# GENERALIZED HYPERGEOMETRIC DISTRIBUTIONS

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## 1. §. Introduction

The Pólya distribution is treated in several textbooks on probability theory and mathematical statistics (see e.g. M. FRÉCHET [4], W. FELLER [2] and M. A. BRICAS [1]). It is well-known that it contains the hypergeometric distribution as a special case. In addition it is known that the generating function of the Pólya distribution is the hypergeometric function multiplied by a constant factor. (See e.g. CH. JORDAN [9], [10], M. FRÉCHET [4] and M. A. BRICAS [1]). For this reason the Pólya distribution is called by BRICAS "generalized hypergeometric distribution." Furthermore the formula of the distribution is given by CH. JORDAN [10] in a form similar to that of the usual hypergeometric distribution, as well as the formula of the generalized hypergeometric distribution of C. D. and A. W. KEMP [12].

In the following we shall see that the generalization made by C. D. and A. W. KEMP is wider: their generalized hypergeometric distribution contains the Pólya distribution as a particular case. The special form by J. G. SKELLAM an J. O. IRWIN are Pólya distributions too.

Some other cases of the distribution treated by C. D. and A. W. KEMP were previously described by CH. JORDAN [10], [11].

The model given by J. O. IRWIN is a case of the Pólya urn-model. It is shown that Pólya's model, resp. its modification for the inverse sampling is appropriate for all cases of the generalized hypergeometric distribution with exception of cases in which both $a$ and $n$ are non-integral. (See formula (2))

Furthermore, it is shown that the generalization of C. D. and A. W. KEMP is incomplete. It excludes for example cases in which $P(0) = 0$, that is, zero does not belong to the possible values of the distribution. So it does not contain even all cases of the usual hypergeometric distribution. It will be shown however that the distributions thus omitted differ from the treated ones by shifting only.

Other rediscoveries of the Pólya distribution are also mentioned.

## 2. §. Comparison of the distributions

The usual form of the Pólya distribution is as follows (see W. FELLER [3], M. A. BRICAS [1]):

$$(1) \quad P(r) = \binom{m}{r} \frac{p(p+\gamma)\ldots(p+r\gamma-\gamma)\,q(q+\gamma)\ldots(q+m\gamma-r\gamma-\gamma)}{1\cdot(1+\gamma)(1+2\gamma)\ldots(1+m\gamma-\gamma)}$$

$$(r = 0, 1, 2, \ldots, m)$$

while $0 < p < 1$, $q = 1 - p$, $\gamma > -1$, $m$ positive integer.

Excluding the case of the binomial distribution ($\gamma = 0$) and taking $a = -p/\gamma$, $b = -q/\gamma$ we obtain the form (see CH. JORDAN [10])

$$(2) \quad P(r) = \frac{\binom{a}{r}\binom{b}{n-r}}{\binom{a+b}{n}}$$

$$(r = 0, 1, 2, \ldots, n)$$

where the following relations must hold : $a$ and $b$ are real numbers of the same sign, $n$ is a positive integer, for positive $a$ and $b$ $n < a + b$; for positive, non-integer $a$ resp. $b$ $n < a + 1$ resp. $n < b + 1$.

Thus it can be seen that the Pólya distribution is characterized by the formula of the hypergeometric distribution, permiting non-integral values for $a$ and $b$.

This formula is the starting-point of C. D. and A. W. KEMP. In their generalization, also the third parameter, $n$ may be an arbitrary real number and $a, b$ may have different signs.

(For interpreting the ratio of factorials of negative integers they define

$$(3) \quad \frac{(-x)!}{(-x-y)!} = (-1)^y \frac{(x+y-1)!}{(x-1)!}$$

for positive integer values of $x$ and $y$.)

So the Pólya distribution is a special case of the generalized hypergeometric distribution of C. D. and A. W. KEMP.

Investigating the classification of the hypergeometric distribution given by C. D. and A. W. KEMP it can be seen, that Pólya distributions result in following cases :

Type I. A(i)
Type I. A(ii) for integer $n$ only
Type II. A.

Additional types can be regarded as Pólya distributions if we change the parameters.

The substitution

$$(4) \quad a_1 = n \; , \quad n_1 = a \; , \quad b_1 = a + b - n$$

interchanges Type I. A (i) with Type I. A (ii); Type II. A with Type III. A; Type II. B with Type III. B ; the types non-mentioned are unaltered.

So we see that all cases of Types I. A., II. A., III. A can be regarded as Pólya distributions.

The comparison of the restrictions shows however that there are several cases of Pólya distributions not contained in any of the types of C. D. and A. W. KEMP. We shall return to this question in § 7. extending the class of generalized hypergeometric distributions with additional types, including thus the whole class of Pólya distributions.

### 3. §. Formulae of the distribution

The formula of the distribution can be written in several different forms. The formulae given by C. D. and A. W. KEMP differ from well-known forms of the Pólya distribution. There are various forms given by other authors, mentioned in §. 5. Yet there can be given additional forms.

Separate formulae are given below for each type. These formulae are equivalent, each of them providing the complete generalization with exception of (1), (6), (7), (8) ; the use of negative factorials being inconvenient, they can be suitably used for one type only.

C. D. and A. W. KEMP classify the generalized hypergeometric distributions into four general types. In fact there are three different types only, since — as mentioned before — Type II and III are identical by substitution (4).

*Type I.*

The formula given by C. D. and A. W. KEMP :

$$P(r) = \frac{\binom{a}{r}\binom{b}{n-r}}{\binom{a+b}{n}} = \frac{\binom{n}{r}\binom{a+b-n}{a-r}}{\binom{a+b}{a}} = \frac{a!\,n!\,b!\,(a+b-n)!}{r!\,(a-r)!\,(n-r)!\,(b-n+r)!\,(a+b)!}\,. \tag{5}$$

In case of integral $n$ (or $a$) also formula (1) can be used. In formula (1) $p = a/(a+b)$, $q = b/(a+b)$, $\gamma = -1/(a+b)$, $m = n$ or $p = n/(a+b)$, $q = (a+b-n)/(a+b)$, $\gamma = -1/(a+b)$, $m = a$.

If in addition $a$ and $b$ (resp. $n$ and $b$) are rational, the well-known formula of the Pólya urn-model is also appropriate :

$$P(r) = \binom{m}{r} \frac{\prod_{i=0}^{r-1} (M + iR) \prod_{i=0}^{m-r-1} (N - M + iR)}{\prod_{i=0}^{m-1} (N + iR)} \tag{6}$$

while $m = n$, $M = -Ra$, $N = -R(a+b)$ (resp. $m = a$, $M = -Rn$, $N = -R(a+b)$), $R$ is a negative-integer, $-R$ being the common denominator of $a$, $n$ and $b$.

*Type II—III.*

Using the relation (3) we obtain the following forms:

$$P(r)=\frac{\binom{c+r-1}{r}\binom{d+m-r-1}{m-r}}{\binom{c+d+m-1}{m}}=$$

$$=\frac{(c+r-1)!\,(d+m-r-1)!\,m!\,(c+d-1)!}{r!\,(m-r)!\,(c-1)!\,(d-1)!\,(c+d+m-1)!}$$

which can be written in a form given by E. J. GUMBEL and H. VON SCHELLING [5] for the distribution of the number of exceedances (see §. 5.):

$$P(r)=\frac{\binom{N}{c}c\binom{m}{r}}{(m+N)\binom{m+N-1}{c+r-1}}$$

or in a form in which the succession law is given (see §. 5. and J. V. USPENSKY [15]):

$$P(r)=\binom{m}{r}\frac{\int\limits_0^1 z^{c+r-1}(1-z)^{m+N-c-r}\,dz}{\int\limits_0^1 z^{c-1}(1-z)^{N-c}\,dz}.$$

In above formulas $c=-a$, $d=-b$, $m=n$, $N=c+d-1$ (Type II); or $c=-n$, $d=-a-b+n$, $m=a$, $N=c+d-1$ (Type III).

Altering the notations:

$$P(r)=\frac{\binom{c}{m}\binom{d}{r}m}{\binom{c+d}{m+r}(m+r)}=\frac{c!\,d!\,(m+r-1)!\,(c+d-m-r)!}{r!\,(d-r)!\,(c-m)!\,(m-1)!\,(c+d)!}$$

where $c=-a-b-1$, $d=n$, $m=-a$ (Type II); or $c=-a-b-1$, $d=a$, $m=-n$ (Type III).

In case of positive integral $n$ (or $a$) formula (1) is also appropriate (see case of Type I). Are in addition the other two parameters rational, (6) is also suitable. Now $R$ is a positive integer.

Are $n$ and $a$ both integers, following formula is also appropriate :

$$P(r) = \binom{m+r-1}{r} \frac{\prod_{i=0}^{m-1} (p+i\gamma) \prod_{i=0}^{r-1} (q+i\gamma)}{\prod_{i=0}^{m+r-1} (1+i\gamma)} \tag{7}$$

while $m = -a$, $p = (a+b+1)/(a+b-n+1)$, $q = -n/(a+b-n+1)$ $\gamma = 1/(a+b-n+1)$, (Type II. A); or $m = -n$, $p = (a+b+1)/(b+1)$, $q = -a/(b+1)$, $\gamma = 1/(b+1)$ (Type III. A).

Is in addition $b$ rational, following formula is also suitable :

$$P(r) = \binom{m+r-1}{r} \frac{\prod_{i=0}^{m-1} (M+iR) \prod_{i=0}^{r-1} (N-M+iR)}{\prod_{i=0}^{m+r-1} (N+iR)} \tag{8}$$

where $m = -a$, $M = R(a+b+1)$, $N = R(a+b-n+1)$; (Type II. A), or $m = -n$, $M = R(a+b+1)$, $N = R(b+1)$ (Type III. A) ; in both cases $R$ is a negative integer, $-R$ being the denominator of $-b$.

*Type IV.*

Appropriate formulae :

$$P(r) = \frac{(c+r-1)!\,(m+r-1)!\,d!\,(d-c+m)!}{r!\,(d+m+r)!\,(c-1)!\,(m-1)!\,(d-c)!} =$$

$$= \frac{m\binom{c+r-1}{r}\binom{d+c-m}{m}}{(d+m+r)\binom{d+m+r-1}{d}} =$$

$$= \frac{c\binom{m+r-1}{r}\binom{d}{c}}{(d+m+r)\binom{d+m+r-1}{d+m-c}}$$

where $c = -a$, $d = b$, $m = -n$ or $c = -n$, $d = a+b-n$, $m = -a$; further

$$P(r) = \frac{\binom{c+m-1}{m}\binom{d+r-1}{r} m}{\binom{c+d+m+r-1}{m+r}(m+r)}$$

$$= \frac{(d+r-1)!\,(m+r-1)!\,(c+d-1)!\,(c+m-1)!}{r!\,(c+d+m+r-1)!\,(c-1)!\,(d-1)!\,(m-1)!}$$

where $c = a + b + 1$, $d = -n$, $m = -a$, or $c = a + b + 1$, $d = -a$, $m = -n$.

In the case of integral $a$ (or $n$) formula (7) is also appropriate. Are in addition the other two parameters rational, formula (8) is also suitable. In this case $R$ is always a positiv integer.

## 4. §. Model representation

The urn-model mentioned by the authors for the integer case of Type II. A, given by J. O. IRWIN [7], is a special case of the well-known Pólya urn-model. However, Pólya's model is appropriate not only for the integral cases, but for all cases of the Pólya distribution (see e.g. W. FELLER [3], pp. 82—83., p. 128.).

The Pólya urn-model is given below in a form connected with formula (2).

In a set of $n$ successive (dependent) trials the probability of success varies from trial to trial in the following way : In the first trial the probability of success is $a/(a + b)$. If the first $k$ trials resultated in $s$ successes and $k - s$ failures, the (conditional) probability of success in the $(k + 1)$-st trial is $(a - s)/(a + b - k)$. Then the probability of exactly $r$ successes out of $n$ trials is given by formula (2).

The above model illustrates that Pólya's distribution is the generalization of the hypergeometric one in the sense that the parameters $a$ and $b$ may take any real value instead of integers only.

If we modify Pólya's model for the inverse sampling, we get appropriate models for types III. A and IV. in case of integral $n$, for types II. A and IV. in case of integral $a$. The modified model is as follows :

In a set of successive trials, the probability of success varies from trial to trial in the following way. In the first trial the probability of success is $p$. If the first $k$ trials resulted in $s$ successes and $k - s$ failures, the (conditional) probability of success in the $(k + 1)$-st trial is $(p + s\gamma)/(1 + k\gamma)$. The trials are continued until $m$ successes have been obtained ($r$ is now the number of failures). This model leads directly to formula (7).

Are the other parameters rational, the above model can be modified for urn model in a proper sense which leads to formula (8). So we have appropriate models for all cases in which $n$ or $a$ is integral. Are both $n$ and $a$ integral, we have two different models in general.

## 5. §. Other derivations and special cases

C. D. and A. W. KEMP mention two authors (J. G. SKELLAM [18]; J. O. IRWIN [7]) who described some types of their distribution previously. These types belong to the class of Pólya distributions. It follows from the paper of C. D. and A. W. KEMP and from our last § that the main types of the generalized hypergeometric distributions can be derived in three different ways : 1. by extending the formula of the hypergeometric distribution ; 2. by urn models ; 3. by allowing the probability parameter of a binomial (negative binomial) distribution to be a Beta variable.

As it is well-known, the Pólya distribution was introduced in the second way. It is known however, that it can be derived in the third way too

(O. LUNDBERG [14]). Applications of this kind are treated in papers of J. W. HOPKINS [6] and the present author [16] too. I myself ignored at that time that the distribution investigated was of type Pólya. J. W. HOPKINS uses the name "negative hypergeometric distribution" for the distribution adopted from SKELLAM.

CH. JORDAN [10] describes a simple form of the inverse sampling types, namely the case $m = 1$ (see formula (8)).

It seems that the fact that there are three possible derivations of the distribution, is the main cause for its repeated rediscovery. Another cause is that different forms of the law of distribution are possible.

Finally we wish to discuss several problems each of which leads to a special case of the Pólya distribution, namely to the integer case of C. D. and A. W. KEMP II. A (III. A.) type. As mentioned before this type has two different urn-model representations : a direct Pólya sampling model, in which a similar ball is added after each drawing and an inverse sampling model without replacement. It seems to be suitable to restrict the name ,,negative hypergeometric distribution" to this type, as this is the analogon of the negative binomial distribution for sampling without replacement.

The inverse sampling without replacement appears in several statistical problems : e.g. random walk (W. FELLER [2]), waiting time (W. FELLER [3] pp. 35—37.) etc.

A problem of another kind, which leads to this distribution is that of the number of exceedances. (See e.g. E. J. GUMBEL and H. von SCHELLING [5]). It has been shown by the present author [17], that the distribution of the number of exceedances is of this type. Furthermore it is mentioned that the formula of the distribution derived by E. J. GUMBEL any H. von SCHELLING by combining the binomial and Beta distributions can be derived through the Pólya urnmodel too ; and it is shown that Laplace's law of succession (see e.g. J. V. USPENSKY [19]) and the inverse problem (by using Bayes' rule) of sampling without replacement lead to the same distribution.

The moments of Pólya's distribution were treated e.g. by CH. JORDAN [10], M. FRÉCHET [4] and M. A. BRICAS [1], the limiting forms by M. A. BRICAS [1] in detail. I. KOZNIEWSKA [13] determined the first absolute central moment.

## 6. §. The equivalency of the drawings

As well-known, a random variable of binomial probability distribution can be regarded as the sum of independent, equally distributed random variables with the possible values 0 and 1. Similarly, *a random variable of Pólya distribution can be regarded as the sum of the caracteristic random variables of the drawings.* Here however the terms are dependent, but it is known that these characteristic variables have the same apriori distribution. G. PÓLYA [15] showed that they are *equivalent random variables* (see also CH. JORDAN [11], M. FRÉCHET [4]).

The converse of the above theorem does not hold. L. WEISS [20] showed by a counter-example that Pólya's distribution cannot be uniquely derived from the assumption of its variable being the sum of equally distributed random variables on the numbers 0 and 1, having by pairs the same correlation coefficient. Furhermore it follows from his counter-example that even the assumption of equivalency of the terms is insufficient for being the sum Pólya variable.

Here we show the following

**Theorem :** *Every random variable $\xi$ with the possible values* $0, 1, 2, \ldots, n$ *can be written as the sum of n equivalent random variables :*

$$\xi = \zeta_1 + \zeta_2 + \ldots + \zeta_n$$

*while each of the equivalent random variables* $\zeta_1, \zeta_2, \ldots, \zeta_n$ *is distributed on the numbers* 0 *and* 1. *This decomposition of $\xi$ is uniquely determined.*

*Proof:*

It follows from the assumption that if such a decomposition exists, then the probability that $k$ of the variables $\zeta_1, \zeta_2, \ldots, \zeta_n$ in a given order have value 1 and the remaining $n - k$ value 0, is

$$\frac{\mathbf{P}(\xi = k)}{\binom{n}{k}} \tag{9}$$

since there are $\binom{n}{k}$ orders of this kind, each order with the same probability by virtue of equivalency.

The joint distribution of $\zeta_1, \zeta_2, \ldots, \zeta_n$ is uniquely determined by the formula (9). Evidently, any permutation of the variables does not effect the distribution. Thus is follows that $\zeta_1, \zeta_2, \ldots, \zeta_n$ are equivalent random variables.

## 7. §. Completion of the generalization

The generalization given by C. D. and A. W. KEMP is — as mentioned before — incomplete. There are cases in which formula (2) defines a probability distribution, but does not satisfy the restrictions given by C. D. and A. W. KEMP. The classification needs a completion in two directions :

*A)* C. D. and A. W. KEMP consider cases only in which the smallest possible value of the distribution is zero. In addition, it seems reasonable to consider cases in which $n - b$ is a positive integer and formula (5) gives $P(r) > 0$ in the range $n - b \leq r \leq R$ ($R$ positive integer or infinite) and formula (5) sums to unity in this range.

The reason for above assumption is that in each case for which formula (5) gives $P(0) = 0$ but gives positive values for some positive integral values of $r$, $n - b$ must be a positive integer and $P(n - b) \neq 0$ but $P(r) = 0$ if $0 < r < n - b$.

Namely we obtain from (5)

$$\frac{P(0)}{P(r)} = r! \frac{(a-r)!}{a!} \cdot \frac{(b-r)!}{b!} \cdot \frac{(b-n+r)!}{(b-n)!} .$$

The first three factors are always finite and different from 0. Thus $P(0)/P(r) = 0$ if and only if

$$\frac{(b-n+r)!}{(b-n)!} = 0$$

from which follows that $b - n$ must be a negative integer and $r \geqq n - b$.

It will be shown that the probability distribution arising from (5) and having the smallest possible value $n - b$, can be reduced by the transformation

$$(10) \qquad a = b_1\,, \quad b = a_1\,, \quad n = a_1 + b_1 - n_1\,, \quad r = b_1 - n_1 + r_1$$

to those considered above. Namely,

$$P_1(r_1) = P(r) = \frac{\binom{a_1}{r_1}\binom{b_1}{n_1 - r_1}}{\binom{a_1 + b_1}{n_1}}$$

while

$$P_1(0) = P(n-b) > 0\ .$$

The detailed comparison gives that we get new types by the transformation (10) from Types I. A (i), I. A (ii) and IV. of C. D. and A. W. KEMP.

*B)* For Type II. A. the authors exclude the case $b = -1$, similarly for Type III. A. the case $b = n - a - 1$. These exclusions are unjustified. It is true that in the cases mentioned the hypergeometric series are infinite and divergent, but $n$ or $a$ is a positive integer and thus (5) sums to unity in the range $0 \leqq r \leqq n$ resp. $0 \leqq r \leqq a$ which can be proved in the same way as in case of finite series.

An important particular case: If $a = b = -1$, $n$ positive integer, we obtain

$$P(r) = \frac{1}{n+1}$$

$$(r = 0, 1, \ldots, n)$$

that is our random variable is uniformly distributed on the numbers 0, 1, 2, ..., $n$.

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# A HIPERGEOMETRIKUS ELOSZLÁS ÁLTALÁNOSÍTÁSA

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### Kivonat

C. D. and A. W. Kemp [12] a hipergeometrikus eloszlás képletéből (2) kiindulva általánosították az eloszlást arra az esetre, ha a paraméterek nem egész, hanem általában valós számok.

Ebben a cikkben a szerző megmutatja, hogy a C. D. és A. W. Kemp-féle általánosítás a Pólya-féle eloszlást speciális esetenként tartalmazza, továbbá ugyancsak tartalmazza a Pólya-modell inverz (Pascal-féle) megfelelője által származtatott eloszlásokat.

Az említett cikk részletes tárgyalását is több tekintetben kiegészíti, így pl. megmutatja, hogy az indokolatlanul kizár egyes olyan típusokat, amelyek az eredeti feltevésnek megfelelnek.

Megemlít még a szerző a cikkben több ismert vagy az irodalomban tárgyalt eloszlást is, melyekről eddig nem volt ismeretes, hogy Pólya-eloszlások. Ezek : a Laplace-féle következési szabály, továbbá a visszatevéses mintavétel (Bayes-féle) inverz problémájának megoldása, valamint az [5], [6], [7], [16], [18] cikkekben tárgyalt eloszlások.

J. O. Irwin [7] cikke kritikáját (lásd: [20]) kiegészítve, a szerző bebizonyítja, hogy bármely valószínűségi változó, amelynek 0, 1, 2, ..., $n$ a lehetséges értékei, felírható $n$ darab ekvivalens esemény karakterisztikus változónak összegeként.

## ОБОБЩЕННЫЕ ГИПЕРГЕОМЕТРИЧЕСКИЕ РАСПРЕДЕЛЕНИЯ

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### Резюме

C. D. и. A. W. Kemp [12], исходя из формулы гипергеометрического распределения (2), обобщили распределение на тот случай, когда параметры не целые, а, вообще говоря, вещественные числа.

В настоящей статье автор показывает, что это обобщение содержит распределение Pólya как специальный случай, а также содержит распределения, происходящие от обратного (Pascal) аналога модели Pólya.

Дополняется также подробные рассуждения упомянутой статьи, показывается, например, что она необоснованно исключает некоторые типы, которые соответствуют исходным предположениям.

Упоминаются ещё в статье известные или расснотренные в литературе распределения, о которых не было известно, что они являются распределиями Pólya. Это : правило заключения Laplace, решение обратной проблемы выбора образцов с возражением, а также распределения, рассматриваемые в статья [5], [6], [7], [16], [18].

Дополняя критику [20] статьи J. O. Irwin [7], автор доказает, что любая случайная величина, вовможные значения которой являются 0, 1, 2, ... , $n$, может быть представлена в виде суммы характеристических переменных $n$ эквивалентных событий.