

ON HOPF MONOIDS IN DUOIDAL CATEGORIES

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ABSTRACT. Aguiar and Mahajan’s bimonoids A in a duoidal category \mathcal{M} are studied. Under certain assumptions on \mathcal{M} , the Fundamental Theorem of Hopf Modules is shown to hold for A if and only if the unit of A determines an A -Galois extension. Our findings are applied to the particular examples of small groupoids and of Hopf algebroids over a commutative base algebra.

INTRODUCTION

There are several equivalent conditions on a bialgebra A (say, over a commutative ring k) under which we say that it is a Hopf algebra (see e.g. the textbooks [16, 23, 9]):

- (i) The identity map $A \rightarrow A$ has an inverse — called the antipode — in the convolution algebra $\text{End}(A)$.
- (ii) A induces a right Hopf monad $(-)\otimes A$ on the category \mathcal{M}_k of k -modules; in the sense of [8]. That is, the closed structure of \mathcal{M}_k is lifted to the category of right A -modules.
- (iii) A is an A -Galois extension of k . That is, a canonical comonad morphism is an isomorphism.
- (iv) The Fundamental Theorem of Hopf Modules [12] holds. That is, the category of A -Hopf modules is equivalent to the category of k -modules.

In their monograph [2], Marcelo Aguiar and Swapneel Mahajan generalized bialgebras to bimonoids in so-called duoidal categories (termed “2-monoidal” in their work). These are categories equipped with two different monoidal structures. They are required to be compatible in the sense that the functors and natural transformations defining the first monoidal structure, are opmonoidal with respect to the second monoidal structure. Equivalently, the functors and natural transformations defining the second monoidal structure, are monoidal with respect to the first monoidal structure. More details will be recalled in Section 1.2. A bimonoid is a monoid with respect to the first monoidal structure and a comonoid with respect to the second monoidal structure. The compatibility axioms are formulated in terms of the coherence morphisms between the monoidal structures.

A natural question arises how to define a *Hopf* monoid in a duoidal category. There are at least four possibilities listed above.

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The first possibility (i) does not seem applicable. Since the monoid and comonoid structures are defined in different monoidal categories, the notion of convolution monoid is not available. The second possibility (ii) was taken in [5] and it was investigated in relation with the lifting of closed structures. The aim of this paper is to study the remaining possibilities (iii) and (iv). Under certain assumptions on the duoidal category we work in, we prove their equivalence; and that they hold whenever (ii) does.

Let us re-visit for a moment the classical case of a bialgebra A over a commutative ring k . Since a bialgebra carries both an algebra and a coalgebra structure, it induces both a monad and a comonad on the category \mathcal{M}_k of k -modules (with respective Eilenberg-Moore categories \mathcal{M}_A , the category of A -modules; and \mathcal{M}^A , the category of A -comodules). These are in turn related by a mixed distributive law in the sense of [4]. The category of its mixed modules \mathcal{M}_A^A is known as the category of A -Hopf modules. Moreover, there is an associated triangle of functors

$$\begin{array}{ccc}
 & & \mathcal{M}_A^A \\
 & \nearrow^{K=(-)\otimes A} & \uparrow \\
 & & U^A \downarrow \dashv \uparrow F^A=(-)\otimes A \\
 \mathcal{M}_k & \xrightarrow{F_A=(-)\otimes A} & \mathcal{M}_A \\
 & \xleftarrow{U_A} &
 \end{array}$$

in which $U^A K = F_A$ (and U^A and U_A denote forgetful functors).

The Dubuc-Beck theory [4, 10, 18, 3] (shortly recalled in Section 1.1) tells us the sufficient and necessary conditions under which K is an equivalence — that is, the Fundamental Theorem of Hopf Modules holds. The first step is to see that K possesses a right adjoint N . By Dubuc's Adjoint Lifting Theorem [10] (see the dual form of [18, Theorem 2.1]), N is given by certain equalizers which exist by the completeness of \mathcal{M}_k . Then there is a canonical comonad morphism

$$\beta := (F_A U_A \cong U^A K N F^A \xrightarrow{U^A \hat{\epsilon} F^A} U^A F^A),$$

where $\hat{\epsilon}$ denotes the counit of the adjunction $K \dashv N$. With its help, K is an equivalence if and only if β is a natural isomorphism, and F_A preserves the equalizers defining the right adjoint of K and it reflects isomorphisms (see e.g. [11, 13]).

By the above reasoning it is immediate that if K is an equivalence — i.e. above condition (iv) defining a Hopf algebra holds — then β is a natural isomorphism. That is, above condition (iii) defining a Hopf algebra A holds. Conversely, assume that β is a natural isomorphism — i.e. above condition (iii) defining a Hopf algebra holds. Then in this particular situation associated to a k -bialgebra, the equalizers defining the right adjoint of K turn out to be contractible, hence absolute equalizers (with contracting maps constructed in terms of β^{-1}). Such equalizers are preserved by any functor, in particular by F_A . This allows to prove also that F_A reflects isomorphisms, hence K is an equivalence. That is, above condition (iv) defining a Hopf algebra A holds. (Note that since A is a generator in \mathcal{M}_A , $\beta_M \equiv M \otimes_A \beta_A$ is an isomorphism for all right A -modules M if and only if β_A is an isomorphism. The equivalence of this form of condition (iii) and condition (iv) above, can be found e.g. in [9, Section 15.5 (b) \Leftrightarrow (g)].)

In this paper we apply a similar strategy in the case of a bimonoid A in a duoidal category \mathcal{M} . In Section 1 we recall the Dubuc-Beck theory in a nutshell (but with precise references) and some basic facts about duoidal categories and their bimonoids. In Section 2 we study the analogue of the above adjoint triangle and the corresponding comonad morphism β in a somewhat more general setting: We take an A -comodule monoid B and we consider the category of (A, B) -relative Hopf modules. The isomorphism property of the corresponding comonad morphism β defines the notion of A -Galois extension $C \rightarrow B$. In Section 3 we restrict to (non-relative) A -Hopf modules. We assume that idempotent morphisms in \mathcal{M} split and that a canonical functor H — between the category \mathcal{M}^I of comodules over one monoidal unit I , and the category \mathcal{M}_J of modules over the other monoidal unit J in \mathcal{M} — is fully faithful. (In the case when \mathcal{M} is a braided monoidal category, H is a trivial isomorphism.) Under these assumptions we prove the Fundamental Theorem of Hopf Modules which takes now the following form: The canonical comparison functor from \mathcal{M}^I to the category of A -Hopf modules is an equivalence if and only if the canonical comonad morphism β is a natural isomorphism, and if and only if $I \rightarrow A$ is an A -Galois extension. The assumptions made on a duoidal category in our Fundamental Theorem of Hopf Modules, respectively, their dual counterparts, are shown to hold in two duoidal categories described in [2]: In the category of spans (over a fixed base set) and in the category of bimodules (over a fixed commutative, associative and unital algebra). So as an application of the theorem, we obtain that a small category A is a Galois extension of its object set X if and only if the category of A -Hopf modules is equivalent to the slice category \mathbf{set}/X ; and if and only if A is a groupoid. Similarly, if A is a bialgebroid (called a “ \times_R -bialgebra” in [24]) — over a commutative algebra R and such that the unit $R \otimes R \rightarrow A$ takes its values in the center of A —, then the category of A -Hopf modules is equivalent to the category of R -modules if and only if A is a Hopf algebroid (in the sense of [7] and references therein; for the commutative case see also [20]).

1. PRELIMINARIES

1.1. The Dubuc-Beck theory. For later application, in this section we consider the following situation. Let T and S be comonads on respective categories \mathcal{A} and \mathcal{B} . Denote their Eilenberg-Moore categories of comodules (also called coalgebras, see e.g. the dual of the notion in [3, page 88]) by \mathcal{A}^T and \mathcal{B}^S , respectively, with forgetful functors $U^T : \mathcal{A}^T \rightarrow \mathcal{A}$ and $U^S : \mathcal{B}^S \rightarrow \mathcal{B}$. (These functors possess right adjoints, to be denoted by F^T and F^S , respectively; such that $T = U^T F^T$ and $S = U^S F^S$). Assume that there is an adjunction $L \dashv R : \mathcal{B} \rightarrow \mathcal{A}$ (with unit $\nu : \mathcal{A} \rightarrow RL$ and counit $\epsilon : LR \rightarrow \mathcal{B}$). In what follows, we re-collect from the literature some results concerning sufficient and necessary conditions for the existence of an *equivalence* $K : \mathcal{A}^T \rightarrow \mathcal{B}^S$ rendering commutative

$$(1.1) \quad \begin{array}{ccc} \mathcal{A}^T & \xrightarrow{K} & \mathcal{B}^S \\ U^T \downarrow & & \downarrow U^S \\ \mathcal{A} & \xrightarrow{L} & \mathcal{B}. \end{array}$$

Whenever the diagram (1.1) commutes for some functor K , K is said to be a *lifting* of L . By [19, Corollary 5.11] (or by [11, Proposition 1.1 and Theorem 1.2]), the liftings K of L are in a

bijjective correspondence with the so-called *comonad morphisms* of the form (L, λ) (this result is usually attributed to D. Applegate's thesis from 1965). The comonad morphisms [22] from T to S are pairs consisting of a functor $L : \mathcal{A} \rightarrow \mathcal{B}$ and a natural transformation $\lambda : LT \rightarrow SL$ which is compatible with the comonad structures $(\delta^T : T \rightarrow T^2, \varepsilon^T : T \rightarrow \mathcal{A})$ and $(\delta^S : S \rightarrow S^2, \varepsilon^S : S \rightarrow \mathcal{B})$ in the sense of the commutative diagrams

$$\begin{array}{ccc} LT & \xrightarrow{\lambda} & SL \\ L\delta^T \downarrow & & \downarrow \delta^S L \\ LT^2 & \xrightarrow{\lambda_T} SLT & \xrightarrow{S\lambda} S^2L \end{array} \qquad \begin{array}{ccc} LT & \xrightarrow{\lambda} & SL \\ L\varepsilon^T \downarrow & & \downarrow \varepsilon^S L \\ L & \xlongequal{\quad} & L. \end{array}$$

Let us assume that there exists such a comonad morphism $\lambda : LT \rightarrow SL$ hence a lifting K of L . Then there is also an induced comonad morphism from the comonad LTR (with multiplication $LTR \xrightarrow{L\delta^T R} LT^2R \xrightarrow{LT\nu^T R} LTRLTR$ and counit $LTR \xrightarrow{L\varepsilon^T R} LR \xrightarrow{\varepsilon} \mathcal{B}$) to S , see [10, Theorem II.1.1] (and also [11, Theorem 1.2]). It is given by the identity functor \mathcal{B} and the natural transformation

$$(1.2) \quad \beta = (LTR \xrightarrow{\lambda R} SLR \xrightarrow{S\varepsilon} S).$$

Alternatively, using the correspondence between λ and K , it can be re-written as

$$(1.3) \quad \beta = (LTR = U^S K F^T R \xrightarrow{U^S \nu^S K F^T R} U^S F^S U^S K F^T R = SLTR \xrightarrow{SL\varepsilon^T R} SLR \xrightarrow{S\varepsilon} S),$$

where $\nu^S : \mathcal{B}^S \rightarrow F^S U^S$ is the unit of the adjunction $U^S \dashv F^S$.

Since at the end we want K in (1.1) to be an equivalence, it should possess in particular a right adjoint. For the existence of this right adjoint, we obtain sufficient and necessary conditions from Dubuc's Adjoint Lifting Theorem. We apply it in the form which is dual to [18, Theorem 2.1]. Re-draw (1.1) in the form

$$(1.4) \quad \begin{array}{ccc} & & \mathcal{B}^S \\ & \nearrow K & \downarrow U^S \\ \mathcal{A}^T & \xrightarrow{U^T} \mathcal{A} & \xrightarrow{L} \mathcal{B}. \end{array}$$

In the triangular diagram (1.4), both the functor in the bottom row and that in the right vertical are left adjoints. Moreover, U^S is comonadic so in particular of the co-descent type. Hence by the dual form of [18, Theorem 2.1] (see also [10, Theorem A.1] and [11, Proposition 1.3]), K possesses a right adjoint N if and only if for every S -comodule $(B, \rho : B \rightarrow SB)$, there exists the equalizer

$$N(B, \rho) \longrightarrow TRB \xrightleftharpoons[TRS\varepsilon_B \cdot TR\lambda RB \cdot T\nu^T RB \cdot \delta^T RB]{TR\rho} TRSB \quad \text{in } \mathcal{A}^T,$$

providing the object map of N . By the uniqueness of the adjoint up-to natural isomorphism, whenever the right adjoint N of K exists it obeys $NF^S \cong F^T R$, and the counit $\hat{\varepsilon} : KN \rightarrow \mathcal{B}^S$ of

the adjunction $K \dashv N$ renders commutative

$$\begin{array}{ccccc} LTR & \xrightarrow{L\epsilon^T R} & LR & \xrightarrow{\epsilon} & \mathcal{B} \\ \cong \downarrow & & & & \parallel \\ U^S K N F^S & \xrightarrow{U^S \epsilon F^S} & S & \xrightarrow{\epsilon^S} & \mathcal{B}. \end{array}$$

Using this identity together with (1.3) (and with one of the triangle identities on the adjunction $U^S \dashv F^S$), we can re-write (1.2) in the alternative form

$$(1.5) \quad \beta = (LTR \xrightarrow{\cong} U^S K N F^S \xrightarrow{U^S \epsilon F^S} S)$$

whenever the right adjoint N of K exists.

Finally, for any lifting K of L in (1.1), the following assertions are equivalent (see e.g. [11, Theorem 1.7] or [13, Theorem 4.4]).

- The functor K is an equivalence.
- The natural transformation (1.2) is an isomorphism and $\mathcal{A}^T \xrightarrow{U^T} \mathcal{A} \xrightarrow{L} \mathcal{B}$ is comonadic.

Recall that by the dual form of Beck's monadicity theorem (see e.g. [3, page 100, Theorem 3.14]), a left adjoint functor L is comonadic (or co-tripleable) if and only if it reflects isomorphisms and creates the equalizers of L -contractible equalizer pairs.

1.2. Duoidal category. In this section we recall from [2] some information about so-called duoidal (also known as 2-monoidal) categories. These are categories equipped with two related monoidal structures. Bimonoids in duoidal categories and their induced bimonads are also recalled.

Definition 1.1. [2, Definition 6.1] A *duoidal category* is a category \mathcal{M} equipped with two monoidal products \circ and \bullet with respective units I and J , along with morphisms

$$I \xrightarrow{\delta} I \bullet I, \quad J \circ J \xrightarrow{\varpi} J, \quad I \xrightarrow{\tau} J,$$

and, for all objects A, B, C, D in \mathcal{M} , a morphism

$$\zeta_{A,B,C,D} : (A \bullet B) \circ (C \bullet D) \rightarrow (A \circ C) \bullet (B \circ D),$$

called the *interchange law*, which is natural in all of the four occurring objects. These morphisms are required to obey the axioms below.

Compatibility of the units. The monoidal units I and J are compatible in the sense that

$$(J, \varpi, \tau) \text{ is a monoid in } (\mathcal{M}, \circ, I) \text{ and } (I, \delta, \tau) \text{ is a comonoid in } (\mathcal{M}, \bullet, J).$$

Associativity. For all objects A, B, C, D, E, F in \mathcal{M} , the following diagrams commute.

$$(1.6) \quad \begin{array}{ccc} ((A \bullet B) \circ (C \bullet D)) \circ (E \bullet F) & \xrightarrow{\cong} & (A \bullet B) \circ ((C \bullet D) \circ (E \bullet F)) \\ \zeta \circ \text{id} \downarrow & & \downarrow \text{id} \circ \zeta \\ ((A \circ C) \bullet (B \circ D)) \circ (E \bullet F) & & (A \bullet B) \circ ((C \circ E) \bullet (D \circ F)) \\ \zeta \downarrow & & \downarrow \zeta \\ ((A \circ C) \circ E) \bullet ((B \circ D) \circ F) & \xrightarrow{\cong} & (A \circ (C \circ E)) \bullet (B \circ (D \circ F)) \end{array}$$

$$\begin{array}{ccc} ((A \bullet B) \bullet C) \circ ((D \bullet E) \bullet F) & \xrightarrow{\cong} & (A \bullet (B \bullet C)) \circ (D \bullet (E \bullet F)) \\ \zeta \downarrow & & \downarrow \zeta \\ ((A \bullet B) \circ (D \bullet E)) \bullet (C \circ F) & & (A \circ D) \bullet ((B \bullet C) \circ (E \bullet F)) \\ \zeta \bullet \text{id} \downarrow & & \downarrow \text{id} \bullet \zeta \\ ((A \circ D) \bullet (B \circ E)) \bullet (C \circ F) & \xrightarrow{\cong} & (A \circ D) \bullet ((B \circ E) \bullet (C \circ F)) \end{array}$$

Unitality. For all objects A, B in \mathcal{M} , the following diagrams commute.

$$(1.7) \quad \begin{array}{ccc} I \circ (A \bullet B) & \xrightarrow{\delta \circ \text{id}} & (I \bullet I) \circ (A \bullet B) \\ \cong \downarrow & & \downarrow \zeta \\ A \bullet B & \xleftarrow{\cong} & (I \circ A) \bullet (I \circ B) \end{array} \quad \begin{array}{ccc} (A \bullet B) \circ I & \xrightarrow{\text{id} \circ \delta} & (A \bullet B) \circ (I \bullet I) \\ \cong \downarrow & & \downarrow \zeta \\ A \bullet B & \xleftarrow{\cong} & (A \circ I) \bullet (B \circ I) \end{array}$$

$$\begin{array}{ccc} (J \circ J) \bullet (A \circ B) & \xrightarrow{\varpi \bullet \text{id}} & J \bullet (A \circ B) \\ \zeta \uparrow & & \downarrow \cong \\ (J \bullet A) \circ (J \bullet B) & \xrightarrow{\cong} & A \circ B \end{array} \quad \begin{array}{ccc} (A \circ B) \bullet (J \circ J) & \xrightarrow{\text{id} \bullet \varpi} & (A \circ B) \bullet J \\ \zeta \uparrow & & \downarrow \cong \\ (A \bullet J) \circ (B \bullet J) & \xrightarrow{\cong} & A \circ B \end{array}$$

The arrows labelled by \cong in the diagrams above, refer to the associativity and the unit constraints in either monoidal category. The same notation is used in all diagrams throughout the paper. In the formulae, however, we denote the associator by α (the unitors do not happen to occur).

By one of the unitality axioms in (1.7) and unitality of the monoid J , also

$$(1.8) \quad \begin{array}{ccc} (A \bullet I) \circ (B \bullet J) & \xrightarrow{\zeta} & (A \circ B) \bullet (I \circ J) \\ \cong \downarrow & & \downarrow \cong \\ (A \bullet I) \circ B & & (A \circ B) \bullet J \\ (A \bullet \tau) \circ B \downarrow & & \downarrow \cong \\ (A \bullet J) \circ B & \xrightarrow{\cong} & A \circ B \end{array}$$

and some of its symmetrical variants commute, for any objects A and B of \mathcal{M} , see [2, Proposition 6.8].

The simplest examples of duoidal categories are braided monoidal categories. In this case, both monoidal products coincide and the interchange law is induced by the braiding, see [2, Section 6.3]. Generalizing bimonoids in braided monoidal categories, bimonoids can be defined also in duoidal categories — as monoids in the category of comonoids in $(\mathcal{M}, \bullet, J)$, equivalently, as comonoids in the category of monoids in (\mathcal{M}, \circ, I) . Explicitly, this means the following.

Definition 1.2. [2, Definition 6.25] A *bimonoid* in a duoidal category \mathcal{M} is an object A equipped with a monoid structure $(A \circ A \xrightarrow{\mu} A, I \xrightarrow{\eta} A)$ in (\mathcal{M}, \circ, I) , and a comonoid structure $(A \xrightarrow{\Delta} A \bullet A, A \xrightarrow{\varepsilon} J)$ in $(\mathcal{M}, \bullet, J)$, subject to the compatibility axioms

$$\begin{array}{ccccc}
 (A \bullet A) \circ (A \bullet A) & \xrightarrow{\zeta} & (A \circ A) \bullet (A \circ A) & A \circ A \xrightarrow{\varepsilon \circ \varepsilon} J \circ J & I \xrightarrow{\eta} A & I \\
 \uparrow \Delta \circ \Delta & & \mu \bullet \mu \downarrow & \downarrow \mu & \downarrow \delta & \downarrow \Delta \\
 A \circ A & \xrightarrow{\mu} & A & \xrightarrow{\Delta} & A \bullet A & I \bullet I \xrightarrow{\eta \bullet \eta} A \bullet A & A \xrightarrow{\varepsilon} J. \\
 & & & & & \downarrow \eta & \downarrow \tau \\
 & & & & & A & \xrightarrow{\varepsilon} J.
 \end{array}$$

Note that in particular the monoidal units I and J are bimonoids in any duoidal category.

By modules over a bimonoid A in a duoidal category \mathcal{M} , modules over the constituent monoid A in (\mathcal{M}, \circ, I) are meant. It was observed in [2, Section 6.6] that the category of A -modules is monoidal with respect to the monoidal product \bullet and monoidal unit J . This fact can be given the following equivalent formulation.

Proposition 1.3. [5, Theorem 18] *Any bimonoid A in a duoidal category \mathcal{M} induces a bimonad (termed a “Hopf monad” in [15]) $(-)\circ A$ on $(\mathcal{M}, \bullet, J)$.*

Proof. For later reference, we recall the forms of the structure morphisms of the bimonad in the claim. The multiplication and unit of the monad are induced by the multiplication and the unit of the monoid A , respectively. The binary part of the opmonoidal structure is given by

$$(M \bullet M') \circ A \xrightarrow{(M \bullet M') \circ \Delta} (M \bullet M') \circ (A \bullet A) \xrightarrow{\zeta} (M \circ A) \bullet (M' \circ A),$$

for all objects M, M' in \mathcal{M} . The nullary part is provided by $J \circ A \xrightarrow{J \circ \varepsilon} J \circ J \xrightarrow{\varpi} J$. \square

Following the terminology of [8], a bimonad T — on a monoidal category \mathcal{M} with monoidal product \otimes — is called a *right Hopf monad* whenever

$$T(TM \otimes M') \xrightarrow{T_2} T^2M \otimes TM' \xrightarrow{\mu M \otimes TM'} TM \otimes TM'$$

is a natural isomorphism, equivalently, by [8, Theorem 2.15],

$$(1.9) \quad \beta_{Q, M'} := (T(Q \otimes M') \xrightarrow{T_2} TQ \otimes TM' \xrightarrow{\gamma \otimes TM'} Q \otimes TM')$$

is a natural isomorphism (where μ is the multiplication of the monad T , T_2 is the binary part of the opmonoidal structure, M, M' are objects in \mathcal{M} and (Q, γ) is an object in the Eilenberg-Moore category \mathcal{M}^T).

For a bimonoid A in a duoidal category \mathcal{M} , consider the induced bimonad $(-)\circ A$ in Proposition 1.3. The corresponding canonical morphism (1.9) takes the explicit form

$$(1.10) \quad (Q \bullet M') \circ A \xrightarrow{(Q \bullet M') \circ \Delta} (Q \bullet M') \circ (A \bullet A) \xrightarrow{\zeta} (Q \circ A) \bullet (M' \circ A) \xrightarrow{\gamma \bullet (M' \circ A)} Q \bullet (M' \circ A).$$

2. RELATIVE HOPF MODULES

The aim of this section is to develop the notion of Galois extension by a bimonoid in a duoidal category. This requires several steps. As in the case of bialgebras — over a field or, more generally, in a braided monoidal category — we start with defining a comodule monoid over a bimonoid A ; this is a monoid B in the monoidal category of A -comodules. Relative Hopf modules are then B -modules in the category of A -comodules. A comodule monoid is shown to induce a functor to the category of relative Hopf modules. Whenever this functor possesses a right adjoint, we can regard this right adjoint as the functor taking the ‘coinvariant part’ of relative Hopf modules. In particular, we can consider the coinvariant part B^c of B itself. It turns out that it admits a monoid structure and a monoid morphism to B . This defines the notion of an A -extension $B^c \rightarrow B$. Finally, assuming that certain coequalizers exist and are preserved, we associate to any A -extension an adjoint triangle. Whenever the corresponding comonad morphism is iso, we say that the A -extension in question is a Galois extension.

2.1. Comodule monoids and relative Hopf modules. For a bimonoid A in a duoidal category \mathcal{M} , we denote by \mathcal{M}^A the category of A -comodules; that is, the category of comodules over the constituent comonoid A in $(\mathcal{M}, \bullet, J)$. Recall from [2, Section 6.6], that \mathcal{M}^A is a monoidal category via the monoidal product \circ and the monoidal unit I . That is to say, the forgetful functor $\mathcal{M}^A \rightarrow \mathcal{M}$ is strict monoidal.

Definition 2.1. A *right comodule monoid* over bimonoid A in a duoidal category \mathcal{M} , is a monoid B in \mathcal{M}^A . Explicitly, this means that there is a coassociative and counital coaction $\rho : B \rightarrow B \bullet A$ and an associative multiplication $\mu : B \circ B \rightarrow B$ with unit $\eta : I \rightarrow B$ such that the following diagrams commute.

$$(2.1) \quad \begin{array}{ccc} B \circ B & \xrightarrow{\mu} & B & \xrightarrow{\rho} & B \bullet A \\ \rho \circ \rho \downarrow & & & & \uparrow \mu \circ \mu \\ (B \bullet A) \circ (B \bullet A) & \xrightarrow{\zeta} & (B \circ B) \bullet (A \circ A) & & \end{array} \quad \begin{array}{ccc} I & \xrightarrow{\eta} & B \\ \delta \downarrow & & \downarrow \rho \\ I \bullet I & \xrightarrow{\eta \bullet \eta} & B \bullet A \end{array}$$

Any bimonoid A is a comodule monoid over itself, via the coaction provided by the comultiplication. The multiplication and the unit of A are A -comodule morphisms by the first and by the third axiom in Definition 1.2, respectively.

Since a right comodule monoid B over a bimonoid A in a duoidal category \mathcal{M} is defined as a monoid in $(\mathcal{M}^A, \circ, I)$, it induces a monad $(-)\circ B$ on \mathcal{M}^A (lifted from the monad $(-)\circ B$ on \mathcal{M}). Equivalently, the comonad $(-)\bullet A$ on \mathcal{M} lifts to a comonad on the category \mathcal{M}_B of right

B -modules. These liftings correspond to the mixed distributive law (in the sense of [4])

$$(M \bullet A) \circ B \xrightarrow{(M \bullet A) \circ \rho} (M \bullet A) \circ (B \bullet A) \xrightarrow{\zeta} (M \circ B) \bullet (A \circ A) \xrightarrow{(M \circ B) \bullet \mu} (M \circ B) \bullet A,$$

for any object M of \mathcal{M} . (For more on the connection between distributive laws and liftings, we refer to [19].)

Definition 2.2. Consider a bimonoid A in a duoidal category \mathcal{M} and a right A -comodule monoid B . By (right-right) (A, B) -relative Hopf modules we mean modules for the monad $(-) \circ B$ on \mathcal{M}^A ; equivalently, comodules over the comonad $(-) \bullet A$ on \mathcal{M}_B . Explicitly, this means a triple $(X, \gamma : X \circ B \rightarrow X, \rho : X \rightarrow X \bullet A)$, where γ is an associative and unital action, ρ is a coassociative and counital coaction such that the following compatibility condition holds.

$$(2.2) \quad \begin{array}{ccccc} X \circ B & \xrightarrow{\gamma} & X & \xrightarrow{\rho} & X \bullet A \\ \rho \circ \rho \downarrow & & & & \uparrow \gamma \bullet \mu \\ (X \bullet A) \circ (B \bullet A) & \xrightarrow{\zeta} & (X \circ B) \bullet (A \circ A) & & \end{array}$$

Morphisms of (A, B) -relative Hopf modules are morphisms of both A -comodules and B -modules. The category of right-right (A, B) -relative Hopf modules is denoted by \mathcal{M}_B^A .

Clearly, an A -comodule monoid B is itself an (A, B) -relative Hopf module via the action provided by the multiplication. The compatibility between this action and the A -coaction on B holds by the requirement that the multiplication in B is a morphism of A -comodules.

In the case when B is equal to A , the resulting category \mathcal{M}_A^A is called the category of A -Hopf modules.

2.2. Coinvariants. If A is a bialgebra over a field — or more generally, a bimonoid in a braided monoidal category \mathcal{M} — and B is a right A -comodule monoid, then there is a lifting of the functor $(-) \otimes B : \mathcal{M} \rightarrow \mathcal{M}_B$ to $\mathcal{M} \rightarrow \mathcal{M}_B^A$. Whenever appropriate equalizers in \mathcal{M} exist, the lifted functor possesses a right adjoint known as the ‘ A -coinvariants’ functor. In what follows, we look for the analogue of this adjunction in the duoidal setting.

As the first crucial difference from the classical case, the functor $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}_B$ will be lifted now to $\mathcal{M}^I \rightarrow \mathcal{M}_B^A$ (which means a lifting to $\mathcal{M} \rightarrow \mathcal{M}_B^A$, up-to isomorphism, if \mathcal{M} is a braided monoidal category).

Proposition 2.3. Consider a bimonoid A in a duoidal category \mathcal{M} and a right A -comodule monoid B . Then the evident functor $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}_B$ has a lifting to $\mathcal{M}^I \rightarrow \mathcal{M}_B^A$.

Proof. The relevant comonad morphism (in the sense of [22], cf. [19, Corollary 5.11], [11, Proposition 1.1 and Theorem 1.2] or Section 1.1 in the current paper) is given by

$$(2.3) \quad \lambda_M^0 : (M \bullet I) \circ B \xrightarrow{(M \bullet I) \circ \rho} (M \bullet I) \circ (B \bullet A) \xrightarrow{\zeta} (M \circ B) \bullet (I \circ A) \xrightarrow{\cong} (M \circ B) \bullet A,$$

for any object M of \mathcal{M} . It is evidently natural in M . It follows by the first identity in (2.1), by one of the associativity axioms in (1.6) and by naturality of ζ and of the associativity and unit

constraints that λ_M^0 is a morphism of B -modules. Comultiplicativity and counitality of λ^0 ; that is, commutativity of the diagrams

$$\begin{array}{ccc}
(M \bullet I) \circ B & \xrightarrow{\lambda_M^0} & (M \circ B) \bullet A & & (M \bullet I) \circ B & \xrightarrow{\lambda_M^0} & (M \circ B) \bullet A \\
\downarrow (M \bullet \delta) \circ B & & (M \circ B) \bullet \Delta \downarrow & & \downarrow (M \bullet \tau) \circ B & & (M \circ B) \bullet \varepsilon \downarrow \\
(M \bullet (I \bullet I)) \circ B & & (M \circ B) \bullet (A \bullet A) & & (M \bullet J) \circ B & & (M \circ B) \bullet J \\
\downarrow \cong & & \cong \downarrow & & \downarrow \cong & & \cong \downarrow \\
((M \bullet I) \bullet I) \circ B & \xrightarrow{\lambda_{M \bullet I}^0} & ((M \bullet I) \circ B) \bullet A & \xrightarrow{\lambda_{M \bullet A}^0} & ((M \circ B) \bullet A) \bullet A & & M \circ B \xlongequal{\quad} M \circ B
\end{array}$$

follows by coassociativity and counitality of ρ , unitality of the monoid (J, ϖ, τ) , naturality of ζ and of the unit constraints, one of the associativity axioms, see (1.6), and two of the unitality axioms in a duoidal category, see (1.7). \square

For any right I -comodule Z , the A -coaction on $Z \circ B$ induced by the comonad morphism (2.3) is

$$Z \circ B \xrightarrow{\rho \circ B} (Z \bullet I) \circ B \xrightarrow{\lambda_Z^0} (Z \circ B) \bullet A.$$

Next we apply the Adjoint Lifting Theorem (in the form which is dual to [18, Theorem 2.1]) to construct the right adjoint of the lifted functor in Proposition 2.3. In the diagram

$$(2.4) \quad \begin{array}{ccc} \mathcal{M}^I & \xrightarrow{(-) \circ B} & \mathcal{M}_B^A \\ U^I \downarrow & & \downarrow U^A \\ \mathcal{M} & \xrightarrow{(-) \circ B} & \mathcal{M}_B \end{array}$$

U^I and U^A are forgetful functors of comonads and the functor in the bottom row is a left adjoint of a forgetful functor of a monad. Hence all of them possess right adjoints, so the natural transformation λ^0 in (2.3) has a mate

$$\bar{\lambda}_Q = (Q \bullet I \xrightarrow{\cong} (Q \bullet I) \circ I \xrightarrow{(Q \bullet I) \circ \eta} (Q \bullet I) \circ B \xrightarrow{\lambda_Q^0} (Q \circ B) \bullet A \xrightarrow{\gamma \bullet A} Q \bullet A),$$

for any object (Q, γ) of \mathcal{M}_B . In fact, $\bar{\lambda}_Q = Q \bullet \eta$ as the following computation shows.

$$\begin{array}{ccccc}
Q \bullet I & \xrightarrow{\cong} & (Q \bullet I) \circ I & \xrightarrow{(Q \bullet I) \circ \eta} & (Q \bullet I) \circ B \\
& & \downarrow (Q \bullet I) \circ \delta & & \downarrow (Q \bullet I) \circ \rho \\
& & (Q \bullet I) \circ (I \bullet I) & \xrightarrow{(Q \bullet I) \circ (\eta \bullet \eta)} & (Q \bullet I) \circ (B \bullet A) \\
& & \downarrow \zeta & & \downarrow \zeta \\
& & (Q \circ I) \bullet (I \circ I) & \xrightarrow{(Q \circ \eta) \bullet (I \circ \eta)} & (Q \circ B) \bullet (I \circ A) \\
& & \downarrow \cong & & \downarrow \cong \\
& & & & (Q \circ B) \bullet A \\
& & & & \downarrow \gamma \bullet A \\
& & & & Q \bullet A \\
& & & & \uparrow Q \bullet \eta \\
& & & & Q \bullet I
\end{array}$$

The top right square commutes since $\eta : I \rightarrow B$ is a morphism of A -comodules. The region below it commutes by the naturality of ζ . The region at the bottom right commutes by unitality of the

A -action on B . Finally, the region on the left commutes by one of the unitality axioms of a duoidal category, cf.(1.7).

Applying the Adjoint Lifting Theorem (see the dual form of [18, Theorem 2.1] and Section 1.1), we conclude that the functor $(-) \circ B : \mathcal{M}^I \rightarrow \mathcal{M}_B^A$ in Proposition 2.3 possesses a right adjoint $(-)^c : \mathcal{M}_B^A \rightarrow \mathcal{M}^I$ if and only if the equalizer

$$(2.5) \quad X^c \xrightarrow{\iota} X \bullet I \xrightleftharpoons[\varphi^1 := ((X \bullet \eta) \bullet I) \cdot \alpha^{-1} \cdot (X \bullet \delta)]{\varphi^0 := \rho \bullet I} (X \bullet A) \bullet I$$

exists in \mathcal{M}^I for any object (X, γ, ρ) in \mathcal{M}_B^A . We call X^c the A -coinvariant part of X .

2.3. The submonoid of coinvariants. If the equalizers (2.5) exist in \mathcal{M}^I then, in particular, for any right comodule monoid B over a bimonoid A , there is an equalizer

$$(2.6) \quad B^c \xrightarrow{\iota} B \bullet I \xrightleftharpoons[\varphi^1 := ((B \bullet \eta) \bullet I) \cdot \alpha^{-1} \cdot (B \bullet \delta)]{\varphi^0 := \rho \bullet I} (B \bullet A) \bullet I$$

in \mathcal{M}^I .

Proposition 2.4. *For any right comodule monoid B over a bimonoid A in a duoidal category \mathcal{M} , the coinvariant part B^c — whenever it exists — is a monoid in \mathcal{M}^I .*

Proof. First we claim that $B \bullet I$ is a monoid in \mathcal{M}^I . Since B is a monoid in \mathcal{M}^A by definition, and the forgetful functor $\mathcal{M}^A \rightarrow \mathcal{M}$ is strict monoidal, it follows that B is a monoid in \mathcal{M} . Since I is a bimonoid in \mathcal{M} , also the forgetful functor $\mathcal{M}^I \rightarrow \mathcal{M}$ is strict monoidal so in particular opmonoidal. Since the right adjoint of an opmonoidal functor is monoidal, $(-) \bullet I : \mathcal{M} \rightarrow \mathcal{M}^I$ is monoidal. Thus it takes the monoid B in \mathcal{M} to the monoid $B \bullet I$ in \mathcal{M}^I .

Both morphisms

$$\begin{aligned} (B^c \circ B^c \xrightarrow{\iota \circ \iota} (B \bullet I) \circ (B \bullet I) \xrightarrow{\mu} B \bullet I) = \\ (B^c \circ B^c \xrightarrow{\iota \circ \iota} (B \bullet I) \circ (B \bullet I) \xrightarrow{\zeta} (B \circ B) \bullet (I \circ I) \xrightarrow{\cong} (B \circ B) \bullet I \xrightarrow{\mu \bullet I} B \bullet I) \end{aligned}$$

and $(I \xrightarrow{\eta} B \bullet I) = (I \xrightarrow{\delta} I \bullet I \xrightarrow{\eta \bullet I} B \bullet I)$ equalize the parallel morphisms φ^0 and φ^1 in the equalizer (2.6). Hence we obtain the multiplication and unit of B^c by universality. \square

Proposition 2.5. *Let B be a right comodule monoid over a bimonoid A whose coinvariant part B^c exists. Then*

$$\omega = (B^c \xrightarrow{\iota} B \bullet I \xrightarrow{B \bullet \tau} B \bullet J \xrightarrow{\cong} B)$$

is a morphism of monoids.

Proof. The monomorphism $\iota : B^c \rightarrow B \bullet I$ is a morphism of monoids by construction. The fact that also $B \bullet I \xrightarrow{B \bullet \tau} B \bullet J \xrightarrow{\cong} B$ is a morphism of monoids, follows from the commutativity of

$$\begin{array}{ccccc}
(B \bullet I) \circ (B \bullet I) & \xrightarrow{(B \bullet \tau) \circ (B \bullet \tau)} & (B \bullet J) \circ (B \bullet J) & \xrightarrow{\cong} & B \circ B \\
\zeta \downarrow & & \zeta \downarrow & & \downarrow \mu \\
(B \circ B) \bullet (I \circ I) & \xrightarrow{(B \circ B) \bullet (\tau \circ \tau)} & (B \circ B) \bullet (J \circ J) & & \\
\cong \downarrow & & (B \circ B) \circ \varpi \downarrow & & \uparrow \cong \\
(B \circ B) \bullet I & \xrightarrow{(B \circ B) \bullet \tau} & (B \circ B) \bullet J & & \\
\mu \bullet I \downarrow & & \mu \bullet J \downarrow & & \\
B \bullet I & \xrightarrow{B \bullet \tau} & B \bullet J & \xrightarrow{\cong} & B
\end{array}$$

— where the middle square on the left commutes by unitality of the monoid (J, ϖ, τ) and the region at the top right commutes by one of the unitality axioms in (1.7) — and

$$\begin{array}{ccc}
I & & \\
\delta \downarrow & \searrow \eta & \\
I \bullet I & \xrightarrow{I \bullet \tau} & I \bullet J \\
\eta \bullet I \downarrow & & \downarrow \eta \bullet J \\
B \bullet I & \xrightarrow{B \bullet \tau} & B \bullet J \xrightarrow{\cong} B
\end{array}$$

— where the top left triangle commutes by the counitality of the comonoid (I, δ, τ) . \square

The morphism ω in Proposition 2.5 induces a left B^c -action $\gamma := (B^c \circ B \xrightarrow{\omega \circ B} B \circ B \xrightarrow{\mu} B)$. If the coequalizer

$$(2.7) \quad (P \circ B^c) \circ B \xrightleftharpoons[(P \circ \gamma) \cdot \alpha]{\gamma \circ B} P \circ B \xrightarrow{\pi_{P,B}} P \circ_{B^c} B$$

in \mathcal{M} exists — for any right B^c -module (P, γ) — and any power of $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizers (2.7), then there is an adjunction

$$\begin{array}{ccc}
\mathcal{M}_{B^c} & \xrightleftharpoons[\omega^*]{\omega_* = (-) \circ_{B^c} B} & \mathcal{M}_B,
\end{array}$$

see [17]. The functor ω^* takes a right B -module (Q, γ) to the B^c -module $(Q, \gamma \cdot (Q \circ \omega))$, and it acts on the morphisms as the identity map. For any right B^c -module P , the B -action on $P \circ_{B^c} B$ is constructed using the universality of the coequalizer

$$(2.8) \quad \begin{array}{ccc}
((P \circ B^c) \circ B) \circ B & \xrightleftharpoons[(P \circ \gamma) \cdot \alpha \circ B]{(\gamma \circ B) \circ B} & (P \circ B) \circ B \xrightarrow{\pi_{P,B \circ B}} (P \circ_{B^c} B) \circ B \\
& & \cong \downarrow \qquad \qquad \qquad \vdots \gamma \\
& & P \circ (B \circ B) \\
& & P \circ \mu \downarrow \qquad \qquad \qquad \downarrow \\
& & P \circ B \xrightarrow{\pi_{P,B}} P \circ_{B^c} B
\end{array}$$

Thus $\pi_{P,B}$ is a morphism of B -modules by construction.

By Proposition 2.4, B^c is a monoid in \mathcal{M}^I whenever it exists. We denote by $\mathcal{M}_{B^c}^I$ the category of B^c -modules in \mathcal{M}^I .

Proposition 2.6. *For a right comodule monoid B over a bimonoid A in a duoidal category \mathcal{M} , assume that the equalizer (2.6) and the coequalizers (2.7) exist and that any power of $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizers (2.7). Then the functor $(-) \circ_{B^c} B : \mathcal{M}_{B^c} \rightarrow \mathcal{M}_B$ lifts to $\mathcal{M}_{B^c}^I \rightarrow \mathcal{M}_B^A$.*

Proof. In light of [19, Corollary 5.11] or [11, Proposition 1.1 and Theorem 1.2] (see also Section 1.1), we need to construct a morphism $\lambda_P : (P \bullet I) \circ_{B^c} B \rightarrow (P \circ_{B^c} B) \bullet A$ in \mathcal{M}_B , for any right B^c -module P ; and show that λ is in fact a comonad morphism. The morphism λ_P is constructed by using the universality of the coequalizer in \mathcal{M} in the top row:

$$(2.9) \quad \begin{array}{ccccc} ((P \bullet I) \circ B^c) \circ B & \xrightarrow[\text{((P}\bullet\text{I})\circ\gamma).\alpha}]{\gamma \circ B} & (P \bullet I) \circ B & \xrightarrow{\pi_{P \bullet I, B}} & (P \bullet I) \circ_{B^c} B \\ & & \lambda_P^0 \downarrow & & \vdots \downarrow \lambda_P \\ & & (P \circ B) \bullet A & \xrightarrow{\pi_{P, B} \bullet A} & (P \circ_{B^c} B) \bullet A, \end{array}$$

where λ^0 is the comonad morphism (2.3). In order to see that λ_P is well defined, we need to check that $(\pi_{P,B} \bullet A) \cdot \lambda_P^0$ coequalizes the parallel morphisms in the top row. The B^c -action on $P \bullet I$ is given by

$$(P \bullet I) \circ B^c \xrightarrow{(P \bullet I) \circ \rho} (P \bullet I) \circ (B^c \bullet I) \xrightarrow{\zeta} (P \circ B^c) \bullet (I \circ I) \xrightarrow{\cong} (P \circ B^c) \bullet I \xrightarrow{\gamma \bullet I} P \bullet I.$$

With this expression at hand, it follows by the naturality of ζ and of the unit constraints, since ι is a morphism of I -comodules, by the coequalizer property of $\pi_{P,B}$, and since $\delta : I \rightarrow I \bullet I$ is counital; that $(\pi_{P,B} \bullet A) \cdot \lambda_P^0 \cdot (\gamma \circ B)$ is equal to the image of

$$(P \bullet I) \circ B^c \xrightarrow{(P \bullet I) \circ \iota} (P \bullet I) \circ (B \bullet I) \xrightarrow{\zeta} (P \circ B) \bullet (I \circ I) \xrightarrow{\cong} (P \circ B) \bullet I$$

under the functor $(-) \circ B$, composed with

$$((P \circ B) \bullet I) \circ B \xrightarrow{\lambda_{P \circ B}^0} ((P \circ B) \circ B) \bullet A \xrightarrow{\gamma \bullet A} (P \circ B) \bullet A \xrightarrow{\pi_{P, B} \bullet A} (P \circ_{B^c} B) \bullet A.$$

On the other hand, $(\pi_{P,B} \bullet A) \cdot \lambda_P^0 \cdot ((P \bullet I) \circ \gamma) \cdot \alpha$ is equal to the same morphism. This follows by the explicit form of the B^c action on B induced by the morphism ω in Proposition 2.5, by the naturality of ζ and of the unit constraints, by the first compatibility condition in (2.1), by the equalizer property of $\iota : B^c \rightarrow B \bullet I$, by counitality of the comonoid (I, δ, τ) , by one of the associativity axioms in (1.6), and by unitality of the monoid (A, μ, η) . So by universality, the morphism λ_P exists.

λ is natural since λ^0 and π are so. It was proven in Proposition 2.3 that λ_P^0 is a morphism of B -modules. Then so is $(\pi_{P,B} \bullet A) \cdot \lambda_P^0$ hence also λ_P . The compatibilities of λ with the comultiplications and the counits of the comonads $(-) \bullet I$ on \mathcal{M}_{B^c} and $(-) \bullet A$ on \mathcal{M}_B follow by naturality of π in its first argument and the compatibilities of λ^0 in Proposition 2.3. \square

Whenever the assumptions in Proposition 2.6 hold, it follows by Proposition 2.6 that for any object N in $\mathcal{M}_{B^c}^I$, $N \circ_{B^c} B$ is a right A -comodule via the coaction in the bottom row of the following diagram. What is more, $\pi_{N,B}$ is also a morphism of A -comodules by commutativity of the diagram.

$$\begin{array}{ccccc}
(N \bullet I) \circ (B \bullet A) & \xrightarrow{\zeta} & (N \circ B) \bullet (I \circ A) & & \\
\rho \circ \rho \uparrow & & \downarrow \cong & & \\
N \circ B & \xrightarrow{\rho \circ B} & (N \bullet I) \circ B & \xrightarrow{\lambda_N^0} & (N \circ B) \bullet A \\
\pi_{N,B} \downarrow & & \downarrow \pi_{N \bullet I, B} & & \downarrow \pi_{N, B \bullet A} \\
N \circ_{B^c} B & \xrightarrow{\rho \circ_{B^c} B} & (N \bullet I) \circ_{B^c} B & \xrightarrow{\lambda_N} & (N \circ_{B^c} B) \bullet A.
\end{array}$$

Commutativity of the region at the top follows immediately from the form of λ_N^0 , see (2.3).

We have seen that the functor $(-)^c : \mathcal{M}_B^A \rightarrow \mathcal{M}^I$ — whenever it exists — is right adjoint of the functor $(-) \circ B : \mathcal{M}^I \rightarrow \mathcal{M}_B^A$. Our next aim is to show that it factorizes through the right adjoint of $(-) \circ_{B^c} B : \mathcal{M}_{B^c}^I \rightarrow \mathcal{M}_B^A$, via the forgetful functor $\mathcal{M}_{B^c}^I \rightarrow \mathcal{M}^I$.

Proposition 2.7. *For a right comodule monoid B over a bimonoid A in a duoidal category \mathcal{M} , assume that the equalizers (2.5) and the coequalizers (2.7) exist and that any power of $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizers (2.7). Then there is a lifting of $(-)^c : \mathcal{M}_B^A \rightarrow \mathcal{M}^I$ to $\mathcal{M}_{B^c}^A \rightarrow \mathcal{M}_{B^c}^I$, which provides the right adjoint of the functor $(-) \circ_{B^c} B$ in Proposition 2.6.*

Proof. We are to prove that there is an adjoint triangle

$$(2.10) \quad \begin{array}{c}
\mathcal{M}_{B^c}^I \xrightarrow{U^I} \mathcal{M}_{B^c} \xrightarrow{\omega_*} \mathcal{M}_B \\
\downarrow K = (-) \circ_{B^c} B \quad \downarrow U^A \quad \downarrow F^A = (-) \bullet A \\
\mathcal{M}_B^A \xrightarrow{N = (-)^c} \mathcal{M}_B^A \\
\uparrow \omega_* = (-) \circ_{B^c} B \quad \uparrow U^I \quad \uparrow F^I = (-) \bullet I
\end{array}$$

in which $U^A K = \omega_* U^I$. By the Adjoint Lifting Theorem (cf. the dual form of [18, Theorem 2.1], see Section 1.1), the functor N exists, and it is a right adjoint of K , provided that the equalizer

$$(2.11) \quad X^c \longrightarrow X \bullet I \rightrightarrows (X \bullet A) \bullet I$$

exists in $\mathcal{M}_{B^c}^I$, for any (A, B) -relative Hopf module X . The upper one of the parallel arrows is $\varphi^0 = \rho \bullet I$ as in (2.5). The lower one is

$$(2.12) \quad \begin{array}{c}
X \bullet I \xrightarrow{\nu^I} (X \bullet I) \bullet I \xrightarrow{\nu \bullet I} ((X \bullet I) \circ_{B^c} B) \bullet I \xrightarrow{\rho \bullet I} (((X \bullet I) \circ_{B^c} B) \bullet A) \bullet I \xrightarrow{((\epsilon^I \circ_{B^c} B) \bullet A) \bullet I} \\
((X \circ_{B^c} B) \bullet A) \bullet I \xrightarrow{(\epsilon \bullet A) \bullet I} (X \bullet A) \bullet I.
\end{array}$$

Here ν^I is the unit, and ϵ^I is the counit of the adjunction $U^I \dashv F^I$, and ν is the unit and ϵ is the counit of the adjunction $\omega_* \dashv \omega^*$. (Recall that for any right B -module Q , $\epsilon_Q \cdot \pi_{Q,B}$ is equal to the B -action on Q .) The symbol ρ denotes the A -coaction on $(X \bullet I) \circ_{B^c} B$ from Proposition 2.6. A computation — using that $\pi_{X \bullet I, B}$ and the unit $\eta : I \rightarrow B$ are morphisms of A -comodules, naturality of π in the first argument, the relation between the counit ϵ_X and the B -action on X , counitality of the comonoid I and unitality of the B -action on X , and one of the unitality axioms in (1.7) — yields that (2.12) is equal to $\varphi^1 = ((X \bullet \eta) \bullet I) \cdot \alpha^{-1} \cdot (X \bullet \delta)$ as in (2.5).

The B^c -action on $X \bullet I$ is equal to

$$(X \bullet I) \circ_{B^c} \xrightarrow{(X \bullet I) \circ \rho} (X \bullet I) \circ (B^c \bullet I) \xrightarrow{\zeta} (X \circ B^c) \bullet (I \circ I) \xrightarrow{\cong} (X \circ B^c) \bullet I \xrightarrow{(X \circ \omega) \bullet I} (X \circ B) \bullet I \xrightarrow{\gamma \bullet I} X \bullet I.$$

Using the explicit form of ω in Proposition 2.5, the fact that $\iota : B^c \rightarrow B \bullet I$ is a morphism of I -comodules and counitality of the comonoid I , this B^c -action is shown to be equal to

$$(X \bullet I) \circ_{B^c} \xrightarrow{(X \bullet I) \circ \iota} (X \bullet I) \circ (B \bullet I) \xrightarrow{\zeta} (X \circ B) \bullet (I \circ I) \xrightarrow{\cong} (X \circ B) \bullet I \xrightarrow{\gamma \bullet I} X \bullet I.$$

The B^c -action on $X \bullet A$ is equal to

$$(X \bullet A) \circ_{B^c} \xrightarrow{(X \bullet A) \circ \omega} (X \bullet A) \circ B \xrightarrow{(X \bullet A) \circ \rho} (X \bullet A) \circ (B \bullet A) \xrightarrow{\zeta} (X \circ B) \bullet (A \circ A) \xrightarrow{\gamma \bullet \mu} X \bullet A.$$

Using the equalizer property of $\iota : B^c \rightarrow B \bullet I$, counitality of the comonoid I and unitality of the monoid A , this B^c -action turns out to be equal to

$$(X \bullet A) \circ_{B^c} \xrightarrow{(X \bullet A) \circ \iota} (X \bullet A) \circ (B \bullet I) \xrightarrow{\zeta} (X \circ B) \bullet (A \circ I) \xrightarrow{\cong} (X \circ B) \bullet A \xrightarrow{\gamma \bullet A} X \bullet A.$$

With these B^c -actions at hand, $X \bullet \eta : X \bullet I \rightarrow X \bullet A$ is a morphism of B^c -modules by naturality of ζ and functoriality of both monoidal structures. So since $\varphi^0 = \rho \bullet I$ and $(X \bullet \eta) \bullet I$ are in the range of the functor $(-) \bullet I : \mathcal{M}_{B^c} \rightarrow \mathcal{M}_{B^c}$, they are morphisms of B^c -modules. Since $\alpha^{-1} \cdot (X \bullet \delta)$ is the comultiplication of the comonad $(-) \bullet I : \mathcal{M}_{B^c} \rightarrow \mathcal{M}_{B^c}$ evaluated at X , it is a morphism of B^c -modules too. Thus both φ^0 and φ^1 are morphisms of B^c -modules. Since the forgetful functor $\mathcal{M}_{B^c}^I \rightarrow \mathcal{M}^I$ creates equalizers, and (2.11) is an equalizer in \mathcal{M}^I by assumption, we conclude that it is an equalizer in $\mathcal{M}_{B^c}^I$ as needed. \square

2.4. Galois extensions. For a right comodule monoid B over a bimonoid A in a duoidal category \mathcal{M} , assume that the equalizers (2.5) and the coequalizers (2.7) exist and that any power of $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizers (2.7). Then by Proposition 2.7 there is an adjoint triangle (2.10). Hence as in (1.3), there is a corresponding morphism of comonads β :

$$\begin{array}{ccc} \omega_* U^I F^I \omega^* & U^A F^A \omega_* U^I F^I \omega^* \xrightarrow{U^A F^A \omega_* \epsilon^I \omega^*} U^A F^A \omega_* \omega^* \xrightarrow{U^A F^A \epsilon} U^A F^A, \\ \parallel & \parallel \\ U^A K F^I \omega^* \xrightarrow{U^A \nu^A K F^I \omega^*} U^A F^A U^A K F^I \omega^* & \end{array}$$

where ν^A is the unit of the adjunction $U^A \dashv F^A$, and ϵ^I and ϵ are the counits of the adjunctions $U^I \dashv F^I$ and $\omega_* \dashv \omega^*$, respectively. Explicitly, for any right B -module Q , β_Q is the morphism

$$(2.13) \quad (Q \bullet I) \circ_{B^c} B \xrightarrow{\rho} ((Q \bullet I) \circ_{B^c} B) \bullet A \xrightarrow{((Q \bullet \tau) \circ_{B^c} B) \bullet A} ((Q \bullet J) \circ_{B^c} B) \bullet A \xrightarrow{\cong} (Q \circ_{B^c} B) \bullet A \xrightarrow{\epsilon_Q \bullet A} Q \bullet A.$$

Proposition 2.8. *For a right comodule monoid B over a bimonoid A in a duoidal category \mathcal{M} , assume that the equalizers (2.5) and the coequalizers (2.7) exist and that any power of $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}$ preserves the coequalizers (2.7). For any right B -module (Q, γ) , consider the natural transformation*

$$(2.14) \quad \beta_Q^0 := ((Q \bullet I) \circ B \xrightarrow{(Q \bullet I) \circ \rho} (Q \bullet I) \circ (B \bullet A) \xrightarrow{\zeta} (Q \circ B) \bullet (I \circ A) \xrightarrow{\cong} (Q \circ B) \bullet A \xrightarrow{\gamma \bullet A} Q \bullet A).$$

Then β_Q in (2.13) can be characterized as the unique morphism for which $\beta_Q^0 = \beta_Q \cdot \pi_{Q \bullet I, B}$.

Proof. By naturality of π and by $\epsilon_Q \cdot \pi_{Q, B}$ being equal to the B -action on Q , β_Q is the unique morphism for which the diagram

$$\begin{array}{ccc} (Q \bullet I) \circ_{B^c} B & \xrightarrow{\beta_Q} & Q \bullet A \\ \pi_{Q \bullet I, B} \uparrow & & \parallel \\ (Q \bullet I) \circ B & \xrightarrow[\rho]{} ((Q \bullet I) \circ B) \bullet A \xrightarrow{((Q \bullet \tau) \circ B) \bullet A} ((Q \bullet J) \circ B) \bullet A \xrightarrow{\cong} (Q \circ B) \bullet A \xrightarrow{\gamma \bullet A} & Q \bullet A \end{array}$$

commutes. (The A -coaction ρ on $(Q \bullet I) \circ B$ appearing on the left in the bottom row, is induced from the I -coaction on $Q \bullet I$ by the comonad morphism λ^0 in (2.3).) The morphism in the bottom row is equal to (2.14) by the explicit form of the A -coaction on $(Q \bullet I) \circ B$, by the counitality of the comonoid (I, δ, τ) and by naturality of ζ and of the unit constrains. \square

Definition 2.9. Consider a duoidal category \mathcal{M} . A monoid morphism $\omega : C \rightarrow B$ in (\mathcal{M}, \circ, I) is called a *Galois extension* by a bimonoid A in \mathcal{M} if the following conditions hold.

- B is a right comodule monoid over A .
- The A -coinvariant part of any (A, B) -relative Hopf module (i.e. the equalizer (2.5)) exists.
- C fits the equalizer diagram

$$C \xrightarrow{\iota} B \bullet I \xrightleftharpoons[\varphi_1]{\varphi_0} (B \bullet A) \bullet I$$

cf. (2.6) (so that C is the coinvariant part of B), and $\omega : C \rightarrow B$ is the corresponding morphism of monoids in Proposition 2.5.

- The coequalizers (2.7) exist and any power of $(-) \circ B : \mathcal{M} \rightarrow \mathcal{M}$ preserves them.
- The natural transformation β in (2.13) is an isomorphism.

If \mathcal{M} is a braided monoidal category, this reduces to the usual definition of a Galois extension by a bimonoid (see e.g. [21, Definition 3.1]).

3. THE FUNDAMENTAL THEOREM OF HOPF MODULES

In this section we analyze further the particular extension $I \rightarrow A$ by a bimonoid A in a duoidal category \mathcal{M} . Making some assumptions on \mathcal{M} , we relate its Galois property to the Fundamental Theorem of Hopf Modules holding true — that is, to an equivalence between the category of A -Hopf modules and the category \mathcal{M}^I of comodules over the \circ -monoidal unit I . (Clearly, if \mathcal{M} is a braided monoidal category, \mathcal{M}^I is isomorphic to \mathcal{M} .)

3.1. Properties of the Galois morphism. Let A be a bimonoid in a duoidal category \mathcal{M} and regard A as a right A -comodule monoid (via the coaction provided by the comultiplication). The resulting category of A -modules in the category \mathcal{M}^A of A -comodules is denoted by \mathcal{M}_A^A and it is called *the category of A -Hopf modules*.

Lemma 3.1. *Let A be a bimonoid in a duoidal category \mathcal{M} . Then the coinvariant part of A as a Hopf module exists and it is isomorphic to the \circ -monoidal unit I . In fact, the following is a contractible equalizer in \mathcal{M}^I*

$$(3.1) \quad I \xrightarrow{\delta} I \bullet I \xrightarrow{\eta \bullet I} A \bullet I \begin{array}{c} \xrightarrow{\varphi^0 = \Delta \bullet I} \\ \xrightarrow{\varphi^1 = ((A \bullet \eta) \bullet I) \cdot \alpha^{-1} \cdot (A \bullet \delta)} \end{array} (A \bullet A) \bullet I.$$

Proof. The diagram in (3.1) is a fork by the the coassociativity of δ and the third axiom in Definition 1.2. The contracting morphisms are

$$(A \bullet A) \bullet I \xrightarrow{(\varepsilon \bullet A) \bullet I} (J \bullet A) \bullet I \xrightarrow{\cong} A \bullet I \quad \text{and} \quad A \bullet I \xrightarrow{\varepsilon \bullet I} J \bullet I \xrightarrow{\cong} I.$$

This is seen applying the fourth bimonoid axiom in Definition 1.2, counitality of the comonoids I and A , functoriality of both monoidal products and coherence. \square

The most important consequence of Lemma 3.1 is that the relative product \circ_{A^c} reduces to the monoidal product $\circ_I \equiv \circ$ in \mathcal{M} . In particular, the corresponding coequalizer (2.7) is trivial. Hence it exists and it is preserved by any functor.

Lemma 3.1 also implies that in the case of the A -comodule monoid A , the difference between β_Q and β_Q^0 in Proposition 2.8 disappears, they become equal. Moreover, substituting $M' = I$ in (1.10), and $B = A$ in (2.14), the resulting morphisms $\beta_{Q,I}$ and β_Q differ by an isomorphism (a unit constraint in \mathcal{M}). This means that

$$(3.2) \quad \beta_Q = \left((Q \bullet I) \circ A \xrightarrow{(Q \bullet I) \circ \Delta} (Q \bullet I) \circ (A \bullet A) \xrightarrow{\zeta} (Q \circ A) \bullet (I \circ A) \xrightarrow{\cong} (Q \circ A) \bullet A \xrightarrow{\gamma \bullet A} Q \bullet A \right)$$

is an isomorphism for any right A -module (Q, γ) ; that is, β is a natural isomorphism, whenever $(-) \circ A$ is a right Hopf monad on \mathcal{M} . (However, the converse implication needs not be true.) In the rest of this section we study its properties.

Lemma 3.2. *Let A be a bimonoid in a duoidal category \mathcal{M} . Then the natural transformation β in (3.2) obeys the following compatibility with the counit, for any right A -module (Q, γ) .*

$$\begin{array}{ccccc} (Q \bullet I) \circ A & \xrightarrow{(Q \bullet \tau) \circ A} & (Q \bullet J) \circ A & \xrightarrow{\cong} & Q \circ A \\ \beta_Q \downarrow & & & & \downarrow \gamma \\ Q \bullet A & \xrightarrow{Q \bullet \varepsilon} & Q \bullet J & \xrightarrow{\cong} & Q \end{array}$$

Proof. The claim is verified using the counitality of the comonoid A and (1.8). \square

Lemma 3.3. *Let A be a bimonoid in a duoidal category \mathcal{M} . Then the natural transformation β in (3.2) obeys the following compatibility with the unit, for any right A -module (Q, γ) .*

$$\begin{array}{ccc} Q \bullet I & \xrightarrow{\cong} & (Q \bullet I) \circ I \xrightarrow{(Q \bullet I) \circ \eta} (Q \bullet I) \circ A \\ & \searrow^{Q \bullet \eta} & \downarrow \beta_Q \\ & & Q \bullet A \end{array}$$

Proof. This claim follows by the third axiom of a bimonoid in Definition 1.2, unitality of the A -action on Q and one of the unitality axioms in (1.7). \square

Lemma 3.4. *Let A be a bimonoid in a duoidal category \mathcal{M} . Then the natural transformation β in (3.2) obeys the following compatibility with the unit and the comultiplication, for any right A -module (Q, γ) .*

$$\begin{array}{ccccccc} (Q \bullet I) \circ A & \xrightarrow{(Q \bullet \delta) \circ A} & (Q \bullet (I \bullet I)) \circ A & \xrightarrow{\cong} & ((Q \bullet I) \bullet I) \circ A & \xrightarrow{((Q \bullet \eta) \bullet I) \circ A} & ((Q \bullet A) \bullet I) \circ A \\ \beta_Q \downarrow & & & & & & \downarrow \beta_{Q \bullet A} \\ Q \bullet A & \xrightarrow{Q \bullet \Delta} & Q \bullet (A \bullet A) & \xrightarrow{\cong} & (Q \bullet A) \bullet A & & \end{array}$$

Proof. Coassociativity of the comonoid A , one of the associativity axioms in (1.6) and one of the unitality axioms in (1.7) imply the claim. \square

Lemma 3.5. *Let A be a bimonoid in a duoidal category \mathcal{M} . Then the natural transformation β in (3.2) obeys the following compatibility with the unit and the comultiplication, for any object M in \mathcal{M} .*

$$\begin{array}{ccccc} (M \bullet I) \circ A & \xrightarrow{\cong} & ((M \circ I) \bullet I) \circ A & \xrightarrow{((M \circ \eta) \bullet I) \circ A} & ((M \circ A) \bullet I) \circ A \\ (M \bullet I) \circ \Delta \downarrow & & & & \downarrow \beta_{M \circ A} \\ (M \bullet I) \circ (A \bullet A) & \xrightarrow{\zeta} & (M \circ A) \bullet (I \circ A) & \xrightarrow{\cong} & (M \circ A) \bullet A \end{array}$$

Proof. The claim is obtained from the unitality of the monoid A and from coherence and naturality of ζ and of the associativity and unit constraints. \square

3.2. The existence of coinvariants. The aim of this section is to prove — under certain assumptions on a duoidal category \mathcal{M} — that the coinvariant part of any Hopf module over a bimonoid A in \mathcal{M} exists whenever (3.2) is a natural isomorphism.

Proposition 3.6. *Let A be bimonoid in a duoidal category \mathcal{M} such that (3.2) is a natural isomorphism. Then for any A -Hopf module $(X, \gamma : X \circ A \rightarrow X, \rho : X \rightarrow X \bullet A)$, the parallel pair of morphisms*

$$X \bullet I \begin{array}{c} \xrightarrow{\varphi^0 = \rho \bullet I} \\ \xrightarrow{\varphi^1 = ((X \bullet \eta) \bullet I) \cdot \alpha^{-1} \cdot (X \bullet \delta)} \end{array} (X \bullet A) \bullet I$$

is contractible by the functor $H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-) \circ J} \mathcal{M}_J)$.

Proof. Introduce the following morphism to be called θ :

$$\begin{array}{ccc} ((X \bullet A) \bullet I) \circ J & \xrightarrow{(\beta_X^{-1} \bullet I) \circ J} & (((X \bullet I) \circ A) \bullet I) \circ J & \xrightarrow{(((X \bullet I) \circ \varepsilon) \bullet \tau) \circ J} & (((X \bullet I) \circ J) \bullet J) \circ J & \xrightarrow{\cong} \\ & & & & & \\ & & (X \bullet I) \circ (J \circ J) & \xrightarrow{(X \bullet I) \circ \varpi} & (X \bullet I) \circ J & \end{array}$$

It is natural in X by the naturality of β , it is a morphism of J -modules by the associativity of the multiplication $\varpi : J \circ J \rightarrow J$, and it obeys $\theta \cdot (\varphi^1 \circ J) = (X \bullet I) \circ J$ by Lemma 3.3, the fourth bimonoid axiom in Definition 1.2, and the counitality of the comonoid I and unitality of the monoid J . In order to prove the equality $(\varphi^1 \circ J) \cdot \theta \cdot (\varphi^0 \circ J) = (\varphi^0 \circ J) \cdot \theta \cdot (\varphi^0 \circ J)$, note that in the diagram

$$\begin{array}{ccc} (X \bullet I) \circ J & \xrightarrow{\varphi^0 \circ J = (\rho \bullet I) \circ J} & ((X \bullet A) \bullet I) \circ J \\ \downarrow (X \bullet \tau) \circ J & & \downarrow ((X \bullet A) \bullet \tau) \circ J \\ (X \bullet J) \circ J & & ((X \bullet A) \bullet J) \circ J \\ \cong \downarrow & & \cong \downarrow \\ X \circ J & \xrightarrow{\rho \circ J} & (X \bullet A) \circ J \\ & & \downarrow \beta_X^{-1} \circ J \\ ((X \bullet A) \bullet I) \circ A \circ J & \xleftarrow{(\varphi^{0,1} \circ A) \circ J} & ((X \bullet I) \circ A) \circ J \\ \cong \downarrow & & \cong \downarrow \\ ((X \bullet A) \bullet I) \circ (A \circ J) & & (X \bullet I) \circ (A \circ J) \\ \downarrow ((X \bullet A) \bullet I) \circ (\varepsilon \circ J) & & \downarrow (X \bullet I) \circ (\varepsilon \circ J) \\ ((X \bullet A) \bullet I) \circ (J \circ J) & & (X \bullet I) \circ (J \circ J) \\ \downarrow ((X \bullet A) \bullet I) \circ \varpi & & \downarrow (X \bullet I) \circ \varpi \\ ((X \bullet A) \bullet I) \circ J & \xleftarrow{\varphi^{0,1} \circ J} & (X \bullet I) \circ J \end{array} \quad \theta$$

both regions commute by naturality, for either choice φ^0 or φ^1 as $\varphi^{0,1}$. So it suffices to check that

$$(3.3) \quad X \xrightarrow{\rho} X \bullet A \xrightarrow{\beta_X^{-1}} (X \bullet I) \circ A \begin{array}{c} \xrightarrow{\varphi^0 \circ A} \\ \xrightarrow{\varphi^1 \circ A} \end{array} ((X \bullet A) \bullet I) \circ A$$

is a fork. This follows by Lemma 3.4, by coassociativity of the A -coaction on X , and naturality of β together with the fact that the A -coaction on X is a morphism of A -modules. \square

Corollary 3.7. *Let \mathcal{M} be a duoidal category in which idempotent morphisms split and equalizers of $H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-) \circ J} \mathcal{M}_J)$ -contractible equalizer pairs exist. Then for any bimonoid A in \mathcal{M} such that (3.2) is a natural isomorphism, there exists the equalizer*

$$(3.4) \quad X^c \xrightarrow{\iota} X \bullet I \begin{array}{c} \xrightarrow{\varphi^0} \\ \xrightarrow{\varphi^1} \end{array} (X \bullet A) \bullet I.$$

It provides a right adjoint $(-)^c$ of $(-) \circ A : \mathcal{M}^I \rightarrow \mathcal{M}_A^A$.

Proof. By Proposition 3.6, $(H\varphi^0, H\varphi^1)$ is a contractible pair in \mathcal{M}_J . Since idempotent morphisms are assumed to split in \mathcal{M} — hence also in \mathcal{M}_J —, their (contractible) equalizer exists. That is to say, (φ^0, φ^1) is an H -contractible equalizer pair, so their equalizer X^c exists by assumption. The final claim follows by the considerations in Section 2.2, see the text around (2.5). \square

3.3. Fully faithfulness. Let \mathcal{M} be a duoidal category in which idempotent morphisms split and equalizers of $H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-) \circ J} \mathcal{M}_J)$ -contractible equalizer pairs exist. Then combining the results in Section 2 and Section 3.2, for any bimonoid A in \mathcal{M} such that the canonical comonad morphism (3.2) is an isomorphism, we obtain an adjoint triangle

$$(3.5) \quad \begin{array}{ccccc} & & & & \mathcal{M}_A^A \\ & & & & \uparrow \\ & & & & U^A \quad \dashv \quad F^A = (-) \bullet A \\ & & & & \downarrow \\ & & & & \mathcal{M}_A \\ & & & & \uparrow \\ & & & & F_A = (-) \circ A \\ & & & & \downarrow \\ \mathcal{M}^I & \begin{array}{c} \xrightarrow{U^I} \\ \perp \\ \xrightarrow{F^I = (-) \bullet I} \end{array} & \mathcal{M} & \begin{array}{c} \xrightarrow{F_A = (-) \circ A} \\ \perp \\ \xrightarrow{U_A} \end{array} & \mathcal{M}_A \\ & \begin{array}{c} \nearrow K = (-) \circ A \\ \searrow N = (-)^c \end{array} & & & \\ & & & & \end{array}$$

The aim of this section is to find sufficient conditions for K to be fully faithful.

Lemma 3.8. *Let A be a bimonoid in a duoidal category \mathcal{M} such that (3.2) is a natural isomorphism. Then for any right I -comodule Z ,*

$$Z \circ J \xrightarrow{\rho \circ J} (Z \bullet I) \circ J \xrightarrow{\cong} ((Z \circ I) \bullet I) \circ J \xrightarrow{((Z \circ \eta) \bullet I) \circ J} ((Z \circ A) \bullet I) \circ J \begin{array}{c} \xrightarrow{\varphi^0 \circ J} \\ \xrightarrow{\varphi^1 \circ J} \end{array} (((Z \circ A) \bullet A) \bullet I) \circ J$$

is a contractible equalizer in \mathcal{M}_J .

Proof. We know from Proposition 3.6 that for the Hopf module $Z \circ A$, $\theta : (((Z \circ A) \bullet A) \bullet I) \circ J \rightarrow ((Z \circ A) \bullet I) \circ J$ obeys the equalities $\theta.(\varphi^1 \circ J) = ((Z \circ A) \bullet I) \circ J$ and $(\varphi^1 \circ J).\theta.(\varphi^0 \circ J) = (\varphi^0 \circ J).\theta.(\varphi^0 \circ J)$. It follows by the fourth bimonoid axiom in Definition 1.2, by the counitality of the I -coaction on Z and unitality of the monoid J , that

$$\pi := (((Z \circ A) \bullet I) \circ J \xrightarrow{((Z \circ \varepsilon) \bullet \tau) \circ J} ((Z \circ J) \bullet J) \circ J \xrightarrow{\cong} Z \circ (J \circ J) \xrightarrow{Z \circ \varpi} Z \circ J)$$

is a retraction of

$$v := (Z \circ J \xrightarrow{\rho \circ J} (Z \bullet I) \circ J \xrightarrow{\cong} ((Z \circ I) \bullet I) \circ J \xrightarrow{((Z \circ \eta) \bullet I) \circ J} ((Z \circ A) \bullet I) \circ J).$$

Finally, $v.\pi = \theta.(\varphi^0 \circ J)$ by Lemma 3.5, and naturality of the associativity and unit constraints. \square

Theorem 3.9. *Let \mathcal{M} be a duoidal category in which idempotent morphisms split and $H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-) \circ J} \mathcal{M}_J)$ is comonadic. Let A be a bimonoid in \mathcal{M} . If (3.2) is a natural isomorphism, then the functor K in (3.5) is fully faithful.*

Proof. In the diagram

$$\begin{array}{ccccc} (Z \circ A)^c \circ J & \xrightarrow{\iota \circ J} & ((Z \circ A) \bullet I) \circ J & \xrightleftharpoons[\varphi^1 \circ J]{\varphi^0 \circ J} & (((Z \circ A) \bullet A) \bullet I) \circ J \\ \nu_Z \circ J \uparrow & & \parallel & & \parallel \\ Z \circ J & \xrightarrow{v} & ((Z \circ A) \bullet I) \circ J & \xrightleftharpoons[\varphi^1 \circ J]{\varphi^0 \circ J} & (((Z \circ A) \bullet A) \bullet I) \circ J \end{array}$$

in \mathcal{M}_J , the bottom row is an equalizer for any I -comodule Z by Lemma 3.8. By Corollary 3.7, the top row is obtained by applying H to the equalizer of an H -contractible equalizer pair. Since H is comonadic, it preserves such equalizers. Thus the top row is an equalizer too. Recalling from the proof of the Adjoint Lifting Theorem [18] the construction of the unit ν of the adjunction $K \dashv N$, the square on the left of the diagram is seen to commute. Thus the diagram is serially commutative. This proves that $H(\nu_Z) = \nu_Z \circ J$ is an isomorphism. Since H is comonadic, it reflects isomorphisms. This proves that the unit ν of the adjunction $K \dashv N$ is a natural isomorphism. Hence K is fully faithful, see e.g. the dual form of [6, vol. 1 page 114, Proposition 3.4.1]. \square

3.4. The Fundamental Theorem of Hopf modules. Our final task is to find conditions under which the functor K in (3.5) is an equivalence.

Lemma 3.10. *For any bimonoid A in a duoidal category \mathcal{M} , and for any I -comodule Z ,*

$$\vartheta_Z := (Z \circ A \xrightarrow{\rho \circ A} (Z \bullet I) \circ A \xrightarrow{\cong} (Z \bullet I) \circ (J \bullet A) \xrightarrow{\zeta} (Z \circ J) \bullet (I \circ A) \xrightarrow{\cong} (Z \circ J) \bullet A)$$

is equal to $\beta_{Z \circ J}.(\nu_Z \circ A)$. Here ν denotes the unit of the adjunction

$$H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-) \circ J} \mathcal{M}_J) \dashv G := (\mathcal{M}_J \xrightarrow{U_J} \mathcal{M} \xrightarrow{(-) \bullet I} \mathcal{M}^I),$$

β is the natural transformation (3.2) and the A -action on $Z \circ J$ is induced by the counit (a monoid morphism) $\varepsilon : A \rightarrow J$. In particular, if H is fully faithful and (3.2) is a natural isomorphism, then ϑ is a natural isomorphism.

Proof. The claim follows by counitality of the comultiplication $\Delta : A \rightarrow A \bullet A$ and unitality of the multiplication $\varpi : J \circ J \rightarrow J$. \square

Theorem 3.11. *Let \mathcal{M} be a duoidal category in which idempotent morphisms split and $H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-)\circ J} \mathcal{M}_J)$ is fully faithful. Then for any bimonoid A in \mathcal{M} , the following assertions are equivalent.*

- (i) $I \rightarrow A$ is an A -Galois extension.
- (ii) The natural transformation β in (3.2) is an isomorphism.
- (iii) The functor K in (3.5) is an equivalence.

Proof. (i) \Rightarrow (ii). This assertion is trivial.

(ii) \Rightarrow (i). A is an A -comodule monoid by the first and the third axioms of a bimonoid in Definition 1.2. The A -coinvariant part of any A -Hopf module exists by Corollary 3.7. The coinvariant part of A is I , and the corresponding monoid morphism in Proposition 2.5 is the unit $\eta : I \rightarrow A$ of the monoid A , by Lemma 3.1. Then the coequalizers (2.7) are trivial hence they exist and are preserved by any functor — in particular by any power of $(-)\circ A : \mathcal{M} \rightarrow \mathcal{M}$. So if (ii) holds then $\eta : I \rightarrow A$ is an A -Galois extension.

(iii) \Rightarrow (ii). If K in (3.5) is an equivalence then in particular the counit ϵ of the adjunction $K \dashv N$ is a natural isomorphism. Thus β in (3.2) arises as a composite of natural isomorphisms $(Q \bullet I) \circ A \xrightarrow{\cong} (Q \bullet A)^c \circ A \xrightarrow{\epsilon_{Q \bullet A}} Q \bullet A$, cf. (1.5).

(ii) \Rightarrow (iii). Since idempotent morphisms in \mathcal{M} split, they also split in \mathcal{M}^I . Since H is a left adjoint functor and fully faithful (hence separable, in particular) by assumption, it is comonadic by [13, Proposition 3.16]. Thus it follows from Theorem 3.9 that K is fully faithful. So we only need to show that also the counit

$$\epsilon_X = (X^c \circ A \xrightarrow{\iota \circ A} (X \bullet I) \circ A \xrightarrow{(X \bullet \tau) \circ A} (X \bullet J) \circ A \xrightarrow{\cong} X \circ A \xrightarrow{\gamma} X)$$

of the adjunction $K \dashv N$ is a natural isomorphism, for any A -Hopf module X .

Since H is comonadic, it follows by Corollary 3.7 that the image of (3.4) under H is a contractible (thus absolute) equalizer in \mathcal{M}_J — hence also in \mathcal{M} . Thus the bottom row in

$$\begin{array}{ccccc} X^c \circ A & \xrightarrow{\iota \circ A} & (X \bullet I) \circ A & \xrightarrow{\varphi^0 \circ A} & ((X \bullet A) \bullet I) \circ A \\ \vartheta_{X^c, J} \downarrow & & \vartheta_{X \bullet I} \downarrow & & \vartheta_{(X \bullet A) \bullet I} \downarrow \\ (X^c \circ J) \bullet A & \xrightarrow{(\iota \circ J) \bullet A} & ((X \bullet I) \circ J) \bullet A & \xrightarrow{(\varphi^0 \circ J) \bullet A} & (((X \bullet A) \bullet I) \circ J) \bullet A \\ & & & & \xrightarrow{(\varphi^1 \circ J) \bullet A} \end{array}$$

is an equalizer in \mathcal{M} too. The diagram serially commutes by the naturality of ϑ and the vertical arrows are isomorphisms by Lemma 3.10. Then the top row is an equalizer too.

The inverse of ϵ_X is constructed using universality of the equalizer

$$\begin{array}{c}
 X \\
 \swarrow \epsilon_X^{-1} \quad \downarrow \rho \\
 X \bullet A \\
 \downarrow \beta_X^{-1} \\
 X^c \circ A \xrightarrow{\iota \circ A} (X \bullet I) \circ A \xrightarrow[\varphi^1 \circ A]{\varphi^0 \circ A} ((X \bullet A) \bullet I) \circ A
 \end{array}$$

The vertical arrow was shown to equalize the parallel morphisms in the bottom row after (3.3). Hence there is a unique morphism $\epsilon_X^{-1} : X \rightarrow X^c \circ A$ in \mathcal{M} such that $(\iota \circ A) \cdot \epsilon_X^{-1} = \beta_X^{-1} \cdot \rho$. The morphism $\epsilon_X \cdot \epsilon_X^{-1}$ is equal to

$$X \xrightarrow{\rho} X \bullet A \xrightarrow{\beta_X^{-1}} (X \bullet I) \circ A \xrightarrow{(X \bullet \tau) \circ A} (X \bullet J) \circ A \xrightarrow{\cong} X \circ A \xrightarrow{\gamma} X.$$

This is equal to the identity morphism X by Lemma 3.2 and the counitality of the A -coaction on X . Since $\iota \circ A$ is monic, $\epsilon_X^{-1} \cdot \epsilon_X$ is equal to the identity natural transformation $(-)^c \circ A$ if and only if $(\iota \circ A) \cdot \epsilon_X^{-1} \cdot \epsilon_X = \iota \circ A$. That is, if and only if for any Hopf module X ,

$$\begin{aligned}
 (X^c \circ A \xrightarrow{\iota \circ A} (X \bullet I) \circ A \xrightarrow{(X \bullet \tau) \circ A} (X \bullet J) \circ A \xrightarrow{\cong} X \circ A \xrightarrow{\gamma} X \xrightarrow{\rho} X \bullet A) = \\
 (X^c \circ A \xrightarrow{\iota \circ A} (X \bullet I) \circ A \xrightarrow{\beta_X} X \bullet A).
 \end{aligned}$$

This holds by the compatibility condition between the action and the coaction on a Hopf module, the equalizer property of $\iota \circ A$, counitality of the comonoid I and unitality of the monoid A , as well as the explicit form of β_X in (3.2). \square

Remark 3.12. The equivalent conditions in Theorem 3.11 provide an alternative way to define a Hopf monoid A in a duoidal category \mathcal{M} . Although we are not aware of any separating example, the resulting notion does not seem to be equivalent to any of the definition of a Hopf bimonoid in [5] and the property that $(-)^c \circ A$ is a right Hopf monad on (\mathcal{M}, \bullet) (cf. Section 1.2). In fact, it seems to be between these notions: $(-)^c \circ A$ is a right Hopf monad provided that $\beta_{Q, M'}$ in (1.10) is an isomorphism, for any right A -module Q and any object M' of \mathcal{M} . The conditions in Theorem 3.11 assert less: they only say that $\beta_{Q, I}$ is an isomorphism for any right A -module Q and the \circ -monoidal unit I . The definition of a Hopf bimonoid in [5, Definition 9] requires even less: only $\beta_{A, I}$ to be an isomorphism.

In a braided monoidal category \mathcal{M} , there is only one monoidal unit $I = J$. Both of its category of modules and comodules are isomorphic to \mathcal{M} . Thus in this case the functor H in Theorem 3.11 is an isomorphism. In this sense, Theorem 3.11 extends the Fundamental Theorem of Hopf Modules in a braided monoidal category with split idempotents.

More generally, if in a duoidal category \mathcal{M} , the (co)unit $\tau : I \rightarrow J$ is an isomorphism (of monoids and comonoids) then all categories \mathcal{M}^I , \mathcal{M} and \mathcal{M}_J are isomorphic so that the functor H in Theorem 3.11 is an isomorphism. Thus in this case, if idempotent morphisms in \mathcal{M} split, then all assumptions in Theorem 3.11 hold.

3.5. The dual situation. Recall from [2, Section 6.1.2], that also the opposite of a duoidal category is duoidal via the roles of the monoidal structures interchanged. So we can dualize the results in the previous sections without repeating the proofs. It leads to the following.

Theorem 3.13. *Let \mathcal{M} be a duoidal category in which idempotent morphisms split and $G := (\mathcal{M}_J \xrightarrow{U_J} \mathcal{M} \xrightarrow{(-)\bullet I} \mathcal{M}^I)$ is monadic. Let A be a bimonoid in \mathcal{M} . If*

$$(3.6) \quad \varsigma_Q : Q \circ A \xrightarrow{\rho \circ A} (Q \bullet A) \circ A \xrightarrow{\cong} (Q \bullet A) \circ (J \bullet A) \xrightarrow{\zeta} (Q \circ J) \bullet (A \circ A) \xrightarrow{(Q \circ J) \bullet \mu} (Q \circ J) \bullet A$$

is an isomorphism for any A -comodule (Q, ρ) , then the comparison functor $(-)\bullet A : \mathcal{M}_J \rightarrow \mathcal{M}_A^A$ is fully faithful.

Theorem 3.14. *Let \mathcal{M} be a duoidal category in which idempotent morphisms split and $G := (\mathcal{M}_J \xrightarrow{U_J} \mathcal{M} \xrightarrow{(-)\bullet I} \mathcal{M}^I)$ is fully faithful. Then for any bimonoid A in \mathcal{M} , the following assertions are equivalent.*

- (i) *The natural transformation ς in (3.6) is an isomorphism.*
- (ii) *The comparison functor $(-)\bullet A : \mathcal{M}_J \rightarrow \mathcal{M}_A^A$ is an equivalence.*

4. APPLICATIONS AND EXAMPLES

In this section we apply Theorem 3.11 and Theorem 3.14 to the duoidal categories in [2, Example 6.17] and in [2, Example 6.18], respectively. In particular, we prove that the assumptions of these theorems hold in the respective examples.

4.1. The occurrence of idempotent monads. In the examples in the forthcoming sections, we will work with duoidal categories in which the \circ -monoidal unit I induces an idempotent comonad $(-)\bullet I$, or the \bullet -monoidal unit J induces an idempotent monad $(-)\circ J$. Therefore in this section we collect some facts about idempotent (co)monads for later application. As a more general reference, we recommend [6, vol. 2 page 196].

Proposition 4.1. *For a duoidal category \mathcal{M} in which the comultiplication $\delta : I \rightarrow I \bullet I$ on the \circ -monoidal unit I is an isomorphism, the following assertions are equivalent.*

- (i) *The functor $H := (\mathcal{M}^I \xrightarrow{U^I} \mathcal{M} \xrightarrow{(-)\circ J} \mathcal{M}_J)$ is fully faithful.*
- (ii) *For any object M of \mathcal{M} such that $M \bullet \tau$ is an isomorphism, also $(M \circ \tau) \bullet I$ is an isomorphism (where τ is the (co)unit $I \rightarrow J$).*

Proof. Since δ is an isomorphism, $(-)\bullet I : \mathcal{M} \rightarrow \mathcal{M}$ is an idempotent comonad. So \mathcal{M}^I is identified with the full subcategory of \mathcal{M} whose objects are those objects M for which the counit

$$M \bullet I \xrightarrow{M \bullet \tau} M \bullet J \xrightarrow{\cong} M$$

is an isomorphism, equivalently, $M \bullet \tau$ is an isomorphism.

By the dual form of [6, vol. 1 page 114, Proposition 3.4.1], the functor H is fully faithful if and only if the unit

$$\nu_M = \left(M \xrightarrow{\cong} M \bullet J \xrightarrow{(M \bullet \tau)^{-1}} M \bullet I \xrightarrow{\cong} (M \circ I) \bullet I \xrightarrow{(M \circ \tau) \bullet I} (M \circ J) \bullet I \right)$$

of the adjunction $H \dashv (\mathcal{M}_J \xrightarrow{U_J} \mathcal{M} \xrightarrow{(-)\bullet I} \mathcal{M}^I)$ is an isomorphism, for any object M in \mathcal{M}^I ; that is, for any object M in \mathcal{M} such that $M \bullet \tau$ is an isomorphism. This morphism ν_M is an isomorphism if and only if $(M \circ \tau) \bullet I$ is an isomorphism. \square

4.2. The category of spans. In this section we analyze in some detail the duoidal category $\mathbf{span}(X)$ of spans over a given set X . This duoidal category was introduced in [2, Example 6.17], where it was called the “category of directed graphs with vertex set X ”.

The objects of $\mathbf{span}(X)$ are triples (M, t, s) , where M is a set and s and t are maps $M \rightarrow X$, called the *source* and *target* maps, respectively. The morphisms in $\mathbf{span}(X)$ are maps $f : M \rightarrow M'$ such that $s'.f = s$ and $t'.f = t$.

For any spans M and N over X , one monoidal structure is given by the pullback

$$M \circ N = \{(m, n) \in M \times N \mid s(m) = t(n)\} \quad \text{and} \quad I = X$$

and the other monoidal structure is

$$M \bullet N = \{(m, n) \in M \times N \mid s(m) = s(n), t(m) = t(n)\} \quad \text{and} \quad J = X \times X.$$

The interchange law takes the form

$$\zeta : (M \bullet N) \circ (M' \bullet N') \rightarrow (M \circ M') \bullet (N \circ N'), \quad (m, n, m', n') \mapsto (m, m', n, n').$$

The \circ -monoidal unit I is a comonoid with respect to \bullet via the comultiplication

$$\delta : I \rightarrow I \bullet I = \{(x, y) \in X \times X \mid x = y\} \cong I, \quad x \mapsto (x, x) \cong x.$$

The \bullet -monoidal unit J is a monoid with respect to \circ via the multiplication

$$\varpi : J \circ J = \{(x, y, x', y') \mid y = x'\} \rightarrow J, \quad (x, y = x', y') \mapsto (x, y').$$

The counit of the comonoid I and the unit of the monoid J are both given by

$$\tau : I \rightarrow J, \quad x \mapsto (x, x).$$

The monad $(-)\circ J$ and the comonad $(-)\bullet I$ on $\mathbf{span}(X)$ have the respective object maps

$$M \circ J \cong M \times X \quad \text{and} \quad M \bullet I \cong \{m \in M \mid s(m) = t(m)\}.$$

Let us turn to showing that $\mathbf{span}(X)$ satisfies all assumptions made on a duoidal category in Theorem 3.11.

Since $(-)\bullet I$ is an idempotent comonad on $\mathbf{span}(X)$, its category of comodules is isomorphic to the full subcategory of $\mathbf{span}(X)$ whose objects are those spans (Z, s, t) for which the counit $Z \bullet I \xrightarrow{Z \bullet \tau} Z \bullet J \xrightarrow{\cong} Z$ is an isomorphism. An equivalent description of $\mathbf{span}(X)^I$ is the following.

Lemma 4.2. *The category $\mathbf{span}(X)^I$ of I -comodules is isomorphic to the slice category \mathbf{set}/X regarded as the full subcategory of $\mathbf{span}(X)$ whose objects are those spans (Z, s, t) for which $s = t$.*

Proof. For any span Z over X , the map

$$\{z \in Z \mid s(z) = t(z)\} \xrightarrow{\cong} Z \bullet I \xrightarrow{Z \bullet \tau} Z \bullet J \xrightarrow{\cong} Z$$

is just the inclusion map, what proves that $Z \bullet \tau$ is an isomorphism if and only if the source and target maps on Z are equal. \square

Proposition 4.3. *For any span Z over X with equal source and target maps, $(Z \circ \tau) \bullet I$ is an isomorphism.*

Proof. For any span Z over X , the map

$$\{z \in Z \mid s(z) = t(z)\} \cong Z \bullet I \xrightarrow{\cong} (Z \circ I) \bullet I \xrightarrow{(Z \circ \tau) \bullet I} (Z \circ J) \bullet I \cong Z$$

is again the inclusion map. Hence it is an isomorphism whenever the source and target maps of Z are equal. \square

Proposition 4.4. *Idempotent morphisms in $\text{span}(X)^I$ split.*

Proof. For any idempotent morphism $e : M \rightarrow M$ in $\text{span}(X)$, also $\text{Im}(e) := \{e(m) \mid m \in M\}$ is a span over X via the restrictions of the source and target maps of M . Hence the epimorphism $M \rightarrow \text{Im}(e)$, $m \mapsto e(m)$ and the monomorphism $\text{Im}(e) \rightarrow M$, $e(m) \mapsto e(m)$ provide a splitting of e in $\text{span}(X)$. This proves that idempotent morphisms split in $\text{span}(X)$ hence they also split in the full subcategory $\text{span}(X)^I \cong \text{set}/X$, cf. Lemma 4.2. \square

From Proposition 4.3 and Proposition 4.1 we conclude that the functor H in Theorem 3.11 is fully faithful. So taking into account also Proposition 4.4, we see that Theorem 3.11 holds in the duoidal category $\text{span}(X)$. Our next task is to identify its bimonoids for which the canonical comonad morphism (3.2) is a natural isomorphism.

Recall from [2, Example 6.43] that a monoid in $(\text{span}(X), \circ, I)$ is precisely a small category with object set X . So is a bimonoid in $\text{span}(X)$ since the monoidal product \bullet is the categorical product. On any elements a and b of a (bi)monoid such that $s(a) = t(b)$, we denote the multiplication by $\mu(a, b) =: a.b$.

A right module over a bimonoid A in $\text{span}(X)$ is a span Q over X equipped with a map of spans $Q \circ A = \{(q, a) \mid s(q) = t(a)\} \rightarrow Q$, $(q, a) \mapsto q.a$ which is associative and unital in the evident sense. The natural transformation (3.2) takes the explicit form

$$(4.1) \quad \begin{aligned} \beta_Q : (Q \bullet I) \circ A \cong \{(q, a) \mid s(q) = t(q) = t(a)\} &\rightarrow Q \bullet A \cong \{(q, a) \mid s(q) = s(a), t(q) = t(a)\}, \\ (q, a) &\mapsto (q.a, a). \end{aligned}$$

Proposition 4.5. *Let A be a bimonoid in the duoidal category $\text{span}(X)$; that is a small category with object set X . The corresponding canonical comonad morphism (4.1) is an isomorphism if and only if every element in the monoid A is invertible; that is, A is a groupoid.*

Proof. If every element in A is invertible, then we construct the inverse of (4.1) as

$$Q \bullet A \rightarrow (Q \bullet I) \circ A, \quad (q, a) \mapsto (q.a^{-1}, a).$$

Conversely, assume that (4.1) is a natural isomorphism. Then it is an isomorphism, in particular, for $Q = A$. Taking into account the explicit form of (4.1), the inverse of β_A can be written as

$\beta_A^{-1}(b, a) := (b^a, a)$, in terms of some function b^a of a and b (such that $s(a) = s(b)$ and $t(a) = t(b)$), satisfying the conditions

$$(4.2) \quad \begin{aligned} b^a \cdot a &= b, & \text{for } a, b \in A \text{ such that } s(a) = s(b), t(a) = t(b), \\ (b \cdot a)^a &= b, & \text{for } a, b \in A \text{ such that } t(b) = s(b) = t(a). \end{aligned}$$

Introducing the notation $M_{x,y} = \{m \in M \mid s(m) = x, t(m) = y\}$, for any span M over X , β_A induces bijections

$$(\beta_A)_{x,y} : ((A \bullet I) \circ A)_{x,y} \rightarrow (A \bullet A)_{x,y}, \quad (b, a) \mapsto (b \cdot a, a).$$

Any element $c \in A$ such that $s(c) = t(c) = y$, induces two maps

$$\begin{aligned} \varphi_c : ((A \bullet I) \circ A)_{x,y} &\rightarrow ((A \bullet I) \circ A)_{x,y}, & (b, a) &\mapsto (c \cdot b, a) \\ \psi_c : (A \bullet A)_{x,y} &\rightarrow (A \bullet A)_{x,y}, & (b, a) &\mapsto (c \cdot b, a) \end{aligned}$$

rendering commutative the diagram

$$\begin{array}{ccc} ((A \bullet I) \circ A)_{x,y} & \xrightarrow{(\beta_A)_{x,y}} & (A \bullet A)_{x,y} \\ \varphi_c \downarrow & & \downarrow \psi_c \\ ((A \bullet I) \circ A)_{x,y} & \xrightarrow{(\beta_A)_{x,y}} & (A \bullet A)_{x,y}. \end{array}$$

Equivalently, inverting the horizontal arrows,

$$(4.3) \quad c \cdot b^a = (c \cdot b)^a, \quad \text{for } a, b, c \in A \text{ such that } s(a) = s(b), t(a) = t(b) = s(c) = t(c).$$

For any $x \in X$, denote by 1_x the unit morphism at x ; i.e. the image of x under the unit $\eta : I = X \rightarrow A$. For any $c \in A$ such that $s(c) = t(c) = x$, it follows by the first condition in (4.2) that $(1_x)^c \cdot c = 1_x$. By (4.3) and the second condition in (4.2), also $c \cdot (1_x)^c = c^c = 1_x$. So c is invertible with the inverse $(1_x)^c$.

Next we show any morphism from x to y — i.e. any $a \in A$ such that $s(a) = x$ and $t(a) = y$ — is invertible whenever the set $A_{y,x}$ is non-empty; i.e. there is at least one arrow from y to x . Indeed, take $a \in A_{x,y}$ and $b \in A_{y,x}$. Then $a \cdot b \in A_{y,y}$ and $b \cdot a \in A_{x,x}$ are invertible by the previous paragraph; i.e. $(b \cdot a)^{-1} \cdot b \cdot a = 1_x$ and $a \cdot b \cdot (a \cdot b)^{-1} = 1_y$. This implies that a is invertible with the inverse $(b \cdot a)^{-1} \cdot b = b \cdot (a \cdot b)^{-1}$.

Thus the proof is completed if we show that whenever $A_{x,y}$ is a non-empty set then also $A_{y,x}$ must be non-empty. Equivalently, if we show that whenever $A_{x,y}$ is an empty set then also $A_{y,x}$ must be empty. Assuming that $A_{x,y} = \emptyset$ for some $x \neq y \in X$, below we construct an appropriate A -module Q such that the corresponding map β_Q in (4.1) has a non-trivial kernel unless $A_{y,x} = \emptyset$.

Fix $x, y \in X$ such that $A_{x,y} = \emptyset$. Take Q to be the span consisting of two arrows from u to x if $A_{x,u}$ is non-empty and one arrow from u to x if $A_{x,u}$ is empty. That is,

$$Q := \{u \xrightarrow{q_u} x, u \xrightarrow{p_u} x \mid u \in X, A_{x,u} \neq \emptyset\} \cup \{u \xrightarrow{r_u} x \mid u \in X, A_{x,u} = \emptyset\}.$$

Note that if $A_{x,s(a)}$ is non-empty for some $a \in A$, then also $A_{x,t(a)}$ is non-empty (an element is obtained by composing with a). An associative and unital A -action on Q is defined by the

prescriptions

$$\begin{aligned} q_{t(a)}.a = q_{s(a)} & \quad \text{and} \quad p_{t(a)}.a = p_{s(a)} & \quad \text{if } A_{x,s(a)} \neq \emptyset \text{ and } A_{x,t(a)} \neq \emptyset \\ q_{t(a)}.a = r_{s(a)} & \quad \text{and} \quad p_{t(a)}.a = r_{s(a)} & \quad \text{if } A_{x,s(a)} = \emptyset \text{ and } A_{x,t(a)} \neq \emptyset \\ r_{t(a)}.a = r_{s(a)} & & \quad \text{if } A_{x,s(a)} = \emptyset \text{ and } A_{x,t(a)} = \emptyset. \end{aligned}$$

The set $A_{x,x}$ is non-empty since it contains at least the unit arrow 1_x . Hence there are two different elements p_x and q_x in Q . If there is at least one element b in $A_{y,x}$, then it obeys

$$\beta_Q(p_x, b) = (p_x.b, b) = (r_y, b) = (q_x.b, b) = \beta_Q(q_x, b).$$

Thus β_Q has a non-trivial kernel whenever $A_{y,x}$ is non-empty; which contradicts the assumption that β_Q is an isomorphism. So we proved that $A_{y,x}$ is an empty set whenever $A_{x,y}$ is empty. \square

Owing to the fact that the monoidal product \bullet is the categorical product, a comodule for a comonoid A in $\text{span}(X)$ can be described as a span P over X equipped with a map of spans $c : P \rightarrow A$. The corresponding coaction sends $p \in P$ to $(p, c(p))$. A morphism of A -comodules is a map of spans $f : P \rightarrow P'$ such that $c'.f = c$.

A Hopf module over a bimonoid A in $\text{span}(X)$ — that is, over a small category A with object set X — is an A -module Q equipped with a morphism of A -modules $c : Q \rightarrow A$. A morphism of A -Hopf modules is a map of A -modules $f : Q \rightarrow Q'$ such that $c'.f = c$.

From Theorem 3.11 and Proposition 4.5, we obtain the following.

Corollary 4.6. *For a small category A with object set X , the following assertions are equivalent.*

- (i) *A is a groupoid.*
- (ii) *The natural transformation in (4.1) is an isomorphism.*
- (iii) *The canonical comparison functor — from the slice category set/X to the category of A -Hopf modules — is an equivalence.*

4.3. The category of bimodules. Let k be a commutative, associative and unital ring. Throughout the section, the unadorned symbol \otimes denotes the k -module tensor product. Let R be a commutative, associative and unital k -algebra. Its multiplication will be denoted by juxtaposition on the elements. Denote by $\text{bim}(R)$ the category of R -bimodules. In [2, Example 6.18], it was shown to carry a duoidal structure as follows. For any R -bimodules M and N , one of the monoidal structures is provided by the usual R -bimodule tensor product

$$M \bullet N := M \otimes N / \{m \cdot r \otimes n - m \otimes r \cdot n \mid r \in R\} \quad \text{and} \quad J = R.$$

The other monoidal structure is given by an $R \otimes R$ -bimodule tensor product

$$M \circ N := M \otimes N / \{r \cdot m \cdot r' \otimes n - m \otimes r \cdot n \cdot r' \mid r, r' \in R\} \quad \text{and} \quad I = R \otimes R.$$

The interchange law has the form

$$\zeta : (M \bullet N) \circ (M' \bullet N') \rightarrow (M \circ M') \bullet (N \circ N'), \quad (m \bullet n) \circ (m' \bullet n') \mapsto (m \circ m') \bullet (n \circ n').$$

The \circ -monoidal unit I is a comonoid with respect to \bullet via the comultiplication

$$\delta : I \rightarrow I \bullet I, \quad x \otimes y \mapsto (x \otimes 1_R) \bullet (1_R \otimes y).$$

The \bullet -monoidal unit J is a monoid with respect to \circ via the multiplication

$$\varpi : J \circ J \rightarrow J, \quad a \circ b \mapsto ab.$$

The counit of the comonoid I , and the unit of the monoid J are both given by

$$\tau : I \rightarrow J, \quad a \otimes b \mapsto ab.$$

The comonad $(-)\bullet I$ and the monad $(-)\circ J$ on $\mathbf{bim}(R)$ have the respective object maps

$$M \bullet I \cong M \otimes R \quad \text{and} \quad M \circ J \cong M/[M, R].$$

The isomorphism

$$M \circ J \cong M \otimes R / \{x \cdot m \cdot y \otimes r - m \otimes xry \mid x, y \in R\} \cong M/[M, R] = M/\{m \cdot r - r \cdot m \mid r \in R\}$$

is established by the mutually inverse maps

$$\begin{aligned} M \circ J &\rightarrow M/[M, R], & m \circ r = m \cdot r \circ 1_R = r \cdot m \circ 1_R &\mapsto [r \cdot m] = [m \cdot r] \quad \text{and} \\ M/[M, R] &\rightarrow M \circ J, & [m] &\mapsto m \circ 1_R. \end{aligned}$$

In particular, $J \circ J \cong R/[R, R] \cong R = J$, via the isomorphism provided by the multiplication ϖ and its inverse $\varpi^{-1} : r \mapsto r \circ 1_R = 1_R \circ r$. Thus the monad $(-)\circ J$ on $\mathbf{bim}(R)$ is idempotent. So the category $\mathbf{bim}(R)_J$ of its modules is isomorphic to the full subcategory of $\mathbf{bim}(R)$ whose objects are those R -bimodules M for which the unit

$$(4.4) \quad M \xrightarrow{\cong} M \circ I \xrightarrow{M \circ \tau} M \circ J$$

is an isomorphism. Another equivalent description of J -modules can be given as follows.

Lemma 4.7. *The category $\mathbf{bim}(R)_J$ of J -modules is isomorphic to the category $\mathbf{mod}(R)$ of R -modules — regarded as the full subcategory of $\mathbf{bim}(R)$ on whose objects the left and right R -actions coincide.*

Proof. For any R -bimodule M , the map $M \rightarrow M \circ J \cong M/[M, R]$ in (4.4) is the canonical projection. It is an isomorphism if and only if $[M, R] = 0$; that is, the left and right R -actions on M coincide. \square

In what follows, we check that the assumptions of Theorem 3.14 hold in $\mathbf{bim}(R)$.

Proposition 4.8. *If $M \circ \tau$ is an isomorphism for some R -bimodule M , then $(M \bullet \tau) \circ J$ is an isomorphism too.*

Proof. For any R -bimodule M , the map $(M \bullet \tau) \circ J$ is an isomorphism if and only if

$$M \xrightarrow{\cong} M \otimes R/[M \otimes R, R] \xrightarrow{\cong} (M \bullet I) \circ J \xrightarrow{(M \bullet \tau) \circ J} (M \bullet J) \circ J \xrightarrow{\cong} M \circ J \xrightarrow{\cong} M/[M, R]$$

is an isomorphism. The first isomorphism is established by the mutually inverse maps $M \rightarrow M \otimes R/[M \otimes R, R]$, $m \mapsto [m \otimes 1_R]$ and $M \otimes R/[M \otimes R, R] \rightarrow M$, $[m \otimes r] \mapsto r \cdot m$. The displayed map is the canonical projection. So it is an isomorphism if and only if $[M, R] = 0$. Equivalently, by Lemma 4.7, if and only if $M \circ \tau$ is an isomorphism. \square

Idempotent morphisms in any module category split (through the image). We conclude by Proposition 4.8 and the dual form of Proposition 4.1 that the functor G in Theorem 3.14 is fully faithful. So we can apply Theorem 3.14 to the duoidal category $\mathbf{bim}(R)$. Our next task is to identify those bimonoids A in $\mathbf{bim}(R)$ for which the canonical monad morphism ζ in (3.6) is a natural isomorphism.

Recall from [2, Example 6.44] that a monoid A in $\mathbf{bim}(R)$ can be described equivalently as a k -algebra A equipped with algebra homomorphisms s and t from R to the center of A . (The algebra homomorphisms s and t are related to the unit $\eta : I = R \otimes R \rightarrow A$ by $s = \eta(- \otimes 1_R)$ and $t = \eta(1_R \otimes -)$.) The left and right R -actions on A come out as

$$(4.5) \quad r \cdot a = s(r)a = as(r) \quad \text{and} \quad a \cdot r = t(r)a = at(r).$$

A comonoid in $\mathbf{bim}(R)$ is the usual notion of R -coring; that is, an R -bimodule A equipped with a coassociative comultiplication $\Delta : A \rightarrow A \bullet A$ with counit $\varepsilon : A \rightarrow R$, such that both the comultiplication and the counit are R -bimodule maps. For the comultiplication $A \rightarrow A \bullet A$ we use a Sweedler type index notation $a \mapsto a_1 \bullet a_2$, where implicit summation is understood.

Finally, a bimonoid A in $\mathbf{bim}(R)$ is precisely an R -bialgebroid — called a “ \times_R -bialgebra” in [24] — whose unit maps s and t land in the center of A . Explicitly, it obeys the following axioms (see [20, Appendix A1] for the case when also A is a commutative algebra).

- A is a k -algebra equipped with algebra homomorphisms s and t from R to the center of A ,
- the R -bimodule (4.5) carries an R -coring structure,
- the comultiplication $\Delta : A \rightarrow A \bullet A$ and the counit $\varepsilon : A \rightarrow R$ are algebra homomorphisms.

A right comodule over a bimonoid A in $\mathbf{bim}(R)$ is an R -bimodule Q equipped with a coassociative and counital coaction $Q \rightarrow Q \bullet A$ which is a morphism of R -bimodules. For the coaction $Q \rightarrow Q \bullet A$ we use a Sweedler type index notation $q \mapsto q_0 \bullet q_1$, where implicit summation is understood. For any right A -comodule Q , the natural transformation ζ in (3.6) takes the following explicit form.

$$(4.6) \quad \zeta_Q : Q \circ A \rightarrow (Q \circ J) \bullet A \cong Q/[Q, R] \bullet A, \quad q \circ a \mapsto [q_0] \bullet q_1 a.$$

Recall from [7] (and the references therein, in particular [20] in the commutative case) that an R -bialgebroid A as above is said to be a Hopf algebroid — with left bialgebroid structure as above and right bialgebroid structure obtained by interchanging the roles of s and t — if in addition there exists a k -module map $S : A \rightarrow A$ — called the *antipode* — such that, for all $a \in A$ and $r \in R$,

$$(4.7) \quad \begin{aligned} S(as(r)) &= t(r)S(a), & S(t(r)a) &= S(a)s(r), \\ a_1 S(a_2) &= s(\varepsilon(a)), & S(a_1)a_2 &= t(\varepsilon(a)). \end{aligned}$$

Proposition 4.9. *For a bimonoid A in $\mathbf{bim}(R)$ — that is, for an R -bialgebroid A such that the images of the unital maps s and t are central in A — the natural transformation (4.6) is an isomorphism if and only if A is a Hopf algebroid.*

Proof. If A is a Hopf algebroid, then the inverse of (4.6) is given by

$$\zeta_Q^{-1} : Q/[Q, R] \bullet A \rightarrow Q \circ A, \quad [q] \bullet a \mapsto q_0 \circ S(q_1)a.$$

In order to see that it is well defined, note that — since the A -coaction on Q is morphism of R -bimodules, by (4.5) and the first line in (4.7), — $(r \cdot q)_0 \circ S((r \cdot q)_1)a = (q \cdot r)_0 \circ S((q \cdot r)_1)a =$

$q_0 \circ S(q_1)(r \cdot a)$, for all $a \in A$, $q \in Q$ and $r \in R$. Also — by (4.5) and the first line in (4.7), — $q \cdot r \circ S(b)a = q \circ S(r \cdot b)a$, for all $a, b \in A$, $q \in Q$ and $r \in R$. It is indeed the inverse of (4.6) since

$$\varsigma_Q^{-1} \varsigma_Q(q \circ a) = q_0 \circ S(q_1)q_2a = q_0 \circ t(\varepsilon(q_1))a = q_0 \circ a \cdot \varepsilon(q_1) = q_0 \cdot \varepsilon(q_1) \circ a = q \circ a,$$

$$\varsigma_Q \varsigma_Q^{-1}([q] \bullet a) = [q_0] \bullet q_1 S(q_2)a = [q_0] \bullet s(\varepsilon(q_1))a = [q_0] \bullet \varepsilon(q_1) \cdot a = [q_0] \cdot \varepsilon(q_1) \bullet a = [q_0 \cdot \varepsilon(q_1)] \bullet a = [q] \bullet a.$$

Conversely, assume that (4.6) is an isomorphism, for any right A -comodule Q . Then it is an isomorphism in particular for $Q = I \bullet A \cong R \otimes A$ with right R -action $(r \otimes a)r' = r \otimes at(r')$ and A -coaction $r \otimes a \mapsto (r \otimes a_1) \bullet a_2$. So we obtain an isomorphism

(4.8)

$$A \star A := A \otimes A / \{a \cdot r \otimes b - a \otimes b \cdot r \mid r \in R\} \xrightarrow{\cong} (I \bullet A) \circ A \xrightarrow{\varsigma_{I \bullet A}} ((I \bullet A) \circ J) \bullet A \xrightarrow{\cong} A \bullet A,$$

to be denoted by $\widehat{\varsigma}$. The first isomorphism in (4.8) is established by the mutually inverse maps

$$\begin{aligned} A \star A &\rightarrow (I \bullet A) \circ A, & a \star b &\mapsto ((1_R \otimes 1_R) \bullet a) \circ b, & \text{and} \\ (I \bullet A) \circ A &\rightarrow A \star A, & ((r \otimes r') \bullet a) \circ b &\mapsto r' \cdot a \star r \cdot b. \end{aligned}$$

The last isomorphism in (4.8) is established by the mutually inverse maps

$$\begin{aligned} (I \bullet A) \circ J &\cong I \bullet A / [I \bullet A, R] \rightarrow A, & [(r \otimes r') \bullet a] &\mapsto r' \cdot a \cdot r & \text{and} \\ A &\rightarrow (I \bullet A) \circ J \cong I \bullet A / [I \bullet A, R], & a &\mapsto [(1_R \otimes 1_R) \bullet a]. \end{aligned}$$

With these isomorphisms at hand, the explicit form of (4.8) is $\widehat{\varsigma}(a \star b) = a_1 \bullet a_2 b$, for $a, b \in A$. Set $a^+ \star a^- := \widehat{\varsigma}^{-1}(a \bullet 1_A)$; then $\widehat{\varsigma}^{-1}(a \bullet b) = a^+ \star a^- b$, for all $a, b \in A$, since $\widehat{\varsigma}$ and thus also its inverse are right A -module maps. Put $S(a) := t(\varepsilon(a^+))a^-$, for all $a \in A$. It is well-defined since ε is a right R -module map, since t is multiplicative and by (4.5). We claim that S is an antipode of A .

Since $\widehat{\varsigma}$ is a morphism of right A -modules, so is its inverse. Hence

$$(t(r)a)^+ \star (t(r)a)^- = \widehat{\varsigma}^{-1}(t(r)a \bullet 1_A) = \widehat{\varsigma}^{-1}(a \bullet s(r)) = a^+ \star a^- s(r), \quad \forall a \in A, r \in R.$$

Since the comultiplication on A is a morphism of left R -modules, so is $\widehat{\varsigma}$. Hence also its inverse is a morphism of left R -modules in the sense that

$$(as(r))^+ \star (as(r))^- = a^+ s(r) \star a^-, \quad \forall a \in A, r \in R.$$

With these identities at hand,

$$\begin{aligned} S(t(r)a) &= t(\varepsilon((t(r)a)^+))(t(r)a)^- = t(\varepsilon(a^+))a^- s(r) = S(a)s(r) & \text{and} \\ S(as(r)) &= t(\varepsilon((as(r))^+))(as(r))^- = t(\varepsilon(a^+ s(r)))a^- = t(r)t(\varepsilon(a^+))a^- = t(r)S(a), \end{aligned}$$

for any $a \in A, r \in R$. The penultimate equality in the second line holds since ε is a morphism of left R -modules and t is multiplicative.

From $\widehat{\varsigma}^{-1} \cdot \widehat{\varsigma} = A \star A$, it follows that

$$(4.9) \quad a_1^+ \star a_1^- a_2 = a \star 1_A, \quad \forall a \in A.$$

Since $\widehat{\varsigma}$ is a left A -comodule map; i.e. $(\Delta \bullet A) \cdot \widehat{\varsigma} = (A \bullet \widehat{\varsigma}) \cdot (\Delta \star A)$, also $\widehat{\varsigma}^{-1}$ is a left A -comodule map. That is,

$$(4.10) \quad a^+ \bullet a^+ \star a^- = a_1 \bullet a_2^+ \star a_2^-, \quad \forall a \in A.$$

Composing both sides of the equality $\mu = (A \star A \xrightarrow{\widehat{\zeta}} A \bullet A \xrightarrow{\varepsilon \bullet A} J \bullet A \xrightarrow{\cong} A)$ by $\widehat{\zeta}^{-1}$, we obtain

$$(4.11) \quad a^+ a^- = s(\varepsilon(a)), \quad \forall a \in A.$$

Then by (4.9) it follows that

$$S(a_1)a_2 = t(\varepsilon(a_1^+)a_1^-)a_2 = t(\varepsilon(a))1_A = t(\varepsilon(a)),$$

and by (4.10) and (4.11),

$$a_1 S(a_2) = a_1 t(\varepsilon(a_2^+)a_2^-) = a_1^+ t(\varepsilon(a_2^+)a_2^-) = a_1^+ a^- = s(\varepsilon(a)),$$

for any $a \in A$. This proves that A is a Hopf algebroid. \square

A right module over a bimonoid A in $\mathbf{bim}(R)$ is, equivalently, a right module over the constituent k -algebra. It is an R -bimodule via the actions induced by s and t . A morphism of A -modules in $\mathbf{bim}(R)$ is a morphism of modules over the constituent k -algebras; it is automatically a morphism of R -bimodules.

A right comodule of a bimonoid A in $\mathbf{bim}(R)$ is an R -bimodule Q equipped with a coassociative and counital coaction $Q \rightarrow Q \bullet A$ which is a morphism of R -bimodules. A morphism of A -comodules in $\mathbf{bim}(R)$ is an R -bimodule map which is compatible with the coactions in the evident sense.

A Hopf module M over a bimonoid A in $\mathbf{bim}(R)$ is a right A -module which is also an A -comodule via the left and right R -actions induced by s and t , respectively; such that the compatibility condition $(m \cdot a)_0 \bullet (m \cdot a)_1 = m_0 \cdot a_1 \bullet m_1 a_2$ holds, for all $m \in M$ and $a \in A$. A morphism of A -Hopf modules is a morphism of modules over the constituent k -algebras — hence also a morphism of R -bimodules — which is compatible with the coactions in the evident sense.

From Theorem 3.14 and Proposition 4.9, we obtain the following.

Corollary 4.10. *Let R be a commutative algebra over a commutative ring k . Let A be an R -bialgebroid whose unit $R \otimes R \rightarrow A$ takes its values in the center of A — equivalently, let A be a bimonoid in the duoidal category $\mathbf{bim}(R)$. Then the following assertions are equivalent.*

- (i) *A is a Hopf algebroid (via the given left bialgebroid structure and the right bialgebroid structure obtained by interchanging the roles of the unital maps s and t).*
- (ii) *The natural transformation (4.6) is an isomorphism.*
- (iii) *The canonical comparison functor — from the category of R -modules to the category of A -Hopf modules — is an equivalence.*

Note added. Soon after we had submitted the first version of this paper (on the 5th of December in 2012), two closely related papers [1] and [14] appeared in the arXiv (although [1] was submitted for publication much earlier). Their relation to our work is analyzed in [14], here we shortly recall that on the request of the referee.

In [1], Aguiar and Chase study the following situation. They consider a bimonad T on a monoidal category \mathcal{C} (which can be taken to be e.g. the bimonad $(-) \circ A$ induced on (\mathcal{M}, \bullet) by a bimonoid A in a duoidal category \mathcal{M}); a T -comodule monad S (which can be chosen to be $(-) \circ A$ as well); and an arbitrary comonoid c in \mathcal{C} (which can be taken to be e.g. the \circ -monoidal unit I in the duoidal category \mathcal{M}). Associated to these data, there is a category of generalized Hopf

modules (with our choices it comes out as the category \mathcal{M}_A^A of Hopf modules in Definition 2.2); and a comparison functor K from the category of c -comodules to this category of generalized Hopf modules (it reduces to the same functor K in (3.5) in our case). In [1, Theorem 5.8], under certain assumptions on S and c , it is proven that K is an equivalence if and only if a canonical ‘Galois morphism’ (reducing to (3.2) with our choices) is an isomorphism.

Comparing the assumptions in [1, Theorem 5.8] with the dual forms of the Beck criteria (see [3, page 100, Theorem 3.14], cf. Section 1.1), they imply in turn that the composite of the forgetful functor U_c corresponding to the category of c -comodules, and of the left adjoint F_S of the forgetful functor corresponding to the Eilenberg-Moore category of S -modules, is comonadic (whence the conclusion follows by [11, Theorem 1.7] or [13, Theorem 4.4], see Section 1.1). Indeed, $F_S U_c$ is a composite of two left adjoint functors hence it is left adjoint. Since S is assumed to be conservative, so is F_S and thus also the composite functor $F_S U_c$. Taking an $F_S U_c$ -contractible equalizer pair (f, g) , it is taken by U_c to an F_S -contractible equalizer pair $(U_c f, U_c g)$. By assumption, their equalizer is created by F_S . Moreover, by the assumption that $(-) \otimes c$ and $(-) \otimes c \otimes c$ preserve the equalizer of $(U_c f, U_c g)$, it follows that U_c creates the equalizer of (f, g) .

Applying [1, Theorem 5.8] to the comparison functor K in (3.5), this means that *assumptions are made on A* , which imply the comonadicity of the functor in the bottom row of (3.5). As a conceptual difference, in Section 3 of this paper we make no assumption on A . We prove the comonadicity of the functor in the bottom row of (3.5) from assumptions *on the duoidal category \mathcal{M} alone*.

In Mesablishvili and Wisbauer’s paper [14], a slight generalization and an alternative proof of our Theorem 3.11 is presented. In their generalized version, the functor in the bottom row of the diagram below is only required to be separable (not necessarily fully faithful). In their proof, they avoid the explicit construction of the inverse of the comparison functor K in (3.5). Instead, they observe that there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}^I & \xrightarrow{U^I} & \mathcal{M} & \xrightarrow{(-) \circ A} & \widetilde{\mathcal{M}}_A \\ \parallel & & & & \downarrow (-) \circ A J \\ \mathcal{M}^I & \xrightarrow{U^I} & \mathcal{M} & \xrightarrow{(-) \circ J} & \mathcal{M}_J \end{array}$$

where $\widetilde{\mathcal{M}}_A$ stands for the Kleisli category of the monad $(-) \circ A$ (i.e. the category of free right A -modules). Since the left adjoint functor in the bottom row is separable, and idempotent morphisms in \mathcal{M} (and thus in \mathcal{M}^I) split by assumption, it follows by [14, Proposition 1.13] that the functor $U^I(-) \circ A$ in the top row reflects isomorphisms and any $U^I(-) \circ A$ -contractible pair possesses a contractible equalizer in \mathcal{M}^I . These properties imply that composing the functor in the top row with the fully faithful embedding $\widetilde{\mathcal{M}}_A \rightarrow \mathcal{M}_A$, we obtain a comonadic functor: that in the bottom row of (3.5). In light of [11, Theorem 1.7] or [13, Theorem 4.4] (see Section 1.1), this provides an alternative proof of Theorem 3.11 (although not yielding the explicit form of the inverse of the comparison functor K in (3.5)).

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