ON THE CONVERGENCE OF SEQUENCES OF RANDOM VARIABLES (A REMARK ON A PROBLEM OF A. PRÉKOPA)

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Introduction

In the paper [1] the following problem is mentioned.

Let $\xi_1, \xi_2, \ldots$ denote a sequence of independent random variables. Let us suppose, that

$$\sum_{k=1}^{\infty} \xi_k$$

converges with probability 1 regardless of the order of summation. Let $A_i\ (i = 1, 2, \ldots)$ be sets of natural numbers with the property that if $B_n = A_n + A_{n+1} + \ldots$ then

$$\prod_{n=1}^{\infty} B_n = 0 .$$

In [1] it is shown that for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\{|\xi(A_n)| > \varepsilon\} = 0$$

where

$$\xi(A_n) = \sum_{k \in A_n} \xi_k$$

and the problem is formulated whether the stronger relation

$$\mathbf{P}\{\lim_{n\to\infty} \xi(A_n) = 0\} = 1 \tag{1}$$

holds. In view of the relation

$$\xi(B_n) = \xi(B_n - A_n) + \xi(A_n)$$

where on the right-hand side there stand independent random variables, it can be seen that (1) would be true if for every sequences $\xi_n, \eta_n, \zeta_n = \xi_n + \eta_n$ of random variables satisfying the conditions

a) $\mathbf{P}\{\lim_{n\to\infty} \zeta_n = 0\} = 1$

b) $\xi_n \Rightarrow 0\,, \quad \eta_n \Rightarrow 0$

c) for every n, ξ_n and η_n are independent,

there would follow:

$$\mathbf{P}\{\lim_{n\to\infty} \xi_n = 0\} = \mathbf{P}\{\lim_{n\to\infty} \eta_n = 0\} = 0 \ .$$

[The convergence to 0 with probability 1 of the sequence $\xi(B_n)$ follows immediately from the relation

$$\xi(B_n) = \sum_{k \in B_n} \xi_k$$

where the series on the right-hand side converges with probability 1 (Cf. [2] p. 118., Corollary 1.).]

We shall show, however, that this assertion in general does not hold. In this paper we consider not only this last problem but the following more general one:

If $\zeta_n = \xi_n + \eta_n$ and ξ_n, η_n are independent random variables for every n ($n = 1, 2, \ldots$), further ζ_n converges to 0 in some sense, what kind of convergence to 0 is implied for ξ_n and η_n by this relation if the convergence of ξ_n and η_n to 0 in some other sense is supposed.

We take into account the following kinds of convergence:

1. *Uniform convergence.* ξ_n converges to 0 uniformly, if for every $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that $\mathbf{P}\{|\xi_n| < \varepsilon\} = 1$ provided that $n \geqq n_0(\varepsilon)$.

2. *Convergence with probability 1.* ξ_n converges to 0 with probability 1, if

$$\mathbf{P}\{\lim_{n\to\infty} \xi_n = 0\} = 1 \ .$$

3. *Convergence in the mean (en moyenne).* ξ_n converges to 0 in the mean, if $\mathbf{M}(\xi_n^2) \to 0$

4. *Weak convergence.* ξ_n converges to 0 weakly if $\mathbf{M}(\xi_n \zeta) \to 0$ where ζ is an arbitrary random variable with a finite variance.

In the cases 3. and 4. we assume that the mean and variance of all the random variables in question exist.

5. *Stochastic convergence.* ξ_n converges to 0 stochastically ($\xi_n \Rightarrow 0$), if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\{|\xi(A_n)| > \varepsilon\} \to 0 \ .$$

1. §. First case: $\mathbf{P}\{\lim_{n\to\infty} \zeta_n = 0\} = 1$

I. Let $\zeta_n = \xi_n + \eta_n$. We shall prove that there exist such sequences of random variables ξ_n, η_n that the following conditions hold:

a) $\mathbf{P}\{\lim_{n\to\infty} \zeta_n = 0\} = 1$

b) $\xi_n \Rightarrow 0\ , \quad \eta_n \Rightarrow 0$

c) ξ_n, η_n are independent,

but

d) $\mathbf{P}\{\lim_{n\to\infty} \xi_n = 0\} = \mathbf{P}\{\lim_{n\to\infty} \eta_n = 0\} = 0\ .$

Let the sample space be the interval $[0, 1]$ and the probability measure the Lebesgue measure. In this case the random variables are Lebesgue measurable real functions. We define the variables ξ_n, η_n as follows.

Let I_n, K_n, L_n be the following intervals:

$$I_n = [(l-1)\cdot 10^{-2(k+1)}\ ,\quad l\cdot 10^{-2(k+1)}]$$

$$K_n = [1 - 10^{-(k+1)}\ ,\ 1]$$

$$L_n = \begin{cases} \left[1 - 10^{-(k+1)} - \dfrac{1 - 10^{-(k+1)} - 10^{-2(k+1)}}{10^{k+1}+1}\ ,\quad 1 - 10^{-(k+1)}\right], & \text{if } 1 \le l \le 5 \sum\limits_{j=k+1}^{2k+1} 10^j \\ \left[0\ ,\quad \dfrac{1 - 10^{-(k+1)} - 10^{-2(k+1)}}{10^{k+1}+1}\right], & \text{if } 5 \sum\limits_{j=k+1}^{2k=1} 10^j < l \le 9 \sum\limits_{j=k+1}^{2k+1} 10^j \end{cases}$$

where $1 \le l \le 9 \sum\limits_{j=k+1}^{2k+1} 10^j$ for $k = 0, 1, 2, \dots$

and

$$n = \begin{cases} 9 \sum\limits_{i=0}^{k-1} \sum\limits_{j=i+1}^{2i+1} 10^j + l\ , & \text{if} \quad k = 1, 2, \dots \\ l & \text{if} \quad k = 0 \end{cases}$$

Let $C_n = I_n + K_n$, $D_n = I_n + L_n$ and we define the random variables ξ_n and $-\eta_n$ as the characteristic functions (in the sense of the theory of sets) of the sets C_n and D_n, respectively.

Now we prove that ξ_n and η_n satisfy the above conditions. Condition *a)* is satisfied since $\xi_n + \eta_n \ne 0$ holds only on the set $K_n + L_n$. In order to show the fulfilment of Condition *b)* it is sufficient to prove that $\xi_n \Rightarrow 0$. In fact, if $0 < \varepsilon < 1$, then

$$\mathbf{P}\{|\xi_n| > \varepsilon\} = \mathbf{P}\{|\xi_n| = 1\} = |I_n| + |K_n| = 10^{-(k+1)} + 10^{-2(k+1)} = \frac{1 + 10^{k+1}}{10^{2(k+1)}} \to 0$$

what was to be proved. Let us now consider Condition *c)*. The random variables ξ_n and η_n can take on only the values 0, 1 and 0, -1, respectively. Thus ξ_n and η_n are independent if

$$\mathbf{P}\{\xi_n = 1,\ \eta_n = -1\} = \mathbf{P}\{\xi_n = 1\}\,\mathbf{P}\{\eta_n = -1\},$$

$$\mathbf{P}\{\xi_n = 1,\ \eta_n = 0\} = \mathbf{P}\{\xi_n = 1\}\,\mathbf{P}\{\eta_n = 0\}$$

$$\mathbf{P}\{\xi_n = 0,\ \eta_n = -1\} = \mathbf{P}\{\xi_n = 0\}\,\mathbf{P}(\eta_n = -1\},$$

$$\mathbf{P}\{\xi_n = 0,\ \eta_n = 0\} = \mathbf{P}\{\xi_n = 0\}\,\mathbf{P}\{\eta_n = 0\}\ .$$

It is sufficient to prove that the first relation holds as the fulfilment of one of these relations implies that of the remaining ones. Since

$$\mathbf{P}\{\xi_n = 1\} = \frac{10^{k+1} + 1}{10^{2(k+1)}}\ ,$$

$$\mathbf{P}(\eta_n = -1) = |I_n| + |L_n| = 10^{-2(k+1)} + \frac{1 - 10^{-(k+1)} - 10^{-2(k+1)}}{1 + 10^{(k+1)}} = \frac{1}{1 + 10^{(k+1)}}\ ,$$

$$\mathbf{P}\{\xi_n = 1,\ \eta_n = -1\} = |I_n| = 10^{-2(k+1)}\ ,$$

(where $|I_n|$, $|K_n|$, $|L_n|$ denote the lengths of the intervals I_n, K_n, L_n, respectively) our assertion holds. From the definition of the sequences ξ_n and η_n it is easy to see that *d)* holds.

II. This example shows that there exist such sequences ξ_n, η_n which satisfy the following stronger conditions:

a) $\mathbf{P}\{\lim_{n\to\infty} \zeta_n = 0\} = 1$

b)* $\mathbf{M}(\xi_n^2) \to 0,\ \mathbf{M}(\eta_n^2) \to 0$

c) ξ_n, η_n are independent for every n,

but

d) $\mathbf{P}\{\lim_{n\to\infty} \xi_n = 0\} = \mathbf{P}\{\lim_{n\to\infty} \eta_n = 0\} = 0\ .$

III. It is easy to see that Conditions *a), b), c)* neither imply that $\mathbf{M}(\xi_n^2) \to 0$ nor that ξ_n converges to 0 weakly. In fact, if $\xi_n^* = 10^{4k}\,\xi_n$, $\eta_n^* = 10^{4k}\,\eta_n$, where

$$1 \leq l \leq 9 \sum_{j=k+1}^{2k+1} 10^j \qquad k = 0, 1, 2, \ldots\ .$$

and

$$n = \begin{cases} \displaystyle\sum_{i=0}^{k-1} \sum_{j=i+1}^{2i+1} 10^j + l & \text{if}\quad k = 1, 2, \ldots \\ l & \text{if}\quad k = 0 \end{cases}$$

then ξ_n^* satisfy Conditions *a), b), c)* and

$$\mathbf{M}(\xi_n^*) \to \infty$$

i. e. ξ_n does not converge weakly to 0.

2. §. Second case: $\mathbf{M}(\zeta_n^2) \to 0$

I. We shall show that if ξ_n, η_n satisfy the following conditions :

a) $\mathbf{M}(\zeta_n^2) \to 0$

b) $\mathbf{M}(\xi_n) \to 0$, $\mathbf{M}(\eta_n) \to 0$

c) ξ_n, η_n are independent for every n

then

$$\mathbf{M}(\xi_n^2) \to 0, \quad \mathbf{M}(\eta_n^2) \to 0 \ .$$

Proof : Since

$$\mathbf{M}(\zeta_n^2) = \mathbf{M}[(\xi_n + \eta_n)^2] = \mathbf{M}(\xi_n^2) + \mathbf{M}(\eta_n^2) + 2\,\mathbf{M}(\xi_n \eta_n) = \mathbf{M}(\xi_n^2) + \mathbf{M}(\eta_n^2) + \\ + 2\,\mathbf{M}(\xi_n)\,\mathbf{M}(\eta_n)$$

and

$$\mathbf{M}(\zeta_n^2) \to 0, \quad \mathbf{M}(\xi_n) \to 0, \quad \mathbf{M}(\eta_n) \to 0$$

we get

$$\mathbf{M}(\xi_n^2) \to 0, \quad \mathbf{M}(\eta_n^2) \to 0 \ .$$

It is clear that if we replace Condition *c)* by the following condition :

c)* ξ_n, η_n are uncorrelated

then our statement remains true.

II. Let us now define the Condition *b*)* as

b)* $\xi_n \Rightarrow 0$, $\eta_n \Rightarrow 0$.

We shall show that if Conditions *a), b*), c*)* hold then the relations $\mathbf{M}(\eta_n^2) \to 0$, $\mathbf{M}(\xi_n^2) \to 0$ remain true.

Proof : By I. of this §. is sufficient to prove that $\mathbf{M}(\xi_n) \to 0$. Our conditions imply that

$$\mathbf{M}(\zeta_n^2) = \mathbf{M}[(\xi_n + \eta_n)^2] \geqq \mathbf{D}^2(\xi_n) + \mathbf{D}^2(\eta_n)$$

and thus $\mathbf{D}^2(\xi_n) \to 0$, $\mathbf{D}^2(\eta_n) \to 0$. Moreover, since $\mathbf{D}^2(\xi_n) \to 0$ and $\xi_n \Rightarrow 0$, we conclude $\mathbf{M}(\xi_n) \to 0$.

3. §. Third case: ζ_n converges to 0 uniformly

I. We shall prove that if ξ_n, η_n satisfy the following conditions :

a) ζ_n converges to 0 uniformly

b) $\xi_n \Rightarrow 0$, $\eta_n \Rightarrow 0$

c) ξ_n, η_n are independent

then ξ_n and η_n converge to 0 uniformly.

Proof: Let us suppose that ξ_n does not converge to 0 uniformly. In this case there exist an ε with the property that for every positive integer n_0 there can be found an $n \geqq n_0$ so that

$$\mathbf{P}\{|\xi_n| > 2\varepsilon\} = \delta_n > 0 \ .$$

This implies that one of the relations

$$\mathbf{P}\{\xi_n > 2\varepsilon\} \geqq \frac{\delta_n}{2}\,, \quad \mathbf{P}\{\xi_n < -2\varepsilon\} \geqq \frac{\delta_n}{2}$$

holds. We may suppose that $\mathbf{P}\{\xi_n > 2\,\varepsilon\} \geqq \delta_n/2$, the other case can be treated similarly. Since $\xi_n + \eta_n$ converges to 0 uniformly, there exists an N such that

$$\mathbf{P}\{|\xi_n + \eta_n| \leqq \varepsilon\} = 1$$

if $n \geq N$. Let $n_0 = N$; then there exists an n such that

$$\mathbf{P}\{\xi_n > 2\,\varepsilon\} \geqq \frac{\delta_n}{2}\,, \quad \mathbf{P}\{|\xi_n + \eta_n| \leqq \varepsilon\} = 1 \ .$$

By comparing these inequalities it follows that if $\xi_n > 2\varepsilon$ then $\eta_n < -\varepsilon$ with probability 1. Thus

$$\mathbf{P}\{\eta_n < -\varepsilon \,|\, \xi_n > 2\,\varepsilon\} = 1 \ .$$

On the other hand, ξ_n and η_n are independent, hence

$$\mathbf{P}\{\eta_n < -\varepsilon \,|\, \xi_n > 2\varepsilon\} = \mathbf{P}\{\eta_n < -\varepsilon\} \ .$$

Thus we conclude

$$\mathbf{P}\{\eta_n < -\varepsilon\} = 1$$

which contradicts to $\eta_n \Rightarrow 0$.

From the proof it is clear that Condition *c)* can be weakened i. e. it is sufficient to assume that

$$\lim_{n\to\infty} \frac{G_n(y) - H_n(x, y)}{1 - F_n(x)} < 1 \quad \text{if} \quad y < 0,\ x > 0 \ ,$$

where

$$F_n(x) = \mathbf{P}\{\xi_n < x\},\ G_n(y) = \mathbf{P}\{\eta_n < y\},\ H_n(x, y) = \mathbf{P}\{\xi_n < x, \eta_n < y\} \ .$$

II. Finally we show that if Condition *b)* is replaced by the following one

b)* $\mathbf{M}(\xi_n) \to 0, \quad \mathbf{M}(\eta_n) \to 0$,

then the above statement remains true. In other terms if ξ_n and η_n have the properties

a) ζ_n converges to 0 uniformly

b)* $\mathbf{M}(\xi_n) \to 0, \quad \mathbf{M}(\eta_n) \to 0$

c) ξ_n and η_n are independent for every n

then the sequences ξ_n and η_n converge to 0 uniformly.

The proof of this statement can be accomplished word by word in the same way in which the assertion in I. of this §. was proved.

We summarize the preceedings in the following table:

condition \ concludesion		u'	p'	m'	w'
u	w'	+	+	+	+
u	s'	+	+	+	+
p	m'	—	—	+	+
p	s'	—	—	—	—
m	w'	—	—	+	+
m	s'	—	—	+	+

In the above table the signes + and — are written according as the statement on the head of a column follows or not from the statement at the left hand side of the row and the letters have the following meaning:

u) $\zeta_n = \xi_n + \eta_n$ converges to 0 uniformly
p) ζ_n converges to 0 with probability 1
m) ζ_n converges to 0 in the mean
u') ξ_n and η_n converge to 0 uniformly
p') ξ_n and η_n converge to 0 with probability 1
m') ξ_n and η_n converge to 0 in the mean
w') ξ_n and η_n converge to 0 weakly
s') ξ_n and η_n converge to 0 stochastically.

(Received: 20. III. 1957.)

REFERENCES

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VALÓSZÍNŰSÉGI VÁLTOZÓK SOROZATÁNAK KONVERGENCIÁJÁRÓL (MEGJEGYZÉS PRÉKOPA A. EGY PROBLÉMÁJÁHOZ)

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Kivonat

Legyen $\{\xi_n\}$ és $\{\eta_n\}$ két, valószínűségi változókból álló sorozat, és jelölje ζ_n a $\xi_n + \eta_n$ összeget. Tegyük fel, hogy ξ_n és η_n minden n-re függetlenek. A dolgozatban megvizsgáljuk a következő feltételek egymáshoz való viszonyát:

u) minden $\varepsilon > 0$-hoz létezik oly $n_0(\varepsilon)$, hogy $\mathbf{P}\{|\zeta_n| < \varepsilon\} = 1$, ha $n \geqq n_0$, azaz ζ_n egyenletesen tart 0-hoz

p) $\mathbf{P}\{\lim_{n\to\infty} \zeta_n = 0\} = 1$

m) $\mathbf{M}(\zeta_n^2) \to 0$

u') ξ_n és η_n egyenletesen tart 0-hoz

p') $\mathbf{P}\{\lim_{n\to\infty} \xi_n = 0\} = \mathbf{P}\{\lim_{n\to\infty} \eta_n = 0\} = 1$

m') $\mathbf{M}(\xi_n^2) \to 0$, $\mathbf{M}(\eta_n^2) \to 0$

w') $\mathbf{M}(\xi_n \zeta) \to 0$, $\mathbf{M}(\eta_n \zeta) \to 0$ tetszőleges véges szórású ζ valószínűségi változóra

s') ξ_n és η_n sztochasztikusan 0-hoz tart.

A dolgozat eredményeit a mellékelt táblázat szemlélteti, ahol a +, illetve — jel azt jelenti, hogy az illető sor fejlécében szereplő feltételekből következik, illetve nem következik az illető oszlop fejlécében szereplő állítás.

О СХОДИМОСТИ ПОСЛЕДОВАТЕЛЬНОСТЕЙ, СОСТОЯЩИХ ИЗ СЛУЧАЙНЫХ ВЕЛИЧИН

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Резюме

Пусть $\{\xi_n\}$ и $\{\eta_n\}$ — две последовательности, состоящие из случайных величин, и пусть ζ_n обозначает сумму $\xi_n + \eta_n$. Предположим, что ξ_n и η_n независимы при всех значениях от n. В работе исследуется связь следующих условий:

u) для всякого $\varepsilon > 0$ существует такое $n_0(\varepsilon)$, что $\mathbf{P}\{|\zeta_n| < \varepsilon\} = 1$, если $n \geqq n_0$, т. е. ζ_n равномерно стремится к нулью.

p) $\mathbf{P}\{\lim_{n\to\infty} \zeta_n = 0\} = 1$.

m) $\mathbf{M}\{\zeta_n^2\} \to 0$.

u') ξ_n и η_n равномерно стремится к нулью.

p') $\mathbf{P}\{\lim_{n\to\infty} \xi_n = 0\} = \mathbf{P}\{\lim_{n\to\infty} \eta_n = 0\} = 1$.

m') $\mathbf{M}\{\xi_n^2\} \to 0$, $\mathbf{M}\{\eta_n^2\} \to 0$.

w') $\mathbf{M}\{\xi_n \zeta\} \to 0$, $\mathbf{M}\{\eta_n \zeta\} \to 0$ для любой случайной величины с конечной дисперсией.

s') ξ_n и η_n стохастически стремятся к нулью.

Результаты статьи находятся в приложенной таблице, где знаки + или — означают, что из условий фигурирующих в виньетке одной строки следует предложение или нетфигурирующих в виньетке соответствующего столбца.