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# EDGE DISTURBANCES OF THE SHALLOW HYPERBOLIC PARABOLOIDAL SHELL BOUNDED BY FOUR GENERATRICES

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**The hyperbolic paraboloid shell bounded by four generatrices subjected at one of its edges to bending moments is investigated with the aid of the elastic bending theory of shallow shells. Inside the shell the decreasing curve of the bending moment is determined. The result is compared on the one hand with that of the Bleich-Salvadori solution, and on the other, with the decreasing curve of the bending moment of the plain plate.**

## 1. Introduction

The doubly curved shell structures bear the distributed loads by membrane forces, provided the supports are able to resist the membrane forces arising at the edges. Thus, the bending moments acting in these types of shells are not necessary for ensuring the equilibrium. They arise from the deformation incompatibilities occurring at the edges, and are rapidly dying out with an increasing distance from the edges.

This decrease of the edge disturbances has been investigated for barrel-vaults, elliptic paraboloid and saddle-shaped hyperbolic paraboloid shells (translational surfaces) [2], [5], [4]. However, the decrease of the edge disturbances of the hyperbolic paraboloid shells bounded by four generatrices is not yet clarified. As far as the author's knowledge goes, up to the present it was only DUDDECK [3] who treated a similar problem: he determined the stress pattern of a hyperbolic paraboloid simply supported at the straight edges and subjected to an uniformly distributed load. The edge disturbances themselves have been analysed by BLEICH and SALVADORI [1] on the basis of approximate assumptions. Their results will be compared with those of the present solution.

The aim of this paper is to establish the decreasing bending moment diagram of a hyperbolic paraboloid (hypar), loaded by bending moments along a straight edge, and to determine the distance beyond which it may practically be taken to be equal to zero. Namely, this type of load may be considered as a basic case, on the base of which one can form a notion of the decrease of other types of edge disturbances and can estimate the width of the strip to be reinforced.

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## 2. Notations

(See also Fig. 1)

| | |
|---|---|
| $x, y$ | orthogonal coordinetes; |
| $z\ (x, y)$ | ordinate of the shell surface; |
| $r, s, t$ | second order partial derivatives of the shell surface; |
| $n_x, n_y, n_{xy}$ | membrane forces; |
| $q_x, q_y$ | flexural shearing forces; |
| $p_z$ | load component parallel to $z$; |
| $m_x, m_y, m_{xy}$ | bending and twisting moments; |
| $u, v$ | displacements parallel to $x$ and $y$, respectively; |
| $w$ | displacement normal to the surface; |
| $E, G = E/[2(1+\nu)]$ | moduli of elasticity in tension and shear, respectively; |
| $\nu$ | Poisson's ratio (in the deductions: $\nu = 0$); |
| $h$ | thickness of the shell; |
| $a, b$ | side lengths of the plan projection of the hyperbolic paraboloid; |
| $f$ | height of the corner of the hypar above the $xy$ plane; |
| $\alpha^n = n\pi/a$ | $(n = 1, 2, 3, \ldots)$. |

Differentiation with respect to $x$ is denoted by prime, differentiation with respect to $y$ is denoted by dot.

## 3. Deduction of the governing equations

The investigations will be made on the base of the theory of shallow shells [4], [7], the calculations thus being simpler and the differential equations have constant coefficients.

The general equilibrium equations of the shallow shells consist of three projection and two moment equations [4]:

$$n_x' + n_{xy}^{\cdot} = 0\,, \tag{1a}$$

$$n_{xy}' + n_y^{\cdot} = 0\,, \tag{1b}$$

$$rn_x + 2\,sn_{xy} + tn_y + q_x' + q_y^{\cdot} + p_z = 0\,, \tag{1c}$$

$$m_x' + m_{xy}^{\cdot} - q_x = 0\,, \tag{1d}$$

$$m_{xy}' + m_y^{\cdot} - q_y = 0\,. \tag{1e}$$

(The third moment equation becomes meaningless for the case of shallow shells; the reason for it is explained in [4].)

In these equations it is implied that the load has only a vertical component, $p_z$, parallel to the $z$-axis.

The internal forces are connected to the displacement components by Hooke's law as follows:

$$n_x = Eh(u' - rw) \tag{2a}$$

$$n_y = Eh(v^{\cdot} + tw)\,, \tag{2b}$$

$$n_{xy} = \frac{Eh}{2}(u^{\cdot} + v' - 2sw)\,, \tag{2c}$$

$$m_x = -\frac{Eh^3}{12} w'' , \tag{2d}$$

$$m_y = -\frac{Eh^3}{12} w^{\cdot\cdot} , \tag{2e}$$

$$m_{xy} = -\frac{Eh^3}{12} w'^{\cdot} . \tag{2f}$$

Substituting the set of equations (2) into (1), and eliminating the shear forces $q_x, q_y$ from (1c) with the aid of (1d) and (1e), we arrive at three differential equations of the displacement components $u, v, w$:

$$2u'' + u^{\cdot\cdot} + v'^{\cdot} - 2rw' - 2sw^{\cdot} = 0 , \tag{3a}$$

$$u'^{\cdot} + v'' + 2v^{\cdot\cdot} - 2tw^{\cdot} - 2sw' = 0 , \tag{3b}$$

$$\begin{aligned} & ru' + s(u^{\cdot} + v') + tv^{\cdot} - (r^2 + 2s^2 + t^2)w - \\ & - \frac{h^2}{12}(w^{IV} + 2w''^{\cdot\cdot} + w^{::}) + p_z = 0 . \end{aligned} \tag{3c}$$

These three equations together with the boundary conditions fully determine the three displacement functions $u, v$ and $w$.

We can also arrive to a different set of equations equivalent to the group of Eq. (3) in such a way that we consider the internal forces $n_x$, $n_y$, $n_{xy}$ as second derivatives of a stress function $F(x, y)$:

$$n_x = F^{\cdot\cdot}, \tag{4a}$$

$$n_y = F'', \tag{4b}$$

$$n_{xy} = -F'^{\cdot}, \tag{4c}$$

whereby the equilibrium equations (1a, 1b) are automatically satisfied. There still remains Eq. (1c) which after substituting Eqs (1d, 1e) and (2d to 2f) contains only two unknown functions, $F(x, y)$ and $w(x, y)$:

$$\frac{Eh^3}{12}(w^{IV} + 2w''^{\cdot\cdot} + w^{::}) - (tF'' - 2sF'^{\cdot} + rF^{\cdot\cdot}) = p_z . \tag{5a}$$

However, we must still deduce from Eqs (2a to 2c) a *compatibility equation* starting out from the equality of the mixed second derivatives, expressing that the three displacements $(u, v, w)$ are continuous functions, that is, the shell surface gets neither torn, nor crumpled:

$$(F^{IV} + 2F''^{\cdot\cdot} + F^{::}) + Eh(tw'' - 2sw'^{\cdot} + rw^{\cdot\cdot}) = 0 . \tag{5b}$$

Thus we obtained for $F$ and $w$ two partial differential equations of the fourth order [4], [7].

Advantage of the set of Eqs (5a, 5b) is that it contains only two unknown functions. It has the disadvantage, however, that only the statical boundary conditions and the displacement boundary conditions in terms of $w$ may be easily applied in connection with them. If the displacement boundary conditions also include constraints in terms of the displacements $u$ or $v$, then it is commonly prefarable to use the set of Eqs (3a to 3c). In this paper it will also be started from the set of Eqs (3), and the three displacement functions $u$, $v$, $w$ should be determined.

In the following, the load component $p_z$ will be omitted from the equations because for the case of the edge disturbance at hand the shell is regarded as unloaded.

## 4. Application of the equations to the hypar shell

### *The boundary conditions to be considered*

The equation of the shell surface illustrated in Fig. 1 is:

$$z = \frac{f}{ab} xy. \tag{6}$$

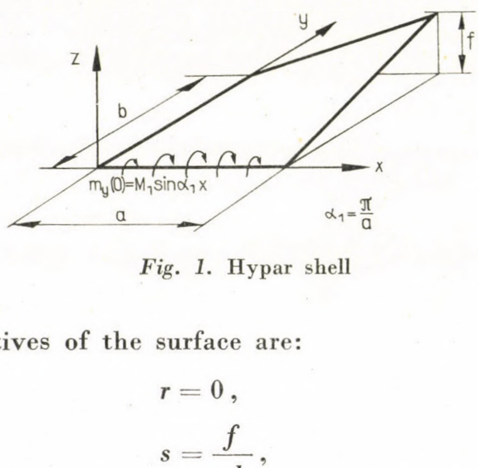


*Fig. 1.* Hypar shell

The second derivatives of the surface are:

$$r = 0, \tag{6a}$$

$$s = \frac{f}{ab}, \tag{6b}$$

$$t = 0 \tag{6c}$$

whereby the set of Eqs (3) will have the following simpler form:

$$2u'' + u^{\cdot\cdot} + v'^{\cdot} - \frac{2f}{ab} w^{\cdot} = 0, \tag{7a}$$

$$u^{\cdot\cdot}+v''+2\,v^{\cdot\cdot}-\frac{2f}{ab}w'=0\,, \tag{7b}$$

$$u^{\cdot}+v'-\frac{2f}{ab}w-\frac{h^2\,ab}{12f}(w^{\mathrm{IV}}+2\,w''^{\cdot\cdot}+w^{::})=0\,. \tag{7c}$$

We assume a simple support at every edge which allows free rotation and precludes lateral thrust. The edge $y = 0$ will be subjected to the effect of a bending moment with a law of distribution $m_y = M_1 \sin \pi x/a = M_1 \cdot \sin \alpha_1 x$. Therefore, deflection, bending moment normal to the edge, and lateral thrust must be equal to zero along all edges. This involves the following boundary conditions:

At the edge $y = 0$:

$$w = 0\,, \tag{8a}$$

$$w^{\cdot\cdot} = \frac{12}{Eh^3} M_1 \sin \frac{\pi}{a} x\,, \tag{8b}$$

$$n_y = 0\,,\ \text{that is,}\ v^{\cdot} = 0\,. \tag{8c}$$

At the edges $x = 0$ and $x = a$:

$$w = 0 \tag{8d}$$

$$m_x = 0,\ \text{that is,}\ w'' = 0\,, \tag{8e}$$

$$n_x = 0,\ \text{that is,}\ u' = 0\,. \tag{8f}$$

We do not consider the conditions of the edge opposite the external bending moment because we assume that the disturbance dies out before reaching the other edge. This assumption is subsequently justified by the moment diagrams obtained as solutions, see Fig. 2.

## 5. Solution of the set of equations

The equations will be solved with the aid of the usual method of the theory of edge disturbances: we establish product functions for $u$, $v$, $w$ in such a way that for the terms depending on $x$ we choose trigonometric functions which, after substituting them into Eqs (7), all yield sine or cosine. These trigonometric terms can thus be omitted, whereby we obtain for the terms depending on $y$ a common set of differential equations. The trigonometric functions depending on $x$ should be selected in such a way as to satisfy — as far as possible — the boundary conditions along the edges $x = 0$ and $x = a$.

In comparison with the bending theory of cylindrical shells [4] the calculation now becomes more intricate, for, contrary to the cylindrical shell, now we cannot find trigonometric functions which, besides the equilibrium equations, also automatically satisfy every boundary condition along $x = 0$ and $x = a$. Some of these boundary conditions can only be satisfied — approximately — by collocation (equalizing in selected points), the more correctly, the greater the number of terms (and edge points) considered.

The three product functions will be chosen as follows:

$$u = \sum_n u_n(y) \sin \alpha_n x \,, \tag{9a}$$

$$v = \sum_n v_n(y) \cos \alpha_n x \,, \tag{9b}$$

$$w = \sum_n w_n(y) \sin \alpha x \,. \tag{9c}$$

In these expressions

$$\alpha_n = n\pi/a \tag{10}$$

and in the Fourier series we consider as many terms as are needed for the degree of accuracy required.

Substituting the $n^{\text{th}}$ term of the expressions (9) into the set of Eqs (7), and simplifying with the trigonometric expressions, we arrive at the following three differential equations:

$$-2\alpha_n^2 u_n + \ddot{u}_n - \alpha_n \dot{v}_n - \frac{2f}{ab} \dot{w}_n = 0, \tag{11a}$$

$$\alpha_n \dot{u}_n - \alpha_n^2 v_n + 2\ddot{v}_n - \frac{2f}{ab} \alpha_n w_n = 0 \,, \tag{11b}$$

$$\dot{u}_n - \alpha_n v_n - \frac{2f}{ab} w_n - \frac{h^2 ab}{12f} (\alpha_n^4 w_n - 2\alpha_n^2 \ddot{w}_n + \ddot{\ddot{w}}_n) = 0 \,. \tag{11c}$$

The solution of this set of homogeneous, linear equations may be written as

$$u_n(y) = U_n e^{\beta_n y}, \tag{12a}$$

$$v_n(y) = V_n e^{\beta_n y}, \tag{12b}$$

$$w_n(y) = W_n e^{\beta_n y}, \tag{12c}$$

where $U_n$, $V_n$ and $W_n$ are, for the moment, unknown constants.

Placing the functions (12) into the group of equations (11) yields the following set of homogeneous, linear equations for the unknowns $U_n$, $V_n$ and $W_n$:

$$(-2\alpha_n^2 + \beta_n^2) U_n - \alpha_n \beta_n V_n - \frac{2f}{ab} \beta_n W_n = 0 \,, \tag{13a}$$

$$\alpha_n \beta_n U_n + (-\alpha_n^2 + 2\beta_n^2) V_n - \frac{2f}{ab} \alpha_n W_n = 0\,, \tag{13b}$$

$$\beta_n U_n - \alpha_n V_n - \left[\frac{2f}{ab} - \frac{h^2 ab}{12f} (\alpha_n^2 - \beta_n^2)^2\right] W_n = 0\,. \tag{13c}$$

This set of equations has a system of nontrivial solutions only in case when its determinant equals zero. Thus $\beta_n$, hitherto unknown, should be defined by this condition (from the so-called characteristic equation). This determinant represents an equation of the fourth order in respect to $\beta_n^2$, therefore, each $n$ is associated with four $\beta_{nj}^2$ values ($j = 1, 2, 3, 4$). From these we only consider those four $\beta_{nj}$-s which give decreasing curves with an increasing distance from the edge $y = 0$, that is only those, the real part of which are *negative*. Two of these four roots ($\beta_{n1}$ and $\beta_{n2}$) are real, and two of them are conjugated complex values:

$$\beta_{n3} = \gamma_n + i \cdot \delta_n \tag{14a}$$

$$\beta_{n4} = \gamma_n - i \cdot \delta_n\,. \tag{14b}$$

We take over the values of $\beta_{n1}$, $\beta_{n2}$, $\gamma_n$ and $\delta_n$ referring to a hyperbolic paraboloid of a *square base* ($a = b$) and to two characteristic ratios of $f/h$ from [3]. In Table I the values of these four quantities, multiplied by the side-length $a$ (thus becoming nondimensional) are given for four or five $n$ terms, respectively.

**Table I**

$f/h = 12{,}5$

| $n =$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $a \cdot \beta_{n1} =$ | —7,400 | —13,99 | —20,30 | —26,62 |
| $a \cdot \beta_{n2} =$ | —0,349 | — 5,00 | —11,12 | —17,35 |
| $a \cdot \gamma_n =$ | —3,875 | — 9,50 | —15,71 | —21,99 |
| $a \cdot \delta_n =$ | +4,765 | + 4,76 | + 4,69 | + 4,63 |

$f/h = 50$

| $n$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $a \cdot \beta_{n1} =$ | —10,90 | —18,23 | —24,77 | —31,17 | —37,50 |
| $a \cdot \beta_{n2} =$ | — 0,0892 | — 2,165 | — 7,09 | —12,98 | —19,15 |
| $a \cdot \gamma_n =$ | — 5,495 | —10,20 | —15,93 | —22,08 | —28,32 |
| $a \cdot \delta_n =$ | + 8,36 | + 9,78 | + 9,61 | + 9,51 | + 9,45 |

It is worth while to observe that in case of $n = 1$, $\beta_{n2} < 1$ which means that the stresses diminish but slowly with the increasing distance from the edge, and the more slowly, the greater the ratio $f/h$ is, that is, the greater is the membrane stiffness in comparison to its bending rigidity. (Had the bending rigidity been equal to zero, so it would also be $\beta_{n2}$, and we should arrive at the case of the membrane forces propagating undiminished along the straight generatrices.)

As a matter of course, every $\beta_{nj}$ is associated with different $U_{nj}$, $V_{nj}$ and $W_{nj}$. Thus, because of $j = 1,2,3,4$, we shall have $4 \cdot 3n = 12n$ unknown constants if we consider $n$ number of terms in the Fourier series (9) in the $x$-direction.

The displacement functions (9a to 9c) obtain then the following forms:

$$u = \sum_n \left( \sum_{j=1}^{4} U_{nj}\, e^{\beta_{nj} y} \right) \sin \alpha_n x, \tag{15a}$$

$$v = \sum_n \left( \sum_{j=1}^{4} V_{nj}\, e^{\beta_{nj} y} \right) \cos \alpha_n x, \tag{15b}$$

$$w = \sum_n \left( \sum_{j=1}^{4} W_{nj}\, e^{\beta_{nj} y} \right) \sin \alpha_n x . \tag{15c}$$

The unknown values, $12n$ in number, are related to each other by the equilibrium Eqs (13) in such a way that from every three values $U_{ni}$, $V_{ni}$ and $W_{ni}$ only one remains free. Therefore, only $4n$ of the boundary conditions may be satisfied.

From the boundary conditions (8a to 8f) given in section 4 the assumed functions (9) automatically satisfy the conditions (8d) and (8e). Therefore, the remaining four boundary conditions are just enough for the determination of the four unknowns associated with every $n$.

Had the boundary condition (8f) not existed, we could satisfy the boundary conditions separately for every $n$, that is, the distribution of the external moment (8b) considered (Fig. 1) would only require the consideration of the term $n = 1$. The boundary condition (8f), however, can only "forcedly", in separate points, be imposed upon the $u_n$-functions having the form of (9a), and only in case if we consider several $n$ terms. This is why four or five $n$ terms should be considered, though the higher $n$ terms yield bending moments which are dying out more rapidly.

In the following, the functions (12) will be rewritten in pure real forms wherein the $U_{n3}$, $U_{n4}$; $V_{n3}$, $V_{n4}$ and $W_{n3}$, $W_{n4}$ are conjugated complex constant values. On the basis of the known complex relation

$$e^{iz} = \cos z + i \sin z \tag{16}$$

$(i = \sqrt{-1})$ we can, for example, write down that

$$\begin{aligned} & U_{n3} \cdot e^{\beta_{n3} y} + U_{n4} e^{\beta_{n4} y} = \\ & = e^{\gamma_n y} [(U_{n3} + U_{n4}) \cos \delta_n y + (U_{n3} - U_{n4}) i \sin \delta_n y] = \\ & = U_{n5} e^{\gamma_n y} \cos \delta_n y + U_{n6} e^{\gamma_n y} \sin \delta_n y . \end{aligned} \tag{17}$$

Therefore, the new real constants are:

$$U_{n5} = U_{n3} + U_{n4} , \tag{18a}$$

$$U_{n6} = i(U_{n3} - U_{n4}) \tag{18b}$$

and in the same way

$$V_{n5} = V_{n3} + V_{n4} , \tag{18c}$$

$$V_{n6} = i(V_{n3} - V_{n4}) , \tag{18d}$$

$$W_{n5} = W_{n3} + W_{n4} , \tag{18e}$$

$$W_{n6} = i(W_{n3} - W_{n4}) . \tag{18f}$$

From the set of the homogeneous equilibrium equations (13a to 13c) we use the first two equations because they have a simpler form. For the cases $j = 3$ and $j = 4$ also these have to be rewritten into real form. For example for $j = 3$, substituting the new real constants by making use of (18a to 18f) and writing down also the root $\beta_{n3}$ according to (14a), Eq. (13) takes the form

$$\begin{aligned} & (-2\alpha_n^2 + \gamma_n^2 + 2i\gamma_n \delta_n - \delta_n^2) \frac{U_{n5} - iU_{n6}}{2} - \\ & - \alpha_n(\gamma_n + i\delta_n) \frac{V_{n5} - iV_{n6}}{2} - \\ & - \frac{2f}{ab} (\gamma_n + i\delta_n) \frac{W_{n5} - iW_{n6}}{2} = 0 . \end{aligned} \tag{19a}$$

The equations associated with $j = 4$ may be rewritten in the same way, and once adding and once subtracting them, we arrive at two real equations.

Finally we shall have four boundary conditions:

(8a):
$$W_{n1} + W_{n2} + W_{n3} + W_{n4} = 0 ,$$

that is, in a real form, making use of (18e):

$$W_{n1} + W_{n2} + W_{n5} = 0 . \tag{20a}$$

(8b):

$$\text{if } n = 1: \ \beta_{11}^2 W_{11} + \beta_{12}^2 W_{22} + \beta_{13}^2 W_{13} + \beta_{14}^2 W_{14} = \frac{12}{Eh^2} M_1 .$$

That is, in a real form:

$$\beta_{11}^2 W_{11}+\beta_{12}^2 W_{12}+(\gamma_1^2+2\,i\gamma_1\,\delta_1-\delta_1^2)\,\frac{W_{15}-W_{16}}{2}+ \\ +(\gamma_1^2-2\,i\gamma_1\,\delta_1-\delta_1^2)\,\frac{W_{15}+iW_{16}}{2}=\frac{12}{Eh^3}\,M_1\,. \tag{20b}$$

It is to be seen that the imaginary terms have dropped out.

For the case $n<1$, $\gamma_n$ and $\delta_n$ should be written instead of $\gamma_1$ and $\delta_1$, and zero at the right-hand side.

(8c):

$$\beta_{n1}V_{n1}+\beta_{n2}V_{n2}+\beta_{n3}V_{n3}+\beta_{n4}V_{n4}=0$$

and rewritten to the real form:

$$\beta_{n1}V_{n1}+\beta_{n2}V_{n2}+(\gamma_n+i\delta_n)\,\frac{V_{n5}-iV_{n6}}{2}+(\gamma_n-i\delta_n)\,\frac{V_{n5}+iV_{n6}}{2}=0\,. \tag{20c}$$

The boundary condition (8f) cannot be written down for the whole edge line, only to $n$ points of it, that is, to the points

$$y_k=(k-1)\,\frac{b}{n} \tag{21}$$

where

$$k=1,2,\ldots n\,. \tag{22}$$

Thus the equation

$$\sum_{n=1}^{n}\left(\sum_{j=1}^{4}U_{nj}\;e^{\beta_{nj}y_k}\right)\alpha_n=0 \tag{22a}$$

must be valid for every $k$. This, expanded and rewritten in a real form, becomes for every $k$:

$$\sum_{n=1}^{n}\Bigg\{\Bigg[U_{n1}\cdot e^{\beta_{n1}y_k}+U_{n2}\cdot e^{\beta_{n2}y_k}+ \\ +\frac{U_{n5}-iU_{n6}}{2}\,e^{\gamma_n y_k}(\cos\delta_n y_k+i\sin\delta_n y_k)+ \\ +\frac{U_{n5}+iU_{n6}}{2}\,e^{\gamma_n y_k}(\cos\delta_n y_k-i\sin\delta_n y_k)\Bigg]\alpha_n\Bigg\}=0 \tag{23}$$

from which the imaginary terms also fall out.

On the basis of what has been said above, the equilibrium Eqs (13a, 13b) must be rewritten for all of the $n$-s and $j$-s in the real form corresponding to (19a), as well as the boundary conditions (20a to 20c) for all of the $n$-s, and finally the boundary condition (23) for all of the $k$-s. Thus we obtain exactly the $12n$ equations for the determination of the $12n$ unknown constants.

## 6. Numerical results

We can solve the set of equations obtained in the foregoing section by making use of the $\beta$-roots given to the hyperbolic paraboloid of a *square base* ($a = b$) by DUDDECK [3], reproduced in Table I for two values of the ratio $f/h$ by considering four or five $n$ terms, respectively. This leads to a set of

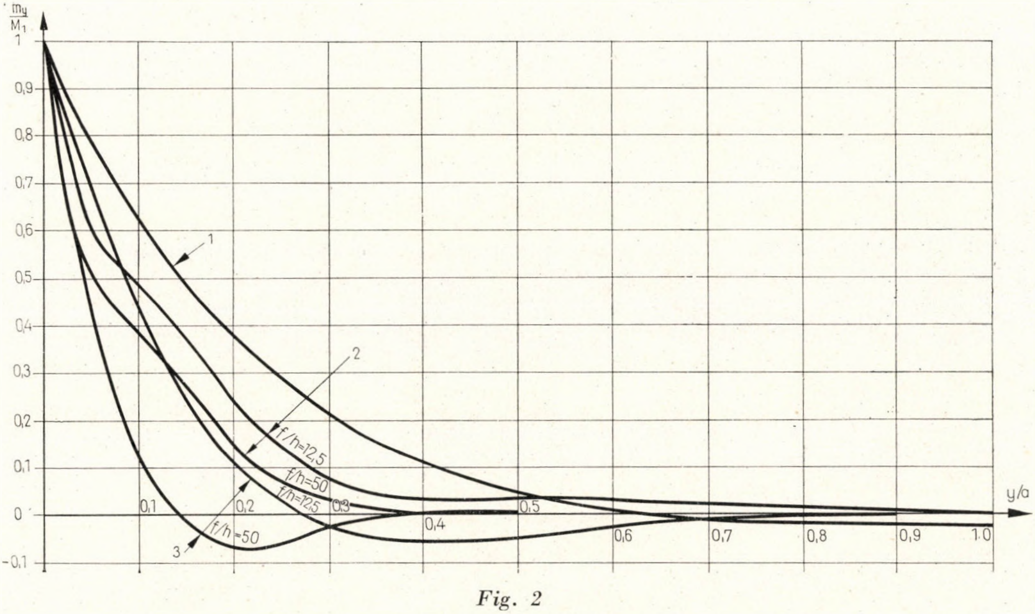


*Fig. 2*

equations with 48 or 60 unknowns, respectively. The numerical calculations were carried out at the Centre of Computation of the Ministry of Heavy Industry with the aid of the computer National Elliot 803 B.

In the equations, besides the ratio $f/h$, also the ratio $f/a$ is included. But carrying out the computation with several values of $f/a$, it was found that every $f/a$-ratio associated with the same value of $f/h$, resulted in the very same moment diagram.

Since we are interested first of all in the decrease of the edge moment inside the shell, we only plotted the diagram of the bending moment $m_y$

versus $y$, along the section $x = a/2$ (Fig. 2). The formula of this moment is

$$m_y = E\frac{h^3}{12} w^{\cdot\cdot} = E\frac{h^3}{12}\sum_n \{\beta_{n1}^2 W_{n1} \cdot e^{\beta_{n1}y} + \beta_{n2}^2 W_{n2}\, e^{\beta_{n2}y} +$$
$$+ e^{\gamma_n y}[(\gamma_n^2 - \delta_n^2) W_{n5} + 2\gamma_n \delta_n W_{n6}] \cos\delta_n y + \tag{24}$$
$$+ e^{\gamma_n y}[(\gamma_n^2 - \delta_n^2) W_{n6} - 2\gamma_n \delta_n W_{n5}] \sin\delta_n y\} \sin\alpha_n x\,.$$

## 7. Approximate calculation of the edge disturbances

According to the suggestion of [1], by neglecting the displacement components in the plane of the shell ($u = v = 0$), we can obtain a simple *approximate solution* for the calculation of the edge disturbances.

Our presented solution permits to check the accuracy of this approximate method. From the equations of equilibrium (7a to 7c) we omit the first two equations retaining only the third one, leaving out the terms containing $u$ and $v$. Thereby Eq. (7c) becomes

$$24\left(\frac{f}{ab}\right)^2 \frac{1}{h^2} w + w^{\mathrm{IV}} + 2\, w''^{\cdot\cdot} + w^{::} = 0\,. \tag{25}$$

We take again for $w$ a series of product functions of the form

$$w = \sum_n W_n\, e^{\beta_n y} \sin\alpha_n x\,. \tag{26}$$

After substituting it into Eq. (25) we obtain for $\beta_n$ the following characteristic equation

$$\varepsilon^4 + (\alpha_n^2 - \beta_n^2)^2 = 0\,. \tag{27}$$

Here we will introduce the abbreviation:

$$\varepsilon^4 = 24\left(\frac{f}{ab}\right)^2 \frac{1}{h^2}\,. \tag{28}$$

From Eq. (27) we obtain four $\beta_{nj}$-s ($j = 1, 2, 3, 4$):

$$\beta_{nj} = \alpha_n\,[\pm(A + iB) \pm (A - iB)] \tag{29}$$

where

$$A = +\sqrt{0.5\left(\sqrt{1 + \frac{\varepsilon^4}{\alpha_n^4}} + 1\right)} \tag{30a}$$

and

$$B = +\sqrt{0.5\left(\sqrt{1 + \frac{\varepsilon^4}{\alpha_n^4}} - 1\right)}\,. \tag{30b}$$

From the four $\beta_{nj}$-s we only retain the two roots which give a decreasing bending moment with increasing $y$, and rewrite $w$ to the real form

$$w = \sum_n (W_{n1} e^{-(A+iB)\alpha_n y} + W_{n2} e^{-(A-iB)\alpha_n y}) \sin \alpha_n x = \\ = \sum_n e^{-A\alpha_n y} (W_{n3} \cos B\alpha_n y + W_{n4} \sin B\alpha_n y) \sin \alpha_n x . \tag{31}$$

The explanation of the new constants $W_{n3}$ and $W_{n4}$ is

$$W_{n3} = W_{n1} + W_{n2} , \tag{32a}$$

$$W_{n4} = i(W_{n1} - W_{n2}) \tag{32b}$$

and $W_{n1}$ and $W_{n2}$ are the complex constants associated with $\beta_{n1}$ and $\beta_{n2}$, respectively.

To the two constants ($W_{n3}$, $W_{n4}$) two equations of boundary conditions are needed. As a matter of course, they can now be expressed only by $w$. Thus, from the boundary conditions (8) used earlier, the following should be satisfied (Fig. 1):

$$y = 0 ; \; w = 0 , \tag{33a}$$

$$w^{\cdot\cdot} = \frac{M_1 \sin \alpha_1 x}{E(h^3/12)} . \tag{33b}$$

Since we do not need to take into account the boundary conditions $x = 0$ and $x = a$, it is sufficient to consider from the series of $w$ (31) merely the term $n = 1$, corresponding to the external moment. Therefore, from the boundary condition (33a) we obtain

$$W_{13} = 0. \tag{34}$$

To the boundary condition (33b) the second derivative of $w$ with respect to $y$ is needed:

$$w^{\cdot\cdot} = W_{14} e^{-A\alpha_1 y} [(A^2 - B^2) \alpha_1^2 \sin B\alpha_1 y - 2 AB\alpha_1^2 \cos B\alpha_1 y] \sin \alpha_1 x . \tag{35}$$

Substituting this into (33b) yields

$$W_{14} = - \frac{M_1}{2 AB\alpha_1^2 E(h^3/12)} . \tag{36}$$

Now we can write down the expression of the bending moment in the $y$-direction which is the most significant from our viewpoint

$$m_y = E\frac{h^3}{12}w^{\cdot\cdot} = M_1 \sin\alpha_1 x\left[\left(-\frac{A}{2B}+\frac{B}{2A}\right)\sin B\alpha_1 y+\cos B\alpha_1 y\right]e^{-A\alpha_1 y}. \quad (37)$$

The variation of $m_y$ versus $y$ is also represented in Fig. 2.

## 8. Decrease of the edge moment of the plain plate

For comparison, the decrease of the edge moment in a plate, freely supported at three edges and infinitely long in the fourth direction (Fig. 3)

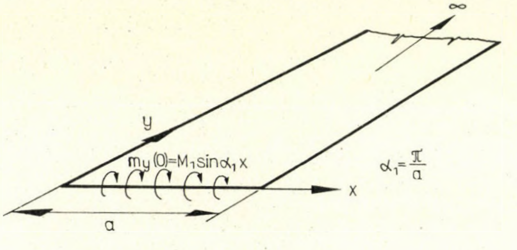


*Fig. 3.* Plain plate

should be established on the basis of [6]. Omitting the deduction, here only the expression of the $m_y$ should be given:

$$m_y = M_1 \sin\alpha_1 x\left(\frac{\alpha_1}{2}y-1\right)e^{-\alpha_1 y}. \quad (38)$$

The corresponding moment diagram is also represented in Fig. 2.

## 9. Comparison and evaluation

Analysing Fig. 2, the following could be established.

The diagrams clearly show that the higher the corner point of the shell ($f$) lies in comparison to its thickness $h$, the more rapidly the edge moment diminishes. The extreme case, $f = 0$, is realized by the plain plate. However, as was mentioned above, the ratio $f/a$ is indifferent, therefore, in a steeper and thicker shell the moment can diminish in the same way as in a flatter and thinner one, provided their $f/h$-ratio should be the same.

From Fig. 2 we can establish by interpolation, practically for all of the hypars of any geometric ratio, the width of the zone of the edge disturbance, that is, the width to be reinforced.

The moment diagrams calculated both with $f/h = 12{,}5$ and $f/h = 50$ follow a somewhat irregular trace. This could be corrected by considering some more terms in the series of the deformation functions. In the author's opinion, however, we should not profit at all thereby, because the rate of the diminution of the moment would remain the same, even in case of a greater number of terms considered and, in turn, the amount of the computation work would considerably increase.

The moment diagram gives negligibly small moment values even in case of plates in a distance equal to $a$. Thus the assumption is justified that the effect of the edge opposite to the disturbance on the internal forces might be neglected in a shell of a square base.

Finally, it is clearly to be seen that the moment diagram obtainable by the approximate assumption $u = v = 0$ shows a much more rapid decrease than the more exact computation. The reason for this might be that the assumption $u = v = 0$ represents a much stronger constraint on the shell than the real boundary conditions (especially $n_x = 0$, being valid for the edges $x = 0$ and $x = a$). As a result, the shell will be, so to say, more rigid, and we obtain an unreal rapid decrease of the moment. Therefore, the application of the above approximation cannot be recommended.

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**Randstörungen der windschiefviereckförmigen flachen hyperbolischen Paraboloidschale.** Es wird die an einem Rand durch Biegemoment beanspruchte, windschiefviereckförmige hyperbolische Paraboloidschale aufgrund der elastischen Biegetheorie der flachen Schalen untersucht, und die Abklingungskurve des den Rand belastenden Moments ermittelt. Das Ergebnis wird einerseits der Bleich-Salvadorischen Näherungslösung, andererseits der Momentenabklingungskurve der ebenen Platte gegenübergestellt.

**Краевые возмущения плоской гиперболической параболоидной оболочки в виде искаженного четырехугольника** *(Л. Коллар, М. Сетч).* В работе исследуется нагруженная на одном краю изгибающим моментом гиперболическая параболоидная оболочка в виде искаженного четырехугольника на основе теории упругого изгиба плоских оболочек. Определяется кривая затухания момента, нагружающего край, внутри самой оболочки. Полученный результат сравнивается, с одной стороны, с приближенным решением Блейха — Сальвадора, а с другой стороны, с кривой затухания момента плоской пластины.