Some computational results on small 3-nets embedded in a projective plane over a field

G. P. Nagy and N. Pace.

Abstract

In this paper, we investigate dual 3-nets realizing the groups $C_3 \times C_3$, $C_2 \times C_4$, Alt_4 and that can be embedded in a projective plane $PG(2,\mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field. We give a symbolically verifiable computational proof that every dual 3-net realizing the groups $C_3 \times C_3$ and $C_2 \times C_4$ is algebraic, namely, that its points lie on a plane cubic. Moreover, we present two computer programs whose calculations show that the group Alt_4 cannot be realized if the characteristic of $\mathbb{K}$ is zero.

1 Introduction

In a projective plane a 3-*net* consists of three pairwise disjoint classes of lines such that every point incident with two lines from distinct classes is incident with exactly one line from each of the three classes. If one of the classes has finite size, say n, then the other two classes also have size n, called the *order* of the 3-net. In this paper we are considering 3-nets in a projective plane $PG(2,\mathbb{K})$ over an algebraically closed field $\mathbb{K}$ which are coordinatized by a group. Such a 3-net, with line classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and coordinatizing group $G = (G, \cdot)$, is equivalently defined by a triple of bijective maps from G to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, say

$$\alpha : G \to \mathcal{A},\ \beta : G \to \mathcal{B},\ \gamma : G \to \mathcal{C}$$

such that $a \cdot b = c$ if and only if $\alpha(a), \beta(b), \gamma(c)$ are three concurrent lines in $PG(2,\mathbb{K})$, for any $a, b, c \in G$. If this is the case, the 3-net in $PG(2,\mathbb{K})$ is said to *realize* the group G. Recently, finite 3-nets realizing a group in the complex plane have been investigated in connection with complex line arrangements and resonance theory, see [1, 7, 9] and the references therein.

Since key examples arise naturally in the dual plane of $PG(2, \mathbb{K})$, it is convenient to work with the dual concept of a 3-net. Formally, a *dual* 3*-net* of order n in $PG(2, \mathbb{K})$ consists of a triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ with $\Lambda_1, \Lambda_2, \Lambda_3$ pairwise disjoint point-sets of size n, called *components*, such that every line meeting two distinct components meets each component in precisely one point. A dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizing a group is *algebraic* if its points lie on a plane cubic. Moreover, we say that the dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is *of conic-line type (triangular)* if the components are contained in the union of a line and a nonsingular conic (in the union of three lines).

In our computer-aided investigation, combinatorial methods are used to study finite 3-nets realizing the groups $\mathbf{C}_2 \times \mathbf{C}_4, \mathbf{C}_3 \times \mathbf{C}_3$, and Alt_4. These results are fundamental for the complete classification of 3-nets embedded in a projective plane over a field, see [7]. Indeed, large groups could be dealt with theoretical results, but small groups having elements of order less than 5 needed a more explicit computation. This is main motivation of this paper.

We can summarize our results in the following theorem.

Theorem 1.1. *Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order n which realizes a group G in the projective plane $PG(2, \mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic p, where $p = 0$ or $p \geq 5$. We also assume that $n < p$ whenever $p > 0$. The following statements hold.*

(I) *If $G \cong \mathbf{C}_3 \times \mathbf{C}_3$ or $G \cong \mathbf{C}_2 \times \mathbf{C}_4$, then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic.*

(II) *If $p = 0$, then the group Alt_4 cannot be realized.*

The proof is divided in three parts, see Sections 2,3, or 4, according as $G \cong \mathbf{C}_3 \times \mathbf{C}_3$, $G \cong \mathbf{C}_2 \times \mathbf{C}_4$, or $G \cong \mathrm{Alt}_4$. Our notation and terminology are standard, see [6]. In view of Theorem 1.1, $\mathbb{K}$ denotes an algebraically closed field of characteristic p where either $p = 0$ or $n < p$ where n denotes the order of the dual 3-net.

2 $G \cong \mathbf{C}_3 \times \mathbf{C}_3$

We denote by $G = \{0, \dots, 8\}$ the elementary abelian group of order 9 given by the multiplication table

$$\begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\
2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\
3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\
4 & 5 & 3 & 7 & 8 & 6 & 1 & 2 & 0 \\
5 & 3 & 4 & 8 & 6 & 7 & 2 & 0 & 1 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
7 & 8 & 6 & 1 & 2 & 0 & 4 & 5 & 3 \\
8 & 6 & 7 & 2 & 0 & 1 & 5 & 3 & 4
\end{array}.$$

Let H be the subgroup $\{0, 1, 2\}$ of G.

Let $\mathbb{K}$ be an algebraically closed field whose characteristic is either 0 or more than 9. In this paper, all points are points of the projective plane over $\mathbb{K}$. We denote by ω a cubic root of unity in $\mathbb{K}$.

It is easy to see that realizations of the cyclic group of order 3 are precisely the Pappus configurations. The point set $\mathcal{P}$ of a triangular dual 3-net realizing $\mathbf{C}_3$ consists of 9 points such that any line intersects $\mathcal{P}$ in 1 or 3 points. In other words, $\mathcal{P}$ forms an $AG(2,3)$, where $AG(n,q)$ denotes the affine geometry over the finite field $\mathbb{F}_q$. It is also well known that any $AG(2,3)$ embedded in $PG(2,\mathbb{K})$ is a *Hesse configuration,* that is, the points are the inflection points of a nonsingular cubic curve.

Lemma 2.1. *Let*

$$\Delta' = \{0_1, 1_1, 2_1, 0_2, 1_2, 2_2, 0_3, 1_3, 2_3\}$$

be a realization of H. Then there is a unique cyclic collineation α of order three mapping

$$0_1, 1_1, 2_1, 0_2, 1_2, 2_2, 0_3, 1_3, 2_3 \text{ to } 1_1, 2_1, 0_1, 1_2, 2_2, 0_2, 2_3, 0_3, 1_3,$$

respectively. α is never central. The cubic curves containing Δ form a pencil. All these cubics are invariant under α. □

Lemma 2.2. *Let X be a set of nine points in a projective plane such that for all $A, B \in X$, the line AB contains a third point of X. Then, X is either contained in a line, or form an $AG(2,3)$.* □

In the sequel, we denote by $\Delta = \{0_1, \dots, 8_1, 0_2, \dots, 8_2, 0_3, \dots, 8_3\}$ a realization of G. We denote by Δ' the subset of Δ realizing the subgroup $H = \{0, 1, 2\}$. We will often use that the points of Δ can be re-indexed and the blocks $\{i_1\}$, $\{j_2\}$, $\{k_3\}$ can be interchanged.

Lemma 2.3. *There is a line which intersects Δ in exactly two points.*

Proof. Assume that no line intersects Δ in exactly two points. As $\mathbb{K}^*$ has no elementary abelian subgroup of order 9, Δ cannot be triangular or of conic-line type. Theorem 5.1 of [1] implies that none of the blocks $\{i_1\}$, $\{j_2\}$, $\{k_3\}$ is contained in a line. Lemma 2.2 implies that these blocks must form an $AG(2,3)$. Moreover, each line intersecting Δ in more than two points, intersect Δ in precisely three points. This means that with respect to the line intersections, Δ forms a Steiner triple system. As any three points of Δ generate a subsystem of order 9, Δ is in fact a Hall triple system, cf. [2, pages 496–499]. As $|\Delta| = 27$, we obtain that Δ is an embedding of $AG(3,3)$ in a projective plane, which is not possible by [8] if $\mathrm{char}(\mathbb{K}) \neq 3$. □

By re-indexing Δ, we can suppose that the line $0_1 1_1$ intersects Δ in $\{0_1, 1_1\}$, that is, $0_1, 1_1, 2_1$ are not collinear. Let α be the cyclic collineation of order three corresponding to the subnet Δ' realizing $H = \{0, 1, 2\}$. We will choose our projective coordinate system such that the following hold:

(1) $0_1 = (1, 0, 0)$, $1_1 = (0, 1, 0)$ and $2_1 = (0, 0, 1)$.

(2) $F = (1, 1, 1)$ is a fixed point of α.

(3) If the lines $0_1 0_2$, $1_1 1_2$, $2_1 2_2$ are concurrent then $F = (1, 1, 1)$ is their intersection.

Notice that the lines $i_1 j_1$ contain no fixed point of α, hence (2) does not conflict with (1). Furthermore, if $0_1 0_2$, $1_1 1_2$, $2_1 2_2$ are concurrent then their intersection is a fixed point of α.

The collineation α has the matrix form

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As 0_3 is not on the lines 1_i1_j, 0_3 has coordinates of the form $(a, b, 1)$ with $a, b \neq 0$. Then, we can compute the coordinates of the points $1_3 = (b, 1, a)$, $2_3 = (1, a, b)$ and $0_2 = (b, ba, a), 1_2 = (a, b, ba), 2_2 = (ba, a, b)$.

Let β be the cyclic collineation of order 3 corresponding to the subnet $\{0_1, 1_1, 2_1, 3_2, 4_2, 5_2, 3_3, 4_3, 5_3\}$. The matrix of β has the form

$$\begin{pmatrix} 0 & 0 & u \\ v & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

for some nonzero $u, v \in \mathbb{K}$. The point 3_2 has nonzero coordinates $(x, y, 1)$. Then, we have $4_2 = (u, vx, y)$, $5_2 = (uy, uv, vx)$ and

$$3_3 = (uy, vxy, vx),\ 4_3 = (xy, vx, y),\ 5_3 = (uvx, vuy, vxy).$$

For all points i_1, $i \in \{3, \ldots, 8\}$, there are three lines of the form j_2k_3, $j, k \in \{0, \ldots, 5\}$ such that $i_1 \in j_2k_3$. The fact that the corresponding line triples are concurrent, can be expressed by the equations

$$\begin{aligned}
\hat{f}_3 &= \det(0_2 \times 3_3, 1_2 \times 4_3, 2_2 \times 5_3) = 0, \\
\hat{f}_4 &= \det(0_2 \times 4_3, 1_2 \times 5_3, 2_2 \times 3_3) = 0, \\
\hat{f}_5 &= \det(0_2 \times 5_3, 1_2 \times 3_3, 2_2 \times 4_3) = 0, \\
f_6 &= \det(3_2 \times 0_3, 4_2 \times 1_3, 5_2 \times 2_3) = 0, \\
f_7 &= \det(3_2 \times 1_3, 4_2 \times 2_3, 5_2 \times 0_3) = 0, \\
f_8 &= \det(3_2 \times 2_3, 4_2 \times 0_3, 5_2 \times 1_3) = 0.
\end{aligned}$$

The values a, b, u, v, x, y determine Δ uniquely. The f_i's ($i \in \{3, \ldots, 8\}$) are polynomial expressions of these values. In fact, we will look at a, b, u, v, x, y as indeterminates over $\mathbb{K}$ and at the f_i's as elements of $\mathbb{K}[a, b, u, v, x, y]$. The polynomials f_7, f_8, f_9 have degree three in x, y, while for $i = 4, 5, 6$, the polynomials $\hat{f}_i$ have the form $\hat{f}_i = abvxyf_i$, where f_i is in $\mathbb{K}[a, b, u, v, x, y]$. The degree of f_4, f_5, f_6 in x, y is three.

Generally speaking, we are looking for specializations such that the corresponding configuration gives rise to a proper realization of G.

Lemma 2.4. *If any of the equations $u = 1$, $v = 1$, $u = v$ holds then Δ is algebraic.*

Proof. If $u = v = 1$ then $\alpha = \beta$. Let Γ be the cubic curve containing Δ' and 3_2. The equation of Γ can be computed explicitly, and one sees that if $u = v = 1$ then $3_3 \in \Gamma$. As by Lemma 2.1 Γ is invariant under $\alpha = \beta$, we have $4_2, 5_2, 4_3, 5_3 \in \Gamma$, too. Γ cannot be completely reducible since then, some i_1 would be collinear with some j_2, k_2. Suppose that $\Gamma = \ell \cup C$ with line ℓ and irreducible conic C. Then ℓ, C are α-invariant and the 1_i's are in 3. If $0_2 \in 3$ then $0_3, 1_3, 2_3 \in \ell$, and all 2_j's are in 3 and all 3_k's are in ℓ. As $\{0_1, 1_1, 2_1\}$, $\{0_2, 1_2, 2_2\}$, $\{0_3, 1_3, 2_3\}$, $\{3_2, 4_2, 5_2\}$, $\{3_3, 4_3, 5_3\}$ are all orbits of $\langle\alpha\rangle$ and the lines $0_2 3_3, 1_2 4_3, 2_2 5_3$ are concurrent, we have that $\{3_1, 4_1, 5_1\}$, $\{6_1, 7_1, 8_1\}$ are $\langle\alpha\rangle$-orbits contained in C. Continuing the process, we conclude that $\Delta \subset \Gamma$ (which is of course not possible). The same result is obtained if we start from $j_2 \in C$ or $k_3 \in C$.

Suppose now that Γ is irreducible. Denote by Γ^* the set of nonsingular points. The $\langle\alpha\rangle$-orbits are all cosets of a subgroup H^* of $(\Gamma^*, +)$ of order 3. Then, simple arithmetic on Γ^* yields that $\{3_1, 4_1, 5_1\}$, $\{6_1, 7_1, 8_1\}$ are H^* cosets of Γ^*. Repeating this argument, we obtain $\Delta \subset \Gamma$ again.

It remains to show that any of the equations $u = 1$, $v = 1$, $u = v$ implies the other two. For that we observe the following equations of rational expressions:

$$\begin{aligned}
\left.\frac{f_6}{\det(3_2 \times 0_3, 4_2 \times 1_3, 0_2 \times 4_3)}\right|_{u=1} &= \frac{v-1}{ay-1}, \\
\left.\frac{f_6}{\det(0_2 \times 5_3, 1_2 \times 3_3, 3_2 \times 0_3)}\right|_{v=1} &= \frac{u-1}{(ua-x)by}, \\
\left.\frac{f_6}{\det(0_2 \times 3_3, 1_2 \times 4_3, 3_2 \times 0_3)}\right|_{u=v} &= \frac{v-1}{(b-y)ax}.
\end{aligned}$$

In either case, the denominators at the left hand side cannot be zero as the corresponding lines are not concurrent. This proves that one equation implies another one, and two imply the third. This finishes the proof. □

The proof of the following lemma contains some elementary, but heavy computation. This computation can be formally verified by any computer algebra dealing with Groebner bases within a few seconds.

Lemma 2.5. *If $a^3 = b^3 = 1$ then $v = 1$. In particular, Δ is algebraic.*

Proof. We observe that $a^3 = b^3 = 1$ holds if and only if the lines $0_1 0_2$, $1_1 1_2$, $2_1 2_2$ and the lines $0_1 1_2$, $1_1 2_2$, $2_1 0_2$ are concurrent. (In other words, Δ' forms a dual $AG(2,3)$.) Remember that in this case, our coordinate system is chosen such that $F(1,1,1)$ is the fixed point $0_1 0_2 \cap 1_1 1_2 \cap 2_1 2_2$. As $0_1 = (1,0,0)$ and $0_3 \in O_1 O_2 = 0_1 F$, we have $b = 1$. By conjugaction in $\mathbb{K}$, we can assume $a = \omega$ w.l.o.g.

Now, we can find polynomials $s_i, t_i, p_1, p_2, q_1, q_2$, $i \in \{3, \ldots, 8\}$, in the indeterminates a, b, x, y, u, v with integer coefficients such that

$$\begin{aligned} \sum s_i f_i + p_1(a-\omega) + p_2(b-1) &= 18xv(v-1)(vx-y^2)(vx-\omega y^2), \\ \sum t_i f_i + q_1(a-\omega) + q_2(b-1) &= 18(v-1)(uy^3 - v^2x^3). \end{aligned}$$

Assume $v \neq 1$. Since $\det(0_1, 3_2, 4_2) = y^2 - vx \neq 0$, we have $vx - \omega y^2 = uy^3 - v^2x^3 = 0$. This implies $u = \omega^2 xy$ and $v = \omega y^2/x$. Straightforward computation shows that the collineation γ given by the matrix

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

fixes $0_1, 1_1, 2_1$ and maps the points $0_2, \ldots, 5_2, 0_3, \ldots, 5_3$ to the points

$$1_2, 2_2, 0_2,\ 5_2, 3_2, 4_2,\ 1_3, 2_3, 0_3,\ 5_3, 3_3, 4_3,$$

respectively. As $5 \cdot 2 = 6$ and $5 \cdot 3 = 4$ in G, we have

$$\gamma(3_1) = \gamma(0_2 3_3 \cap 1_2 4_3) = 1_2 5_3 \cap 2_2 3_3 = 4_1.$$

Similarly, $\gamma(4_1) = 5_1$ and $\gamma(5_1) = 3_1$. Thus, γ permutes the lines $3_1 0_3$, $4_1 1_3$, $5_1 2_3$ cyclically. As these lines intersect in 6_2, 6_2 is a fixed point of γ, which is not possible. □

We are now prepared to prove the main result.

Theorem 2.6. Δ *is algebraic.*

Proof. We can consider the f_i's as polynomials in the indeterminates a, b, u, v, x, y. Fix the values a, b, u, v and let $F_i(X, Y)$ be the polynomials in two variables such that $F_i(x, y) = f_i(a, b, u, v, x, y)$. Define the linear series L generated by the F_i's.

Recall that β is the collineation of order 2 mapping the points 0_1, 1_1, 2_1, 3_2, 4_2, 5_2, 3_3, 4_3, 5_3 to 1_1, 2_1, 0_1, 4_2, 5_2, 3_2, 5_3, 3_3, 4_3, respectively. From the definition of the f_i's one sees that the substitution $X' = u/Y$, $Y' = vX/Y$ induces a linear automorphism of L of degree 3. We will denote this induced map by β, as well.

Define the polynomials

$$\begin{aligned} E_1 &= uvX + \omega^2 uY^2 + \omega vX^2Y, \qquad & E_2 &= vX^2 + \omega^2 uY + \omega Y^2 X, \\ \bar{E}_1 &= uvX + \omega uY^2 + \omega^2 vX^2Y, \qquad & \bar{E}_2 &= vX^2 + \omega uY + \omega^2 Y^2 X, \end{aligned}$$

and

$$\begin{aligned} Q_1 &= \omega F_7 - F_8, \qquad & Q_2 &= \omega^2 F_4 - F_5, \\ \bar{Q}_1 &= \omega^2 F_7 - F_8, \qquad & \bar{Q}_2 &= \omega F_4 - F_5 \end{aligned}$$

of L. Then E_1, E_2, Q_1, Q_2 are eigenvectors of β with eigenvalue $\omega uv/Y^3$ and $\bar{E}_1, \bar{E}_2, \bar{Q}_1, \bar{Q}_2$ are eigenvectors of β with eigenvalue $\omega^2 uv/Y^3$. We have the following resultant values:

$$R_{E_1,E_2}(Y) = \omega^2 uvY(uv - Y^3)^2, R_{\bar{E}_1,\bar{E}_2}(Y) = \omega uvY(uv - Y^3)^2.$$

This shows that the intersection of $E_1 = 0, E_2 = 0$ and the intersection of $\bar{E}_1 = 0, \bar{E}_1 = 0$ consist of the points $0_1, 1_1, 1_1$ (with multiplicity 0) and the fixed points of β (with multiplicity 2). In particular, E_1, E_2 and $\bar{E}_1, \bar{E}_2$ are linearly independent.

Straightforward calculation shows that

$$\begin{aligned} Q_1 &= G_{11}E_1 + G_{12}E_2, \qquad & Q_2 &= G_{21}E_1 + G_{22}E_2, \\ \bar{Q}_1 &= \bar{G}_{11}\bar{E}_1 + \bar{G}_{12}\bar{E}_2, \qquad & \bar{Q}_2 &= \bar{G}_{21}\bar{E}_1 + \bar{G}_{22}\bar{E}_2, \end{aligned}$$

where

$$\begin{aligned} G_{11} &= (\omega^2 b + ab^2\omega + a^2)(\omega^2 + v\omega + u), \\ G_{12} &= (b^2\omega + \omega^2 a + a^2 b)(uv + \omega^2 u + v\omega), \\ G_{21} &= \omega(\omega^2 b + ab^2\omega + a^2)(\omega + \omega^2 v + u), \\ G_{22} &= (b^2\omega + \omega^2 a + a^2 b)(\omega u + \omega^2 v + uv), \\ \bar{G}_{11} &= (\omega b + \omega^2 ab^2 + a^2)(\omega + \omega^2 v + u), \\ \bar{G}_{12} &= (a\omega + a^2 b + \omega^2 b^2)(\omega u + \omega^2 v + uv), \\ \bar{G}_{21} &= \omega^2(\omega b + \omega^2 ab^2 + a^2)(\omega^2 + v\omega + u), \\ \bar{G}_{22} &= (a\omega + a^2 b + \omega^2 b^2)(uv + \omega^2 u + v\omega). \end{aligned}$$

Assume that Q_1, Q_2 are linearly independent. Then $E_1, E_2 \in \langle Q_1, Q_2 \rangle \leq L$. Therefore, $\Gamma_1 \cap \cdots \cap \Gamma_4$ is contained in the zero set of $E_1 = E_2 = 0$, a contradiction. We can similarly show that $\bar{Q}_1, \bar{Q}_2$ must be linearly dependent. This implies

$$\begin{aligned}
0 &= G_{11}G_{22} - G_{12}G_{21} \\
&= (2+\omega^2)(b^2\omega + \omega^2 a + ba^2)(\omega ab^2 + \omega^2 b + a^2)(u-v)(u-1)(v-1), \\
0 &= \bar{G}_{11}\bar{G}_{22} - \bar{G}_{12}\bar{G}_{21} \\
&= (2+\omega)(a\omega + ba^2 + \omega^2 b^2)(\omega b + \omega^2 ab^2 + a^2)(u-v)(u-1)(v-1).
\end{aligned}$$

The resultants of the polynomials $(b^2\omega + \omega^2 a + ba^2)(\omega ab^2 + \omega^2 b + a^2)$, $(a\omega + ba^2 + \omega^2 b^2)(\omega b + \omega^2 ab^2 + a^2)$ with respect to a, b are $9b^7(b^3-1)^6$ and $9a^7(a^3-1)^6$. Thus, Δ is algebraic by Lemmas 2.4 and 2.5. □

The Maple 13 program performing the computations of this section is attached in Appendix A. We use Buchberger's algorithm in order to explicitely construct the polynomials of Lemma 2.5. Thus, any computer algebra which can do symbolical calculation with rational polynomials can be used to verify the results. This convinces us about the correctness of our computation.

3 $G \cong \mathbf{C}_2 \times \mathbf{C}_4$

The main ingredient of the proof is Lame's Theorem [7, Proposition 2.3]. A classical *Lame configuration* consists of two triples of distinct lines in $PG(2, \mathbb{K})$, say ℓ_1, ℓ_2, ℓ_3 and r_1, r_2, r_3, such that no line from one triple passes through the common point of two lines from the other triple. For $1 \leq j, k \leq 3$, let R_{jk} denote the common point of the lines ℓ_j and r_k. There are nine such common points, and they are called the points of the Lame configuration.

Proposition 3.1. Lame's Theorem. *If eight points from a Lame configuration lie on a plane cubic then the ninth also does.*

The group $C_2 \times C_4$ can be given by the multiplication table

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ 5 & 6 & 7 & 8 & 2 & 1 & 4 & 3 \\ 6 & 5 & 8 & 7 & 1 & 2 & 3 & 4 \\ 7 & 8 & 5 & 6 & 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 & 3 & 4 & 1 & 2 \end{array}.$$

The triple $\{i_1, j_2, k_3\}$ is collinear if and only if $i * j = k$.

The following 6-tuples of collinear points form a Lamé configuration:

$$\begin{aligned} U_1 &= \{1_1,1_2,1_3\},\{3_1,5_2,7_3\},\{6_1,7_2,3_3\} + \{1_1,7_2,7_3\},\{3_1,1_2,3_3\},\{6_1,5_2,1_3\}, \\ U_2 &= \{1_1,3_2,3_3\},\{3_1,7_2,5_3\},\{6_1,5_2,1_3\} + \{1_1,5_2,5_3\},\{3_1,3_2,1_3\},\{6_1,7_2,3_3\}, \\ U_3 &= \{1_1,1_2,1_3\},\{3_1,7_2,5_3\},\{8_1,5_2,3_3\} + \{1_1,5_2,5_3\},\{3_1,1_2,3_3\},\{8_1,7_2,1_3\}, \\ U_4 &= \{1_1,3_2,3_3\},\{3_1,5_2,7_3\},\{8_1,7_2,1_3\} + \{1_1,7_2,7_3\},\{3_1,3_2,1_3\},\{8_1,5_2,3_3\}. \end{aligned}$$

Let C be a cubic curve through the points

$$1_1, 1_2, 1_3, 3_1, 7_2, 7_3, 6_1, 7_2, 3_3.$$

Then $|C \cap U_1|, |C \cap U_2| \geq 8$, hence, C passes through the nineth points 3_2 and 5_2. It follows that $|C \cap U_3|, |C \cap U_4| \geq 8$. Thus, C contains

$$U_1 \cup U_2 \cup U_3 \cup U_4 = \{1_1, 3_1, 6_1, 8_1, 1_2, 3_2, 5_2, 7_2, 1_3, 3_3, 5_3, 7_3\}.$$

It is straightforward to check that any of the following Lamé configurations intersects C in at least 8 points:

$$\begin{aligned} &\{1_1,5_2,5_3\},\{3_1,1_2,3_3\},\{5_1,3_2,7_3\} + \{1_1,3_2,3_3\},\{3_1,5_2,7_3\},\{5_1,1_2,5_3\}, \\ &\{1_1,1_2,1_3\},\{3_1,5_2,7_3\},\{7_1,3_2,5_3\} + \{1_1,5_2,5_3\},\{3_1,3_2,1_3\},\{7_1,1_2,7_3\}, \\ &\{1_1,5_2,5_3\},\{6_1,7_2,3_3\},\{8_1,2_2,7_3\} + \{1_1,7_2,7_3\},\{6_1,2_2,5_3\},\{8_1,5_2,3_3\}, \\ &\{1_1,5_2,5_3\},\{6_1,4_2,7_3\},\{8_1,7_2,1_3\} + \{1_1,7_2,7_3\},\{6_1,5_2,1_3\},\{8_1,4_2,5_3\}, \\ &\{1_1,1_2,1_3\},\{6_1,7_2,3_3\},\{8_1,3_2,6_3\} + \{1_1,3_2,3_3\},\{6_1,1_2,6_3\},\{8_1,7_2,1_3\}, \\ &\{1_1,1_2,1_3\},\{6_1,3_2,8_3\},\{8_1,5_2,3_3\} + \{1_1,3_2,3_3\},\{6_1,5_2,1_3\},\{8_1,1_2,8_3\}. \end{aligned}$$

Hence, C contains the further points $5_1, 7_1, 2_2, 4_2, 6_3, 8_3$. Finally, we consider the Lamé configurations

$$\{1_1, 1_2, 1_3\}, \{2_1, 5_2, 6_3\}, \{6_1, 2_2, 5_3\} + \{1_1, 5_2, 5_3\}, \{2_1, 2_2, 1_3\}, \{6_1, 1_2, 6_3\},$$
$$\{3_1, 1_2, 3_3\}, \{4_1, 7_2, 6_3\}, \{6_1, 2_2, 5_3\} + \{3_1, 7_2, 5_3\}, \{4_1, 2_2, 3_3\}, \{6_1, 1_2, 6_3\},$$
$$\{1_1, 5_2, 5_3\}, \{7_1, 6_2, 3_3\}, \{8_1, 3_2, 6_3\} + \{1_1, 6_2, 6_3\}, \{7_1, 3_2, 5_3\}, \{8_1, 5_2, 3_3\},$$
$$\{1_1, 8_2, 8_3\}, \{5_1, 3_2, 7_3\}, \{6_1, 7_2, 3_3\} + \{1_1, 7_2, 7_3\}, \{5_1, 8_2, 3_3\}, \{6_1, 3_2, 8_3\},$$
$$\{1_1, 1_2, 1_3\}, \{7_1, 7_2, 2_3\}, \{8_1, 2_2, 7_3\} + \{1_1, 2_2, 2_3\}, \{7_1, 1_2, 7_3\}, \{8_1, 7_2, 1_3\},$$
$$\{3_1, 2_2, 4_3\}, \{5_1, 1_2, 5_3\}, \{6_1, 7_2, 3_3\} + \{3_1, 1_2, 3_3\}, \{5_1, 7_2, 4_3\}, \{6_1, 2_2, 5_3\}.$$

As before, one sees that any of them has at least 8 points in common with C, thus, C passes through all the points of the embedding of $C_2 \times C_4$.

4 $G \cong \mathrm{Alt}_4$ $(p = 0)$

Let the group Alt(4) be given on the underlying set $\{1, \ldots, 12\}$ by the Cayley table

1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	7	8	5	6	12	11	10	9
3	4	1	2	8	7	6	5	10	9	12	11
4	3	2	1	6	5	8	7	11	12	9	10
5	6	7	8	9	10	11	12	1	2	3	4
6	5	8	7	11	12	9	10	4	3	2	1
7	8	5	6	12	11	10	9	2	1	4	3
8	7	6	5	10	9	12	11	3	4	1	2
9	10	11	12	1	2	3	4	5	6	7	8
10	9	12	11	3	4	1	2	8	7	6	5
11	12	9	10	4	3	2	1	6	5	8	7
12	11	10	9	2	1	4	3	7	8	5	6

We have that the points $1_1, \ldots, 12_1$, $1_2, \ldots, 12_2$, $1_3, \ldots, 12_3$ of the complex projective plane form a realization of Alt(4), if for all $i, j, k = 1, \ldots, 12$, i_1, j_2, k_3 are collinear if and only if $i * j = k$.

Proposition 4.1. Alt(4) *cannot be realized on the complex projective plane.*

Proof. We see that $\{1, 2, 3, 4\}$ is an elementary Abelian normal subgroup and $\{1, 5, 9\}$ is a subgroup. Without a loss of generality, we can assume that

$$1_1 = [1, 0, 0], \quad 1_2 = [0, 1, 0], \quad 1_3 = [1, -1, 0],$$
$$2_1 = [0, 1, 1], \quad 2_2 = [1, 0, 1], \quad 2_3 = [0, 0, 1].$$

As $3_1, 1_2, 3_3$ are collinear, we can take 3_1 and 3_3 in the form $3_1 = [a, b, c]$ and $3_3 = [a, b+1, c]$. This enables us to compute the remaining points

$$\begin{array}{lll} 3_1 = [a, b, c], & 3_2 = [a-1, 1+b, c], & 3_3 = [a, b+1, c], \\ 4_1 = [a-1, 1+b, c-1], & 4_2 = [a, b, c-1], & 4_3 = [a-1, b, c-1]. \end{array}$$

These points indeed form a realization of $\mathbf{C}_2 \times \mathbf{C}_2$. Similarly, we choose 5_1 and 9_1 as generic points $[d_1, d_2, 1]$, $[d_4, d_5, 1]$. Then $5_3, 9_3$ have the form $[d_1, d_3, 1]$, $[d_4, d_6, 1]$, respectively and computation yields

$$5_2 = [d_4 + d_5 - d_3, d_3, 1], 9_2 = [d_1 + d_2 - d_6, d_6, 1].$$

Define the points

$$\begin{array}{lll} 5_1 = 9_2 1_3 \cap 1_2 5_3, & 5_2 = 9_1 1_3 \cap 1_1 5_3, & 5_3 = 5_1 1_2 \cap 1_1 5_2, \\ 6_1 = 9_2 4_3 \cap 2_2 5_3, & 6_2 = 9_1 2_3 \cap 4_1 5_3, & 6_3 = 5_1 2_2 \cap 4_1 5_2, \\ 7_1 = 9_2 2_3 \cap 3_2 5_3, & 7_2 = 9_1 3_3 \cap 2_1 5_3, & 7_3 = 5_1 3_2 \cap 2_1 5_2, \\ 8_1 = 9_2 3_3 \cap 4_2 5_3, & 8_2 = 9_1 4_3 \cap 3_1 5_3, & 8_3 = 5_1 4_2 \cap 3_1 5_2, \\ 9_1 = 5_2 1_3 \cap 1_2 9_3, & 9_2 = 5_1 1_3 \cap 1_1 9_3, & 9_3 = 9_1 1_2 \cap 1_1 9_2, \\ 10_1 = 5_2 3_3 \cap 2_2 9_3, & 10_2 = 5_1 2_3 \cap 3_1 9_3, & 10_3 = 9_1 2_2 \cap 3_1 9_2, \\ 11_1 = 5_2 4_3 \cap 3_2 9_3, & 11_2 = 5_1 3_3 \cap 4_1 9_3, & 11_3 = 9_1 3_2 \cap 4_1 9_2, \\ 12_1 = 5_2 2_3 \cap 4_2 9_3, & 12_2 = 5_1 4_3 \cap 2_1 9_3, & 12_3 = 9_1 4_2 \cap 2_1 9_2. \end{array}$$

Let us denote by d_{ijk} the determinant of the 3×3 matrix with rows i_1, j_2, k_3. We define the sets $X = \{d_{ijk} \mid k = i * j\}$ and

$$\begin{array}{rcllllll} Y & = & \{(1,2,6), & (9,1,10), & (1,10,8), & (5,9,2), & (9,5,2), & (1,10,2), \\ & & (1,3,11), & (1,4,12), & (1,5,7), & (9,1,12), & (9,5,4), & (5,9,4), \\ & & (1,5,6), & (1,6,11), & (1,9,12), & (1,12,4), & (1,5,8), & (1,7,3), \\ & & (1,7,12), & (5,9,3), & (9,1,11), & (1,12,7), & (1,9,11), & (5,1,6), \\ & & (5,1,7), & (1,6,2), & (1,3,7), & (1,8,4), & (1,11,6), & (5,1,8), \\ & & (9,5,3), & (1,8,10), & (1,11,3), & (1,4,8), & (1,9,10), & (1,2,10)\}. \end{array}$$

For all $(i, j, k) \in Y$, $i \cdot j \neq k$, thus for any proper complex realization of Alt(4), we can substitute complex numbers in the variables $a, b, c, d_1, \cdots, d_6$ such that all polynomials in X are zero and all for all $(i, j, k) \in Y$, $d_{ijk} \neq 0$. Put $g = \Pi_{(i,j,k) \in Y} d_{ijk}$ and define

$$X' = \{f / \gcd(f, g) \mid f \in X\}.$$

It is still true that Alt(4) has a realization if and only if one can substitue complex numbers in $a, b, c, d_1, \cdots, d_6$ such that all polynomials in X' vanish. Groebner basis computation shows that the ideal generated by X' contains the polynomials $d_1 - d_4$, $d_2 - d_5$, implying $5_1 = 9_1$, a contradiction. □

The Groebner basis computation of this section is too heavy for most of computer programs of this type. We found two programs which is able to compute the Groebner basis: the F4 algorithm [4] in the computer algebra system Maple 13 and the *modStd* library [5] of Singular [3]. These programs do not store the cofactors of the Groebner bases, hence one cannot verify the result symbolically. However, two completely different implementations deliver the same result, thus, we can trust this computation as well.

The Maple 13 implementation is attached in Appendix B and the Singular implementation is attached in Appendix C. The computations take less that 3 minutes, and less that 3 hours, respectively.

A Maple code for the case $G = C_3 \times C_3$

This appendix contains

```
########################################################
# Maple 13 program for computing with dual 3-nets
# G = C3 x C3
########################################################
# Preparation

with(LinearAlgebra):

isect:=proc(a,b,c,d)
  evala(CrossProduct(CrossProduct(a,b),CrossProduct(c,d))):
end proc:
idet:=proc(a,b,c,d,e,f)
  evala(Determinant(<CrossProduct(a,b)|CrossProduct(c,d)|CrossProduct(e,f)>)):
end proc:

alias(omega=RootOf(X^2+X+1));

########################################################
# Part 1: Constructing the points, the transformations and the equations.
# One defines the transformations, the base points and the equations which
# correspond to certain collinearities. The unknowns $a,b,u,v,x,y$ are seen as
# fixed elements of the base field.
# In the program, Px_i denotes the point $x_i$ of the dual $3$-net.

alpha:=<<0,1,0>|<0,0,1>|<1,0,0>>;
beta:=<<0,v,0>|<0,0,1>|<u,0,0>>;
```

```
P0_1:=<1,0,0>: P1_1:=<0,1,0>: P2_1:=<0,0,1>:
P3_2:=<x,y,1>; P4_2:=beta.P3_2; P5_2:=beta.P4_2;
P0_3:=<a,b,1>; P1_3:=(alpha^(-1)).P0_3; P2_3:=(alpha^(-1)).P1_3;
                    # We turn Lambda_3 in the opposite direction!

P0_2:=isect(P0_1,P0_3,P1_1,P1_3);
P1_2:=isect(P0_1,P1_3,P1_1,P2_3);
P2_2:=isect(P0_1,P2_3,P1_1,P0_3);
P3_3:=isect(P0_1,P3_2,P1_1,P5_2);
P4_3:=isect(P0_1,P4_2,P1_1,P3_2);
P5_3:=isect(P0_1,P5_2,P1_1,P4_2);

f:=[  0,0,
  idet(P0_2,P3_3,P1_2,P4_3,P2_2,P5_3)/(a*b*v*x*y),  # P3_1
  idet(P0_2,P4_3,P1_2,P5_3,P2_2,P3_3)/(a*b*v*x*y),  # P4_1
  idet(P0_2,P5_3,P1_2,P3_3,P2_2,P4_3)/(a*b*v*x*y),  # P5_1
  idet(P3_2,P0_3,P4_2,P1_3,P5_2,P2_3),              # P6_1
  idet(P3_2,P1_3,P4_2,P2_3,P5_2,P0_3),              # P7_1
  idet(P3_2,P2_3,P4_2,P0_3,P5_2,P1_3)               # P8_1
]:
f:=factor(f):

#######################################################
# Part 2 (Lemma 2.4): Any of u=1, v=1, u=v implies the other two equations.

factor(subs(u=1,f[6]/idet(P0_2,P4_3,P3_2,P0_3,P4_2,P1_3)));
factor(subs(v=1,f[6]/idet(P0_2,P5_3,P1_2,P3_3,P3_2,P0_3)));
factor(subs(u=v,f[6]/idet(P0_2,P3_3,P1_2,P4_3,P3_2,P0_3)));

#######################################################
# Part 3 (Lemma 2.5): If a^3=b^3=1 then v=1.
# We can assume a=omega and b=1.

gb:=Groebner[Basis]([op(f),a-omega,b-1], plex(u,v,x,y,a,b),output=extended):
factor(gb[1][3]),factor(gb[1][5]);

# We construct the cofactors explicitely:
s:=evala(18*gb[2][3][1..8]):
t:=evala(18*gb[2][5][1..8]):
q:=evala(18*[gb[2][5][9],gb[2][5][10]]):
p:=evala(18*[gb[2][3][9],gb[2][3][10]]):

factor(add(f[i]*s[i],i=3..8)+p[1]*(a-omega)+p[2]*(b-1));
factor(add(f[i]*t[i],i=3..8)+q[1]*(a-omega)+q[2]*(b-1));

# This shows that all cofactors have coefficients in Z[omega]:
seq(denom(factor(_u)),_u in [op(s),op(t),op(p),op(q)]);

#######################################################
# Part 4: Computation with the beta-invariant polynomials.
# From now on, we consider $a,b,u,v$ as fixed elements of the base field
# and $X,Y$ as indetereminates.
# We define the action of $\beta$ on the polynomial ring in two variables.

F:=map(_x->subs({x=X,y=Y},_x),f): map(_x->degree(_x,{X,Y}),F);

betaonpoly:=proc(U) return factor(subs({X=u*Y/(v*X),Y=u/X},U)): end proc:
```

```
# This shows that the nontrivial solutions of F[3]=F[4]=F[5]=0
# and F[6]=F[7]=F[8]=0 are $\beta$-invariant:

map(_x->factor(_x),
  [betaonpoly(F[3])/F[5],betaonpoly(F[4])/F[3],betaonpoly(F[5])/F[4]]
);
map(_x->factor(_x),
  [betaonpoly(F[6])/F[7],betaonpoly(F[7])/F[8],betaonpoly(F[8])/F[6]]
);

# We define the E[i]'s, barE[i]'s, Q[i]'s and barQ[i]'s
# and show that their curves are $\beta$-invariant:

E:=[u*v*X+omega^2*u*Y^2+omega*v*X^2*Y,v*X^2+omega^2*u*Y+omega*Y^2*X]:
barE:=[u*v*X+omega*u*Y^2+omega^2*v*X^2*Y,v*X^2+omega*u*Y+omega^2*Y^2*X]:

Q:= [omega * F[6]-F[7],omega^2 * F[3]-F[4]]:
barQ:= [omega^2 * F[6]-F[7],omega * F[3]-F[4]]:

seq(factor(betaonpoly(_u)/_u), _u in [op(E),op(Q)]);
seq(factor(betaonpoly(_u)/_u), _u in [op(barE),op(barQ)]);

# We define the G[i,j]'s and barG[i,j]'s.
# We show that they are indeed coefficients of Q[i]'s and barQ[i]'s.

G[2,1]:=omega*(omega^2*b+a*b^2*omega+a^2)*(omega+omega^2*v+u);
G[2,2]:=(b^2*omega+omega^2*a+a^2*b)*(omega*u+omega^2*v+u*v);
G[1,1]:=(omega^2*b+a*b^2*omega+a^2)*(omega^2+v*omega+u);
G[1,2]:=(b^2*omega+omega^2*a+a^2*b)*(u*v+omega^2*u+v*omega);
barG[1,1]:=(omega*b+omega^2*a*b^2+a^2)*(omega+omega^2*v+u);
barG[1,2]:=(omega*a+a^2*b+omega^2*b^2)*(omega*u+omega^2*v+u*v);
barG[2,1]:=omega^2*(omega*b+omega^2*a*b^2+a^2)*(omega^2+v*omega+u);
barG[2,2]:=(omega*a+a^2*b+omega^2*b^2)*(u*v+omega^2*u+v*omega);

map(_x->evalb(factor(_x)), [
  Q[1]=G[1,1]*E[1]+G[1,2]*E[2],
  Q[2]=G[2,1]*E[1]+G[2,2]*E[2],
  barQ[1]=barG[1,1]*barE[1]+barG[1,2]*barE[2],
  barQ[2]=barG[2,1]*barE[1]+barG[2,2]*barE[2]
]);

# We compute the factors of the determinants of the G[i,j]'s and barG[i,j]'s:

map(_x->evalb(factor(_x)), [
  G[1,1]*G[2,2]-G[1,2]*G[2,1]=
    (2+omega^2)*(b^2*omega+omega^2*a+b*a^2)*(omega*a*b^2+omega^2*b+a^2)*(u-v)*(u-1)*(v-1),
  barG[1,1]*barG[2,2]-barG[1,2]*barG[2,1]=
    (2+omega)*(a*omega+b*a^2+omega^2*b^2)*(omega*b+omega^2*a*b^2+a^2)*(u-v)*(u-1)*(v-1)
]);

ra:=resultant(
  (-b+a*b^2*omega-omega*b+a^2)*(-b^2*omega+a+a*omega-b*a^2),
  (-omega*b+a*b^2+a*b^2*omega-a^2)*(a*omega+b*a^2-b^2-b^2*omega),
  a):
rb:=resultant(
  (-b+a*b^2*omega-omega*b+a^2)*(-b^2*omega+a+a*omega-b*a^2),
```

```
  (-omega*b+a*b^2+a*b^2*omega-a^2)*(a*omega+b*a^2-b^2-b^2*omega),
  b):
factor(ra/(b^3-1)^6);
factor(rb/(a^3-1)^6);
```

B Maple code for the case $G = \mathrm{Alt}_4$

This appendix contains the implementation of the computations of Section 4, using the F4 algorithm [4] in the computer algebra system Maple 13. This program does not store the cofactors of the Groebner bases, hence one cannot verify the result symbolically. The computation takes less than 3 minutes.

```
########################################################
# Maple 13 program for computing with dual 3-nets
# G = Alt(4)
########################################################
# Preparation

with(LinearAlgebra);
isect:=proc(a,b,c,d)
  evala(CrossProduct(CrossProduct(a,b),CrossProduct(c,d))):
end proc:

ct:=Matrix(
[ [ 1,2,3,4,5,6,7,8,9,10,11,12 ],
  [ 2,1,4,3,7,8,5,6,12,11,10,9 ],
  [ 3,4,1,2,8,7,6,5,10,9,12,11 ],
  [ 4,3,2,1,6,5,8,7,11,12,9,10 ],
  [ 5,6,7,8,9,10,11,12,1,2,3,4 ],
  [ 6,5,8,7,11,12,9,10,4,3,2,1 ],
  [ 7,8,5,6,12,11,10,9,2,1,4,3 ],
  [ 8,7,6,5,10,9,12,11,3,4,1,2 ],
  [ 9,10,11,12,1,2,3,4,5,6,7,8 ],
  [ 10,9,12,11,3,4,1,2,8,7,6,5 ],
  [ 11,12,9,10,4,3,2,1,6,5,8,7 ],
  [ 12,11,10,9,2,1,4,3,7,8,5,6 ]
 ]);

d:=array(1..6);

########################################################
# Part 1: We define the points of the dual 3-net
# using a,b,c,d[1],...d[6] as indeterminates.

P:=[  [<1,0,0>,<0,1,0>,<1,-1,0>],
      [<0,1,1>,<1,0,1>,<0,0,1>],
      [<a,b,c>,0,<a,1+b,c>],
      [0,0,0],
      [<d[1],d[2],1>,0,<d[1],d[3],1>],
      [0,0,0],
      [0,0,0],
```

```
      [0,0,0],
      [<d[4],d[5],1>,0,<d[4],d[6],1>],
      [0,0,0],
      [0,0,0],
      [0,0,0]  ];

# As P[4,1], P[1,2], P[4,3] are coll, we may assume wlog that
# P[4,1]=<a,b,c> and P[4,3]=<a,1+b,c>.
# Similar argument for P[5,3] and P[9,3], using the fact that
# these points cannot have last coordinate 0.

P[3,2]:=evala(isect(P[3,1],P[1,3],P[1,1],P[3,3])/c);
P[4,2]:=evala(isect(P[3,1],P[2,3],P[2,1],P[3,3])/a);

P[4,1]:=evala(isect(P[3,2],P[2,3],P[2,2],P[3,3])/(1+b));
P[4,3]:=evala(isect(P[1,1],P[4,2],P[2,1],P[3,2])/(1+b-c));

P[5,2]:=isect(P[1,1],P[5,3],P[9,1],P[1,3]);
P[9,2]:=isect(P[1,1],P[9,3],P[5,1],P[1,3]);

#########################################################

P[5,1]:=isect(P[9,2],P[1,3],P[1,2],P[5,3]):
P[5,2]:=isect(P[9,1],P[1,3],P[1,1],P[5,3]):
P[5,3]:=isect(P[5,1],P[1,2],P[1,1],P[5,2]):

P[6,1]:=isect(P[9,2],P[4,3],P[2,2],P[5,3]):
P[6,2]:=isect(P[9,1],P[2,3],P[4,1],P[5,3]):
P[6,3]:=isect(P[5,1],P[2,2],P[4,1],P[5,2]):

P[7,1]:=isect(P[9,2],P[2,3],P[3,2],P[5,3]):
P[7,2]:=isect(P[9,1],P[3,3],P[2,1],P[5,3]):
P[7,3]:=isect(P[5,1],P[3,2],P[2,1],P[5,2]):

P[8,1]:=isect(P[9,2],P[3,3],P[4,2],P[5,3]):
P[8,2]:=isect(P[9,1],P[4,3],P[3,1],P[5,3]):
P[8,3]:=isect(P[5,1],P[4,2],P[3,1],P[5,2]):

P[9,1]:=isect(P[5,2],P[1,3],P[1,2],P[9,3]):
P[9,2]:=isect(P[5,1],P[1,3],P[1,1],P[9,3]):
P[9,3]:=isect(P[9,1],P[1,2],P[1,1],P[9,2]):

P[10,1]:=isect(P[5,2],P[3,3],P[2,2],P[9,3]):
P[10,2]:=isect(P[5,1],P[2,3],P[3,1],P[9,3]):
P[10,3]:=isect(P[9,1],P[2,2],P[3,1],P[9,2]):

P[11,1]:=isect(P[5,2],P[4,3],P[3,2],P[9,3]):
P[11,2]:=isect(P[5,1],P[3,3],P[4,1],P[9,3]):
P[11,3]:=isect(P[9,1],P[3,2],P[4,1],P[9,2]):

P[12,1]:=isect(P[5,2],P[2,3],P[4,2],P[9,3]):
P[12,2]:=isect(P[5,1],P[4,3],P[2,1],P[9,3]):
P[12,3]:=isect(P[9,1],P[4,2],P[2,1],P[9,2]):


#########################################################
# Part 2: We construct the polynomial identities.
```

```
eqs:=[]:
for i from 1 to 12 do
  for j from 1 to 12 do
      aa:=Determinant(<P[i,1]|P[j,2]|P[ct[i,j],3]>):
      eqs:=[op(eqs),aa]:
  end do
end do:

########################################################
# Part 3: We filter out the nonzero factors of the equations.

nepos:=[
[1, 2, 6], [9, 1, 10], [1, 10, 8], [5, 9, 2],
[9, 5, 2], [1, 10, 2], [1, 3, 11], [1, 4, 12],
[1, 5, 7], [9, 1, 12], [9, 5, 4], [5, 9, 4],
[1, 5, 6], [1, 6, 11], [1, 9, 12], [1, 12, 4],
[1, 5, 8], [1, 7, 3], [1, 7, 12], [5, 9, 3],
[9, 1, 11], [1, 12, 7], [1, 9, 11], [5, 1, 6],
[5, 1, 7], [1, 6, 2], [1, 3, 7], [1, 8, 4],
[1, 11, 6], [5, 1, 8], [9, 5, 3], [1, 8, 10],
[1, 11, 3], [1, 4, 8],[1, 9, 10], [1, 2, 10]
];
noneqs:=map(_x->Determinant(<P[_x[1],1]|P[_x[2],2]|P[_x[3],3]>),nepos):

noneqs:=mul(x,x in noneqs):

eqs:=select(x->x<>0,eqs): nops(eqs);
eqs_reduced:=map(x->factor(x/gcd(x,noneqs)),eqs):
map(degree,eqs)-map(degree,eqs_reduced);

########################################################
# Part 4: We compute the Groebner basis of the corresponding ideal.
# This Groebner basis shows that d[1]=d[4], d[2]=d[5], d[3]=d[6].
# The computation takes less that 3 minutes using the F4 algorithm.

gb:=Groebner[Basis](eqs_reduced,tdeg(a,b,c,d[1],d[2],d[3],d[4],d[5],d[6]));
```

C Singular code for the case $G = \mathrm{Alt}_4$

This appendix contains the implementation of the computations of Section 4, using the *modStd* library [5] of SINGULAR [3]. This program does not store the cofactors of the Groebner bases, hence one cannot verify the result symbolically. The computation takes less than 3 hours.

```
////////////////////////////////////////////////////////
// Singular 3.1 program for computing with dual 3-nets
// G = Alt(4)
////////////////////////////////////////////////////////
// Preparation

LIB "modstd.lib";
```

```
intmat ct[12][12]=
   1,2,3,4,5,6,7,8,9,10,11,12 ,
   2,1,4,3,7,8,5,6,12,11,10,9 ,
   3,4,1,2,8,7,6,5,10,9,12,11 ,
   4,3,2,1,6,5,8,7,11,12,9,10 ,
   5,6,7,8,9,10,11,12,1,2,3,4 ,
   6,5,8,7,11,12,9,10,4,3,2,1 ,
   7,8,5,6,12,11,10,9,2,1,4,3 ,
   8,7,6,5,10,9,12,11,3,4,1,2 ,
   9,10,11,12,1,2,3,4,5,6,7,8 ,
   10,9,12,11,3,4,1,2,8,7,6,5 ,
   11,12,9,10,4,3,2,1,6,5,8,7 ,
   12,11,10,9,2,1,4,3,7,8,5,6;

ring r=0,(a,b,c,d(1..6)),dp;

////////////////////////////////////////////////////////
// Part 1: We define the points of the dual 3-net
// using a,b,c,d[1],...d[6] as indeterminates.

// As P[4,1], P[1,2], P[4,3] are coll, we may assume wlog that
// P[4,1]=<a,b,c> and P[4,3]=<a,1+b,c>.
// Similar argument for P[5,3] and P[9,3], using the fact that
// these points cannot have last coordinate 0.

list pt_data=
   1,0,0,         0,1,0,      1,-1,0,      // 1
   0,1,1,         1,0,1,      0,0,1,       // 2
   a,b,c,         0,0,0,      a,1+b,c,     // 3
   0,0,0,         0,0,0,      0,0,0,       // 4
   d(1),d(2),1,   0,0,0,      d(1),d(3),1, // 5
   0,0,0,         0,0,0,      0,0,0,       // 6
   0,0,0,         0,0,0,      0,0,0,       // 7
   0,0,0,         0,0,0,      0,0,0,       // 8
   d(4),d(5),1,   0,0,0,      d(4),d(6),1, // 9
   0,0,0,         0,0,0,      0,0,0,       // 10
   0,0,0,         0,0,0,      0,0,0,       // 11
   0,0,0,         0,0,0,      0,0,0;       // 12

////////////////////////////////////////////////////////
// Procedures for manipulating the points of the dual 3-net:

proc rpt(int x, int i)
{
   return(list(pt_data[(x-1)*9+(i-1)*3+1..(x-1)*9+(i-1)*3+3]));
}

proc setpoint(int x, int i, list u)
{
   pt_data[(x-1)*9+(i-1)*3+1]=u[1];
   pt_data[(x-1)*9+(i-1)*3+2]=u[2];
   pt_data[(x-1)*9+(i-1)*3+3]=u[3];
}

proc divpoint(int x, int i, poly p)
{
```

```
   pt_data[(x-1)*9+(i-1)*3+1]=pt_data[(x-1)*9+(i-1)*3+1]/p;
   pt_data[(x-1)*9+(i-1)*3+2]=pt_data[(x-1)*9+(i-1)*3+2]/p;
   pt_data[(x-1)*9+(i-1)*3+3]=pt_data[(x-1)*9+(i-1)*3+3]/p;
}

proc crossprod(list u,list v)
{
   return(list(u[2]*v[3]-u[3]*v[2],-u[1]*v[3]+u[3]*v[1],u[1]*v[2]-u[2]*v[1]));
}

proc isect(u,v,w,z)
{
   return(crossprod(crossprod(u,v),crossprod(w,z))):
}

proc detpoints(int x, int y, int z)
{
   matrix m[3][3] =
      pt_data[(x-1)*9+0*3+1..(x-1)*9+0*3+3],
      pt_data[(y-1)*9+1*3+1..(y-1)*9+1*3+3],
      pt_data[(z-1)*9+2*3+1..(z-1)*9+2*3+3];
   return(det(m));
}

////////////////////////////////////////////////////////
// Part 2: We set the remaining points of the dual 3-net.

setpoint(3,2,isect(rpt(3,1),rpt(1,3),rpt(1,1),rpt(3,3)));
divpoint(3,2,c); print(rpt(3,2));

setpoint(4,2,isect(rpt(3,1),rpt(2,3),rpt(2,1),rpt(3,3)));
divpoint(4,2,a); print(rpt(4,2));

setpoint(4,1,isect(rpt(3,2),rpt(2,3),rpt(2,2),rpt(3,3)));
divpoint(4,1,1+b); print(rpt(4,1));

setpoint(4,3,isect(rpt(1,1),rpt(4,2),rpt(2,1),rpt(3,2)));
divpoint(4,3,1+b-c); print(rpt(4,3));

////////////////////////////////////////////////////////
setpoint(5,2,isect(rpt(1,1),rpt(5,3),rpt(9,1),rpt(1,3)));
setpoint(9,2,isect(rpt(1,1),rpt(9,3),rpt(5,1),rpt(1,3)));

////////////////////////////////////////////////////////
setpoint(5,1,isect(rpt(9,2),rpt(1,3),rpt(1,2),rpt(5,3)));
setpoint(5,2,isect(rpt(9,1),rpt(1,3),rpt(1,1),rpt(5,3)));
setpoint(5,3,isect(rpt(5,1),rpt(1,2),rpt(1,1),rpt(5,2)));

setpoint(6,1,isect(rpt(9,2),rpt(4,3),rpt(2,2),rpt(5,3)));
setpoint(6,2,isect(rpt(9,1),rpt(2,3),rpt(4,1),rpt(5,3)));
setpoint(6,3,isect(rpt(5,1),rpt(2,2),rpt(4,1),rpt(5,2)));

setpoint(7,1,isect(rpt(9,2),rpt(2,3),rpt(3,2),rpt(5,3)));
setpoint(7,2,isect(rpt(9,1),rpt(3,3),rpt(2,1),rpt(5,3)));
setpoint(7,3,isect(rpt(5,1),rpt(3,2),rpt(2,1),rpt(5,2)));

setpoint(8,1,isect(rpt(9,2),rpt(3,3),rpt(4,2),rpt(5,3)));
```

```
setpoint(8,2,isect(rpt(9,1),rpt(4,3),rpt(3,1),rpt(5,3)));
setpoint(8,3,isect(rpt(5,1),rpt(4,2),rpt(3,1),rpt(5,2)));

setpoint(9,1,isect(rpt(5,2),rpt(1,3),rpt(1,2),rpt(9,3)));
setpoint(9,2,isect(rpt(5,1),rpt(1,3),rpt(1,1),rpt(9,3)));
setpoint(9,3,isect(rpt(9,1),rpt(1,2),rpt(1,1),rpt(9,2)));

setpoint(10,1,isect(rpt(5,2),rpt(3,3),rpt(2,2),rpt(9,3)));
setpoint(10,2,isect(rpt(5,1),rpt(2,3),rpt(3,1),rpt(9,3)));
setpoint(10,3,isect(rpt(9,1),rpt(2,2),rpt(3,1),rpt(9,2)));

setpoint(11,1,isect(rpt(5,2),rpt(4,3),rpt(3,2),rpt(9,3)));
setpoint(11,2,isect(rpt(5,1),rpt(3,3),rpt(4,1),rpt(9,3)));
setpoint(11,3,isect(rpt(9,1),rpt(3,2),rpt(4,1),rpt(9,2)));

setpoint(12,1,isect(rpt(5,2),rpt(2,3),rpt(4,2),rpt(9,3)));
setpoint(12,2,isect(rpt(5,1),rpt(4,3),rpt(2,1),rpt(9,3)));
setpoint(12,3,isect(rpt(9,1),rpt(4,2),rpt(2,1),rpt(9,2)));


////////////////////////////////////////////////////////
// Part 3: We define the nonzero polynomials.

intmat nz_pos[36][3]=
   1,2,6,    9,1,10,   1,10,8,    5,9,2,
   9,5,2,    1,10,2,   1,3,11,    1,4,12,
   1,5,7,    9,1,12,   9,5,4,     5,9,4,
   1,5,6,    1,6,11,   1,9,12,    1,12,4,
   1,5,8,    1,7,3,    1,7,12,    5,9,3,
   9,1,11,   1,12,7,   1,9,11,    5,1,6,
   5,1,7,    1,6,2,    1,3,7,     1,8,4,
   1,11,6,   5,1,8,    9,5,3,     1,8,10,
   1,11,3,   1,4,8,    1,9,10,    1,2,10;

list nz;
for (int i=1; i<=36; i++)
{
  nz=insert(nz,detpoints(nz_pos[i,1],nz_pos[i,2],nz_pos[i,3]));
}

proc nonzero_reduction(poly p)
{
   for (int i=1; i<=36; i++)
   {
      p=p/gcd(p,nz[i]);
   }
   return(p);
}

////////////////////////////////////////////////////////
// Part 4: We construct the polynomial identities.

poly p;
ideal I=0;
for (int i=1; i<=12; i++)
{
   for (int j=1; j<=12; j++)
```

```
    {
       p=detpoints(i,j,ct[i,j]);
       if (p!=0) { I=nonzero_reduction(p),I; }
    }
}

//////////////////////////////////////////////////////
// Part 5: We compute the Groebner basis of the corresponding ideal.
// This Groebner basis shows that d[1]=d[4], d[2]=d[5], d[3]=d[6].
// The computation takes less that 3 hours using the modStd method.

ideal J=modStd(I);
J;
```

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Authors' addresses:

Gábor Péter NAGY
Bolyai Institute
University of Szeged
Aradi vértanúk tere 1
H-6720 Szeged (Hungary).
E–mail: `nagyg@math.u-szeged.hu`

Nicola PACE
Department of Mathematical Sciences
Florida Atlantic University
777 Glades Road
Boca Raton, FL 33431, USA.
E–mail: `nicolaonline@libero.it`