - \tau) \cdot \frac{i}{n}$$

for any linear  $i$ -subspace  $L$  of  $\mathbb{R}^n$ ,  $i = 1, \dots, n - 1$ , invariant under  $G$ , and let  $V(C) = 1$  hold for convex body  $C$  in  $\mathbb{R}^n$  invariant under  $G$ . Then

(i)

$$\int_{S^{n-1}} \log h_C d\mu \geq \log \frac{R(C)^\tau \delta}{n^5}, \text{ and}$$

(ii) if  $\int_{S^{n-1}} \log h_C d\mu \leq \int_{S^{n-1}} \log h_{\tilde{B}} d\mu$ , then

$$R(C) < \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} \text{ and } r(C) > \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}.$$

**Proof.** Let  $E$  be the John ellipsoid of  $C$  (i.e. the maximal volume ellipsoid contained in  $C$ ), and hence  $E$  is invariant under  $G$ , and

$$E \subset C \subset n E. \tag{15}$$

Let  $L_1, \dots, L_m$  be the irreducible linear subspaces invariant under  $G$ . The symmetries of  $E$  yield that there exists a set of principal directions of  $E$  that are part of  $L_1 \cup \dots \cup L_m$ , and for each  $L_i$  there exists  $r_i > 0$  such that  $E \cap L_i = r_i(B^n \cap L_i)$ ,  $i = 1, \dots, m$ . We may assume that  $r_1 \leq \dots \leq r_m$ .

If  $m = 1$ , then (15) yields that  $r_1 B^n \subset C \subset nr_1 B^n$ ; therefore, Proposition 2.2 trivially holds. In particular, let

$$m \geq 2.$$

For

$$Q = \text{conv}\{r_i(B^n \cap L_i)\}_{i=1, \dots, m},$$

$E$  is the Loewner of  $Q$  (i.e. minimal volume ellipsoid containing  $Q$ ), and hence  $Q \subset E \subset \sqrt{n} Q$ , thus (15) yields that  $Q \subset C \subset n^2 Q$ . In particular, writing  $d_i = \dim L_i$  for  $i = 1, \dots, m$ ,  $Q \subset C$  satisfies

$$n^n \prod_{i=1}^m r_i^{d_i} \geq \prod_{i=1}^m r_i^{d_i} \kappa_{d_i} \geq V(Q) \geq n^{-2n} V(C) = n^{-2n} \tag{16}$$



where  $d_1 + \dots + d_m = n$ . We observe that for any  $u \in S^{n-1}$ , there exists  $L_i$  such that  $\|u|L_i\| \geq \frac{1}{\sqrt{m}} > \frac{\delta}{n}$ . For  $i = 1, \dots, m$ , we define

$$\Lambda_i = L_1 \oplus \dots \oplus L_i$$

$$B_i = \left\{ u \in S^{n-1} : \|u|L_i\| \geq \frac{\delta}{n} \text{ and } \|u|L_j\| < \frac{\delta}{n} \text{ for } j > i \right\}.$$

It follows that  $S^{n-1}$  is partitioned into the Borel sets  $B_1, \dots, B_m$ , and as  $B_j \subset \Psi(\Lambda_i \cap S^{n-1}, \delta)$  for  $1 \leq j \leq i \leq m-1$ , we have

$$\mu(B_1) + \dots + \mu(B_i) \leq \frac{(d_1 + \dots + d_i)(1 - \tau)}{n} \quad \text{for } i = 1, \dots, m-1 \quad (17)$$

$$\mu(B_1) + \dots + \mu(B_m) = 1. \quad (18)$$

For  $\zeta = \frac{1-\tau}{n} > \frac{1}{2n}$ , next we define

$$\beta_j = \mu(B_j) - d_j \zeta \quad \text{for } j = 1, \dots, m-1 \quad (19)$$

$$\beta_m = \mu(B_m) - d_m \zeta - \tau \quad (20)$$

where (17) and (18) yield

$$\beta_1 + \dots + \beta_i \leq 0 \quad \text{for } i = 1, \dots, m-1 \quad (21)$$

$$\beta_1 + \dots + \beta_m = 0. \quad (22)$$

It follows from  $r_i B^n \cap L_i \subset Q$  and from the definition of  $B_i$  that  $h_Q(u) \geq r_i \cdot \frac{\delta}{n}$  for  $u \in B_i$ ,  $i = 1, \dots, m$ . We deduce from applying (16), (18), (19), (20), (21), (22),  $r_1 \leq \dots \leq r_m$  and  $\frac{1}{2n} < \zeta < \frac{1}{n}$  that

$$\begin{aligned} \int_{S^{n-1}} \log h_C d\mu &\geq \int_{S^{n-1}} \log h_Q d\mu = \sum_{i=1}^m \int_{B_i} \log h_Q d\mu \\ &\geq \sum_{i=1}^m \mu(B_i) \log r_i + \sum_{i=1}^m \mu(B_i) \log \frac{\delta}{n} = \sum_{i=1}^m \mu(B_i) \log r_i + \log \frac{\delta}{n} \\ &= \sum_{i=1}^m \beta_i \log r_i + \sum_{i=1}^m \zeta d_i \log r_i + \tau \log r_m + \log \frac{\delta}{n} \\ &\geq \sum_{i=1}^m \beta_i \log r_i + \zeta \log \frac{1}{n^{3n}} + \tau \log r_m + \log \frac{\delta}{n} \\ &= (\beta_1 + \dots + \beta_m) \log r_m + \sum_{i=1}^{m-1} (\beta_1 + \dots + \beta_i) (\log r_i - \log r_{i+1}) \\ &\quad - 3n\zeta \log n + \tau \log r_m + \log \frac{\delta}{n} \end{aligned}$$

$$\geq -3 \log n + \tau \log r_n + \log \frac{\delta}{n}$$

where we used  $\zeta < \frac{1}{n}$  at the end. Now  $r_m = R(E) \geq R(C)/n$  and  $\tau < 1$  imply

$$\begin{aligned} -3 \log n + \tau \log r_m + \log \frac{\delta}{n} &\geq -3 \log n + \tau \log \frac{R(C)}{n} + \log \frac{\delta}{n} \\ &\geq -3 \log n + \tau \log R(C) - \log n + \log \delta - \log n \\ &= \log \frac{R(C)^\tau \delta}{n^5}, \end{aligned}$$

proving Proposition 2.2 (i).

For (ii), let  $\tilde{r}_n$  be the radius of  $\tilde{B}$ , and hence  $\Gamma(\frac{n}{2} + 1) < (\frac{n}{2e})^{\frac{n}{2}} \sqrt{2\pi(\frac{n}{2} + 1)} < (\frac{2n}{e})^{\frac{n}{2}}$  (see (14)) implies

$$1 = \tilde{r}_n^n \kappa_n = \tilde{r}_n^n \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} > \tilde{r}_n^n \cdot \left(\frac{e\pi}{2n}\right)^{\frac{n}{2}},$$

and hence

$$\tilde{r}_n < \sqrt{\frac{2n}{e\pi}}. \quad (23)$$

We deduce from (i) and (23) that

$$\log \frac{R(C)^\tau \delta}{n^5} \leq \int_{S^{n-1}} \log h_C d\mu \leq \int_{S^{n-1}} \log h_{\tilde{B}} d\mu < \log \sqrt{\frac{2n}{e\pi}},$$

thus  $R(C) < (\frac{n^6}{\delta})^{\frac{1}{\tau}}$ .

In turn, the bound for  $r(C)$  follows from Lemma 2.1, completing the proof of Proposition 2.2.  $\square$

**Lemma 2.3.** *Let the action of the Coxeter group  $G \subset O(n)$  be irreducible, and let the Borel probability measure  $\mu$  on  $S^{n-1}$  be invariant under  $G$ , and let  $V(C) = 1$  hold for convex body  $C$  in  $\mathbb{R}^n$  invariant under  $G$ . Then*

$$\begin{aligned} \int_{S^{n-1}} \log h_C d\mu &\geq -1; \\ \frac{1}{e} &< r(C) \leq R(C) < n. \end{aligned}$$

**Proof.** As the action of  $G$  is irreducible, it follows that the inscribed ball of  $C$  is the John ellipsoid; namely, the ellipsoid of maximum volume contained in  $C$ . According to Ball [4],  $r(C)$  is at least the inradius  $r_n$  of the regular simplex of volume one, and hence  $n! > (\frac{n}{e})^n \sqrt{2\pi n}$  (see (14)) yields

$$r(C)^n \geq r_n^n = \frac{n!}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}} > \frac{(\frac{n}{e})^n \sqrt{2\pi n}}{2n^{n+\frac{1}{2}}} > \frac{1}{e^n}.$$

On the other hand, as the action of  $G$  is irreducible, it follows that the circumscribed ball of  $C$  is the Loewner ellipsoid; namely, the ellipsoid of minimum volume containing  $C$ . According to Barthe [5] (see also Lutwak, Yang, Zhang [50]),  $R(C)$  is at most the inradius  $R_n$  of the regular simplex of volume one, and hence  $n! < (\frac{n}{e})^n \sqrt{2\pi(n+1)}$  (see (14)) yields

$$R(C)^n \leq R_n^n = \frac{n^n \cdot n!}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}} < \frac{n^{\frac{n}{2}} \cdot (\frac{n}{e})^n \sqrt{2\pi(n+1)}}{(n+1)^{\frac{n+1}{2}}} < \frac{n^n \sqrt{2\pi}}{e^n} < n^n.$$

We conclude  $\frac{1}{e} < r(C) \leq R(C) < n$ .

Finally,  $r(C) > \frac{1}{e}$  implies that  $\log h_C(u) > -1$  for all  $u \in S^{n-1}$ .  $\square$

For a convex body  $K$  with  $V(K) = 1$  and hyperplane symmetries, combining Proposition 2.2 with the consequence  $\int_{S^{n-1}} \log h_K dV_K \leq \int_{S^{n-1}} \log h_{\tilde{B}} dV_K$  of the Logarithmic Minkowski Inequality Theorem 3.2 or using Lemma 2.3 yield the following.

**Corollary 2.4.** *Let  $G \subset O(n)$  be a Coxeter group acting without non-zero fixed points on  $\mathbb{R}^n$ , and let the convex body  $K$  in  $\mathbb{R}^n$ ,  $n \geq 2$ , be invariant under  $G$ . If, for  $\delta, \tau \in (0, \frac{1}{2})$  and  $i = \{1, \dots, n-1\}$ ,*

$$V_K(\Psi(L \cap S^{n-1}, \delta)) < (1 - \tau) \cdot \frac{i}{n} \cdot V(K)$$

for any  $i$ -dimensional subspace  $L$  invariant under  $G$ , then

$$R(K) < \begin{cases} \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} V(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is reducible;} \\ nV(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is irreducible;} \end{cases}$$

$$r(K) > \begin{cases} \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}} V(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is reducible;} \\ \frac{1}{e} \cdot V(K)^{\frac{1}{n}} & \text{if the action of } G \text{ is irreducible.} \end{cases}$$

Another consequence of Proposition 2.2 is a condition yielding that a convex body with hyperplane symmetries is not close to be the direct sum of lower dimensional invariant compact convex sets.

**Proposition 2.5.** *Let  $G \subset O(n)$  be a Coxeter group acting reducibly and without non-zero fixed points on  $\mathbb{R}^n$ , and let the convex body  $K$  in  $\mathbb{R}^n$ ,  $n \geq 2$ , be invariant under  $G$ . If  $\delta, \tau \in (0, \frac{1}{2})$ , and a convex body  $K$  in  $\mathbb{R}^n$  invariant under  $G$  satisfies*

$$V_K(\Psi(L \cap S^{n-1}, \delta)) < (1 - \tau) \cdot \frac{\dim L}{n} \cdot V(K)$$

for any proper coordinate subspace  $L$  invariant under  $G$ , then

$$(1 - \eta)((L \cap K) \oplus (L^\perp \cap K)) \not\subset K$$

for any proper subspace  $L$  invariant under  $G$  where

$$\eta = \frac{\delta\tau}{4n} \cdot \frac{n^{\frac{n}{2}}}{6^n} \left( \frac{\delta}{n^6} \right)^{\frac{n}{\tau}}.$$

**Proof.** We may assume that  $V(K) = 1$ , and define

$$R_0 = \left( \frac{n^6}{\delta} \right)^{\frac{1}{\tau}}$$

$$r_0 = \frac{n^{\frac{n}{2}}}{6^n} \left( \frac{\delta}{n^6} \right)^{\frac{n-1}{\tau}},$$

and hence

$$\eta = \frac{\delta\tau}{4n} \cdot \frac{r_0}{R_0} < \frac{\tau}{4n}, \quad (24)$$

while Proposition 2.2 implies that

$$r_0 B^n \subset K \subset R_0 B^n.$$

We prove Proposition 2.5 by contradiction; therefore, we suppose that there exists a coordinate  $i$ -subspace  $L$ ,  $1 \leq i \leq n-1$ , such that

$$(1 - \eta)((L \cap K) \oplus (L^\perp \cap K)) \subset K. \quad (25)$$

We define

$$\Omega_0 = \left\{ [o, x + y] : x \in (1 - \eta)\partial(L \cap K) \text{ and } y \in (1 - \eta) \left( 1 - \frac{\tau}{2n} \right) (L^\perp \cap K) \right\}.$$

In addition, let

$$\begin{aligned} \Omega &= \{z \in K : \exists t \in (0, 1], \, tz \in \Omega_0\} \\ &= \{z \in K : (1 - \eta)z \in \Omega_0\} \\ \Xi &= \{u \in S^{n-1} : \exists x \in \Omega \cap \partial K, \, h_C(u) = \langle x, u \rangle\}. \end{aligned}$$

We deduce using  $\eta < \frac{\tau}{2n}$  that

$$\begin{aligned} V_K(\Xi) &\geq \mathcal{H}^n(\Omega_0) \\ &= \frac{i}{n} \cdot (1 - \eta)^i \mathcal{H}^i(L \cap K) \cdot (1 - \eta)^{n-i} \left( 1 - \frac{\tau}{2n} \right)^{n-i} \mathcal{H}^{n-i}(L^\perp \cap K) \\ &> (1 - \tau) \frac{i}{n} \cdot \mathcal{H}^i(L \cap K) \cdot \mathcal{H}^{n-i}(L^\perp \cap K) > (1 - \tau) \frac{i}{n}. \end{aligned} \quad (26)$$

Therefore, we contradict (25) by proving

$$\Xi \subset \Psi(L \cap S^{n-1}, \delta). \quad (27)$$

Let  $u \in \Xi$  be an exterior normal at  $z \in \partial K$ . We observe that

$$u = v \cos \beta + w \sin \beta$$

where  $v \in L \cap S^{n-1}$ ,  $w \in L^\perp \cap S^{n-1}$  and  $\beta = \angle(u, v) \in [0, \frac{\pi}{2}]$ . We write  $z = x + y$  for  $x \in L \cap K$  and  $y \in L^\perp \cap K$ . As  $z \in \Xi$ , we have

$$\begin{aligned} (1 - \eta)x + (1 - \eta)y &= (1 - \eta)z \in \Omega_0 \\ &\in (1 - \eta)(L \cap K) + (1 - \eta)\left(1 - \frac{\tau}{2n}\right)(L^\perp \cap K). \end{aligned}$$

In turn, we deduce that

$$y \in \left(1 - \frac{\tau}{2n}\right)(L^\perp \cap K). \quad (28)$$

Let

$$p = (1 - \eta)x + y + \frac{\tau}{4n} \cdot r_0 w,$$

which, using (28),  $r_0 B^n \subset K$ , (24) and (25) satisfies

$$\begin{aligned} p &\in (1 - \eta)(L \cap K) + \left(1 - \frac{\tau}{2n}\right)(L^\perp \cap K) + \frac{\tau}{4n} \cdot (L^\perp \cap K) \\ &= (1 - \eta)(L \cap K) + \left(1 - \frac{\tau}{4n}\right)(L^\perp \cap K) \\ &\subset (1 - \eta)(L \cap K) + (1 - \eta)(L^\perp \cap K) \subset K. \end{aligned}$$

Since  $u$  is exterior normal at  $z = x + y$  where  $w \in L^\perp \cap S^{n-1}$ ,  $v \in L \cap S^{n-1}$  and  $x \in L \cap R_0 B^n$ , we have

$$\begin{aligned} 0 &\geq \langle u, p - z \rangle = \left\langle u, \frac{\tau r_0}{4n} \cdot w - \eta x \right\rangle \\ &= \left\langle v \cos \beta + w \sin \beta, \frac{\tau r_0}{4n} \cdot w - \eta x \right\rangle = \frac{\tau r_0}{4n} \cdot \sin \beta - \langle v, x \rangle \eta \cos \beta \\ &\geq \frac{\tau r_0}{4n} \cdot \sin \beta - R_0 \eta \cos \beta. \end{aligned}$$

We conclude that

$$\|u - v\| \leq \tan \beta \leq \frac{4n\eta}{\tau} \cdot \frac{R_0}{r_0} \leq \delta,$$

which in turn, yields (27) and contradicts (25), proving Proposition 2.5.  $\square$

### 3. On the logarithmic Minkowski conjecture

For origin symmetric convex bodies, the following is an equivalent form of the origin symmetric case of the Logarithmic Brunn-Minkowski conjecture (see Boroczky, Lutwak, Yang, Zhang [17]).

**Conjecture 3.1** (*Logarithmic Minkowski conjecture*). *If  $K$  and  $C$  are convex bodies in  $\mathbb{R}^n$  whose centroid is the origin, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)} \quad (29)$$

with equality if and only if  $K = K_1 + \dots + K_m$  and  $C = C_1 + \dots + C_m$  for compact convex sets  $K_1, \dots, K_m, C_1, \dots, C_m$  of dimension at least one where  $\sum_{i=1}^m \dim K_i = n$  and  $K_i$  and  $C_i$  are dilates,  $i = 1, \dots, m$ .

The argument in Boroczky, Lutwak, Yang, Zhang [18] yields that uniqueness of the solution of the logarithmic-Minkowski problem (1) for any positive even  $C^\infty$   $f$  is equivalent saying that the Logarithmic Minkowski conjecture (29) holds for any  $o$ -symmetric convex bodies  $K$  and  $C$  with  $C_+^\infty$  boundaries with equality if and only if  $K$  and  $C$  are dilates.

In  $\mathbb{R}^2$ , Conjecture 3.1 is verified in Boroczky, Lutwak, Yang, Zhang [17] for origin symmetric convex bodies, but it is still open in general. In higher dimensions, Conjecture 3.1 is proved for with enough hyperplane symmetries (cf. Theorem 3.2) and complex bodies (cf. Rotem [59]).

For origin symmetric convex bodies, Conjecture 3.1 is proved when  $K$  is close to be an ellipsoid by a combination of the local estimates by Kolesnikov, Milman [42] and the use of the continuity method in PDE by Chen, Huang, Li, Liu [21]. Another even more recent proof of this result based on Alexandrov's approach of considering the Hilbert-Brunn-Minkowski operator for polytopes is due to Putterman [58]. Additional local versions of Conjecture 3.1 are due to Kolesnikov, Livshyts [41] and Hosle, Kolesnikov, Livshyts [34].

Following the result on unconditional convex bodies by Saroglou [60], Boroczky, Kalantzopoulos [16] verified the logarithmic Minkowski conjecture for convex bodies with  $n$  independent hyperplane symmetries.

**Theorem 3.2** (*Boroczky, Kalantzopoulos*). *If the convex bodies  $K$  and  $C$  in  $\mathbb{R}^n$  are invariant under linear reflections  $A_1, \dots, A_n$  through  $n$  independent linear  $(n-1)$ -planes  $H_1, \dots, H_n$ , then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} dV_K \geq \frac{V(K)}{n} \log \frac{V(C)}{V(K)},$$

with equality if and only if  $K = K_1 + \dots + K_m$  and  $C = C_1 + \dots + C_m$  for compact convex sets  $K_1, \dots, K_m, C_1, \dots, C_m$  of dimension at least one and invariant under  $A_1, \dots, A_n$  where  $K_i$  and  $C_i$  are dilates,  $i = 1, \dots, m$ , and  $\sum_{i=1}^m \dim K_i = n$ .

The Boroczky, De [12] proved the following stability version of the logarithmic-Minkowski inequality Theorem 3.2 for convex bodies with many hyperplane symmetries.

**Theorem 3.3.** *If the convex bodies  $K$  and  $C$  in  $\mathbb{R}^n$  are invariant under the Coxeter group  $G \subset O(n)$  acting without non-zero fixed points on  $\mathbb{R}^n$ , and*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

for  $\varepsilon > 0$ , then for some  $m \geq 1$ , there exist compact convex sets  $K_1, C_1, \dots, K_m, C_m$  of dimension at least one and invariant under  $G$  where  $K_i$  and  $C_i$  are dilates,  $i = 1, \dots, m$ , and  $\sum_{i=1}^m \dim K_i = n$  such that

$$K_1 + \dots + K_m \subset K \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right) (K_1 + \dots + K_m)$$

$$C_1 + \dots + C_m \subset C \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right) (C_1 + \dots + C_m)$$

where  $c > 1$  is an absolute constant.

If  $K$  is a ball centered at the origin (and hence  $m = 1$ ), then Ivaki [38], Theorem 2.1 proves an improved version of Theorem 3.3 where  $C$  does not need to satisfy any symmetry assumption (only translated in a suitable way) and the error term is of order  $\varepsilon^{\frac{1}{n+1}}$  instead of  $\varepsilon^{\frac{1}{95n}}$ .

#### 4. Proof of Theorem 1.2

For compact convex sets  $K$  and  $C$  in  $\mathbb{R}^n$ , their Hausdorff distance is

$$d_\infty(K, C) = \|h_K - h_C\|_\infty = \min\{r \geq 0 : K \subset C + r B^n \text{ and } C \subset K + r B^n\}.$$

We prove Theorem 1.2 in the following form.

**Theorem 4.1.** *Let  $G \subset O(n)$  be a Coxeter group acting without non-zero fixed points on  $\mathbb{R}^n$ . If  $K$  and  $C$  are convex bodies in  $\mathbb{R}^n$  invariant under  $G$  and satisfy  $V(K) = V(C) = 1$ ,*

$$\begin{aligned} V_K(\Psi(L \cap S^{n-1}, \delta)) &\leq (1 - \tau) \cdot \frac{\dim L}{n}, \\ V_C(\Psi(L \cap S^{n-1}, \delta)) &\leq (1 - \tau) \cdot \frac{\dim L}{n} \end{aligned} \quad (30)$$

for  $\delta, \tau \in (0, \frac{1}{2})$  and for any proper subspace  $L$  invariant under  $G$ , then

$$r_0 < h_K, h_C < R_0; \quad (31)$$

$$d_\infty(K, C) \leq \gamma_0 \cdot d_W(V_K, V_C)^{\frac{1}{95n}} \quad (32)$$

where for some absolute constant  $c > 1$ , we have

- $R_0 = n$ ,  $r_0 = \frac{1}{c}$  and  $\gamma_0 = c^n$  provided the action of  $G$  is irreducible (and hence the condition (30) is irrelevant);
- $R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$ ,  $r_0 = \frac{n^{\frac{2}{\tau}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$  and  $\gamma_0 = \frac{c^n}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}}$  provided the action of  $G$  is reducible.

We need the simple statements Lemma 4.2 and (33).

**Lemma 4.2.** *If  $K$  is a convex body with  $K \subset R B^n$  for  $R > 0$ , then  $|h_K(u) - h_K(v)| \leq R\|u - v\|$  for  $u, v \in S^{n-1}$ .*

**Proof.** Let  $x_0 \in \partial K$  satisfy that  $h_K = \langle u, x_0 \rangle$ , and hence

$$h_K(u) - h_K(v) \leq \langle u, x_0 \rangle - \langle v, x_0 \rangle = \langle u - v, x_0 \rangle \leq \|u - v\| \cdot R.$$

Since similar argument shows that  $h_K(v) - h_K(u) \leq \|v - u\| \cdot R$ , we conclude the lemma.  $\square$

We also note that if  $\mu, \nu$  are Borel probability measures on  $S^{n-1}$ , and  $f : S^{n-1} \rightarrow \mathbb{R}$  and  $\omega > 0$  satisfy that  $|f(u) - f(v)| \leq \omega\|u - v\|$  for  $u, v \in S^{n-1}$ , then

$$\left| \int_{S^{n-1}} f d\mu - \int_{S^{n-1}} f d\nu \right| \leq \omega \cdot d_W(\mu, \nu). \quad (33)$$

**Proof of Theorem 4.1.** Let  $d_W(V_K, V_C) = \varepsilon$ . In order to apply Corollary 2.4, we set

$$R_0 = \begin{cases} \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} & \text{if the action of } G \text{ reducible;} \\ n & \text{if the action of } G \text{ irreducible;} \end{cases}$$

$$r_0 = \begin{cases} \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}} & \text{if the action of } G \text{ reducible;} \\ \frac{1}{e} & \text{if the action of } G \text{ irreducible} \end{cases}$$

and deduce (31) from Corollary 2.4. In particular, if  $u, v \in S^{n-1}$ , then first (31), secondly Lemma 4.2 and (31) imply that if  $u, v \in S^{n-1}$ , then

$$|\log h_K(u) - \log h_K(v)| \leq \frac{|h_K(u) - h_K(v)|}{r_0} \leq \frac{R_0}{r_0} \cdot \|u - v\|$$

$$|\log h_C(u) - \log h_C(v)| \leq \frac{|h_C(u) - h_C(v)|}{r_0} \leq \frac{R_0}{r_0} \cdot \|u - v\|$$

where

$$\frac{R_0}{r_0} = \begin{cases} \frac{6^n}{n^{\frac{n}{2}}} \left(\frac{n^6}{\delta}\right)^{\frac{n}{\tau}} & \text{if the action of } G \text{ reducible;} \\ en & \text{if the action of } G \text{ irreducible.} \end{cases} \quad (34)$$

For

$$\varepsilon = d_W(V_K, V_C),$$



we deduce from applying first (33) and  $d_W(V_K, V_C) = \varepsilon$ , then from the Logarithmic Minkowski Inequality Theorem 3.2, and using again (33) and  $d_W(V_K, V_C) = \varepsilon$  that

$$\begin{aligned} \int_{S^{n-1}} \log h_C dV_K &\leq \int_{S^{n-1}} \log h_C dV_C + \frac{R_0}{r_0} \cdot \varepsilon \leq \int_{S^{n-1}} \log h_K dV_C + \frac{R_0}{r_0} \cdot \varepsilon \\ &\leq \int_{S^{n-1}} \log h_K dV_K + \frac{2R_0}{r_0} \cdot \varepsilon. \end{aligned}$$

It follows from Theorem 3.3 that for some  $m \geq 1$ , there exist  $\theta_1, \dots, \theta_m > 0$  and compact convex sets  $K_1, \dots, K_m > 0$  invariant under  $G$  such that  $\sum_{i=1}^m \dim K_i = n$  and

$$\begin{aligned} K_1 \oplus \dots \oplus K_m \subset K \subset \left(1 + c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) (K_1 \oplus \dots \oplus K_m) \\ \theta_1 K_1 \oplus \dots \oplus \theta_m K_m \subset C \subset \left(1 + c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) (\theta_1 K_1 \oplus \dots \oplus \theta_m K_m) \end{aligned} \quad (35)$$

where  $c_0 > 1$  is an absolute constant.

If the action of  $G$  is irreducible, then  $m = 1$ , and hence

$$\left(1 + c_1^n \left(\frac{R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right)^{-1} K \subset C \subset \left(1 + c_1^n \left(\frac{R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) K$$

for some absolute constant  $c_1 > 1$ . In turn,  $K, C \subset R_0 B^n$  (cf. (31)),  $R_0 = n$  and  $\left(\frac{R_0}{r_0}\right)^{\frac{1}{95n}} < 2$  (cf. (34)) yield (32) as

$$d_\infty(K, C) \leq R_0 \cdot c_1^n \left(\frac{R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}} \leq (2c_1)^n \cdot \varepsilon^{\frac{1}{95n}}.$$

Next, let the action of  $G$  be reducible. First, we assume that

$$\varepsilon < c_2^{95n^2} (\delta\tau)^{95n} \left(\frac{\delta}{n^6}\right)^{\frac{96n^2}{\tau}} \quad (36)$$

where  $c_2 \in (0, 1)$  is a suitably small absolute constant such that if  $\varepsilon > 0$  satisfies (36), then

$$c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}} < \frac{\delta\tau}{4n} \cdot \frac{r_0}{R_0} (< 1) \quad (37)$$

holds for the  $c_0$  in (35) (cf. (34)). Therefore, on the one hand, we have

$$\left(1 - c_0^n \left(\frac{2R_0}{r_0} \cdot \varepsilon\right)^{\frac{1}{95n}}\right) ((K \cap L_1) \oplus \dots \oplus (K \cap L_m)) \subset K$$

for  $L_i = \text{lin } K_i$ ,  $i = 1, \dots, m$ , and, on the other hand, we deduce from (37) and Proposition 2.5 that  $m = 1$ . In particular,

$$\left(1 - c_3^n \left(\frac{n^6}{\delta}\right)^{\frac{1}{95\tau}} \varepsilon^{\frac{1}{95n}}\right) K \subset C \subset \left(1 + c_3^n \left(\frac{n^6}{\delta}\right)^{\frac{1}{95\tau}} \varepsilon^{\frac{1}{95n}}\right) K$$

for an suitable absolute constant  $c_3 > 1$ , and hence  $K, C \subset R_0 B^n$  implies

$$d_\infty(K, C) \leq R_0 \cdot c_3^n \left(\frac{n^6}{\delta}\right)^{\frac{1}{95\tau}} \varepsilon^{\frac{1}{95n}}.$$

We conclude Theorem 4.1 under the condition (36).

Finally, we assume that the condition (36) does not hold; namely,

$$\varepsilon \geq c_2^{95n^2} (\delta\tau)^{95n} \left(\frac{\delta}{n^6}\right)^{\frac{96n^2}{\tau}}.$$

Since  $o \in K, C \subset R_0 B^n$ , we have

$$\begin{aligned} d_\infty(K, C) &\leq R_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}} \leq c_2^{-n} (\delta\tau)^{-1} \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau} (1 + \frac{96n}{95})} \varepsilon^{\frac{1}{95n}} \\ &\leq \frac{c_2^{-n}}{\tau} \cdot \delta^{-1} \cdot \left(\frac{n^6}{\delta}\right)^{\frac{2n}{\tau}} \varepsilon^{\frac{1}{95n}} \leq \frac{c_2^{-n}}{\tau} \cdot \delta^{-\frac{3n}{\tau}} n^{\frac{12n}{\tau}} \varepsilon^{\frac{1}{95n}}, \end{aligned}$$

proving Theorem 4.1.  $\square$

**Proof of Theorem 1.2.** According to Theorem 1.1, there exist convex bodies  $K$  and  $C$  invariant under  $G$  such that  $h_1(u) = h_K(u)$  and  $h_2(u) = h_C(u)$  for  $u \in S^{n-1}$ . In turn, we conclude (8) from (31), and (7) from (31) and (32).  $\square$

After verifying Theorem 1.2, we consider the case when  $\mu_1(S^{n-1}) \neq \mu_2(S^{n-1})$ .

**Proof of Corollary 1.3.** We may assume that

$$1 = M = \mu_1(S^{n-1}) \leq \mu_2(S^{n-1}).$$

For  $\varepsilon = d_{\text{bL}}(\mu_1, \mu_2) \leq 1$ , it follows from (9) that

$$1 \leq \mu_2(S^{n-1}) \leq 1 + \varepsilon. \quad (38)$$

We consider the probability measure  $\tilde{\mu}_2 = \mu_2(S^{n-1})^{-1} \cdot \mu_2$ . Since  $d_{bL}(\mu_2, \tilde{\mu}_2) \leq \varepsilon$  by (10), the triangle inequality yields  $d_{bL}(\mu_1, \tilde{\mu}_2) \leq 2\varepsilon$  by (38), and readily

$$\tilde{\mu}_2(\Psi(L \cap S^{n-1}, \delta)) \leq (1 - \tau) \cdot \frac{\dim L}{n}$$

for any proper subspace  $L$  invariant under  $G$ . In addition,

$$\tilde{h}_2 = \mu_2(S^{n-1})^{\frac{-1}{n}} \cdot h_2$$

is the invariant Alexandrov solution of the Logarithmic Minkowski Problem (2).

We deduce from Theorem 1.2 that

$$\begin{aligned} \|h_1 - \tilde{h}_2\|_\infty &\leq \tilde{\gamma}_0 \cdot (2\varepsilon)^{\frac{1}{95n}} \\ r_0 &\leq h_1, \tilde{h}_2 \leq \tilde{R}_0 \end{aligned}$$

where for some absolute constant  $\tilde{c} > 1$ , we have

- $\tilde{R}_0 = n$ ,  $r_0 = \frac{1}{\varepsilon}$  and  $\tilde{\gamma}_0 = \tilde{c}^n \cdot \varepsilon^{\frac{1}{95n}}$  provided the action of  $G$  is irreducible;
- $\tilde{R}_0 = \left(\frac{n^6}{\delta}\right)^{\frac{1}{\tau}}$ ,  $r_0 = \frac{n^{\frac{n}{2}}}{6^n} \left(\frac{\delta}{n^6}\right)^{\frac{n-1}{\tau}}$  and  $\tilde{\gamma}_0 = \frac{\tilde{c}^n}{\tau} \cdot \delta^{\frac{-3n}{\tau}} n^{\frac{12n}{\tau}}$  provided the action of  $G$  is reducible.

Therefore,  $h_2 = \mu_2(S^{n-1})^{\frac{1}{n}} \cdot \tilde{h}_2$  and (38) imply Corollary 1.3 with  $c = 2\tilde{c}$  and  $R_0 = 2\tilde{R}_0$ .  $\square$

## 5. Partial converses Theorem 1.5 and Theorem 1.6 of Theorem 1.2

In this section, we prove the two partial converses Theorem 1.5 and Theorem 1.6 of Theorem 1.2 by verifying Theorem 5.1 and Theorem 5.2.

Our argument for Theorem 5.1 is based on Hug, Schneider [37], which paper proved that if  $R > 0$  and  $K$  and  $C$  are convex bodies in  $\mathbb{R}^n$  satisfying  $K, C \subset RB^n$ , then

$$d_{bL}(S_K, S_C) \leq \tilde{\gamma}(R, n) \cdot \sqrt{d_\infty(K, C)} \quad (39)$$

where  $\tilde{\gamma}(R, n) > 0$  depends on  $R$  and  $n$ . Theorem 1.5 directly follows from the following theorem (see the explanation after (2)).

**Theorem 5.1.** *If  $R > 0$  and  $K$  and  $C$  are convex bodies in  $\mathbb{R}^n$  satisfying  $o \in \text{int}K, \text{int}C$  and  $K, C \subset RB^n$ , then*

$$d_{bL}(V_K, V_C) \leq \gamma(R, n) \cdot \sqrt{d_\infty(K, C)}$$

where  $\gamma(R, n) > 0$  depends on  $R$  and  $n$ .

**Proof.** Let  $\varepsilon = d_\infty(K, C) \leq R$ . By the symmetry of  $K$  and  $C$ , it is sufficient to prove that if  $f \in \text{Lip}_1$  with  $\|f\|_\infty \leq 1$ , then

$$\int_{S^{n-1}} f dV_K - \int_{S^{n-1}} f dV_C \leq \gamma(R, n) \cdot \sqrt{\varepsilon}$$

where  $\gamma(R, n) > 0$  depends on  $R$  and  $n$ , which is equivalent to say that

$$\int_{S^{n-1}} f \cdot h_K dS_K - \int_{S^{n-1}} f \cdot h_C dS_C \leq n\gamma(R, n) \cdot \sqrt{\varepsilon}. \quad (40)$$

It follows from  $d_\infty(K, C) \leq \varepsilon$  that

$$h_K \leq h_C + \varepsilon.$$

We deduce from  $C \subset RB^n$  and Lemma 4.2 that  $h_C \in \text{Lip}_R$ , and hence  $f \cdot h_C \in \text{Lip}_{2R}$ . For  $g = \frac{1}{2R} f \cdot h_C$ , it follows that  $g \in \text{Lip}_1$  and  $\|g\|_\infty \leq 1$ , thus  $\|f\|_\infty \leq 1$ ,  $K \subset RB^n$  and the result (39) by Hug, Schneider [37] yield

$$\begin{aligned} \int_{S^{n-1}} f h_K dS_K - \int_{S^{n-1}} f h_C dS_C &\leq \int_{S^{n-1}} f(h_C + \varepsilon) dS_K - \int_{S^{n-1}} f h_C dS_C \\ &= \varepsilon \cdot \int_{S^{n-1}} f dS_K + \\ &\quad 2R \left( \int_{S^{n-1}} g dS_K - \int_{S^{n-1}} g dS_C \right) \\ &\leq \varepsilon \cdot R^{n-1} n \kappa_n + 2R \cdot \tilde{\gamma}(R, n) \cdot \sqrt{\varepsilon}. \end{aligned}$$

We conclude (40) from  $\varepsilon < 2R$ , and in turn Theorem 5.1.  $\square$

Convex bodies whose centroid is the origin and having almost equality in Theorem 1.1 (ii) were characterized by Böröczky, Henk [15]. More precisely, if  $\varepsilon \in (0, \tilde{\varepsilon}_0)$  and the convex body  $K \subset \mathbb{R}^n$  has its centroid at the origin, and satisfies

$$V_K(L \cap S^{n-1}) \geq (1 - \varepsilon) \cdot \frac{d}{n} \cdot V(K)$$

for a linear  $d$ -space  $L$  with  $1 \leq d < n$ , then

$$(1 - \tilde{\gamma} \cdot \varepsilon^{\frac{1}{5n}})(C + M) \subset K \subset C + M \quad (41)$$

for some compact convex set  $C \subset L^\perp$ , and complementary  $d$ -dimensional compact convex set  $M$  where  $\tilde{\varepsilon}_0, \tilde{\gamma} > 0$  depend on the dimension  $n$ .

The paper [15] also verified two observations that we need in the sequel. For a convex body  $Q$  in  $\mathbb{R}^n$ , we write  $\sigma(Q)$  to denote the centroid, and  $\|x\|_{Q-Q}$  to denote the norm of an  $x \in \mathbb{R}^n$

with respect to the origin symmetric convex body  $Q - Q$ ; namely,  $\|x\|_{Q-Q} = \min\{t \geq 0 : x \in t(Q - Q)\}$ .

For convex bodies  $K, \tilde{K}$  in  $\mathbb{R}^n$ , writing  $K \Delta \tilde{K}$  to denote the symmetric difference, Lemma 3.4 in [15] says that if  $V(K \Delta \tilde{K}) \leq t V(\tilde{K})$  for  $t \in (0, \frac{1}{4^n e})$ , then

$$\|\sigma(\tilde{K}) - \sigma(K)\|_{\tilde{K}-\tilde{K}} \leq 4nt. \quad (42)$$

The second observation, Lemma 3.3 in [15] states that if  $z \in \mathbb{R}^n$ , then

$$V(\tilde{K} \Delta (z + \tilde{K})) \leq 2n\|z\|_{\tilde{K}-\tilde{K}} V(\tilde{K}). \quad (43)$$

The following statement exhibits why we need a condition of the type of (30) in Theorem 4.1.

**Theorem 5.2.** *Let  $n \geq 2$ ,  $R > \sqrt{n}$ , and let the convex body  $K \subset RB^n$  with  $V(K) = 1$  have its centroid at the origin. There exist constants  $\varepsilon_0, \gamma > 0$  depending on the dimension  $n$ , such that, if  $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$  and  $\delta \in (0, \varepsilon]$  and*

$$V_K(\Psi(L \cap S^{n-1}, \delta)) \geq (1 - \varepsilon) \cdot \frac{d}{n}$$

for a linear  $d$ -space  $L$  with  $1 \leq d < n$ , then

$$d_\infty(K, C + M) \leq \gamma R^{n+1} \varepsilon^{\frac{1}{5n}}$$

for some compact convex set  $C \subset L^\perp$ , and complementary  $d$ -dimensional compact convex set  $M$ .

If, in addition,  $K$  is invariant under a group  $G \subset O(n)$  leaving  $L$  invariant and acting without fixed point on  $S^{n-1}$ , then we may assume that  $C = K|L^\perp$  and  $M = K|L$ .

**Proof.** We assume that  $\varepsilon \in (0, \frac{\varepsilon_0}{R^n})$  where  $\varepsilon_0 > 0$  depending on  $n$  is small enough to make the argument work.

We deduce from Lemma 2.1 that  $r B^n \subset K$  for

$$r = \frac{n^{\frac{n}{2}}}{5^n R^{n-1}}.$$

We plan to cut off a rim from  $K$  in order to apply (41). For

$$\eta = \frac{4 \cdot 5^n R^n}{n^{\frac{n}{2}}} \cdot \delta,$$

we claim that if  $u \in S^{n-1}$  is an exterior normal at  $x \in \partial K$  with  $x|L \in (1 - \eta)(K|L)$ , then

$$u \notin \Psi(L \cap S^{n-1}, \delta). \quad (44)$$

Let  $\alpha \in [0, \frac{\pi}{2}]$ ,  $v \in S^{n-1} \cap L$  and  $w \in S^{n-1} \cap L^\perp$  such that  $u|L = v \cos \alpha$  and  $u|L^\perp = w \sin \alpha$ , and hence  $u = v \cos \alpha + w \sin \alpha$ .

Next let  $y \in \partial K$  be such that  $v$  is an exterior normal at  $y$ . Since  $x|L \in (1 - \eta)(K|L)$ , we have

$$\langle x, v \rangle \leq (1 - \eta)\langle y, v \rangle \leq \langle y, v \rangle - \eta r.$$

It follows that

$$0 \leq h_K(u) - \langle y, u \rangle = \langle x - y, v \cos \alpha + w \sin \alpha \rangle \leq -\eta r \cos \alpha + 2R \sin \alpha,$$

and hence  $\tan \alpha \geq \frac{\eta r}{2R} = \frac{\eta \cdot n^{\frac{2}{n}}}{2 \cdot 5^n R^n}$ , proving (44).

We define

$$\tilde{K} = \{x \in K : x|L \in (1 - \eta)(K|L)\},$$

thus (44) implies that

$$\begin{aligned} V_{\tilde{K}}(L \cap S^{n-1}) &\geq (1 - \eta)^n V_K(\Psi(L \cap S^{n-1}, \delta)) \geq (1 - \gamma_1(R^n \delta + \varepsilon)) \cdot \frac{d}{n} \\ &\geq (1 - 2\gamma_1 R^n \varepsilon) \cdot \frac{d}{n} \cdot V(\tilde{K}) \end{aligned} \quad (45)$$

for  $\gamma_1 > 0$  depending on  $n$ . Since  $(1 - \eta)K \subset \tilde{K}$ , it follows that

$$V(K \Delta \tilde{K}) \leq \gamma_2 R^n V(\tilde{K}) \cdot \varepsilon$$

for  $\gamma_2 > 0$  depending on  $n$ . According to (42) based on [15], the centroid  $\sigma(\tilde{K})$  of  $\tilde{K}$  satisfies

$$\|\sigma(\tilde{K})\|_{\tilde{K}-\tilde{K}} \leq 4n\gamma_2 R^n \cdot \varepsilon; \quad (46)$$

It follows from (43) based on [15] the convex body  $K_0 = \tilde{K} - \sigma(\tilde{K})$  satisfies that  $\sigma(K_0) = o$  and

$$V(K_0 \Delta \tilde{K}) \leq 8n^2 \gamma_2 R^n V(\tilde{K}) \cdot \varepsilon,$$

and hence

$$V_{K_0}(L \cap S^{n-1}) \geq V_{\tilde{K}}(L \cap S^{n-1}) - V(K_0 \Delta \tilde{K}) \geq (1 - \gamma_3 R^n \varepsilon) \cdot \frac{d}{n} \cdot V(K_0)$$

for  $\gamma_3 > 0$  depending on  $n$ . We deduce from (41) based on [15] and  $V(K_0) \leq 1$  that there exist some compact convex set  $C_0 \subset L^\perp$ , and complementary  $d$ -dimensional compact convex set  $M_0$  such that

$$(1 - \gamma_4 R^{\frac{1}{5}} \cdot \varepsilon^{\frac{1}{5n}})(C_0 + M_0) \subset K_0 \subset C_0 + M_0 \quad (47)$$

where  $\gamma_4 > 0$  depends on the dimension  $n$ . Since

$$K_0 + \sigma(\tilde{K}) \subset K \subset (1 - \eta)^{-1}(K_0 + \sigma(\tilde{K})),$$

we deduce from  $\tilde{K} - \tilde{K} = K_0 - K_0$ , (46) and (47) that there exist some compact convex set  $C \subset L^\perp$ , and complementary  $d$ -dimensional compact convex set  $M$  such that

$$(1 - \gamma_5 R^n \cdot \varepsilon^{\frac{1}{5n}})(C + M) \subset K \subset C + M \quad (48)$$

where  $\gamma_5 > 0$  depends on the dimension  $n$ . As  $K \subset R B^n$ , we have  $d_\infty(K, C + M) \leq \gamma_5 R^{n+1} \varepsilon^{\frac{1}{5n}}$ .

Finally, if  $K$  is invariant under a group  $G \subset O(n)$  leaving  $L$  (and hence also  $L^\perp$ ) invariant and acting without fixed point on  $L \cap S^{n-1}$ , then let  $G' \subset O(n)$  be the group whose elements are of the form  $\Phi|_L \oplus \text{id}_{L^\perp}$  for  $\Phi \in G$  that acts without non-zero fixed point on  $L$ , and let  $G'' \subset O(n)$  be the group whose elements are of the form  $\Phi|_{L^\perp} \oplus \text{id}_L$  for  $\Phi \in G$  that acts without non-zero fixed point on  $L^\perp$ . Now for any  $x \in K|L^\perp$ , the section  $K \cap (x + L)$  is invariant under  $G'$ , and hence the centroid  $\sigma(K \cap (x + L))$  of  $K \cap (x + L)$  is invariant under  $G'$ , which in turn yields that  $x|L^\perp = \sigma(K \cap (x + L)) \in K$ . Therefore,  $K|L^\perp = K \cap L^\perp$ . Since similar argument implies  $K|L = K \cap L$ , we may choose  $C = K|L^\perp$  and  $M = K|L$ .  $\square$

**Proof of Theorem 1.6.** According to the remarks after (2), there exists a convex body  $K$  invariant under  $G$  such that  $h = h_K$  and  $\mu = V_K$ . Since the centroid of  $K$  is invariant under the action of  $G$  that does not have non-zero fixed points, it follows that the centroid  $\sigma(G)$  of  $G$  is the origin.

We deduce from Theorem 5.2 that there exists  $\gamma_0(R, n) > 0$  depending on  $R$  and  $n$  such that

$$d_\infty(K, C_0 + M_0) \leq \gamma_0(R, n) \cdot \varepsilon^{\frac{1}{5n}}$$

where  $C_0 = K|L^\perp$  and  $M_0 = K|L$ . Rescaling  $C_0$  and  $M_0$ , we obtain convex compact  $C_1 \subset L^\perp$  and  $M_1 \subset L$  invariant under  $G$  such that

$$d_\infty(K, C_1 + M_1) \leq \gamma_1(R, n) \cdot \varepsilon^{\frac{1}{5n}} \quad \text{and} \quad V(C_1 + M_1) = 1$$

where  $\gamma_1(R, n) > 0$  depends on  $R$  and  $n$ . For  $Q = C_1 + M_1$ , it follows from Theorem 5.1 that

$$d_W(V_K, V_Q) \leq \gamma(R, n) \cdot \varepsilon^{\frac{1}{10n}}$$

where  $\gamma(R, n) > 0$  depends on  $R$  and  $n$ .

Let  $d = \dim L$ , and let  $\varrho > 0$  be the maximal radius of a  $d$ -dimensional ball centered at the origin and contained in  $M_1$ . We deduce that  $d_\infty(\frac{t+\varrho}{\varrho} M_1, M_1) \geq t$  for any  $t > 1$ , and hence

$$Q_t = \frac{t+\varrho}{\varrho} \cdot M_1 + \left( \frac{\varrho}{t+\varrho} \right)^{\frac{d}{n-d}} \cdot C_1$$

satisfies

$$d_\infty(K, Q_t) \geq t;$$

$$V(Q_t) = 1 \quad \text{and} \quad V_{Q_t} = V_Q;$$

$$d_W(V_K, V_{Q_t}) \leq \gamma(R, n) \cdot \varepsilon^{\frac{1}{10n}}.$$

Therefore, we choose  $h_t = h_{Q_t}$  and  $\mu_t = V_{Q_t}$ .  $\square$

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