

FIRM FROBENIUS MONADS AND FIRM FROBENIUS ALGEBRAS

GABRIELLA BÖHM AND JOSÉ GÓMEZ-TORRECILLAS

Dedicated to

Toma Albu and Constantin Năstăsescu on the occasion of their 70th birthdays

ABSTRACT. *Firm Frobenius algebras* are firm algebras and counital coalgebras such that the comultiplication is a bimodule map. They are investigated by categorical methods based on a study of adjunctions and lifted functors. Their categories of comodules and of firm modules are shown to be isomorphic if and only if a canonical comparison functor from the category of comodules to the category of non-unital modules factorizes through the category of firm modules. This happens for example if the underlying algebra possesses local units, e.g. the firm Frobenius algebra arises from a co-Frobenius coalgebra over a base field; or if the comultiplication splits the multiplication (hence the underlying coalgebra is coseparable).

Key Words: firm Frobenius monad, firm Frobenius adjunction, firm module, comodule, separable functor

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1. INTRODUCTION

The classical notion of *Frobenius algebra* due to Brauer and Nesbitt [8] can be reformulated in terms of the existence of a suitable coalgebra structure on the algebra [2]. Thus, a Frobenius algebra over a commutative ring k is a k -module carrying the structures of an associative and unital algebra and a coassociative and counital coalgebra. These structures are required to be compatible in the sense that the comultiplication is a bimodule map (with respect to the actions provided by the multiplication). Equivalently, the multiplication is a bicomodule map (with respect to the coactions provided by the comultiplication). As discussed by Abrams in [2], this compatibility condition results in an isomorphism between the category of modules and the category of comodules over a Frobenius algebra.

In [21], Frobenius algebras were treated by Street in the broader framework of monoidal (bi)categories. The behavior of the module and comodule categories was given a deep conceptual explanation and a number of equivalent characterizations of Frobenius monoids was given. Applying it to the monoidal category of functors $\mathbb{A} \rightarrow \mathbb{A}$ for an arbitrary category \mathbb{A} , the notion of *Frobenius monad* is obtained. By [21, Theorem 1.6], a Frobenius monad is a monad $M : \mathbb{A} \rightarrow \mathbb{A}$, with multiplication $\mu : M^2 \rightarrow M$ and unit $\eta : \mathbb{A} \rightarrow M$ such that any of the following equivalent assertions holds.

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- (i) There exist natural transformations $\varepsilon : M \rightarrow \mathbb{A}$ and $\varrho : \mathbb{A} \rightarrow M^2$ such that $M\varepsilon \cdot M\mu \cdot \varrho M = M = \varepsilon M \cdot \mu M \cdot M\varrho$.
- (ii) There exists a comonad structure $\delta : M \rightarrow M^2$, $\varepsilon : M \rightarrow \mathbb{A}$ such that $\mu M \cdot M\delta = \delta \cdot \mu = M\mu \cdot \delta M$.
- (iii) The forgetful functor from the category of M -modules to \mathbb{A} possesses a right adjoint $A \xrightarrow{\phi} A' \mapsto (MA, \mu A) \xrightarrow{M\phi} (MA', \mu A')$.

It is immediate from (ii) that an endofunctor $(-) \otimes R$ on the category of modules over a commutative ring k is a Frobenius monad if and only if R is a Frobenius k -algebra. Characterization (i) yields a description of Frobenius algebras in terms of a *functional* $\varepsilon_k : R \rightarrow k$ and a *Casimir element* $\varrho_k(1) \in R \otimes R$.

The above classical notion of Frobenius algebra is essentially self-dual: the algebra and coalgebra structures play symmetric roles. So if allowing the algebra to be non-unital, it is not immediately clear what properties remain true.

Our approach to non-unital Frobenius algebras in this paper is based on Street's categorical treatment in [21]. Generalizing non-unital algebras, we start with discussing *non-unital monads*; that is endofunctors equipped with an associative multiplication possibly without a unit. While there is an evident notion of their *non-unital modules*, our definition of a *firm module* is slightly more sophisticated. It leads to the notion of a *firm monad* which is a non-unital monad whose free modules are firm.

A *non-unital Frobenius monad* is defined as a (coassociative and counital) comonad equipped also with an associative but not necessarily unital multiplication satisfying the compatibility conditions in (ii) above. Associated to it, there is the usual Eilenberg-Moore category of (coassociative and counital) comodules over the constituent comonad and the categories of non-unital, and of firm modules over the constituent non-unital monad. Generalizing the equivalence (ii) \Leftrightarrow (iii) above, we investigate in what sense the corresponding forgetful functors possess adjoints.

This analysis leads in particular to a canonical comparison functor from the category of comodules to the category of non-unital modules. It is proven to factorize through the category of firm modules if and only if the underlying monad is firm and the category of firm modules and the category of comodules are isomorphic. We collect situations when this happens. In particular, we show that any adjunction in which the left adjoint is separable, induces a firm Frobenius monad. For firm Frobenius monads of this kind, the category of firm modules and the category of comodules are shown to be isomorphic.

We apply our results to algebras over commutative rings. This yields a generalization of Abrams' theorem [2] to *firm* Frobenius algebras. In particular, the isomorphism between the category of firm modules and the category of comodules follows from our theory in the following situations.

- For firm Frobenius algebras arising from coseparable coalgebras (and even from coseparable corings) over any base ring. This provides an alternative proof of [7, Proposition 2.17].
- For firm Frobenius algebras with local units.
- In particular, for firm Frobenius algebras arising from co-Frobenius coalgebras over a field. This provides an alternative proof of [11, Theorem 2.3] and [6, Proposition 2.7].

A firm algebra R with a *non-degenerate* multiplication is shown to be a firm Frobenius algebra if and only if there exists a generalized Casimir element in the multiplier algebra of $R \otimes R$ (cf. conditions (i) above).

2. NON-UNITAL MONADS AND FIRM MODULES

2.1. Non-unital monad. By a *non-unital monad* on a category \mathbb{A} we mean a pair of a functor $M : \mathbb{A} \rightarrow \mathbb{A}$ and a natural transformation $\mu : M^2 \rightarrow M$ obeying the associativity condition

$$\begin{array}{ccc} M^3 & \xrightarrow{M\mu} & M^2 \\ \mu M \downarrow & & \downarrow \mu \\ M^2 & \xrightarrow{\mu} & M. \end{array}$$

2.2. Non-unital module. By a *non-unital module* over a non-unital monad $M : \mathbb{A} \rightarrow \mathbb{A}$ we mean a pair of an object A and a morphism $\alpha : MA \rightarrow A$ (called the *M-action*) in \mathbb{A} obeying the associativity condition

$$(2.1) \quad \begin{array}{ccc} M^2 A & \xrightarrow{M\alpha} & MA \\ \mu A \downarrow & & \downarrow \alpha \\ MA & \xrightarrow{\alpha} & A. \end{array}$$

Morphisms of non-unital M -modules are morphisms $A \rightarrow A'$ in \mathbb{A} which are compatible with the M -actions in the evident sense. These objects and morphisms define the category \mathbb{A}_M of non-unital M -modules.

In terms of the forgetful functor $U_M : \mathbb{A}_M \rightarrow \mathbb{A}$ and the functor

$$F_M : \mathbb{A} \rightarrow \mathbb{A}_M, \quad A \xrightarrow{\phi} A' \mapsto (MA, \mu A) \xrightarrow{M\phi} (MA', \mu A'),$$

we can write $M = U_M F_M$. Although there is a natural transformation

$$\alpha : F_M U_M(A, \alpha) \rightarrow (A, \alpha),$$

in the absence of a unit this does not provide an adjunction.

2.3. Firm module. We say that a non-unital module (A, α) over a non-unital monad $M : \mathbb{A} \rightarrow \mathbb{A}$ is *firm* if α is an epimorphism in \mathbb{A} and the fork

$$(2.2) \quad M^2 A \begin{array}{c} \xrightarrow{\mu A} \\ \xrightarrow{M\alpha} \end{array} MA \xrightarrow{\alpha} A$$

is a coequalizer in \mathbb{A}_M . The full subcategory of firm modules in \mathbb{A}_M will be denoted by $\mathbb{A}_{(M)}$, and we will denote by J the full embedding $\mathbb{A}_{(M)} \rightarrow \mathbb{A}_M$. When M is a usual (unital) monad, then the category of firm M -modules is just the usual Eilenberg-Moore category of unital M -modules.

If the functor underlying a non-unital monad is known to preserve epimorphisms, then a simpler criterion for firmness can be given.

Lemma 1. *Let $M : \mathbb{A} \rightarrow \mathbb{A}$ be a non-unital monad and (A, α) be a non-unital M -module. If (2.2) is a coequalizer in \mathbb{A} and $M\alpha$ is an epimorphism in \mathbb{A} , then (A, α) is a firm M -module.*

Proof. By assumption, α is a (regular) epimorphism in \mathbb{A} . Consider a morphism κ of non-unital M -modules from $(MA, \mu A)$ to any non-unital M -module (X, ξ) such that $\kappa \cdot \mu A = \kappa \cdot M\alpha$. Since (2.2) is a coequalizer in \mathbb{A} , there is a unique morphism $\tilde{\kappa} : A \rightarrow X$ in \mathbb{A} satisfying $\tilde{\kappa} \cdot \alpha = \kappa$. The subdiagrams of

$$\begin{array}{ccccc}
MA & \xrightarrow{\alpha} & & & A \\
\downarrow M\tilde{\kappa} & \swarrow M\alpha & M^2A & \xrightarrow{\mu A} & MA & \searrow \alpha \\
& & \swarrow M\kappa & & & \searrow \kappa \\
MX & \xrightarrow{\xi} & & & X
\end{array}$$

commute since κ is a morphism of non-unital M -modules and α is associative. So by the assumption that $M\alpha$ is an epimorphism in \mathbb{A} , also the exterior commutes proving that $\tilde{\kappa}$ is a morphism in \mathbb{A}_M hence (2.2) is a coequalizer in \mathbb{A}_M . \square

2.4. Firm monad. We say that a non-unital monad M on a category \mathbb{A} is a *firm monad* if the functor $F_M : \mathbb{A} \rightarrow \mathbb{A}_M$ in Section 2.2 factorizes through some functor $F_{(M)} : \mathbb{A} \rightarrow \mathbb{A}_{(M)}$ via the inclusion $J : \mathbb{A}_{(M)} \rightarrow \mathbb{A}_M$. That is, for any object A of \mathbb{A} , μA is an epimorphism in \mathbb{A} and

$$(2.3) \quad M^3A \begin{array}{c} \xrightarrow{\mu MA} \\ \xrightarrow{M\mu A} \end{array} M^2A \xrightarrow{\mu A} MA$$

is a coequalizer in \mathbb{A}_M . Then in terms of the forgetful functor $U_{(M)} : \mathbb{A}_{(M)} \rightarrow \mathbb{A}$, the equality $M = U_{(M)}F_{(M)}$ holds and there is a natural transformation

$$\alpha : F_{(M)}U_{(M)}(A, \alpha) = (MA, \mu A) \rightarrow (A, \alpha).$$

However, in general this does not extend to an adjunction.

3. NON-UNITAL MONADS VERSUS ADJUNCTIONS

3.1. Non-unital adjunction. By a *non-unital adjunction* we mean a pair of functors $U : \mathbb{B} \rightarrow \mathbb{A}$ and $F : \mathbb{A} \rightarrow \mathbb{B}$ together with a natural transformation $\varphi : FU \rightarrow \mathbb{B}$.

Associated to any non-unital adjunction $\varphi : FU \rightarrow \mathbb{B}$, there is a non-unital monad $(UF, U\varphi F)$. Conversely, associated to any non-unital monad $M : \mathbb{A} \rightarrow \mathbb{A}$, there is a non-unital adjunction $F_M U_M \rightarrow \mathbb{A}_M$ as in Section 2.2.

For any non-unital adjunction $\varphi : FU \rightarrow \mathbb{B}$, there is an induced functor

$$L_{UF} : \mathbb{B} \rightarrow \mathbb{A}_{UF}, \quad B \xrightarrow{\phi} B' \mapsto (UB, U\varphi B) \xrightarrow{U\phi} (UB', U\varphi B').$$

3.2. Firm adjunction. We say that a non-unital adjunction $\varphi : FU \rightarrow \mathbb{B}$ is *firm* if the functor $L_{UF} : \mathbb{B} \rightarrow \mathbb{A}_{UF}$ in Section 3.1 factorizes through some functor $L_{(UF)} : \mathbb{B} \rightarrow \mathbb{A}_{(UF)}$ via the inclusion $J : \mathbb{A}_{(UF)} \rightarrow \mathbb{A}_{UF}$. That is, for any object B in \mathbb{B} , $U\varphi B$ is an epimorphism in \mathbb{A} and

$$(3.1) \quad UFUFUB \begin{array}{c} \xrightarrow{U\varphi FUB} \\ \xrightarrow{UFU\varphi B} \end{array} UFUB \xrightarrow{U\varphi B} UB$$

is a coequalizer in \mathbb{A}_{UF} .

For an adjunction $F \dashv U$ in the usual (unital and counital) sense, (3.1) is a split coequalizer in \mathbb{A} (in the sense of [4, page 93]). Hence $F \dashv U$ is a firm adjunction by Lemma 1.

Associated to any firm adjunction $\varphi : FU \rightarrow \mathbb{B}$, there is a firm monad $(UF, U\varphi F)$. Conversely, associated to any firm monad $M : \mathbb{A} \rightarrow \mathbb{A}$, there is a firm adjunction $F_{(M)}U_{(M)} \rightarrow \mathbb{A}_{(M)}$ as in Section 2.4.

4. FROBENIUS STRUCTURES

4.1. Non-unital Frobenius monad. By a *non-unital Frobenius monad* we mean a functor $M : \mathbb{A} \rightarrow \mathbb{A}$ which carries a non-unital monad structure $\mu : M^2 \rightarrow M$ and a comonad structure $(\delta : M \rightarrow M^2, \varepsilon : M \rightarrow \mathbb{A})$ such that the following diagram commutes.

$$(4.1) \quad \begin{array}{ccc} M^2 & \xrightarrow{M\delta} & M^3 \\ \delta M \downarrow & \searrow \mu & \downarrow \mu M \\ & M & \\ & \searrow \delta & \\ M^3 & \xrightarrow{M\mu} & M^2 \end{array}$$

A *firm Frobenius monad* is a non-unital Frobenius monad which is a firm monad.

Let $M : \mathbb{A} \rightarrow \mathbb{A}$ be a non-unital Frobenius monad. As in the case of any non-unital monad M , we denote by \mathbb{A}_M the category of non-unital M -modules and we denote by $\mathbb{A}_{(M)}$ the full subcategory of firm M -modules. The usual Eilenberg-Moore category of counital comodules for the comonad M will be denoted by \mathbb{A}^M with corresponding forgetful functor $U^M : \mathbb{A}^M \rightarrow \mathbb{A}$ and its right adjoint $F^M : \mathbb{A} \rightarrow \mathbb{A}^M$ with $A \xrightarrow{\phi} A' \mapsto (MA, \delta A) \xrightarrow{M\phi} (MA', \delta A')$.

4.2. Non-unital Frobenius adjunction. We say that a (firm) non-unital adjunction $\varphi : FU \rightarrow \mathbb{B}$ is *Frobenius* if U is the left adjoint of F (in the usual sense, with unit $\eta : \mathbb{B} \rightarrow FU$ and counit $\varepsilon : UF \rightarrow \mathbb{A}$). Then UF is a (firm) non-unital Frobenius monad, with multiplication $U\varphi F$, comultiplication $U\eta F$ and counit ε . The aim of the next sections is to prove the converse: to associate a (firm) non-unital Frobenius adjunction to any (firm) non-unital Frobenius monad.

4.3. Firm Frobenius monads versus adjunctions to firm modules.

Proposition 1. *Any firm Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$ determines a firm Frobenius adjunction $F_{(M)}U_{(M)} \rightarrow \mathbb{A}_{(M)}$ as in Section 2.4 such that $U_{(M)}F_{(M)} = M$ as firm Frobenius monads.*

Proof. The counit of the adjunction $U_{(M)} \dashv F_{(M)}$ is the counit $\varepsilon : U_{(M)}F_{(M)} = M \rightarrow \mathbb{A}$ of the comonad M , and the unit $\eta : \mathbb{A}_{(M)} \rightarrow F_{(M)}U_{(M)}$ is defined via universality of the coequalizer (in $\mathbb{A}_{(M)}$) in the top row of

$$\begin{array}{ccccc} M^2 A & \xrightarrow{\mu A} & M A & \xrightarrow{\alpha} & A \\ \delta M A \downarrow & \searrow M\alpha & \downarrow \delta A & & \downarrow \eta(A, \alpha) \\ M^3 A & \xrightarrow{M\mu A} & M^2 A & \xrightarrow{M\alpha} & M A, \\ & \searrow M^2 \alpha & & & \downarrow \eta(A, \alpha) \end{array}$$

for any firm M -module (A, α) . The square on the left commutes serially (in the sense of [4, page 72]) by the Frobenius condition (4.1) and by naturality of δ . The bottom row is a fork by the associativity of α . By the Frobenius property (4.1) of M , δA is a morphism of non-unital M -modules, hence so is $M\alpha \cdot \delta A$. This

proves the existence and the uniqueness of the morphism of non-unital M -modules $\eta(A, \alpha) : (A, \alpha) \rightarrow (MA, \mu A)$.

Naturality of η follows from commutativity of the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\eta(A, \alpha)} & MA & & \\
 \downarrow \phi & \swarrow \alpha & MA \xrightarrow{\delta A} M^2 A & \searrow M\alpha & \downarrow M\phi \\
 & & \downarrow M\phi & & \downarrow M^2\phi \\
 & & MA' \xrightarrow{\delta A'} M^2 A' & \searrow M\alpha' & \\
 & \swarrow \alpha' & & & \\
 A' & \xrightarrow{\eta(A', \alpha')} & MA' & &
 \end{array}$$

for any morphism of firm M -modules $\phi : (A, \alpha) \rightarrow (A', \alpha')$, because α is an epimorphism (both in $\mathbb{A}_{(M)}$ and \mathbb{A}).

It remains to check that η and ε satisfy the triangular identities. Given an object A of \mathbb{A} , $\eta F_{(M)} A = \eta(MA, \mu A)$ is the unique morphism rendering commutative the diagram

$$\begin{array}{ccc}
 M^2 A & \xrightarrow{\mu A} & MA \\
 \delta MA \downarrow & & \downarrow \eta F_{(M)} A \\
 M^3 A & \xrightarrow{M\mu A} & M^2 A.
 \end{array}$$

So by the Frobenius condition (4.1), $\eta F_{(M)} A = \delta A$. Thus, $F_{(M)} \varepsilon \cdot \eta F_{(M)} = M \varepsilon \cdot \delta = F_{(M)}$. Since α is an epimorphism in \mathbb{A} for all $(A, \alpha) \in \mathbb{A}_{(M)}$, the other triangle condition follows by the commutativity of

$$\begin{array}{ccccc}
 A = U_{(M)}(A, \alpha) & \xrightarrow{U_{(M)} \eta(A, \alpha)} & MA & \xrightarrow{\varepsilon U_{(M)}(A, \alpha)} & U_{(M)}(A, \alpha) = A \\
 \alpha \uparrow & & M\alpha \uparrow & & \alpha \uparrow \\
 MA & \xrightarrow{\delta A} & M^2 A & \xrightarrow{\varepsilon MA} & MA.
 \end{array}$$

□

For any firm Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$, the unit $\eta : \mathbb{A}_{(M)} \rightarrow F_{(M)} U_{(M)}$ of the adjunction $U_{(M)} \dashv F_{(M)}$ in Proposition 1 induces a functor $K_{(M)}$ rendering commutative

$$(4.2) \quad \begin{array}{ccc}
 & & \mathbb{A}^M \\
 & \nearrow K_{(M)} & \downarrow U^M \\
 \mathbb{A}_{(M)} & \xrightarrow{U_{(M)}} & \mathbb{A},
 \end{array}$$

since $U_{(M)} F_{(M)} = M$ as comonads. Explicitly, $K_{(M)}$ is given by

$$(A, MA \xrightarrow{\alpha} A) \xrightarrow{\phi} (A', MA' \xrightarrow{\alpha'} A') \mapsto (A, A \xrightarrow{\eta(A, \alpha)} MA) \xrightarrow{\phi} (A', A' \xrightarrow{\eta(A', \alpha')} MA').$$

4.4. Non-unital Frobenius monads versus adjunctions to comodules. In order to associate a non-unital Frobenius adjunction to any, not necessarily firm, non-unital Frobenius monad, we shall work with the category of comodules instead of the categories of firm or non-unital modules in the previous sections.

Let $M : \mathbb{A} \rightarrow \mathbb{A}$ be a non-unital Frobenius monad. For any M -comodule (A, α) , throughout the paper the notation

$$(4.3) \quad \bar{\alpha} := MA \xrightarrow{M\alpha} M^2 A \xrightarrow{\mu A} MA \xrightarrow{\varepsilon A} A$$

will be used.

Lemma 2. *Let $M : \mathbb{A} \rightarrow \mathbb{A}$ be a non-unital Frobenius monad. Using the notation in (4.3), for any M -comodule (A, α) the following assertions hold.*

- (1) *The coaction α obeys $\mu A \cdot M\alpha = \alpha \cdot \bar{\alpha}$.*
- (2) *The identity $\alpha \cdot \bar{\alpha} = M\bar{\alpha} \cdot \delta A$ holds. That is, $\bar{\alpha}$ is a morphism of M -comodules $(MA, \delta A) \rightarrow (A, \alpha)$.*

Proof. (1). The claim follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
 MA & \xrightarrow{M\alpha} & M^2A & \xrightarrow{\mu A} & MA & \xrightarrow{\varepsilon A} & A \\
 M\alpha \downarrow & & M^2\alpha \downarrow & & \downarrow M\alpha & & \downarrow \alpha \\
 M^2A & \xrightarrow{M\delta A} & M^3A & \xrightarrow{\mu MA} & M^2A & & \\
 \mu A \downarrow & & \delta A & \nearrow & \searrow \varepsilon MA & & \\
 MA & \xrightarrow{\hspace{10em}} & MA & & MA & & MA
 \end{array}$$

(2). By the naturality and the counitality of δ and the Frobenius condition (4.1), $M\bar{\alpha} \cdot \delta A = \mu A \cdot M\alpha$. So the claim follows by part (1). \square

Proposition 2. *Any non-unital Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$ determines a non-unital Frobenius adjunction $F^M U^M \rightarrow \mathbb{A}^M$ such that $U^M F^M = M$ as non-unital Frobenius monads.*

Proof. For any comonad M , $U^M \dashv F^M$ and $U^M F^M = M$ as comonads. A non-unital adjunction $F^M U^M \rightarrow \mathbb{A}^M$ is provided by the M -comodule morphisms $\bar{\alpha} : (MA, \delta A) = F^M U^M(A, \alpha) \rightarrow (A, \alpha)$ in Lemma 2 (2). In light of (4.3), their naturality follows by the naturality of μ and ε . The equality $U^M F^M = M$ of non-unital monads follows by $\bar{\delta} A = \varepsilon MA \cdot \mu MA \cdot M\delta A = \varepsilon MA \cdot \delta A \cdot \mu A = \mu A$, cf. (4.3). \square

For any non-unital Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$, corresponding to the non-unital adjunction $F^M U^M \rightarrow \mathbb{A}^M$ in Proposition 2, there is an induced functor $L_M : \mathbb{A}^M \rightarrow \mathbb{A}_M$ as in Section 3.1. It renders commutative the diagram

$$(4.4) \quad \begin{array}{ccc}
 & & \mathbb{A}_M \\
 & \nearrow L_M & \downarrow U_M \\
 \mathbb{A}^M & \xrightarrow{U^M} & \mathbb{A}
 \end{array}$$

and sends $(A, A \xrightarrow{\alpha} MA) \xrightarrow{\phi} (A', A' \xrightarrow{\alpha'} MA')$ to $(A, \bar{\alpha}) \xrightarrow{\phi} (A', \bar{\alpha}')$, cf. (4.3).

5. MODULES AND COMODULES OF A FIRM FROBENIUS MONAD

The aim of this section is to see when the functor $K_{(M)}$, associated in (4.2) to a firm Frobenius monad M , is an isomorphism.

Proposition 3. *For any firm Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$, the functor $K_{(M)}$ in (4.2) is fully faithful.*

Proof. For any firm M -module (A, α) and the functor L_M in (4.4),

$$L_M K_{(M)}(A, \alpha) = (A, MA \xrightarrow{M\eta(A, \alpha)} M^2A \xrightarrow{\mu A} MA \xrightarrow{\varepsilon A} A).$$

Since $\eta(A, \alpha)$ is a morphism of non-unital M -modules, and by one of the triangle identities on the adjunction $U_{(M)} \dashv F_{(M)}$, $\varepsilon A \cdot \mu A \cdot M\eta(A, \alpha) = \varepsilon A \cdot \eta(A, \alpha) \cdot \alpha = \alpha$; that is, $L_M K_{(M)} = J$. Since J is faithful, we get that $K_{(M)}$ is faithful, too. In order to see that $K_{(M)}$ is full, take a morphism

$$K_{(M)}(A, \alpha) = (A, \eta(A, \alpha)) \xrightarrow{\phi} K_{(M)}(A', \alpha) = (A', \eta(A', \alpha'))$$

in \mathbb{A}^M . Then

$$L_M K_{(M)}(A, \alpha) = (A, \alpha) \xrightarrow{L_M \phi = \phi} L_M K_{(M)}(A', \alpha) = (A', \alpha')$$

belongs to the full subcategory $\mathbb{A}_{(M)}$, and applying to it $K_{(M)}$, we re-obtain ϕ . \square

Theorem 1. *For a non-unital Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$, the following assertions are equivalent.*

- (1) *The non-unital Frobenius adjunction $F^M U^M \rightarrow \mathbb{A}^M$ in Proposition 2 is a firm Frobenius adjunction. That is, for any M -comodule (A, α) , $\bar{\alpha}$ in (4.3) is an epimorphism in \mathbb{A} and there is a coequalizer*

$$M^2 A \begin{array}{c} \xrightarrow{\mu A} \\ \xrightarrow{M\bar{\alpha}} \end{array} MA \xrightarrow{\bar{\alpha}} A \quad \text{in } \mathbb{A}_M.$$

- (2) *M is a firm Frobenius monad and the functor $K_{(M)}$ in (4.2) is an isomorphism.*

Proof. (1) \Rightarrow (2). Since M arises from a firm Frobenius adjunction in (1), it is a firm Frobenius monad. Let $\mathbb{A}^M \xrightarrow{L_{(M)}} \mathbb{A}_{(M)} \xrightarrow{J} \mathbb{A}_M$ be a factorization of L_M . We claim that $L_{(M)}$ provides the inverse of $K_{(M)}$. We know from the proof of Proposition 3 that $JL_{(M)}K_{(M)} = L_M K_{(M)} = J$, so that $L_{(M)}K_{(M)} = \mathbb{A}_{(M)}$. For any M -comodule (A, α) , $L_{(M)}(A, \alpha) = (A, \bar{\alpha})$ in (4.3) is firm by assumption. So $K_{(M)}L_{(M)}(A, \alpha) = (A, \eta(A, \bar{\alpha}))$, and the proof is complete if we show $\eta(A, \bar{\alpha}) = \alpha$. Since $\bar{\alpha}$ is an epimorphism in \mathbb{A} , this follows by commutativity of the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta(A, \bar{\alpha})} & & & MA \\ & \searrow \bar{\alpha} & MA & \xrightarrow{\delta A} & M^2 A & \xrightarrow{M\bar{\alpha}} & MA \\ & & \downarrow M\alpha & & \downarrow M^2 \alpha & & \parallel \\ & & M^2 A & \xrightarrow{\delta MA} & M^3 A & & \\ & \swarrow \mu A & & & \downarrow M\mu A & & \\ MA & \xrightarrow{\delta A} & & & M^2 A & \xrightarrow{M\varepsilon A} & MA \\ & \searrow & & & & & \end{array}$$

where the region on the left commutes by Lemma 2 (1).

(2) \Rightarrow (1) Since $L_M K_{(M)} = J$, $L_M = JK_{(M)}^{-1}$ is the desired factorization. \square

Next we look for situations when the equivalent conditions in Theorem 1 hold.

Proposition 4. *For a non-unital Frobenius monad $M : \mathbb{A} \rightarrow \mathbb{A}$, assume that there exists a natural section (i.e. right inverse) ν of the multiplication $\mu : M^2 \rightarrow M$*

rendering commutative

$$\begin{array}{ccc} M^2 & \xrightarrow{\mu} & M \\ \nu M \downarrow & & \downarrow \nu \\ M^3 & \xrightarrow{M\mu} & M^2. \end{array}$$

Then the equivalent conditions in Theorem 1 hold.

Proof. We will show that for every M -comodule (A, α) , and $\bar{\alpha}$ as in (4.3), the diagram

$$\begin{array}{ccccc} & \xleftarrow{\nu A} & & \xleftarrow{M\varepsilon A \cdot \nu A \cdot \alpha} & \\ & \mu A & \searrow & \bar{\alpha} & \\ M^2 A & \xrightarrow{M\bar{\alpha}} & M A & \xrightarrow{\bar{\alpha}} & A \end{array}$$

is a contractible coequalizer in \mathbb{A} (in the sense of [4, page 93]). By assumption, $\mu A \cdot \nu A = M A$. By Lemma 2 (1), and naturality of ν , we get that the diagram

$$\begin{array}{ccccccc} M A & \xlongequal{\quad} & M A & \xrightarrow{\nu A} & M^2 A & \xlongequal{\quad} & M^2 A \\ \downarrow \bar{\alpha} & & \downarrow M\alpha & \downarrow \nu M A & \downarrow M^2 \alpha & & \downarrow M\bar{\alpha} \\ & & M^2 A & \xrightarrow{\nu M A} & M^3 A & & \\ & & \downarrow \mu A & & \downarrow M\mu A & & \\ A & \xrightarrow{\alpha} & M A & \xrightarrow{\nu A} & M^2 A & \xrightarrow{M\varepsilon A} & M A \end{array}$$

is commutative. Thus, $M\bar{\alpha} \cdot \nu A = M\varepsilon A \cdot \nu A \cdot \alpha \cdot \bar{\alpha}$. Using that ν is natural, the Frobenius condition and the assumption, also the following diagram is seen to commute.

$$\begin{array}{ccccc} M & \xrightarrow{\delta} & M^2 & & M^2 \\ \mu \swarrow & & M^2 & \xrightarrow{M\delta} & M^3 & \searrow \mu M \\ \nu \downarrow & & \nu M \downarrow & & \downarrow \nu M^2 & \downarrow \nu M \\ & & M^3 & \xrightarrow{M^2 \delta} & M^4 & \\ M\mu \swarrow & & & & & \searrow M\mu M \\ M^2 & \xrightarrow{M\delta} & M^3 & & M^3 \end{array}$$

Since μ is a (split) natural epimorphism by assumption, also the outer rectangle commutes, what implies commutativity of

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & M A & \xrightarrow{\nu A} & M^2 A & \xrightarrow{M\varepsilon A} & M A \\ \downarrow \alpha & & M\alpha \downarrow & & & & M\alpha \downarrow \\ M A & \xrightarrow{\delta A} & M^2 A & \xrightarrow{\nu M A} & M^3 A & \xrightarrow{M\varepsilon M A} & M^2 A \\ \parallel & & & & M\delta A \uparrow & & \\ M A & \xrightarrow{\nu A} & M^2 A & & & & \end{array}$$

Composing both equal paths around this diagram by $\varepsilon A \cdot \mu A$, using that ν is a section of μ and the counitality of α , we obtain $\bar{\alpha} \cdot M\varepsilon A \cdot \nu A \cdot \alpha = A$. Since in this way $\bar{\alpha}$ is a split epimorphism, it is taken by M to a (split) epimorphism. So we conclude by Lemma 1 that $L_M(A, \alpha) = (A, \bar{\alpha})$ is a firm M -module. \square

Every Frobenius pair of functors in the sense of [12] gives obviously a (unital) Frobenius adjunction and, therefore, a (unital) Frobenius monad. A more interesting situation is described in the following corollary. Separable functors were introduced and studied in [18].

Corollary 1. *Let $U : \mathbb{B} \rightarrow \mathbb{A}$ be a separable functor possessing a right adjoint F . Then UF carries the structure of a firm Frobenius monad such that the comparison functor $K_{(UF)} : \mathbb{A}_{(UF)} \rightarrow \mathbb{A}^{UF}$ is an isomorphism.*

Proof. By Rafael's theorem [20], there exists a retraction (i.e. left inverse) φ of the unit $\eta : \mathbb{B} \rightarrow FU$ of the adjunction. Then $\varphi : FU \rightarrow \mathbb{B}$ is a non-unital Frobenius adjunction so that UF is a non-unital Frobenius monad. The claim follows by applying to it Proposition 4, putting $\nu := U\eta F$. \square

6. APPLICATION: FIRM FROBENIUS ALGEBRAS OVER COMMUTATIVE RINGS

6.1. Firm Frobenius algebra. Let k be an associative, unital, commutative ring and denote the category of k -modules by \mathbb{M}_k . It is a monoidal category via the k -module tensor product \otimes and the neutral object k .

Any associative algebra R — possibly without a unit — over k may be equivalently defined as a non-unital monad $(-) \otimes R : \mathbb{M}_k \rightarrow \mathbb{M}_k$. The category of non-unital modules for this monad — equivalently, the category of non-unital modules for the algebra R — will be denoted by \mathbb{M}_R . For any (non-unital) right R -module (A, α) and left R -module (B, β) , we denote by $A \otimes_R B$ the coequalizer of $\alpha \otimes B$ and $A \otimes \beta$ in \mathbb{M}_k and we call it the *R -module tensor product*.

Proposition 5. *For a non-unital k -algebra R and a non-unital right R -module $(A, A \otimes R \xrightarrow{\alpha} A)$, the following assertions are equivalent.*

- (1) *(A, α) is a firm module for the non-unital monad $(-) \otimes R : \mathbb{M}_k \rightarrow \mathbb{M}_k$.*
- (2) *The action $\alpha : A \otimes R \rightarrow A$ projects to a bijection $A \otimes_R R \rightarrow A$.*

Proof. Since $(-) \otimes R$ is a right exact endofunctor on \mathbb{M}_k , we get that coequalizers exist in \mathbb{M}_R and the forgetful functor $\mathbb{M}_R \rightarrow \mathbb{M}_k$ creates them. Hence (2.2) is a coequalizer in \mathbb{M}_R if and only if it is a coequalizer in \mathbb{M}_k so if and only if (2) holds. This proves (1) \Rightarrow (2). If (2) holds then α is surjective, hence it is an epimorphism in \mathbb{M}_k proving (2) \Rightarrow (1). \square

Similarly, $(-) \otimes R : \mathbb{M}_k \rightarrow \mathbb{M}_k$ is a firm monad if and only if the multiplication map $R \otimes R \rightarrow R$ projects to an isomorphism $R \otimes_R R \rightarrow R$. That is, if and only if R is a firm ring in the sense of [19].

By a *non-unital Frobenius k -algebra* we mean a k -module R such that $(-) \otimes R : \mathbb{M}_k \rightarrow \mathbb{M}_k$ is a non-unital Frobenius monad. Explicitly, this means that R is a non-unital k -algebra with multiplication $\mu : R \otimes R \rightarrow R$ and a k -coalgebra with comultiplication $\Delta : R \rightarrow R \otimes R$ and counit $\epsilon : R \rightarrow k$, such that μ is a morphism of R -bicomodules, equivalently, Δ is a morphism of R -bimodules, that is, the following diagram commutes.

$$\begin{array}{ccc}
 R \otimes R & \xrightarrow{R \otimes \Delta} & R \otimes R \otimes R \\
 \Delta \otimes R \downarrow & \searrow \mu & \downarrow \mu \otimes R \\
 R \otimes R \otimes R & \xrightarrow{R \otimes \mu} & R \otimes R
 \end{array}$$

A *firm Frobenius k -algebra* is a non-unital Frobenius k -algebra which is a firm algebra.

6.2. The Casimir multiplier. In the case of a unital Frobenius algebra R , the R -bilinear comultiplication is tightly linked to a so-called Casimir element in $R \otimes R$ (the image under Δ of the unit 1). In our present setting, this is only possible if we allow the Casimir element to be a multiplier [14].

Let R be a non-unital k -algebra with a non-degenerate multiplication. That is, assume that any of the conditions ($sr = 0, \forall s \in R$) and ($rs = 0, \forall s \in R$) implies $r = 0$. A *multiplier* on R is a pair (λ, ϱ) of k -module endomorphisms of R such that

$$(6.1) \quad \varrho(r)s = r\lambda(s), \quad \text{for all } r, s \in R.$$

By [14, 1.5], λ is a right R -module map and ρ is a left R -module map.

The k -module $\mathcal{M}(R)$ of all multipliers is a unital associative algebra with the multiplication $(\lambda, \varrho)(\lambda', \varrho') = (\lambda\lambda', \varrho'\varrho)$, where juxtaposition in the components means composition of maps. Throughout, we denote by 1 the unit element (id, id) of $\mathcal{M}(R)$. There exists an injective homomorphism of algebras from R to $\mathcal{M}(R)$ sending $r \in R$ to the multiplier (λ_r, ρ_r) , where $\lambda_r(s) = rs$ and $\rho_r(s) = sr$, for all $r, s \in R$. The image of R becomes a two-sided ideal of $\mathcal{M}(R)$: a multiplier $\omega = (\lambda, \varrho)$ acts on an element $r \in R$ by $\omega r = \lambda(r)$, and $r\omega = \varrho(r)$ (so that (6.1) can be rewritten as $(r\omega)s = r(\omega s)$, for all $s, r \in R$ and $\omega \in \mathcal{M}(R)$; allowing for a simplified writing $r\omega s$). This yields inclusions $R \otimes R \subseteq \mathcal{M}(R) \otimes \mathcal{M}(R) \subseteq \mathcal{M}(R \otimes R)$. By the non-degeneracy of the multiplication of R , two multipliers ω and ω' on R are equal if and only if $\omega r = \omega' r$ for all $r \in R$ and if and only if $r\omega = r\omega'$ for all $r \in R$.

Proposition 6. *Let R be a firm algebra with non-degenerate multiplication over a commutative ring k . Then R is a firm Frobenius algebra if and only if there exists a multiplier $e \in \mathcal{M}(R \otimes R)$ and a linear map $\epsilon : R \rightarrow k$ such that, for all $r \in R$, $(r \otimes 1)e = e(1 \otimes r)$ is an element of $R \otimes R$ and*

$$(\epsilon \otimes R)((r \otimes 1)e) = r = (R \otimes \epsilon)(e(1 \otimes r)).$$

Proof. Given an R -bilinear comultiplication $\Delta : R \rightarrow R \otimes R$, define $\tilde{\Delta} : \mathcal{M}(R) \rightarrow \mathcal{M}(R \otimes R)$ by

$$(6.2) \quad \begin{aligned} \tilde{\Delta}(\omega)(s \otimes r) &= \Delta(\omega r)(s \otimes 1) \\ (s \otimes r)\tilde{\Delta}(\omega) &= (1 \otimes r)\Delta(s\omega), \quad \text{for } \omega \in \mathcal{M}(R), \quad s, r \in R. \end{aligned}$$

The following computation shows that $\tilde{\Delta}(\omega)$ is a multiplier:

$$\begin{aligned} ((s' \otimes r')\tilde{\Delta}(\omega))(s \otimes r) &= (1 \otimes r')\Delta(s'\omega)(s \otimes r) = (1 \otimes r')\Delta(s'\omega r)(s \otimes 1) \\ &= (s' \otimes r')\Delta(\omega r)(s \otimes 1) = (s' \otimes r')(\tilde{\Delta}(\omega)(s \otimes r)), \end{aligned}$$

where in the second equality we used that Δ is right R -linear, and in the third one that it is left R -linear. Put $e = \tilde{\Delta}(1) \in \mathcal{M}(R \otimes R)$. Take $s' \otimes r' \in R \otimes R$ and $r \in R$. Using once more that Δ is left and right R -linear,

$$(r \otimes 1)e(s' \otimes r') = (r \otimes 1)\Delta(r')(s' \otimes 1) = \Delta(rr')(s' \otimes 1) = \Delta(r)(s' \otimes r').$$

Therefore, $(r \otimes 1)e = \Delta(r)$ and so $(\epsilon \otimes R)((r \otimes 1)e) = r$. Symmetrically, $e(1 \otimes r) = \Delta(r)$ and so $(R \otimes \epsilon)(e(1 \otimes r)) = r$.

Conversely, assume the existence of $e \in \mathcal{M}(R \otimes R)$ and $\epsilon : R \rightarrow k$ as in the claim. Define

$$\Delta(r) = (r \otimes 1)e = e(1 \otimes r) \in R \otimes R, \quad \text{for all } r \in R.$$

This is clearly an R -bilinear comultiplication with the counit ϵ . It remains to prove that Δ is coassociative. Since R is firm, we know that μ is an epimorphism. Therefore, the coassociativity of Δ follows from the commutativity of the diagram

$$\begin{array}{ccccc}
R & \xrightarrow{\Delta} & R \otimes R & & \\
\Delta \downarrow & \swarrow \mu & R \otimes R \xrightarrow{R \otimes \Delta} R \otimes R \otimes R & \xrightarrow{\mu \otimes R} & R \otimes R \\
& & \Delta \otimes R \downarrow & \downarrow \Delta \otimes R \otimes R & \\
& & R \otimes R \otimes R \xrightarrow{R \otimes R \otimes \Delta} R \otimes R \otimes R \otimes R & \xrightarrow{R \otimes \mu \otimes R} & R \otimes R \\
& \swarrow R \otimes \mu & & & \Delta \otimes R \downarrow \\
R \otimes R & \xrightarrow{R \otimes \Delta} & R \otimes R \otimes R & & R \otimes R \otimes R
\end{array}$$

□

6.3. Firm modules and comodules. The following extension of Abrams' classical theorem on unital Frobenius algebras [2] is an immediate consequence of Theorem 1.

Theorem 2. *Let R be a non-unital Frobenius algebra over a commutative ring k . Then the following assertions are equivalent.*

- (1) *Any right R -comodule N is a firm right R -module via the action $n \cdot r := n_0 \epsilon(n_1 r)$ (where Sweedler's implicit summation index notation $n \mapsto n_0 \otimes n_1$ is used for the coaction).*
- (2) *R is a firm Frobenius k -algebra and the category $\mathbb{M}_{(R)}$ of firm right R -modules and the category \mathbb{M}^R of right R -comodules are isomorphic via the following mutually inverse functors. The functor $\mathbb{M}_{(R)} \rightarrow \mathbb{M}^R$ sends a firm right R -module M to*

$$(M, M \xrightarrow{\cong} M \otimes_R R \xrightarrow{M \otimes_R \Delta} M \otimes_R R \otimes R \xrightarrow{\cong} M \otimes R).$$

The functor $\mathbb{M}^R \rightarrow \mathbb{M}_{(R)}$ sends an R -comodule (N, ρ) to

$$(N, N \otimes R \xrightarrow{\rho \otimes R} N \otimes R \otimes R \xrightarrow{N \otimes \mu} N \otimes R \xrightarrow{N \otimes \epsilon} N).$$

On the morphisms both functors act as the identity maps.

6.4. Example: coseparable coalgebra. A (coassociative and counital) coalgebra C over a commutative ring k is said to be *coseparable* if there is a C -bicomodule retraction (i.e. left inverse) of the comultiplication. Equivalently, the forgetful functor U from the category of (say, right) C -comodules \mathbb{M}^C to \mathbb{M}_k is separable [9, Corollary 3.6]. Since U is left adjoint to $(-) \otimes_k C$, Corollary 1 yields a firm Frobenius monad $(-) \otimes_k C$ on \mathbb{M}_k . Therefore C admits the structure of a firm ring. The multiplication is given by the bilinear retraction of the comultiplication (see [10] for a direct proof). By Corollary 1, the category of firm modules $\mathbb{M}_{(C)}$ and the category of comodules \mathbb{M}^C are isomorphic.

The above reasoning can be repeated for a coring C over an arbitrary (associative and unital) algebra A . By [9, Corollary 3.6], C is a coseparable coring if and only if the (left adjoint) forgetful functor from the category of (say, right) C -comodules to the category of (right) A -modules is separable. So by Corollary 1, C possesses a firm ring structure, described already in [10]. In this way Corollary 1 extends [7, Proposition 2.17].

6.5. Example: graded rings. Let G be an arbitrary group. For any commutative ring k , consider the k -coalgebra R with free k -basis $\{p_g : g \in G\}$, comultiplication $\Delta(p_g) = p_g \otimes p_g$ and counit $\epsilon(p_g) = 1$ for $g \in G$. This is in fact a coseparable coalgebra via the bicomodule section of Δ — hence associative multiplication — determined by $p_g \otimes p_h \mapsto p_g$ if $g = h$ and $p_g \otimes p_h \mapsto 0$ otherwise, for $g, h \in G$.

For any G -graded unital k -algebra $A = \bigoplus_{g \in G} A_g$, the linear map

$$R \otimes A \rightarrow A \otimes R, \quad p_g \otimes a_h \mapsto a_h \otimes p_{gh}, \quad \text{for } g, h \in G \text{ and } a_h \in A_h,$$

is an entwining structure (a.k.a. mixed distributive law) and, therefore, it defines an A -coring structure on $A \otimes R$ (see [9, Proposition 2.2]). The A -bimodule structure on $A \otimes R$ is determined by

$$c(a \otimes p_g)b_h = cab_h \otimes p_{gh}, \quad \text{for } a, c \in A, \quad b_h \in A_h, \quad g, h \in G,$$

and its comultiplication and counit are the linear extensions of $a \otimes p_g \mapsto (a \otimes p_g) \otimes_A (1 \otimes p_g)$ and $a \otimes p_g \mapsto a$, respectively. The category of right comodules over this coring is isomorphic to the category of entwined (or mixed) modules [9, Proposition 2.2] which, in the present situation, is isomorphic to the category $\mathbf{gr}\text{-}A$ of G -graded right A -modules.

Thanks to the coseparability of the coalgebra R , the A -coring $A \otimes R$ is coseparable: its comultiplication has a bicomodule section

$$\mu : (A \otimes R) \otimes_A (A \otimes R) \rightarrow A \otimes R, \quad (a \otimes p_g) \otimes_A (b \otimes p_h) \mapsto ab_{g^{-1}h} \otimes p_h,$$

for $a, b \in A, g, h \in G$. Therefore, the discussion at the end of Example 6.4 shows that $A \otimes R$ is a firm k -algebra via the multiplication induced by μ . (In fact, this firm algebra even has local units.) Moreover, the category $\mathbf{gr}\text{-}A$ is isomorphic to the category $\mathbb{M}_{(A \otimes R)}$ of firm modules. Since the firm algebra $A \otimes R$ is isomorphic to the smash product $A \sharp G^*$ in [5], this isomorphism $\mathbb{M}_{(A \otimes R)} \cong \mathbb{M}_{(A \sharp G^*)} \cong \mathbf{gr}\text{-}A$ reproduces the main result of [5].

6.6. Example: Frobenius algebra with local units. We say that an algebra R is an algebra with *local units* if there is a set \mathbf{E} of idempotent elements in R such that for every finite set $r_1, \dots, r_n \in R$ there is $e \in \mathbf{E}$ obeying $er_i = r_i = r_i e$ for every $i = 1, \dots, n$. (This definition of local units is the one used in [15], and it can be traced back to [1] and [3]. It is more general than [1, Definition 1.1], since we do not assume that the elements of \mathbf{E} commute. In fact, the present notion generalizes that of [1] since, when the idempotents of \mathbf{E} commute, it is enough to require that for each element $r \in R$ there exists $e \in \mathbf{E}$ such that $er = r = re$, see [1, Lemma 1.2].) Every algebra R with local units is firm, and a right R -module M is firm if and only if for every $m \in M$ there is $e \in \mathbf{E}$ such that $m \cdot e = m$.

Corollary 2. *Let R be an algebra with local units over a commutative ring. If R is a firm Frobenius algebra, then the category of firm right R -modules and the category of right R -comodules are isomorphic.*

Proof. According to Theorem 2, we need to prove that any right R -comodule M is a firm right R -module via the action $m \cdot r := m_0 \epsilon(m_1 r)$: Given $m \in M$, we know that $m_0 \otimes m_1 = \sum_i m_i \otimes r_i$ for finitely many $m_i \in M, r_i \in R$. Let $e \in \mathbf{E}$ such that $r_i e = r_i$ for every i . Then $m \cdot e = \sum_i m_i \epsilon(r_i e) = \sum_i m_i \epsilon(r_i) = m$. \square

6.7. Example: co-Frobenius coalgebra over a field. Let C be a coalgebra over a field k . The dual vector space C^* is an associative and unital k -algebra via the convolution product $(\phi * \psi)(c) = \phi(c_1)\psi(c_2)$, for all $c \in C$ and $\phi, \psi \in C^*$, where for the comultiplication the Sweedler-Heynemann index notation is used, implicit summation understood. Every right/left C -comodule becomes then a left/right C^* -module, in particular, C becomes a C^* -bimodule. The coalgebra C is said to be *left/right co-Frobenius* if there exists a monomorphism of left/right C^* -modules $C \rightarrow C^*$, see [17].

Proposition 7. *For a coalgebra C over a field k , the following assertions are equivalent.*

- (1) *C is a left and right co-Frobenius coalgebra.*
- (2) *The given coalgebra structure of C extends to a non-unital Frobenius algebra with a non-degenerate multiplication.*
- (3) *The given coalgebra structure of C extends to a firm Frobenius algebra whose multiplication admits local units.*

Proof. (1) \Rightarrow (3) is proved, in fact, in [11]. The main line of the reasoning can be summarized as follows. When C is semiperfect (that is, the categories of left and right C -comodules have enough projectives), then the left and right rational ideals of the convolution algebra C^* coincide [13, Corollary 3.2.16]. Let $\text{Rat}(C^*)$ denote their common value. By [13, Corollary 3.2.17], $\text{Rat}(C^*)$ is a ring with local units. Now, if C is left and right co-Frobenius, then C is semiperfect [17] and, by [16, Theorem 2.1], there is an isomorphism of say, right C^* -modules $C \cong \text{Rat}(C^*)$. Then we may pull-back the multiplication of $\text{Rat}(C^*)$ to C so that C becomes a k -algebra with local units. By [11, Theorem 2.2], the resulting multiplication is a morphism of C^* -bimodules. Using the relation between the C^* -actions and the C -coactions on C , we conclude that the opposite of this multiplication is a C -bicomodule map. Hence C is also a firm Frobenius algebra.

(3) \Rightarrow (2) is trivial.

(2) \Rightarrow (1). A right C^* -module map $C \rightarrow C^*$ is provided by $c \mapsto \epsilon(c-)$. Indeed, for $c, d \in C$ and $\varphi \in C^*$,

$$\epsilon((c \leftarrow \varphi)d) = \varphi(c_1)\epsilon(c_2d) = \varphi(cd) = \epsilon(cd_1)\varphi(d_2) = (\epsilon(c-)\varphi)(d).$$

It is injective since if $\epsilon(cd) = 0$ for all $d \in C$, then $0 = \epsilon(cd_1)d_2 = cd$, hence $c = 0$ by the non-degeneracy of the multiplication. Symmetrically, a monomorphism of left C^* -modules is provided by $c \mapsto \epsilon(-c)$. \square

Note that the right C^* -module map $C \rightarrow C^*$, $c \mapsto \epsilon(c-)$ in the proof of Proposition 7 is anti-multiplicative by

$$\epsilon(cd_1)\epsilon(c'd_2) = \epsilon(c'\epsilon(cd_1)d_2) = \epsilon(c'cd), \quad \text{for all } c, c', d \in C.$$

So we conclude by Corollary 2 that if for a coalgebra C over a field the equivalent assertions in Proposition 7 hold, then the categories \mathbb{M}^C and ${}_{(\text{Rat}(C^*))}\mathbb{M} \cong \mathbb{M}_{(C)}$ are isomorphic. Therefore, Corollary 2 extends [11, Theorem 2.3] and [6, Proposition 2.7].

Non-degenerate algebras over a field equipped with a so-called separability idempotent, were discussed recently in [22]. Using the terminology of the current paper, they are in fact coalgebras obeying the equivalent properties in Proposition 7 and the additional requirement that their comultiplication splits the multiplication

(‘separability’ of the Frobenius structure); cf. Section 6.4. In particular, they have local units, answering in affirmative an open question in [23].

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WIGNER RESEARCH CENTRE FOR PHYSICS, BUDAPEST, H-1525 BUDAPEST 114, P.O.B. 49,
HUNGARY

E-mail address: `bohm.gabriella@wigner.mta.hu`

DEPARTAMENTO DE ÁLGEBRA, UNIVERSIDAD DE GRANADA, E-18071 GRANADA, SPAIN

E-mail address: `gomezj@ugr.es`