REMARK ON THE THEORY OF QUASIANALYTIC FUNCTION-CLASSES

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Investigating the solubility and uniqueness-problems of the heat-equation

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} = 0$$

GEVRAY and HADAMARD were led to the following question in 1912:

What are the necessary and sufficient conditions for the sequence M_n so that if $f_1(x)$ and $f_2(x)$ are infinitely often derivable for $a \leqq x \leqq b$ with

(1) $$\max_{a \leq x \leq b} |f_j^{(n)}(x)| \leqq k_j^n M_n , \quad j = 1, 2 ; \quad n = 0, 1, \ldots$$

$(k_j = k_j(f))$ *and with an* $a < x_0 < b$

(2) $$f_1^{(n)}(x_0) = f_2^{(n)}(x_0) , \qquad n = 0, 1, \ldots$$

then for $a \leqq x \leqq b$

$$f_1(x) \equiv f_2(x)$$

follows?

Clearly the functions analytical for $a \leqq x \leqq b$ form such a class, with

$$M_n = n!$$

This question was after the first results of DENJOY completely solved only in 1926 by CARLEMAN.

The problem of quasianaliticity can be formulated more generally as follows. We are asking for possibly large classes A of functions in $[a, b]$ with the property that if $f_1(x)$ and $f_2(x)$ from A „in the neighbourhood of $x = x_0$ $(a < x_0 < b)$ identically behave", then for the whole interval $a \leqq x \leqq b$

$$f_1(x) = f_2(x)$$

holds „essentially". In the class of infinitely often derivable functions (2) is a possible way of realising the „identical behaviour in the neighbourhood of $x = x_0$". The necessity of other interpretation of the phrase „identical behaviour in the neighbourhood" emerged from the investigations of S. BERNSTEIN and S. MANDELBROJT; the significance of such classes is indicated by the fact that they contain also nowhere derivable functions. Since our remarks refer to quasianalyticity in MANDELBROJT's sense, we shall confine ourselves to that. In his formulation „identical behaviour of $f_1(x)$ and $f_2(x)$ in the neighbourhood of $x = x_0$" means that with a fixed $\alpha > 0$

$$\liminf_{h \to +0} e^{h^{-\alpha}} \int_{x_0-h}^{x_0} |f_1(t) - f_2(t)|\, dt < \infty \tag{3}$$

and the phrase „$f_1(x)$ is essentially identical with $f_2(x)$ for $a \leq x \leq b$" means that $f_1(x) = f_2(x)$ almost everywhere there. MANDELBROJT proved that the class of the functions $f(x)$ with the period 2π having the Fourier-expansion

$$f(x) \sim \sum_{j=1}^{\infty} (a_j \cos m_j x + b_j \sin m_j x)$$

forms a quasianalytical class in his sense if

$$\sum_{j=1}^{\infty} m_j^{-\delta} < \infty$$

with $\delta < \alpha/(\alpha + 1)$ and this is no longer true (see S. MANDELBROJT and N. WIENER [2]), when $\delta = \alpha/(\alpha + 1)$. His theorem was extended to functions

$$f(x) \sim \sum_{j=1}^{\infty} (a_j \cos \lambda_j x + b_j \sin \lambda_j x) \tag{4}$$

with increasing λ-sequence by B. J. LEVIN [3].

In MANDELBROJT's theorem the class was characterized by the Fourier-*exponents* of the function. In this case of Bernstein-quasianaliticity N. LEVINSON [4] proved a theorem, which amounts to a characterisation of the corresponding class by the Fourier-*coefficients*. So the question arose whether or not a quasianalytical class in MANDELBROJT's sense can be characterized by the Fourier-*coefficients* of its functions. I solved this problem (see [5], [6]) which is in a sense dual to that of MANDELBROJT, even among the functions (4), in the case when we have a little „weaker" quasianaliticity-definition than MANDELROJT's. More exactly, I proved the following theorem.

Let $\alpha > 0$ be fixed and

$$f(x) = \sum_{j=1}^{\infty} a_j e^{i t_j x} \tag{5}$$

with arbitrary real exponents t_j so that

(6) $$\limsup_{\omega\to\infty} e^{\frac{2}{a}\omega\log\omega} \sum_{j>\omega} |a_j| < \infty \,.$$

If for any two functions $f_1(x)$ and $f_2(x)$ of this class we have for a real x_0

(7) $$\liminf_{h\to+0} e^{h^{-a}} \max_{x_0-h\leq x\leq x_0} |f_1(x) - f_2(x)| < \infty$$

then $f_1(x) = f_2(x)$ on the whole real axis.

The main tool of my proof was the inequality

(8) $$\max_{a\leq x\leq a+d} \left| \sum_{j=1}^{n} b_j e^{i\mu_j x} \right| \geq \left(\frac{d}{6\,(a+d)} \right)^n \left| \sum_{j=1}^{n} b_j \right|$$

for any real μ_j-s, complex b_j-s and positive a and d. To prove the theorem (5)—(6)—(7) with MANDELBROJT's quasianaliticity (3) instead of (7) one would need an inequality similar to (8) but replacing the left side by

(9) $$\int_a^{a+d} \left| \sum_{j=1}^{n} b_j e^{i\mu_j x} \right| dx$$

a problem of independent interest.

The theorem (5)—(6)—(7) raises the further question whether or not the condition (6) can be relaxed. The aim of this note is to show that (6) cannot be replaced by the weaker condition

(10) $$\limsup_{\omega\to\infty} e^{\omega^{1-\varepsilon}} \sum_{j>\omega} |a_j| < \infty \,,$$

however small we choose the positive number ε. This shows that our theorem is not far from being best-possible.

To show this it is sufficient to give an $f_0(x)$ of the form (5) satisfying (10), further

(11) $$\liminf_{h\to+0} e^{h^{-a}} \max_{-h\leq x\leq 0} |f_0(x)| < \infty$$

and $f_0(x) \not\equiv 0$. For this sake we choose fixing our a the parameter β so large that

(12) $$\beta = 2\left[\frac{1}{\varepsilon}\right] \qquad (> \max(2, a)\,;\quad \beta \text{ is even})$$

and consider

(13) $$f_0(x) = \exp\left(-\frac{1}{\sin^\beta x}\right).$$

Then $f_0(x) \not\equiv 0$, is of the form (5) and (11) is obviously fulfilled ; thus we have only to verify (10). Hence we have to investigate the sum

$$\sum_{j>\omega} \left| \int_0^\pi \exp\left(-\frac{1}{\sin^\beta x}\right) \cos jx\, dx \right| \equiv \sum_{j>\omega} |I_j| \ . \tag{14}$$

For a fixed index j we integrate partially μ times, μ being determined only later. This gives evidently

$$|I_j| \leqq \frac{\pi}{j^\mu} \max_{0\leqq x\leqq \pi/2} \left| \frac{d^\mu}{dx^\mu} e^{-\frac{1}{\sin^\beta x}} \right| .$$

Using Cauchy's estimation with the circle $|z—x| = x/40\beta$ we obtain

$$|I_j| \leqq \pi \frac{\mu!}{j^\mu} \max_{0\leqq x\leqq \pi/2} \left\{ \left(\frac{40\,\beta}{x}\right)^\mu \max_{|z-x|=x/40\beta} \exp\left[-\mathcal{Re}\left(\frac{1}{\sin^\beta z}\right)\right]\right\} =$$

$$= \pi \frac{\mu!}{j^r} \max_{0\leqq x\leqq \pi/2} \left\{ \left(\frac{40\,\beta}{x}\right)^\mu \max_{|z-x|=x/40\beta} e^{-\frac{\cos(\beta \arg \sin z)}{|\sin z|^\beta}} \right\} . \tag{15}$$

But for $z = \xi + i\eta$

$$|\arg \sin z| = \left| \operatorname{arc\,tg} \frac{\cos \xi(e^\eta - e^{-\eta})}{\sin \xi(e^\eta + e^{-\eta})} \right| ;$$

since in our case

$$|\eta| \leqq \frac{x}{40\beta}, \quad x\left(1 - \frac{1}{40\beta}\right) \leqq \xi \leqq x\left(1 + \frac{1}{40\beta}\right)$$

and $0 \leqq x \leqq \pi/2$, we have for all positive sufficiently small ε

$$\left| \operatorname{cotg} \xi \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}} \right| \leqq \frac{1}{\frac{1}{\pi}\xi} \frac{3\eta}{1} \leqq 3\pi \frac{x}{40\beta} \frac{1}{x\left(1 - \frac{1}{40\beta}\right)} < \frac{\pi}{10\beta} .$$

Thus

$$\left| \operatorname{arc\,tg} \left(\operatorname{cotg} \xi \frac{e^\eta - e^{-\eta}}{e^\beta + e^{-\eta}} \right) \right| \leqq \frac{\pi}{10\beta}$$

$$\cos(\beta \arg \sin z) > \frac{1}{2} . \tag{16}$$

Further we have for our z-s with a suitable positive numerical c_1 (> 1)

$$|\sin z| \leqq c_1 |z| \leqq c_1 \left(1 + \frac{1}{20\beta}\right) x$$

i. e. from (15) and (16) we have

$$(17) \qquad |I_j| \leqq \mu!\, \pi \cdot \left(\frac{40\beta}{j}\right)^{\mu} \max_{x \geqq 0} \left\{ \frac{1}{x^{\mu}} \exp\left(- \frac{1}{2\, c_1^{\beta} \left(1 + \frac{1}{20\,\beta}\right)^{\beta} x^{\beta}} \right) \right\}.$$

Putting

$$\frac{1}{2 c_1^{\beta} \left(1 + \frac{1}{20\beta}\right)} = c_2(\beta)$$

the maximum of the function

$$x^{-\mu} \exp\left(- \frac{c_2(\beta)}{x^{\beta}} \right)$$

for $x \geqq 0$ is evidently assumed at

$$x = \left(\frac{\beta\, c_2(\beta)}{\mu} \right)^{\frac{1}{\beta}}$$

and its value is

$$\left(\frac{\mu}{c_2(\beta) \cdot \beta} \right)^{\frac{\mu}{\beta}} e^{-\frac{\mu}{\beta}}.$$

Since roughly

$$\mu! < \mu^{\mu},$$

we have from (17)

$$|I_j| \leqq \pi \left(\frac{40\beta\, e^{-\frac{1}{\beta}}}{j} \right)^{\mu} \mu^{\mu} \left(\frac{\mu}{\beta\, c_2(\beta)} \right)^{\frac{\mu}{\beta}} =$$

$$= \pi \left(\frac{40\beta\, e^{-\frac{1}{\beta}}}{\beta^{\frac{1}{\beta}}\; c_2(\beta)^{\frac{1}{\beta}}} \cdot \frac{1}{j} \right)^{\mu} \mu^{\left(1 + \frac{1}{\beta}\right)\mu} = \pi \left(\frac{1}{j}\, e^{c_3(\beta)}\, \mu^{\left(1 + \frac{1}{\beta}\right)} \right)^{\mu}$$

with a positive $c_3(\beta)$. Now choosing

$$\mu = \mu_0 \equiv \left[e^{-\frac{2\, c_3(\beta)}{1 + 1/\beta}} \cdot j^{\frac{1}{1 + 1/\beta}} \right]$$

we obtain the estimation

$$|I_j| \leqq \pi e^{-c_3(\beta)\mu_0} \leqq \pi \exp\left\{-c_3(\beta)\left(e^{-\frac{2e_3(\beta)}{1+\frac{1}{\beta}}} \cdot j^{\frac{\beta}{1+\beta}} - 1\right)\right\} =$$

$$= c_4(\beta) \exp\left\{-c_5(\beta)\, j^{\frac{\beta}{1+\beta}}\right\} < c_6(\varepsilon) \exp\left\{-c_7(\varepsilon) \cdot j^{1-\frac{\varepsilon}{2}}\right\}$$

with positive $c_6(\varepsilon)$ and $c_7(\varepsilon)$. From this (10) evidently follows.

It seems to be possible and would be of interest to construct examples still nearer to our theorem than $f_0(x)$. Possibly

$$f_1(x) = e^{-e^{\frac{1}{\sin^2 x}}}$$

would furnish a better example but this would need a more suitable analysis.

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MEGJEGYZÉS A KVÁZI-ANALITIKUS FÜGGVÉNYOSZTÁLYOK ELMÉLETÉHEZ

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Kivonat

A szerző egy régebbi dolgozatában (lásd: [5] és [6]) a következő tételt bizonyította be:

Ha fix $\alpha > 0$ mellett az

$$f(x) = \sum_{j=1}^{\infty} a_j e^{i t_j x}$$

üggvényeknek A_α-osztálya olyan, hogy

$$\limsup_{\omega \to \infty} e^{\frac{2}{\alpha}\omega \log \omega} \sum_{j \geqq \omega} |a_j| < \infty, \tag{1}$$

akkor abból, hogy egy valós x_0-ra

$$\liminf_{h \to +0} e^{h^{-\alpha}} \max_{x_0 - h \leqq x \leqq x_0} |f_1(x) - f_2(x)| < \infty,$$

következik, hogy $f_1(x) = f_2(x)$ minden x-re.

E dolgozatban a szerző azon kérdéssel foglalkozik, hogy az (1) feltétel menynyire enyhíthető. Mint alkalmas példán megmutatja, (1)-et a nem lényegesen gyengébb

$$\limsup_{\varepsilon \to \infty} e^{\omega^{1-\varepsilon}} \sum_{j \geqq \omega} |a_j| < \infty$$

feltétellel helyettesítve, ahol ε tetszőleges kis fix pozitív szám, a tétel már nem lesz igaz.

ЗАМЕЧАНИЕ К ТЕОРИИ КЛАССОВ КВАЗИАНАЛИТИЧЕСКИХ ФУНКЦИЙ

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Резюме

Автор в одной из своих предыдущих работ (см. [5] и [6]) доказал следующую теорему:

Если при фиксированном $\alpha > 0$ класс A_α состоит из функций

$$f(x) = \sum_{j=1}^{\infty} a_j e^{i t_j x}$$

дли которых

(1) $$\limsup_{\omega \to \infty} e^{\frac{2}{\alpha}\omega \log \omega} \sum_{j \geqq \omega} |a_j| < \infty$$

то из того, что для некоторого вещественного x_0

$$\liminf_{h \to +0} e^{h^{-\alpha}} \max_{x_0 - h \leqq x \leqq x_0} |f_1(x) - f_2(x)| < \infty ,$$

следует, что $f_1(x) = f_2(x)$ при всех x.

В настоящей работе автор занимается следующим вопросом: насколько может быть ослаблено условие (1)? Как он показывает на подходящем примере, если заменить условие (1) незначительно более слабым условием

$$\limsup_{\omega \to \infty} e^{\omega^{1-\varepsilon}} \sum_{j \geqq \omega} |a_j| < \infty$$

где любое сколь угодно малое положительное число, теорема уже не будет верна.