ON SECONDARY STOCHASTIC PROCESSES GENERATED BY A MULTIDIMENSIONAL POISSON PROCESS[1)]

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Introduction

In an earlier paper [4] the author deduced some theorems concerning secondary stochastic processes generated by a one-dimensional Poisson process. In the present paper a more general case will be investigated. We suppose that the underlying process is a homogeneous Poisson process defined on an m-dimensional space. We shall establish theorems which are generalizations of the theorems formulated in [4]. The proofs are based on the method of [4].

§. 1. Homogeneous Poisson process defined on a Euclidean space of finite dimension

Let us consider the field $\mathfrak{S}$ of all Borel-measurable sets S of a Euclidean space of finite dimension. Denote by $\mu(S)$ the Lebesgue measure defined on the sets $S \in \mathfrak{S}$. For each set S, with $\mu(S) < \infty$, let there be defined a random variable $\xi(S)$ with the following properties:

1° $\xi(S)$ *assumes only non-negative integer values and* $\mathbf{P}\{\xi(S)=0\} \neq 1$ if $\mu(S) > 0$.

2° The probability distribution of $\xi(S)$ *depends only on the measure* $\mu(S)$.

3° If S_1 *and* S_2 *are disjoint sets, then* $\xi(S_1)$ *and* $\xi(S_2)$ *are independent random variables and we have* $\xi(S_1+S_2)=\xi(S_1)+\xi(S_2)$.

4° $$\lim_{\mu(S)\to 0} \frac{\mathbf{P}\{\xi(S)\geq 1\}}{\mathbf{P}\{\xi(S)=1\}} = 1 .$$

Another definition of the multidimensional Poisson process has been given e.g. by C. Ryll-Nardzewski [3].

Theorem 1.: *Under the assumptions 1°—4° we have*

$$\mathbf{P}\{\xi(S)=k\} = e^{-p\mu(S)} \frac{[p\mu(S)]^k}{k!} \tag{1}$$

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1) This is an address delivered at the Colloquium on Stochastic Processes, Balatonvilágos, September 13—15, 1956.

for all $S \in \mathfrak{S}$, with $\mu(s) < \infty$, where p is a positive constant.

Proof: Consider a decomposition of S:

$$S = S_n^{(1)} + S_n^{(2)} + \ldots + S_n^{(n)}$$

where $S_n^{(i)}$ $(i = 1, 2, \ldots, n)$ are disjoint sets and $\mu(S_n^{(i)}) = \mu(S)/n$. Let S_n be one of the sets $S_n^{(i)}$. Now, by 3° we have

$$\mathbf{P}\{\xi(S) = 0\} = [\mathbf{P}\{\xi(S_n) = 0\}]^n \tag{2}$$

without supposing the mutual independence of the random variables $\xi(S_n^{(i)})$ $(i = 1, 2, \ldots, n)$. As $\mathbf{P}\{\xi(S_n) = 0\} = 1 - \mathbf{P}\{\xi(S_n) = 1\} - \mathbf{P}\{\xi(S_n) > 1\}$ and $\mu(S_n) = \mu(S)/n$, taking 4° into consideration, we obtain that

$$\lim_{n\to\infty} n\,\mathbf{P}\{\xi(S_n) = 1\} = -\log \mathbf{P}\{\xi(S) = 0\}\ . \tag{3}$$

This limit cannot be infinite. For $\mathbf{P}\{\xi(S) = 0\} = 0$ would imply $\mathbf{P}\{\xi(S) = 0\} = 0$ for all sets S. Consequently also it would follow that $\mathbf{P}\{\xi(S) = k\} = 0$ for all sets S and for all k, which is impossible. The limit cannot be equal to 0, for $\mathbf{P}\{\xi(S) = 0\} = 1$ would imply the same relation for all sets S. But the case $\mathbf{P}\{\xi(S) = 0\} = 1$ is excluded.

Using the condition 3°, we have

$$\mathbf{M}\{e^{it\xi(S)}\} = [\mathbf{M}\{e^{it\xi(S_n)}\}]^n\ . \tag{4}$$

Clearly, we have

$$\mathbf{M}\{e^{it\xi(S_n)}\} = \mathbf{P}\{\xi(S_n) = 0\} + \mathbf{P}\{\xi(S_n) = 1\}\,e^{it} + \mathbf{P}\{\xi(S_n) > 1\}\,\vartheta$$

where $|\vartheta| \leqq 1$. Now putting $\mathbf{P}\{\xi(S_n) = 0\} = 1 - \mathbf{P}\{\xi(S_n) = 1\} - \mathbf{P}\{\xi(S_n) > 1\}$, from (4) it results that

$$\mathbf{M}\{e^{it\xi(S)}\} = \exp\left\{(e^{it} - 1)\lim_{n\to\infty} n\,\mathbf{P}\{\xi(S_n) = 1\}\right\} \tag{5}$$

or by virtue of (3),

$$\mathbf{M}\{e^{it\xi(S)}\} = \left(\mathbf{P}\{\xi(S) = 0\}\right)^{(1-e^{it})}\ . \tag{6}$$

Consequently $\xi(S)$ has a Poisson distribution. The expectation $\mathbf{M}\{\xi(S)\}$ exists and by (6) we have

$$\mathbf{M}\{\xi(S)\} = -\log \mathbf{P}\{\xi(S) = 0\}\ . \tag{7}$$

The expectation $\mathbf{M}\{\xi(S)\}$ is a non-negative additive set function, which depends only on $\mu(S)$. Consequently $\mathbf{M}\{\xi(S)\} = p\mu(S)$ with a positive p. The cases $p = 0$ and $p = \infty$ are excluded. Finally

$$\mathbf{M}\{e^{it\xi(S)}\} = e^{-p\mu(S)(1-e^{it})}\ . \tag{8}$$

which proves (1).

In the following we shall call a set of random variables $\{\xi(S)\}$ which satisfies 1°—4° *a homogenous Poisson process.*

Remark 1. If $n > 2$ and $S_1, S_2, \ldots, S_n$ are disjoint sets, then $\xi(S_1)$, $\xi(S_2), \ldots, \xi(S_n)$ are not necessarily mutually independent random variables. However, it is easy to construct a set of random variables $\{\xi(S)\}$ which satisfies beside 1°—4° also the following condition :

5° If for an arbitrary n, $S_1, S_2, \ldots, S_n$ are disjoint sets then the random variables $\xi(S_1), \xi(S_2), \ldots \xi(S_n)$ are mutually independent.

The stochastic process $\{\xi(S)\}$ can be interpreted as follows : Let us consider random points (random events) distributed in the space. Denote by $\xi(S)$ the number of the random points or random events taking place in the set S. For a realization $\{\xi(S)\}$ a point P is one of the random points if $\lim_{S\to P} \xi(S) \geqq 1$ in such a way that $P \in S$, where S is an open set.

We prove two lemmas :

Lemma 1. *Let us consider the Poisson process $\{\xi(S)\}$ fulfilling 1°—4°. Let $\mu(S) > 0$. Under the condition $\xi(S) = 1$ the random point in S is distributed uniformly in S.*

Proof: Let $S = S_1 + S_2$, where S_1 and S_2 are disjoint sets. Then we have

$$\mathbf{P}\{\xi(S_1) = 1 \mid \xi(S) = 1\} = \frac{\mathbf{P}\{\xi(S_1) = 1, \xi(S_2) = 0\}}{\mathbf{P}\{\xi(S) = 1\}} =$$

$$= \frac{\mathbf{P}\{\xi(S_1) = 1\}\,\mathbf{P}\{\xi(S_2) = 0\}}{\mathbf{P}\{\xi(S) = 1\}} = \frac{\mu(S_1)}{\mu(S)},$$

as was to be proved.

Lemma 2. *Let us consider a Poisson process $\{\xi(S)\}$ fulfilling 1°—5°. Let $\mu(S) > 0$. Under the condition $\xi(S) = k$, the k random points in S are distributed independently and uniformly in S.*

Proof: For an arbitrary n, let $S = S_1 + S_2 + \ldots + S_n$ where $S_1, S_2, \ldots, S_n$ are any disjoint sets and $k = k_1 + k_2 + \ldots + k_n$ where $k_1, k_2, \ldots, k_n$ are any non-negative integers. Then we have

$$\mathbf{P}\{\xi(S_1) = k_1, \xi(S_2) = k_2, \ldots, \xi(S_n) = k_n \mid \xi(S) = k\} =$$

$$= \frac{\mathbf{P}\{\xi(S_1) = k_1, \xi(S_2) = k_2, \ldots, \xi(S_n) = k_n\}}{\mathbf{P}\{\xi(S) = k\}} =$$

$$= \frac{\mathbf{P}\{\xi(S_1) = k_1\} \cdot \mathbf{P}\{\xi(S_2) = k_2\} \ldots \mathbf{P}\{\xi(S_n) = k_n\}}{\mathbf{P}\{\xi(S) = k\}} =$$

$$= \frac{k!}{k_1!\, k_2! \ldots k_n!} \left(\frac{\mu(S_1)}{\mu(S)}\right)^{k_1} \left(\frac{\mu(S_2)}{\mu(S)}\right)^{k_2} \cdots \left(\frac{\mu(S_n)}{\mu(S)}\right)^{k_n}.$$

This completes the proof.

Remark 2. If we assume more generally that $\mu(S)$ is any non-atomic measure other than the Lebesgue one, then similar theorems are valid as above. In this case $\{\xi(S)\}$ is called a *non-homogeneous Poisson process.*

§. 2. Secondary stochastic processes generated by a Poisson process

Let us consider a homogeneous Poisson process $\{\xi(S)\}$ defined on the Euclidean space of m dimensions. For the sake of brevity we introduce the vector notation $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$, $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ etc., for denoting the points of the space. Let us suppose that every event in the Poisson process gives rise to a signal depending on a random parameter. If $\boldsymbol{y}$ is the point representing an event and a is the value of the corresponding parameter, then denote the magnitude of this signal at the point $\boldsymbol{x}$ by $f(\boldsymbol{x},\boldsymbol{y},a)$. Suppose that the parameters belonging to different events are mutually independent random variables with a common distribution function $H(a)$ and further that the different signals linearly superpose. In the following we suppose that $f(\boldsymbol{x},\boldsymbol{y},a)$ is a Baire function.

Let us consider the random variable

$$\eta(\boldsymbol{x}\,;S) = \sum_{\boldsymbol{y}_\nu \in S} f(\boldsymbol{x}, \boldsymbol{y}_\nu, \alpha_\nu) \tag{9}$$

which represents at the point $\boldsymbol{x}$ the sum of the signals arising from the random events occurring in the set S. Here $\boldsymbol{y}_\nu$ denote the different random points and α_ν the random parameters. In the case when $S = \boldsymbol{R}_m$ (the whole space), let us write $\eta(\boldsymbol{x})$ instead of $\eta(\boldsymbol{x}_m;\, \boldsymbol{R}_m)$. These sums do not necessarily converge. If they do, we say that the process $\eta(\boldsymbol{x}\,;S)$ resp. $\eta(\boldsymbol{x})$ exists. If $\mu(S) < \infty$ then the process $\eta(\boldsymbol{x}\,;S)$ exists with probability 1. If $S = \boldsymbol{R}_m$ we obtain the following

Theorem 2. *If for all $\boldsymbol{x}$ we have*

$$\int_{\boldsymbol{R}_m} \left[\int_{-\infty}^{\infty} |f(\boldsymbol{x},\boldsymbol{y},a)|\, dH(a) \right] d\boldsymbol{y} < \infty \tag{10}$$

then the process $\{\eta(\boldsymbol{x})\}$ exists with probability 1.

Proof: Let us decompose the space $\boldsymbol{R}_m$ as the countable union of disjoint sets with finite measures: $\boldsymbol{R}_m = S_1 + S_2 + \ldots + S_n + \ldots$. Then we have

$$\eta(\boldsymbol{x}) = \sum_{n=1}^{\infty} \eta(\boldsymbol{x}\,;S_n)\ . \tag{11}$$

As

$$\mathbf{M}\{|\eta(\boldsymbol{x}\,;S_n)|\} \leq p \int_{S_n} \left[\int_{-\infty}^{\infty} |f(\boldsymbol{x},\boldsymbol{y},a)|\, dH(a) \right] d\boldsymbol{y}$$

and

$$\sum_{n=1}^{\infty} \mathbf{M}\{|\eta(\boldsymbol{x}\,;S_n)|\} \leq p \int_{\boldsymbol{R}_m} \left[\int_{-\infty}^{\infty} |f(\boldsymbol{x},\boldsymbol{y},a)|\, dH(a) \right] d\boldsymbol{y} < \infty\ ,$$

it follows from the known theorem of Beppo Levi or from the known inequality of Markov concerning non-negative random variables that the process $\{\eta(\boldsymbol{x})\}$ exists with probability 1. Evidently, $\eta(\boldsymbol{x})$ is independent of the partition of $\boldsymbol{R}_m$.

Theorem 3. *Let $\mu(S)$ be finite. The characteristic function of the random variable $\eta(\boldsymbol{x}; S)$ has the following form:*

$$\mathsf{M}\{e^{it\eta(\boldsymbol{x};S)}\} = \exp\left\{p \int_S \left[\int_{-\infty}^{\infty} e^{itf(\boldsymbol{x},\boldsymbol{y},a)}\, dH(a) - 1\right] d\boldsymbol{y}\right\}. \tag{12}$$

Proof: Let be $S = S_1 + S_2 + \ldots + S_n$ where $S_1, S_2, \ldots, S_n$ are disjoint sets with the same measure. According to the condition 3° concerning $\{\xi(S)\}$ we can write

$$\mathsf{M}\{e^{it\eta(\boldsymbol{x};S)}\} = \prod_{k=1}^{n} \mathsf{M}\{e^{it\eta(\boldsymbol{x};S_k)}\} \tag{13}$$

without supposing the mutual independence of the random variables $\eta(\boldsymbol{x}; S_k)$ $(k = 1, 2, \ldots, n)$. By the theorem of total expectation and Lemma 1 we obtain

$$\mathsf{M}\{e^{it\eta(\boldsymbol{x};S_k)}\} = \sum_{j=0}^{\infty} \mathsf{P}\{\xi(S_k) = j\}\, \mathsf{M}\{e^{it\eta(\boldsymbol{x};S_k)} \mid \xi(S_k) = j\} =$$

$$= \mathsf{P}\{\xi(S_k) = 0\} + \mathsf{P}\{\xi(S_k) = 1\} \frac{1}{\mu(S_k)} \int_{S_k} \left[\int_{-\infty}^{\infty} e^{itf(\boldsymbol{x},\boldsymbol{y},a)}\, dH(a)\right] d\boldsymbol{y} + \mathsf{P}\{\xi(S_k) > 1\}\, \vartheta$$

where $|\vartheta| \leqq 1$. Taking into consideration that $\mathsf{P}\{\xi(S_k) = 0\} = 1 - \mathsf{P}\{\xi(S_k) = 1\} - \mathsf{P}\{\xi(S_k) > 1\}$ and $\mu(S_k) = \mu(S)/n$ we obtain by taking logarithms

$$\log \mathsf{M}\{e^{it\eta(\boldsymbol{x};S_k)}\} = p \int_{S_k} \left[\int_{-\infty}^{\infty} e^{itf(\boldsymbol{x},\boldsymbol{y},a)}\, dH(a) - 1\right] d\boldsymbol{y} + C_k \left[\frac{p\,\mu(S)}{n}\right]^2,$$

where $|C_k| < 4$. Letting $n \to \infty$ we obtain by (13)

$$\log \mathsf{M}\{e^{it\eta(\boldsymbol{x};S)}\} = p \int_S \left[\int_{-\infty}^{\infty} e^{itf(\boldsymbol{x},\boldsymbol{y},a)}\, dH(a) - 1\right] d\boldsymbol{y}$$

which proves (12).

Remark 3. Assuming also 5° for the process $\{\xi(S)\}$ we may prove Theorem 3 in a simpler way using Lemma 2. For by the theorem of total expectation we get

$$\mathsf{M}\{e^{it\eta(\boldsymbol{x};S)}\} = \sum_{j=0}^{\infty} \mathsf{P}\{\xi(S) = j\}\, \mathsf{M}\{e^{it\eta(\boldsymbol{x};S)} \mid \xi(S) = j\}$$

and by Lemma 2

$$\mathsf{M}\{e^{it\eta(\boldsymbol{x};S)} \mid \xi(S) = j\} = [\mathsf{M}\{e^{it\eta(\boldsymbol{x};S)} \mid \xi(S) = 1\}]^j$$

where

$$\mathbf{M}\{e^{it\eta(x;S)}|\xi(S)=1\}=\frac{1}{\mu(S)}\int_S\left[\int_{-\infty}^{\infty}e^{itf(x,y,a)}\,dH(a)\right]dy\ .$$

Carrying out the corresponding substitutions we get (12), what was to be proved.

Now we shall prove the following limit theorem:

Theorem 4. *Let us suppose that* $\mathbf{D}\{\eta(x;S)\}$ *exists; then we have*

$$\lim_{p\to\infty}\mathbf{P}\left\{\frac{\eta(x;S)-\mathbf{M}\{\eta(x;S)\}}{\mathbf{D}\{\eta(x;S)\}}\le x\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du\ . \tag{14}$$

Proof: By Lemma 2

$$\eta(x;S)=\sum_{y_\nu\in S}f(x,y_\nu,a_\nu)$$

can be considered as a sum of a random number of identically distributed independent random variables, where the number of the variables is independent of the variables themselves. The number of the variables follows a Poisson distribution with mean $p\mu(S)$. If $p\to\infty$ then we obtain the limiting distribution (14) by the theorem stated by H. Robbins [2] (Cf. R. L. Dobrushin [1]).

Remark 4. Theorem 3 may be easily proved also for the case when the underlying process is a non-homogeneous Poisson process. In this case we have

$$\mathbf{M}\{e^{it\eta(x;S)}\}=\exp\left\{p\int_S\left[\int_{-\infty}^{\infty}e^{itf(x,y,a)}\,dH(a)-1\right]\mu(dy)\right.\ .$$

Theorem 5. *If for all x we have (10), then*

$$\mathbf{M}\{e^{it\eta(x)}\}=\exp\left\{p\int_{R_m}\left[\int_{-\infty}^{\infty}e^{itf(x,y,a)}\,dH(a)-1\right]dy\right\}\ . \tag{15}$$

Proof: (15) follows from the existence of the random variable $\eta(x)$. However, we may prove it directly by the known theorem of P. Lévy and H. Cramér concerning the convergence of a sequence of characteristic functions.

Remark 5. Let us denote the s-th semi-invariant of $\eta(x)$ by $\Lambda_s\{\eta(x)\}$ $(s=1,2,\ldots)$. By (15) we get

$$\Lambda_s\{\eta(x)\}=\frac{1}{i^s}\left(\frac{d^s\log\mathbf{M}\{e^{it\eta(x)}\}}{dt^s}\right)_{t=0}=p\int_{R_m}\left[\int_{-\infty}^{\infty}(f(x,y,a))^s\,dH(a)\right]dy \tag{16}$$

if this exists at all. By (16) we obtained a new generalization of the formulae of N. CAMPBELL well-known in physics. Especially we have

$$\mathbf{M}\{\eta(\boldsymbol{x})\} = \Lambda_1\{\eta(\boldsymbol{x})\} \text{ and } \mathbf{D}^2\{\eta(\boldsymbol{x})\} = \Lambda_2\{\eta(\boldsymbol{x})\}.$$

Remark 6. Let us suppose that $f(\boldsymbol{x}, \boldsymbol{y}, a)$ depends only on the difference $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$. In this case let us put $f(\boldsymbol{x}, \boldsymbol{y}, a) = g(\boldsymbol{r}, a)$. Then the distribution function of $\eta(\boldsymbol{x})$ is independent of $\boldsymbol{x}$ and its characteristic function is

$$\mathbf{M}\{e^{it\eta(\boldsymbol{x})}\} = \exp\left\{p \int_{R_m} \left[\int_{-\infty}^{\infty} e^{itg(\boldsymbol{r}, a)}\, dH(a) - 1 \right] d\boldsymbol{r} \right\}. \tag{17}$$

In such cases we shall call the process $\{\eta(\boldsymbol{x})\}$ a *homogeneous* one.

If further, $g(\boldsymbol{r}, a)$ depends only on $r = |\boldsymbol{r}|$, then let us put $g(\boldsymbol{r}, a) = h(r, a)$. In this case the characteristic function of $\eta(\boldsymbol{x})$ is

$$\mathbf{M}\{e^{it\eta(\boldsymbol{x})}\} = \exp\left\{ \frac{p\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} \int_0^{\infty} r^{m-1} \left[\int_{-\infty}^{\infty} e^{ith(r, a)}\, dH(a) - 1 \right] dr \right\}. \tag{18}$$

We shall say in this case that the process $\eta(\boldsymbol{x})$ is a *homogeneous* and *isotropic* one.

Theorem 6. *If $\{\eta(\boldsymbol{x})\}$ is a homogeneous process and $\mathbf{M}\{(\eta(\boldsymbol{x}))^2\} < \infty$, then the correlation function $R(\boldsymbol{r}) = \boldsymbol{R}\{\eta(\boldsymbol{x}), \eta(\boldsymbol{x} + \boldsymbol{r})\}$ exists and is independent of $\boldsymbol{x}$. We have*

$$R(\boldsymbol{r}) = \frac{\int_{R_m} \left[\int_{-\infty}^{\infty} g(\boldsymbol{y}, a)\, g(\boldsymbol{y} + \boldsymbol{r}, a)\, dH(a) \right] d\boldsymbol{y}}{\int_{R_m} \left[\int_{-\infty}^{\infty} (g(\boldsymbol{y}, a))^2\, dH(a) \right] d\boldsymbol{y}}. \tag{19}$$

Proof: Let $\boldsymbol{r}$ be fixed and $\eta^*(\boldsymbol{x}) = \eta(\boldsymbol{x}) + \eta(\boldsymbol{x} + \boldsymbol{r})$. Then $\{\eta^*(\boldsymbol{x})\}$ is also a homogeneous process and $\mathbf{M}\{(\eta^*(\boldsymbol{x}))^2\} < \infty$. $\{\eta^*(\boldsymbol{x})\}$ differs from the process $\{\eta(\boldsymbol{x})\}$ merely by taking the signal $g^*(\boldsymbol{y}, a) = g(\boldsymbol{y}, a) + g(\boldsymbol{y} + \boldsymbol{r}, a)$ instead of $g(\boldsymbol{y}, a)$. Then by (16) we have

$$\mathbf{D}^2\{\eta^*(\boldsymbol{x})\} = \int_{R_m} \left[\int_{-\infty}^{\infty} (g^*(\boldsymbol{y}, a))^2\, dH(a) \right] d\boldsymbol{y}$$

i. e.

$$\mathbf{D}^2\{\eta(\boldsymbol{x}) + \eta(\boldsymbol{x} + \boldsymbol{r})\} = \int_{R_m} \left[\int_{-\infty}^{\infty} (g(\boldsymbol{y}, a) + g(\boldsymbol{y} + \boldsymbol{r}, a))^2\, dH(a) \right] d\boldsymbol{y}.$$

On the other hand, evidently

$$\mathbf{D}^2\{\eta(\boldsymbol{x}) + \eta(\boldsymbol{x} + \boldsymbol{r})\} = 2\, \mathbf{D}^2\{\eta(\boldsymbol{x})\}\,[1 + R(\boldsymbol{r})].$$

Comparing the latter two formulae we get $R(\boldsymbol{r})$.

Remark 7. If $\{\eta(\boldsymbol{x})\}$ is a homogeneous and isotropic process, then the correlation function $R(\boldsymbol{r})$ depends only on $r = |\boldsymbol{r}|$ and in this case let us put $R(r) = R(\boldsymbol{r})$ for which we have

$$(20)\quad R(r) = \frac{\dfrac{\pi^{\frac{m-1}{2}}}{\Gamma\left(\dfrac{m-1}{2}\right)}\displaystyle\int_0^\infty y^{m-1}\left\{\int_0^\pi\left[\int_{-\infty}^\infty h(y,a)\,h(\sqrt{r^2+y^2-2ry\cos\varphi},a)\,dH(a)\right]|\cos\varphi|^{m-2}d\varphi\right\}dy}{\dfrac{\pi^{\frac{m}{2}}}{\Gamma\left(\dfrac{m}{2}\right)}\displaystyle\int_0^\infty y^{m-1}\left[\int_{-\infty}^\infty (h(y,a))^2\,dH(a)\right]dy}\,.$$

The spectral function of the process $\{\eta(\boldsymbol{x})\}$. As the correlation function of a stationary stochastic process can be expressed by the known formula of A. J. Khintchine, similarly the correlation function of $\{\eta(\boldsymbol{x})\}$ may be expressed as follows:

$$(21)\qquad R(\boldsymbol{r}) = \int_{R_m} e^{i\boldsymbol{\lambda}\boldsymbol{r}}\,dF(\boldsymbol{\lambda})$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$, $\boldsymbol{\lambda}\boldsymbol{r} = \lambda_1 r_1 + \lambda_2 r_2 + \ldots + \lambda_m r_m$ and $F(\boldsymbol{\lambda}) = F(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a distribution function of m dimensions.

Theorem 7. *If*

$$(22)\qquad \Gamma(\boldsymbol{\lambda}, a) = \left(\frac{1}{2\pi}\right)^{\frac{m}{2}}\int_{R_m} e^{-i\boldsymbol{\lambda}\boldsymbol{x}}\,g(\boldsymbol{x}, a)\,d\boldsymbol{x}\,,$$

the Fourier transform of $g(\boldsymbol{x}, a)$ *exists and* $|\Gamma(\boldsymbol{\lambda}, a)|^2$ *is Stieltjes-integrable with respect to* $H(a)$ *then* $F(\boldsymbol{\lambda})$ *has a density function* $f(\boldsymbol{\lambda})$ *and we have*

$$(23)\qquad f(\boldsymbol{\lambda}) = \frac{\displaystyle\int_{-\infty}^\infty |\Gamma(\boldsymbol{\lambda}, a)|^2\,dH(a)}{\displaystyle\int_0^\infty\left[\int_{-\infty}^\infty (g(\boldsymbol{y}, a))^2\,dH(a)\right]d\boldsymbol{y}}\,.$$

Proof: $f(\boldsymbol{\lambda})$ may be determined from (21) by Fourier inversion:

$$f(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^m}\int_{R_m} e^{-i\boldsymbol{\lambda}\boldsymbol{r}}\,R(\boldsymbol{r})\,d\boldsymbol{r}\,.$$

Remark 8. If $\eta(x)$ is a homogeneous and isotropic process, then

$$R(r) = 2^{\frac{m-2}{2}} \Gamma\left(\frac{m}{2}\right) \int_0^\infty \frac{J_{\frac{m-2}{2}}(\lambda r)}{(\lambda r)^{\frac{m-2}{2}}} dF(\lambda) \tag{24}$$

where $F(\lambda)$ is a distribution function of a non-negative random variable and $J_{\frac{m-2}{2}}(z)$ is the Bessel function of order $(m-2)/2$. If

$$\Gamma(\lambda, a) = \int_0^\infty \frac{r^{\frac{m}{2}} J_{\frac{m-2}{2}}(\lambda r)}{\lambda^{\frac{m-2}{2}}} h(r, a)\, dr$$

exists and $|\Gamma(\lambda, a)|^2$ is integrable with respect to $H(a)$, then the density function $f(\lambda) = F'(\lambda)$ exists and we have

$$f(\lambda) = \frac{\lambda \int\limits_{-\infty}^{\infty} \left[\int\limits_0^\infty r^{\frac{m}{2}} J_{\frac{m-2}{2}}(\lambda r) h(r, a)\, dr\right]^2 dH(a)}{\int\limits_0^\infty r^{m-1}\left[\int\limits_{-\infty}^{\infty} (h(r, a))^2\, dH(a)\right] dr}. \tag{25}$$

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TÖBBDIMENZIÓS POISSON-FOLYAMAT ÁLTAL SZÁRMAZTATOTT MÁSODLAGOS FOLYAMATOKRÓL

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Kivonat

Egy véges dimenziójú euklideszi tér Borel-féle S részhalmazain legyen értelmezve egy $\{\xi(S)\}$ homogén Poisson folyamat, amelyre

$$\mathbf{P}\{\xi(S) = k\} = e^{-p\mu(S)} \frac{[p\,\mu(S)]^k}{k!},$$

ahol $\mu(S)$ az S halmaz Lebesgue mértéke és p pozitív állandó.

A Poisson folyamat eseményei előfordulási pontjainak sokaságát jelölje $\{\boldsymbol{y}_\nu\}$. Tegyük fel, hogy a Poisson folyamat minden egyes eseménye létrehoz egy jelet. Jelölje az $\boldsymbol{y}_\nu$ ponthoz tartozó jel nagyságát $\boldsymbol{x}$ pontban $f(\boldsymbol{x}, \boldsymbol{y}_\nu, \alpha_\nu)$, ahol α_ν egy véletlen paraméter. Feltesszük, hogy az $\{\alpha_\nu\}$ paraméterek egyforma eloszlású független valószínűségi változók. A szerző az

$$\eta(\boldsymbol{x}; S) = \sum_{\boldsymbol{y}_\nu \in S} f(\boldsymbol{x}, \boldsymbol{y}_\nu, \alpha_\nu) \tag{1}$$

sztochasztikus folyamat vizsgálatával foglalkozik. Meghatározza az $\eta(\boldsymbol{x}; S)$ változó eloszlását és az $\{\eta(\boldsymbol{x}; S)\}$ folyamat korrelációs függvényét és spektrális eloszlását, midőn S az egész tér.

О ПРОЦЕССАХ ПОРОЖДЕННЫХ МНОГОМЕРНЫМ ПРОЦЕССАМ POISSON-A

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Резюме

Пусть $\{\xi(S)\}$ однородный процесс Poisson-a, определенный на Borel-евских подмножествах S некоторого конечномерного эвклидового пространства, и пусть

$$\mathbf{P}\{\xi(S) = k\} = e^{-p\mu(S)} \frac{[p\mu(S)]^k}{k!}$$

где $\mu(S)$ Lebesgue-овская мера множества S и p положительное число.

Пусть $\{\boldsymbol{y}_\nu\}$ означает множество точек *нахождения* событий процесса Poisson.

Предположим, что всякое событие процесса Poisson-a создаёт сигнал.

Пусть $f(\boldsymbol{x}, \boldsymbol{y}_\nu, \alpha_\nu)$ означает в точке $\boldsymbol{x}$ величину сигнала, принадлежащего точке $\boldsymbol{y}_\nu$, где α_ν случайный параметр. Будем предполагать, что параметры $\{\alpha_\nu\}$ независимые, одинаково распределённые случайные величины. Автор занимается исследованием стохастического процесса

$$\eta(\boldsymbol{x}; S) = \sum_{\boldsymbol{y}_\nu \in S} f(\boldsymbol{x}, \boldsymbol{y}_\nu, \alpha_\nu).$$

Определены распределение случайной величины $\eta(\boldsymbol{x}; S)$, корреляционная и спектральная функции процесса $\{\eta(\boldsymbol{x}; S)\}$, когда S является польным пространством.